Oligopoly Competition in Fake Reviews

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Abstract

How do fake reviews alter oligopoly market outcomes? We model fake reviews in strategic quantity and price competitions, in which a firm writes fake reviews. Each firm has private information of its product quality, which consumers need to infer before purchasing. The consumers observe a noisy review rating for each firm, which is the combination of authentic and fake reviews, thus the subject of strategic manipulation. A firm's fake review writing action, though costly, could inflate the rating of its product, raise consumers' willingness to pay, and upwardly shift the firm's demand function. Given the inflated demand functions, firms then engage in static quantity or price competition. By focusing on a linear strategy with private information, we establish a monotone equilibrium fake review strategy. When consumers rationally conjecture firms' costly fake review generations, expected prices and quantities are fake-review proof, while other outcomes are altered. Counter-intuitively, expected consumer surplus increases with fake reviews due to their signaling role. When some portions of consumers naively believe observed ratings are genuine, strategic substitutability emerges among firms' fake review actions, and equilibrium oligopoly market outcomes are distorted by fake reviews.

Key words: Fake Review, Oligopoly Market, Rating, Signal Jamming

JEL codes: D83, L13, L15, L51, L86

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1 Introduction

Despite considerable public concerns about fake reviews, there are few theoretical studies modeling review manipulation with an oligopolistic market structure. The pioneering and empirical study is Mayzlin et al. (2014) , which reports a stylized Hotelling competition fake review model in its Appendix. In their appendix, the authors assume that no economic agents (i.e., neither of the two firms at both ends of the Hotelling space, nor the consumers) know the true product quality until the end of the game (thus, information is incomplete but symmetric among all agents), leaving room for further model development. In our study, we propose asymmetric information regarding product quality, both between firms and consumers, extending the existing literature.

Despite the large social concerns over manipulated reviews, notably among online ratings, which have relatively lower manipulation costs, the literature so far has provided quite restricted models for manipulated (i.e., fake) reviews. Specifically, the number of economic articles that model strategic fake reviews is scarce, and this may hinder the regulatory discussions for anti-competitive policies. By proposing a manipulated review model under the classical Cournot and Bertrand competition, this article aims to contribute to the literature and assist in policy discussions. Following the tradition of Industrial Organization, we apply the classical Cournot and Bertrand oligopolistic models to manipulated reviews. Notably, for each firm, we introduce the asymmetric information of producer type (i.e., product quality for each firm). The novel modeling contribution is that, although our revenue functions are quadratic in terms of effort to generate fake reviews, the derived fake-review-writing strategy is linear with respect to the firm's private information type; thus a linear strategy is established.

The empirical literature suggests that the low-quality seller provides more fake reviews. By contrast, in this paper, the incentive to provide fake reviews is positively correlated with the seller's quality. This result is robust to the competitive structure in the second stage (price/quantity competition) and the number of firms. This implies that the incentive for low-quality seller to provide fake reviews would come from its cost structure or future incentive to shirk consumers.

Focusing on book reviews and book sales on Amazon.com, Reimers and Waldfogel (2021) report that crowd ratings on the platform have a ten times larger impact on consumer welfare, compared to those provided by professional reviewers. As online crowd ratings are relatively cheap to manipulate (which is frequently done through anonymous accounts) compared to the cost of manipulating professional reviews, Reimers and Waldfogel (2021) findings further raise concerns about the role of fake reviews under differentiated-product oligopoly competition.

The topic of fake reviews is intrinsically based on differentiated products: A firm engages in a fake review activity in order to make its product look relatively more attractive than that of rival firms. Accordingly, we model differentiated product oligopolies, in which buyers substitute among products.[1]

In our analysis of the fake-review writing strategies among firms, the key equilibrium property is the unresponsive best response to other firms' fake-review writing strategies. This key property stems from the rational representative consumer, who observes a rating for each product, conjectures true product quality of each project, and makes consumption choices. In equilibrium, given the nature of costly fake review writing activity of firm i, the rational representative consumer downwardly assesses the observed rating (of firm i's product) and evaluates the expected product quality. Based on this rational conjecturing process, at the simultaneous fake review writing action stage, firm i chooses its fake review writing action, anticipating such consumer's downwardly assessing evaluation processes for other firms' products, and choosing its fake review action unrelated to other firms' fake review actions. In other words, firm i engages in a fake-review writing competition only against an average type of competing firms, founded on representative consumer's rational conjecturing processes. This unresponsive best response equilibrium property results in fake review partial equivalence, which is the main finding of this study.

In the latter half of this article, we explore the behavioral analyses of fake reviews with a representative partially naive consumer. Our motivation is as follows: A consumer may naively believe that observed (and publicly displayed) ratings are genuine and does not fully perceive the existence of fake reviews. If some portion of the market consists of such a naive consumer, a firm could exploit fake reviews to extract more profit. Particularly, the effect of fake reviews on a representative naive consumer's potential welfare loss would be of interest to market authorities and policymakers. In addition, to the best of our knowledge, welfare analyses of the fake reviews in a varying-demand oligopoly market with a representative naive consumer have not been investigated in the literature. To answer these behavioral questions, we extend our benchmark model in the following two steps.

First, focusing on a consumer's conjecturing process (guessing true product quality from an observed product rating), we parsimoniously extend our benchmark model of a representative rational consumer to that of a representative partially naive consumer. In our behavioral model, a naive consumer cannot comprehend firms' fake-review writing activities, and they naively believe that an observed rating genuinely comes from the true product quality distribution without strategic manipulation. One notable advantage in our defini-

[1]Nevertheless, we also investigate the case where products become close to homogeneous.

tion of a behavioral consumer is its nesting property: The proposed representative naive consumer nests that of the fully rational consumer as an extreme case, as well as nesting the fully naive consumer as another extreme case, enabling researchers to pursue comparative statics.

Second, given the existence of a representative partially naive consumer, we reconsider a firm's optimal fake-review writing strategies. Under the behavioral model setting, fake reviews have a new role: an exploitation role to extract surplus from a representative partially naive consumer. As a naive consumer tends to be attracted by a product with inflated rating score with fake reviews, strategic substitutability across fake review actions emerges.

The remaining sections are organized as follows: Section 2 reviews the literature related to fake reviews. Section 3 explains our main fake review models under Cournot quantity and Bertrand price competition, respectively. Section 4 reports the results, such as equilibrium properties and welfare analyses. Section 5 extends the model to an n-firm oligopoly. Section 6 summarizes and concludes the study.

2 Literature Review

This article is related to the learning literature. Specifically, our linear review score model equation stems from the linear output formula in the seminal paper by Holmström (1999), which is a standard specification in this literature. By further employing the linear review writing strategies, we contribute to this literature by expanding it to the manipulated reviews under the standard Cournot and Bertrand oligopolistic competitions for obtaining policy insights.

This study could be categorized as an oligopolistic-market application of costly-signal literature, which is initiated by the pioneering study of Spence (1978). In this literature, an economic agent has private information, and she can credibly convey it to the other agents only through a costly signal. In our model, the costly manipulated review writing activity could be interpreted as a costly (and noisy) signal. Specifically, in our model, it is increasingly costly to write manipulated reviews, following the tradition in the signaling literature. However, unlike those in the literature, a gain from manipulated reviews crucially depends on the strategic market interactions, either strategic substitute and complement form, which we will clarify. (This sentence might be removed later).

Moreover, this study is also closely linked to the signal jamming literature, which was initiated by the pioneering contributions of Milgrom and Roberts (1982) and Fudenberg and Tirole (1986). In this literature, an economic agent (e.g., a firm) takes costly action not to reveal and often obfuscate its true nature, and rational consumers form expectations based on such a costly signal-jamming action. The manipulated review action, the focus of this article, could be interpreted as a costly non-revealing action, at least in (but not limited to) the sense that it condemns the consumers to conjecture a firm's true nature from an observed review score. To the best of our knowledge, such behavioral implication (of manipulated reviews) under the standard oligopolistic competition environment has not been reported in the literature.

This study is also related to advertisement literature, which is initiated by Milgrom and Roberts (1986). Focusing on separating equilibria, there is a commonality between this study and those in the advertisement literature: Based on observed signals, rational consumers backwardly conjecture true product quality. Nevertheless, there is one critical difference between this study and the literature. While rational consumers in the traditional advertisement literature know the true quality of the product (and know the equilibrium amounts of advertisement), upon their purchase, in a separating equilibrium, the rational consumers in our model are still exposed to the stochastic payoffs/utilities. These stochastic payoffs stem from the post-action review shocks (i.e., review noise). In other words, the consumers in the traditional advertisement literature have perfect learning, while those in our model experience imperfect learning.

Moreover, upon the analysis of the behavioral consumers, who can only partially recognize manipulated review activities, this perfect-versus-imperfect-learning difference generates subtly different policy implications.

Our paper is also related to the theoretical literature on the properties of fake and promotional reviews. The influential study of Mayzlin (2006) reports a promotional chat (i.e., fake review) model, notably with a random-message-draw specification. Using an online chatting bulletin board (equivalent to the product review section of a platform), her model focuses on a setting with two duopoly firms, binary quality types, an exogenously set price, and unit-demand consumers. A firm self-promotes through fake reviews on the online forum, referred to as promotional chatting message activity. The fake review affects the probability that a consumer randomly draws a positive review for this firm's product from the forum. Moreover, Mayzlin (2006) reports the properties of the promotional-chat equilibrium, characterizing the persuasiveness (or credibility) of word-of-mouth promotional chat.[2]

As to the positive correlation between product quality and seller promotion behavior, the seminal studies of Nelson (1970) and Nelson (1974) pioneer the concept of costly promotional activities among firms conveying product quality information. Regarding economic modeling, Milgrom and Roberts (1986) make contributions to the theoretical literature by formalizing a market with asymmetric information between a seller and consumers. In their model, a seller attempts to solve the asymmetric information through promotional activities, notably by using costly advertisements as a product quality signal.

In most promotion-related literature studies, the promotion is modeled as an advertisement, which is directly observed by consumers. In a separating equilibrium, after observing high advertising effort (i.e., a large advertisement expenditure), consumers rationally infer that such high advertising effort comes from a high-quality seller because such a high effort does not pay off for a low-quality seller. This inference among consumers is because the advertisement induces repeated purchases for high-quality products but not for low-quality products.

By contrast, in our study, unlike the above-mentioned promotion-related studies (including most of the advertisement studies), the seller's effort level is not directly observed by

[2] As a comparison, Mayzlin's random-message-draw probability is different from our signal-jamming construction: The former emphasizes the likelihood of drawing a positive message, while the latter emphasizes an aggregated (but noisy) review score. In addition, the former focuses on the persuasiveness of the (fake) signals with discrete types, using the same exogenously set price between firms, while the latter focuses on strategic fake reviews and market interactions with price-setting (or quantity-setting), including varying demand.

consumers. In the context of fake reviews, the consumer cannot tell whether a high rating comes from genuine reviews, representing a good experience of the product, or from fake reviews trying to inflate its reputation. For instance, in the context of an online opinion forum, Dellarocas (2006) models a monopolist's word-of-mouth product evaluation system manipulation, inflated by the monopolist's promotional comment-writing behavior, as a one-shot static version of Holmström (1999), showing the possibility of positive correlation between manipulated reviews and product quality.[3,4] However, the role of fake reviews in an oligopolistic market with rational consumers has remained an open question in the literature.

This paper contributes to the literature by analyzing interactions between oligopoly market firms which engage in fake reviews, noisy signaling promotion activities (which are not directly observable to consumers), as well as studying the influence of fake reviews in a horizontally differentiated product market structure. Notably, we derive our main findings without fixing quantities demanded. Our analyses include the degree of substitution between products, the number of firms, and the difference in price and quantity competitions, which have not been reported previously. Most importantly, we report fake review partial equivalence: Compared to no-fake-review market outcomes, some parts of the market outcomes remain the same while other parts are altered with the existence of fake reviews, notably due to rational consumers.

Regarding the empirical literature on fake reviews, there are two seminal papers. First, Mayzlin et al. (2014) exploit a gap in the (hotel) review process of two online platforms: While Tripadvisor allows any user to leave reviews, Expedia only allows customers who actually stayed at the hotel to leave a review. Using online hotel review data from US hotels rated on both Tripadvisor and Expedia, the authors analyze the gap between ratings on the two platforms, indicating the existence of fake reviews. Second, He et al. (2022a) use known fake review data and report market competition implications. In their study, for certain products sold on Amazon.com, sellers purchase fake reviews via other platforms (e.g., a restricted Facebook group). By exploiting this phenomenon, the authors analyse the

[3] In this unobserved promotional activity literature, Yasui (2020) analyzes the monopolist's dynamic behavior in a fake review strategy, as well as its effects on a rating system. Grunewald and Kräkel (2017) also apply a one-shot static version of the Holmström (1999) model with advertisements in a vertically differentiated duopoly market with the fixed total demand, notably by assuming that consumers cannot distinguish between the word-of-mouth product quality information and unobservable advertisement investments made by sellers.

[4] Whether the consumers can distinguish between genuine word-of-mouth product quality information, fake reviews, and advertisements is an interesting behavioral and empirical question. Some recent field experiments suggest that fake reviews and advertisements have distinctively different roles. Sahni and Nair (2020) show that consumers called restaurants more when the listing is revealed as "paid-advertisements" rather than non-paid listings. The study of Akesson et al. (2023) shows that consumers tend to be attracted to low-quality products if their reputation is inflated by fake reviews.

market consequences of fake reviews. They report a positive causal effect of fake reviews on Amazon sales, as well as pricing.

Here, other relevant studies on the topic of ratings and (fake) reviews, both theoretical and empirical ones, should be credited. An incomplete list of such studies is as follows: Chevalier and Mayzlin (2006) is a representative study of online reviews, investigating the effect of online book reviews on sales. Regarding online platform ratings and rating design, the platform economics textbook by Belleflamme and Peitz (2021) section 6.2.1 provides a concise overview. If we consider (potentially fake) ratings as a certification provided by a platform, our study is also closely related to certification design, which was pioneered by Lizzeri (1999), currently an active research area. For example, Zapechelnyuk (2020) reports that under some standard moral-hazard model settings, a simple binary quality certification (pass or fail rule) is optimal. Ratings and reviews are also closely related to dynamic reputation. With a dynamic model, Campbell et al. (2017) carry out a theoretical study on the relation of word-of-mouth and advertisement activities. Chevalier et al. (2018) study the dynamic response to reviews by managers. Hollenbeck (2018) provides an analysis of online reputation mechanisms, focusing on value chains. Related to fake reviews, Yasui (2020) studies a monopolist's dynamic incentives to generate fake reviews, as well as suggesting channels to control them. Based on directly observed paid review activities, He et al. (2022b) propose machine-leaning methods to detect fake reviews. Using detected fake review data, Gandhi and Hollenbeck (2023) model consumer beliefs related to fake reviews, and then conduct structural estimations, reporting the intricate welfare consequences caused by fake reviews. Yoshimoto and Zapechelnyuk (2023) report a dynamic review manipulation model of a monopoly seller, and test theoretical predictions by comparing restaurant reviews provided by online reviewers and professional guidebook reviewers. Lastly, related to our behavioral consumer analysis, Akesson et al. (2023) recently conducted an online field experiment with fake reviews. The authors report that random educational interventions (to mitigate the potential effect of fake reviews) could improve consumer welfare.

Related to quantity and price competition, our oligopoly model expands the classical Cournot and Bertrand competition frameworks set by Singh and Vives (1984) and Vives (1985). Our models differ from these seminal studies at least in the following two points. First, we introduce a costly fake-review writing stage before the oligopoly market competition stage. Second, related to asymmetric information, we propose a rating (or a rating score) for each firm, which is stochastic as well as subject to manipulation by each firm. As a consequence, the welfare analysis is based not on a deterministic utility but on expected utility function, involving variance terms to capture volatilities that a representative consumer faces.

Throughout this article, we report our oligopoly fake review models by primarily focusing on Bertrand price competition, as price-setting competition is more pragmatic in the differentiated product markets. However, we also analyze (and often make a comparison to) Cournot quantity competition, and derivations and descriptions for quantity-setting competitions are reported in the Appendix for the simplicity of expositions.[5]

Lastly, in general, besides the key contributions listed above, there is a shortage of both theoretical and empirical studies of fake reviews, notably studies that can be used to derive market analyses and relevant competition policies. This study aims to fill this gap.

[5]As we will describe in the modeling section and Appendix, price and quantity competition generate a difference in the quadratic function coefficients, for which we use simplified notations with coefficient formulas. We then frequently make the comparison between Bertrand and Cournot equilibrium market outcomes by substituting the profit function of coefficient formulas, in which Bertrand and Cournot models exhibit differences.

3 Model

In this section, we introduce the main model settings. There are n-firms and a representative consumer in a differentiated product market. Each firm produces one brand of differentiated product. The game consists of five stages. Timings, asymmetric information structure, and information dissemination processes are overviewed as follows.

At stage 1, each firm independently draws its product quality type from an i.i.d. distribution. At this point, a product quality type is the private information of each firm. Notably, a firm does not know the competing firms' product quality types. At stage 2, each firm makes a fake review writing effort to inflate the rating score of its product. At this stage, due to the intrinsic nature of fake reviews, neither the competing firms nor the consumers can directly observe a specific firm's fake review writing action. At stage 3, an i.i.d. review shock is realized for each firm's product, and each firm's review score is determined. As further described below, a rating score of each firm's product is the linear combination of true product quality, fake reviews, and realized review shock. Also, the review score of each firm's product becomes public information. At stage 4, each firm engages in an oligopolistic market competition by choosing its price (or quantity).[6] At stage 5, the representative consumer makes a consumption choice by maximizing her/his/their expected utility, conditional on the observed review scores. Then, the profit for each firm is determined. At the end of stage 5, the representative consumer experiences the quality of each product, and her/his/their utility is determined.

Below, by focusing on the simplified two-firm case, each of these steps is explained. In the Appendix, we report a table, which summarizes the notations with labelings and relevant descriptions. The n-firm version of the model is explained in Section 5.

For firm $i \in \{1, 2\}$, its profit is defined as

$$\pi_i = (p_i - c_i)q_i - \frac{\phi_i}{2}F_i^2,$$

where p_i, q_i and F_i are firm i's price, quantity, and fake-review effort level, and c_i and ϕ_i are costs for producing a unit of its product and a cost coefficient for review manipulations. Each firm owns one product and chooses its price or quantity.[7]

Given the qualities of each product, θ_i, we define the *ex-post* utility function for a representative, as in Dixit (1979) without any uncertainty:

[6] We focus on the price-competition version for the sake of simple exposition. Then, we introduce a quantity-competition version later.

[7] When ϕ_i goes infinity, the cost to write fake reviews becomes prohibitively high, and the payoff function is qualitatively equivalent to the one in a standard Cournot or Bertrand oligopoly.

$$U = \theta_1 q_1 + \theta_2 q_2 - \frac{b_1}{2} q_1^2 - \frac{b_2}{2} q_2^2 - s q_1 q_2 - p_1 q_1 - p_2 q_2, \tag{1}$$

where b_i and s are constants. Suppose $b_1 b_1 - s^2 > 0$ for concavity of the utility function. Thus, the (expectation of the) quality contributes to an intercept of the linear differentiated product (inverse) demand function, which could be interpreted as consumers' willingness to pay. Consumers observe ratings of each product, R_i, as signals of quality, but cannot observe the underlying quality itself. Given sellers' prices and ratings, the representative consumer chooses the quantities to purchase to maximize the *interim* expected utility. Details will be described later.

The signal structure and the timing of the game are described below. At stage 1, each firm draws its quality type θ_i from an i.i.d. normal distribution with its mean μ and variance σ_θ^2, that is:[8]

$$\theta_i \sim \mathcal{N}(\mu, \sigma_\theta^2).$$

Although the distribution of θ_i is common knowledge, the drawn type is private information for each firm. In other words, the products are experience goods, so the quality is revealed only after the purchase of the product. We assume $\mu > c_i$ for all i.

At stage 2, the two firms simultaneously choose their efforts to manipulate their reviews, which shifts the rating of their own product upward and could influence demands among consumers. In other words, each firm makes the inflating-review writing effort, F_i.[9]

At stage 3, the information on each product is collected, and ratings for each product are revealed to the public. The rating for firm i is denoted as R_i and defined a la Holmström (1999) as in Dellarocas (2006), by the following equation:

$$R_i = \theta_i + F_i + \epsilon_i, \tag{2}$$

where ϵ_i is a random shock drawn from a normal distribution with its mean zero and variance σ_ϵ^2, that is:

$$\epsilon_i \sim \mathcal{N}(0, \sigma_\epsilon^2).$$

[8]In this study, we assume the normality of the seller's quality (θ_i), as well as using the linear (in θ_i) fake-review strategy. However, some may raise a question about a potentially negative fake review action, which stems from a possibly negative θ_i. This negativity in fake-review action is not a nuisance in this study for the following reasons. First, as reported in the Appendix, we can replace a normal distribution with an elliptical distribution (Cambanis et al., 1981; Gómez et al., 2003; Ball, Forthcoming), specifically by using an elliptical distribution in a nonnegative domain. Note that the elliptical distribution family includes symmetric distributions which do not have infinitely running tails. Particularly, an elliptical distribution can have zero density in the negative domain. See the Appendix for details.

[9]For simplicity, we will focus on the linear strategy equilibrium : The review manipulating effort is the linear function of each firm's type.

In the above linear rating equation, fake reviews can inflate the rating of own product: F_i summarizes the costly fake review-generating activities. Intuitively, R_i reflects a combination of genuine reviews, $R_i + \epsilon_i$, and manipulated reviews, F_i. If the product quality is high, genuine reviews tend to be good ones even though there will be some fluctuation due to the taste heterogeneity of consumers. The seller can inflate such a signal by using the manipulated reviews, F_i.

It is worth noting that the realization of ϵ_i and R_i is after the choice of F_i. In the context of online ratings, it is interpreted that the seller distributes (unincentivized) product samples to consumers and writes fake reviews on top of them upon entry. The sample-experienced consumers write genuine reviews, which are not necessarily "5 stars", while the fake reviewers are incentivized to write "5 stars" reviews.[10]

At stage 4, the firms choose their own prices, $p_i \in [0, \infty)$.

At stage 5, the market clears. The demand function is derived from the maximization of the representative consumer's expected utility, conditional on the ratings of each product, R_i, and the firms' prices, p_i. Note that the consumer can observe R_i, but not the realization of F_i or ϵ_i, so they cannot disentangle the exact value of the quality.

4 Equillibrium

In this article, we focus on a perfect Bayesian equilibrium where the seller's manipulation strategy is linear in its hidden type θ_i and the seller's pricing strategy is independent of θ_i and the consumer believes such independent pricing even if the consumer observes an off-equilibrium price.[11] Formally, it is defined as follows.

Definitnion of Equilibrium Fake review equilibrium is characterized by the following conditions:

1. Expected utility maximization:
   $(q_1(\mathbf{p};\mathbf{R}), q_2(\mathbf{p};\mathbf{R})) = \arg\max_{q_1,q_2} E_c[U|\mathbf{R},\mathbf{p}]$;

[10] Alternatively, in the fashion of traditional non-online product and service reviews, such as newspaper and consumer product magazine reviews, Equation (14) could also be interpreted as a combination of bribed and non-bribed professional reviews. For instance, if a new restaurant opens, the management may offer bribes to some but not all professional reviewers (e.g., local newspaper writers) to write favorable review articles for the restaurant, which is represented by the costly fake review activity, F_i. On the other hand, in general, it is not possible to bribe all reviewers: Other non-bribed reviewers (e.g., town magazine editors) honestly report the true quality θ_i, but their reviews come with a random review shock, ϵ_i.

[11] At this equilibrium, price or quantity depends on the expected qualities of the products conditional on the ratings. Therefore, other prices or quantities are considered to be off the equilibrium path. We suppose that, even in such a case, consumers do not extract information on the quality from the prices or quantities.

2. Profit-maximizing pricing:
   $p_i^*(\mathbf{R}) = \arg\max_{p_i} E_i[(p_i - c_i)q_i(\mathbf{p};\mathbf{R})) - \frac{\phi_i}{2}F_i^2|\theta_i,\mathbf{R}]$ given p_j for $j \neq i$;

3. Profit-maximizing fake review strategy:
   $F_i^* = \arg\max_{F_i} E_i[(p_i^*(\mathbf{R}) - c_i)q_i^*(\mathbf{R}) - \frac{\phi_i}{2}F_i^2|\theta_i]$ where $q_i^*(\mathbf{R}) = q_i(p_1^*(\mathbf{R}), p_2^*(\mathbf{R});\mathbf{R})$;

4. Linear fake review strategy:
   $F_i^* = \alpha_i\theta_i + \gamma_i$ for some constant α_i and γ_i;

5. Passive belief:
   $E_c[\theta_i|\mathbf{R}, p_1, p_2] = E_c[\theta_i|\mathbf{R}, p_1^*(\mathbf{R}), p_2^*(\mathbf{R})]$ for any p_1 and p_2.

The model is solved backward. Conditional on the observed ratings, the expected utility from the products is written as follows:

$$E\left[U|R_1,\, R_2, p_1, p_2\right] = E\left[\theta_1|R_1,\, R_2, p_1, p_2\right]q_1 + E\left[\theta_2|R_1,\, R_2, p_1, p_2\right]q_2 - \frac{b_1}{2}q_1^2 - \frac{b_2}{2}q_2^2 - sq_1q_2 - p_1q_1 - p_2q_2.$$

The consumer's expectation on each product's qualities are calculated by the projection of the product i's rating, R_i, on the quality of product i, θ_i, since θ_i is independent of R_{-i}.[12] Thus, for $i \in \{1, 2\}$, we have the conditional expectation of true product quality θ_i as

$$Y_i \equiv E\left[\theta_i|R_1,\, R_2, p_1, p_2\right] = \mu + \underbrace{\frac{(1+\alpha_i)\,\sigma_\theta^2}{(1+\alpha_i)^2\,\sigma_\theta^2 + \sigma_\epsilon^2}}_{\equiv\lambda_i}(R_i - \underbrace{\{(1+\alpha_i)\,\mu + \gamma\}}_{=E[R_i]}). \tag{3}$$

In the above equation, to simplify the notation, we denote

$$Y_i \equiv E\left[\theta_i|R_1,\, R_2, p_1, p_2\right]$$

and

$$\lambda_i \equiv \frac{(1+\alpha_i)\,\sigma_\theta^2}{(1+\alpha_i)^2\,\sigma_\theta^2 + \sigma_\epsilon^2} = \frac{(1+\alpha_i)}{(1+\alpha_i)^2 + (\sigma_\theta/\sigma_\epsilon)^{-2}}.$$

We broadly call Y_i as firm i's *reputation* and λ_i as consumer's *sensitivity parameter* to firm i's rating.

The quotient term of $\sigma_\theta/\sigma_\epsilon$, a fundamental signal-to-noise ratio, represents the ratio of standard deviations from both true product quality and review shock distributions, a key term to determine equilibrium properties. The economic intuition is that, when this ratio is large, the review score (R_i) is relatively more explanatory in conveying the true quality of

[12] See A.1 for details.

the product. On the other hand, when this ratio is small, a consumer who observes a review score (R_i) is exposed to a relatively higher volatility, upon the consumption and revelation of the true product quality.

Then, the first-order conditions for the utility maximization with respect to q_i's are:

$$\begin{aligned} 0 &= Y_1 - b_1 q_1 - s q_2 - p_1 \\ 0 &= Y_2 - b_2 q_2 - s q_1 - p_2. \end{aligned} \tag{4}$$

Thus, by rearranging eq. (4), we can obtain a linear demand system with its constant terms determined by the reputation of each product, Y_i. Note that the reputation is constant for firms at Stage 4 because R_i is already publicly drawn at Stage 3 and pricing does not affect their reputation, given the consumer's belief that the pricing and manipulative reviews are independent.

At Stage 4, as an equilibrium outcome of price competition, we obtain profit functions for each firm. Specifically, firm 1's profit function is (firm 2's profit function is characterized in a similar manner)

$$\pi_1 = J_1 \times (Y_1 - c_1 + K_1(Y_2 - c_2))^2 - \frac{\phi_1}{2} F_1^2,$$

where

$$\begin{aligned} J_1 &= \frac{b_2 \left(2b_1 b_2 - s^2\right)^2}{\left(b_1 b_2 - s^2\right)\left(4b_1 b_2 - s^2\right)^2}, \text{ and} \\ K_1 &= -\frac{b_1 s}{2b_1 b_2 - s^2}. \end{aligned}$$

Next, at Stage 2, firms are uncertain about the realization of Y_i even after choosing how many fake reviews they utilize. Therefore, the profit function for firm 1 is evaluated with an expectation conditional on own product's quality:[13]

$$E_1\left[\pi_1 | \theta_1\right] = J_1 \times E_1\left[\left(Y_1 - c_1 + K_1(Y_2 - c_2)\right)^2 | \theta_1\right] - \frac{\phi_1}{2} F_1^2.$$

where $E_1[\cdot|\theta_1]$ is the expectation over ϵ_1, ϵ_2, and θ_2. As the firm can manipulate its own reputation Y_i with fake reviews F_i via its rating R_i, we insert $Y_i = \mu + \lambda_i(R_i - \{(1+\alpha)\mu + \gamma_i\})$

[13] By replacing J_i and K_i, we can obtain a similar quadratic profit function for the quantity competition, and then apply the same logic in the following part. See the Appendix for the details.

and $R_i = \theta_i + F_i + \epsilon_i$ into the objective function and rearrange it as follows:

$$\begin{aligned}
E_1\left[\pi_1|\theta_1\right] = & -\frac{\phi_1}{2}F_1^2 \\
& + J_1\left\{\left(\mu + \lambda_1\left(\theta_1 + F_1 - \left\{\left(1+\alpha_1\right)\mu + \gamma_1\right\}\right)\right)^2 + \sigma_{\epsilon_1}^2 \right. \\
& + 2\left(\mu + \lambda_1\left(\theta_1 + F_1 - \left\{\left(1+\alpha_1\right)\mu + \gamma_1\right\}\right)\right)E_1\left[\left(-c_1 + K_1(Y_2 - c_2)\right)|\theta_1\right] \\
& \left. + E_1\left[\left(-c_1 + K_1(Y_2 - c_2)\right)^2|\theta_1\right]\right\}
\end{aligned} \tag{5}$$

In the above equation, ϵ_1 is integrated out. Furhtermore, θ_2 and ϵ_2 only appear in an expectation of Y_2, which is independent of the firm 1's choice, given firm 2's strategy. Therefore, at Stage 2, the above equation is regarded as a deterministic quadratic function for firm 1. By taking the first order condition with respect to F_1, we obtain the optimal fake review strategy for firm 1:

$$F_1 = \frac{2\lambda_1^2}{\left(\frac{\phi_1}{J_1} - 2\lambda_1^2\right)}\theta_1 + \frac{2\lambda_1\left(\mu - \lambda_1\left\{\left(1+\alpha_1\right)\mu + \gamma_1\right\}\right) + 2\lambda_1 E_1\left[\left(-c_1 + K_1(Y_2 - c_2)\right)|\theta_1\right]}{\left(\frac{\phi_1}{J_1} - 2\lambda_1^2\right)}$$

Note that α_1 and λ_1 are determined by matching coefficients:

$$\alpha_1 = \frac{2\lambda_1^2}{(\phi_1/J_1) - 2\lambda_1^2} \tag{6}$$

$$\lambda_1 = \frac{(1+\alpha_1)}{(1+\alpha_1)^2 + (\sigma_\theta/\sigma_\epsilon)^{-2}} \tag{7}$$

The above system of equations is independent of firm 2's strategy or firm 1's production process. The second order condition for the profit maximization is satisfied if and only if $\frac{\phi_1}{J_1} - 2\lambda_1^2 > 0$. Therefore, $\alpha_1 > 0$ holds for firm 1's profit-maximizing strategy. Furthermore, it implies $\lambda_1 > 0$ for firm 1's profit-maximizing strategy.

As to γ_i, by matching the coefficients, we obtain the following:

$$\gamma_1 = \frac{2\lambda_1\left(\mu - \lambda_1\left\{\left(1+\alpha_1\right)\mu + \gamma_1\right\}\right) + 2\lambda_1 E_1\left[\left(-c_1 + K_1(Y_2 - c_2)\right)|\theta_1\right]}{\left(\frac{\phi_1}{J_1} - 2\lambda_1^2\right)}$$

$$\Leftrightarrow \gamma_1 = \frac{2\lambda_1\left(\mu - \lambda_1\left(1+\alpha_1\right)\mu\right) + 2\lambda_1 E_1\left[\left(-c_1 + K_1(Y_2 - c_2)\right)|\theta_1\right]}{(\phi_1/J_1)}$$

Note that $Y_2 = \left(\mu + \lambda_2\left(\theta_2 + F_2 + \epsilon_2 - \left\{\left(1+\alpha_2\right)\mu + \gamma_2\right\}\right)\right)$ and the consumer rationally believes the firm 2's equilibrium strategy $F_2 = \alpha_2\theta_2 + \gamma_2$. Thus, the consumer can correctly discount firm 2's inflated rating in expectation, the firm 1 knows such a discount by the consumer. Thus, $E_1[Y_2|\theta_1]$ is reduced to μ.[14] Then, we obtain

$$\gamma_1 = \frac{2\lambda_1\left(\mu - \lambda_1\left(1+\alpha_1\right)\mu\right) + 2\lambda_1\left(-c_1 + K_1(\mu - c_2)\right)}{(\phi_1/J_1)} \tag{8}$$

Thus, γ_i is also independent of others' fake review strategy. Even though firm i's profit-maximizing amount of fake reviews depends on other products' attractiveness captured by $E[Y_{-i}]$, firm i expects that the consumers rationally discount the boosted ratings in expectation so firm i believes $E[Y_{-i}] = \mu$ holds.

Proposition 1 (Unresponsive Best Responses in Fake Reviews)**.** *If the consumer has a rational and passive belief that sellers implement linear strategies, seller i's best strategy is linear and does not depend on seller j's strategy ($j \neq i$). Furthermore, α_i in such a best strategy is always positive.*

As a result, at any linear equilibrium with a passive consumer's belief, α is positive.

As discussed above, the seller i's α and λ are determined by eqs. (6) and (7), which are independent of seller j's strategy. Furthermore, γ_i is also independent of seller j's strategy as shown in eq. (8). The sign of α is determined from eq.(6).

This unresponsive best-response property of fake reviews in an oligopolistic market, which is directly related to the equilibrium properties, is illustrated as follows. In a two-firm oligopoly market with firms 1 and 2, we consider a situation, in which a fake-review writing cost of firm 2 decreases due to some exogenous economic reason (for example, an exogenously available relatively cheap fake-review writer for firm 2), and all market participants know this cost reduction. Given the lowered fake-review writing cost, firm 2 increases its amount of fake reviews to a new level, at which the marginal expected benefit equates to the marginal cost of fake-review writing, generating a further inflated review rating (of firm 2). However, as the reduction of fake-review writing cost of firm 2 is public information, consumers rationally revise their reasoning of observed review rating of firm 2. Notably, to reflect the fake-review writing cost reduction of firm 2, the consumers further downwardly access the review rating (of firm 2's product) during their inference process. As a result of this rational inference, consumers' willingness to pay, which is the expected intercept of the inverse demand function of firm 2, is not affected by the exogenous cost reduction. Consequently, the inverse demand function of firm 1 is also unaffected by firm 2's fake-review writing cost reduction, and firm 1 does not need to change its fake-review writing strategy, resulting in the unresponsive best-response property.[15]

[14] By contrast, if the consumer does not know the possibility of fake reviews and cannot discount inflated reviews, firm 2's fake review strategy directly changes $E_1[Y_2|\theta_1]$. See Section 6 for details.

[15] By contrasting fake review markets with auction markets, in which there are no consumers, we can

The intuition of the positive α_i is explained as follows. Note that the fake review linearly increases the intercept of its own demand curve Y_i, which linearly increases the equilibrium quantity and price. Thus, the fake review quadratically increases the firm's profit from the market competition. Furthermore, if the seller observes high underlying quality (high θ_i), then the seller expects high reputation (high Y_i) even without fake reviews. Because of the quadratic feature of the profit function, the marginal effect on the profit is higher if it is expected to start from a higher reputation. Thus, the seller with higher quality has a greater incentive to make fake reviews than the seller with lower quality.

Given α_i and γ_i, other parts of the equilibrium are characterized well. The existence and uniqueness of equilibrium hinges on the existence and uniqueness of α and γ satisfying eqs. (6), (7), and (8).

Proposition 2 (Existence and Uniqueness of the equilibium)**.** *The linear equilibrium with a passive consumer's belief always exists and it is unique if $\sigma_\theta > \sigma_\epsilon$*

See the appendix for the proof of the existence and uniqueness. For the following comparative statistics (from Prop.3), we focus on the parameter set where the unique equilibrium is guaranteed. As a preliminary analysis, the following propositions illustrate the effects of parameters on the equilibrium coefficients.

Proposition 3. *The equilibrium coefficient α_i and the consumer's discounting parameter λ_i are affected only by J_i/ϕ_i and $\sigma_\theta/\sigma_\epsilon$:*

1. *α_i is increasing in J_i/ϕ_i and λ_i is decreasing in J_i/ϕ_i.*
2. *α_i and λ_i are increasing in $\sigma_\theta/\sigma_\epsilon$.*

Intuitively, an increase in J_i/ϕ_i implies an increase in the marginal impact on the seller's profit in the market competition stage relative to the marginal cost of providing manipulative behavior. In this case, the signaling property of fake reviews is enhanced (higher α_i) as the number of fake reviews increases. Taking the increased number of fake reviews, the consumer would discount the observed rating even more (lower λ_i).

here further clarify this unresponsive best-response property, as well as the role of rational consumers. We consider a simple two-bidder Bayesian pay-as-bid procurement auction market with bidders A and B. We also consider that bidder B's cost distribution exogenously changes, such as exogenously arrived new cost-reduction technology (i.e., cost reduction in a stochastic dominance fashion), and all market participants know this cost reduction. Knowing such a cost reduction of bidder B, bidder A best-responds with bidding more aggressively by lowering its bidding strategy. Furthermore, given a change in bidder A's revised bidding strategy, bidder B further best-responds, and so on, and the two bidders keep mutually best-responding. Eventually, these consecutive best-response processes reach new equilibrium bidding strategies. In contrast, in a two-firm oligopolistic fake review market, the consumers' rational conjecturing process, which further downwardly evaluates an observed review rating, nullifies the need for revising a best-response.

By contrast, a high $\sigma_\theta/\sigma_\epsilon$ implies a precise rating system without fake reviews. Therefore, the consumer's purchasing behavior is highly responsive to the ratings of each product (high λ_i). By taking this into account, the sellers make more fake reviews and the signal effect of fake reviews is also enhanced (high α).

Proposition 4 (Limits of the Equilibrium)**.** *The equilibrium strategy has the following properties:*

1. *As $s \to 0$, the second stage is reduced to the monopoly for each product (i.e., Dellarocas (2006)).*

2. *As $s \to b_1 = b_2 = b$, $\alpha_i \to \infty$ and $\lambda \to 0$ in price competition, and $\alpha_i, \lambda_i \to \bar{\alpha}, \bar{\lambda}$ for some finite $\bar{\alpha}, \bar{\lambda}$ in quantity competition.*

3. *As $\phi_i \to \infty$, $\alpha_i, \gamma_i \to 0$. (Thus, $E[F_i] \to 0$.)*

4. *As $\sigma_\theta/\sigma_\epsilon \to \infty$, α_i and $\lambda \to \hat{\alpha}, \hat{\lambda}$ for some finite $\hat{\alpha}, \hat{\lambda}$ and $Cor(\theta_i, Y_i) \to 1$*

5. *As $\sigma_\theta/\sigma_\epsilon \to 1$, α_i and $\lambda \to \tilde{\alpha}, \tilde{\lambda}$ for some finite $\tilde{\alpha}, \tilde{\lambda}$*

The differences between price and quantity competition in this model are reduced to differences in parameters of the quadratic profit function J_i and K_i. Thus, we can analyze the difference in the fake review strategies between price and quantity competition by comparing these coefficients in price and quantity competition.

Proposition 5 (Effects of Competitive Structure on Fake Review Strategies)**.** *Given the same demand structure,*

1. *α_i is higher in price competition,*

2. *$E[F_i]$ is higher in quantity competition*

Note that only J_i appears in eq.(6) and none of the parameters for the quadratic profit parameters appears in eq.(7). Because J_i shifts up α and J_i is higher in price competition, α is higher in price competition. (See the appendix for the details.)

Intuitively, the firms' equilibrium quantities are more sensitive to their own reputations in price competition. Once a firm expects its advantageous reputation, it makes fake reviews even more fiercely in price competition. At the same time, if the firm has a low-quality product, it expects a small profit and a small marginal effect of fake reviews. Therefore, α_i is higher in price competition. By contrast, the marginal effect of fake reviews in expectation (with respect to θ_i) is greater in quantity competition. This is due to a larger quadratic profit on average in quantity competition. Thus, the expected number of fake reviews is greater in quantity competition.

At the same time as the competitive structure affects the fake review strategy, the fake review strategy also affects the equilibrium prices and quantities in the second stage via the sellers' reputation Y_i. Recall that

$$Y_i = \mu + \lambda_i(R_i - \underbrace{\{(1+\alpha)\,\mu + \gamma\}}_{E[R_i]})$$

Thus, $E[Y_i] = \mu$ holds as long as the consumer is rational. Because the equilibrium prices and quantities are linear in sellers' reputation (Y_i's)[16] and the rational consumer correctly discounts the boosted ratings, the expected equilibrium prices and quantities are the same as ones without fake reviews if consumers are rational. Variances of prices at equilibrium are also calculated by using the variance of the reputation:

$$\begin{aligned} Var(Y_i) &= \lambda_i^2 Var(R_i) \\ &= \left(\frac{(1+\alpha_i)\sigma_\theta^2}{(1+\alpha_i)^2\sigma_\theta^2 + \sigma_\epsilon^2}\right)^2 Var((1+\alpha_i)\theta_i + \epsilon_i) \\ &= \left(\frac{(1+\alpha_i)^2\sigma_\theta^2}{(1+\alpha_i)^2\sigma_\theta^2 + \sigma_\epsilon^2}\right)\sigma_\theta^2 \end{aligned}$$

As the equilibrium coefficient α_i is positive for any equilibrium, $Var(Y_i)$ is larger with fake reviews than without fake reviews. This also implies that $Var(q_i)$ and $Var(p_i)$ are larger with fake reviews than without fake reviews as q_i and p_i are linear in reputations (Y_1 and Y_2), and the reputations are uncorrelated (i.e., $Cov(Y_1, Y_2) = 0$).

Proposition 6 (Fake Review Invariant Expected Quantities/Prices)**.** *Regardless of Bertrand or Cournot competition in the second stage, the expected equilibrium reputations, prices, and quantities are the same as ones without fake reviews if consumers are rational. The variances of the equilibrium prices and quantities are more than those without fake reviews.*

By comparing how the equilibrium quantity reacts to the reputations of each firm, we can obtain properties on the moments of the equilibrium quantities in price and quantity competitions.

Proposition 7 (Moments of Equilibrium Quantity in Price/Quantity Competition)**.** *The expected equilibrium quantity is larger in price competition than in quantity competition.*

[16] See the Appendix for the detailed derivation.

The expected equilibrium quantity has a larger variance in price competition than in quantity competition.

4.1 Welfare Analysis

In this section, we analyze the effects of fake reviews on *ex ante* surpluses *at equilibrium*. It is reduced to a function of moments of $(Y_i - p_i)$'s by inserting the demand functions into the surplus:[17]

$$E[U] = E\left[\frac{\dot{b}_1}{2}(Y_1 - p_1)^2 + \frac{\dot{b}_2}{2}(Y_2 - p_2)^2 - \dot{s}(Y_1 - p_1)(Y_2 - p_2)\right] \tag{9}$$

$$\begin{aligned} &= \frac{\dot{b}_1}{2}Var\left(Y_1 - p_1\right) + \frac{\dot{b}_2}{2}Var\left(Y_2 - p_2\right)^2 - \dot{s}Cov\left(Y_1 - p_1, Y_2 - p_2\right) \\ &+ \frac{\dot{b}_1}{2}E\left[Y_1 - p_1\right]^2 + \frac{\dot{b}_2}{2}E\left[Y_2 - p_2\right]^2 - \dot{s}E\left[(Y_1 - p_1)\right]E\left[(Y_2 - p_2)\right]) \end{aligned} \tag{10}$$

where $\dot{b}_i = b_j/(b_1 b_2 - s^2)$ for $i = 1, 2$ and $j \neq i$ and $\dot{s} = s/(b_1 b_2 - s^2)$. As the existence of fake reviews does not change the expected reputations, prices, and quantities at equilibrium(Prop. 6), it does not change the last three terms in the above equation. Thus, the fake reviews affect the consumer's surplus only via the variances and covariance of $Y_i - p_i$'s.

Intuitively, the variances in each product's reputation positively contribute to the consumer surplus because the varying reputation means that the ratings are somewhat trustworthy[18] Then, the consumer updates their beliefs on the product qualities, and then, adjusts consumption accordingly. See Dellarocas (2006) and Bonatti and Cisternas (2020) for the same property in monopoly settings.

Furthermore, in contrast to the previous research, the covariance between $(Y_i - p_i)$'s appears in the consumer surplus function. When $(Y_i - p_i)$'s are negatively correlated, the consumer can adjust its consumption level largely, given that the products are substitutes ($s > 0$). When $(Y_i - p_i)$'s are positively correlated, there is not much room for the consumption adjustment even if $(Y_i - p_i)$ for each i fluctuates largely.

Intuitively, the variances in each product's reputation (Y_i) positively contribute to the consumer surplus, because the varying reputation means that the ratings are relatively informative. In other words, when the product's reputation is almost fixed, a consumer faces

[17] See the appendix for the detailed derivation

[18] This is a well-known argument in the information design literature. If the ratings (or, signals in general) are not trustworthy at all, the consumer does not use the ratings when they evaluate the quality of the products. So the conditional expectation would be almost constant at the prior expectation. If the ratings are informative, the conditional expectation given the ratings fluctuates somewhat capturing the fluctuation of the underlying quality level.

difficulties in conjecturing product quality[19] Then, the consumer updates their beliefs on the product qualities, as well as adjusting consumption accordingly. See Dellarocas (2006) and Bonatti and Cisternas (2020) for the same properties in simplified monopoly settings. Furthermore, in contrast to the previous studies, the covariance between $(Y_i - p_i)$'s negatively contributes to the consumer surplus as long as $s > 0$. Note that if $s = 0$, then the two products are not related to each other and the consumer adjusts the consumption level of one product according to its rating alone. However, if the products are substitutes ($s > 0$), then the consumer adjusts the consumption of a product according to the other product's rating as well. When $(Y_i - p_i)$'s are negatively correlated, the consumer can adjust its consumption level largely, given the substitute products' nature ($s > 0$). When $(Y_i - p_i)$'s are positively correlated, there is little room for the consumption adjustment by a consumer, even if $(Y_i - p_i)$ for each i fluctuates largely.

Notably, the above equation holds for any given p_1 and p_2. At the end, we want to evaluate the consumer surplus with equilibrium prices. Because the prices at equilibrium p_i or q_i are written as a linear combination of Y_1 and Y_2, the consumer surplus is further reduced to a linear combination of $Var(Y_1)$, $Var(Y_2)$, $E[Y_1]^2$, $E[Y_2]^2$ and $E[Y_1]E[Y_2]$ (Note that Y_1 and Y_2 are independent of each other. Thus, there is no $Cov(Y_1, Y_2)$ in the consumer surplus at equilibrium) Then, we can show that the variances of the reputations positively contribute to the consumer surplus even after considering the sellers' strategic interaction. Intuitively, p_1 and p_2 at equilibrium are negatively correlated due to the strategic interaction at the price (or quantity) competition stage.[20] Thus, the third term in eq. (10) positively contribute to the consumer surplus.

By taking Proposition 6 and the effects of ratings on the consumer surplus via the variances of the reputation into account, we can evaluate the effect of fake reviews on the consumer surplus.

Proposition 8 (Comparison: With/Without Fake Reviews)**.** *If there exist fake reviews in the market, the expected consumer surplus becomes higher regardless of the market competition than these without fake reviews.*

First, regardless of whether fake reviews exist or not, the expected reputations, prices, and quantities do not change if the consumer is rational (Proposition 6). Thus, the fake review does not change the last three terms in eq. (10). Furthermore, the equilibrium fake

[19] This argument is well known in the information design literature. If the ratings (or, signals in general) are uninformative at all, the consumer does not use the ratings upon their evaluation of product qualities, such as in the extreme case that the conditional expectation would be almost constant at the prior expectation. When ratings are informative, the conditional expectation (given the ratings) fluctuates, modestly reflecting the fluctuation of the underlying quality level.

[20] Note strategic substitute/complement

review strategy has positive α_i (i.e., a higher-quality seller provides more fake reviews). The consumer takes this into account, relies more on ratings, and adjusts its consumption better with fake reviews than without fake reviews.

We can also analyze how the competition structure (price or quantity competition) in the product market affects the consumer surplus, given the existence of fake reviews. Because $E[Y_i - p_i]$ does not change regardless of the existence of fake reviews, the last three terms in eq. (10) correspond to an equilibrium consumer surplus of a deterministic version of Bertrand or Cournot competition with the demand intercept being the unconditional expectation of its quality. Thus, without the variance terms, the welfare comparison is just like as Singh and Vives (1984). In the following proposition, we analyze how the ratings and fake reviews affect the welfare comparison via the variances of reputations and quantities.[21]

Proposition 9 (Consumer Surplus in Bertrand and Cournot Competition with Fake Reviews)**.** *With fake reviews, the expected consumer surplus is higher in price competition than quantity competition*

Besides the fact the expected quantities are larger in price competition (Singh and Vives (1984)), the quantities fluctuate more in price competition as the ratings are more informative in price competition. The reason for the large fluctuation is decomposed into two parts. First, given the variance of the reputation, the equilibrium prices and quantities fluctuate more in price competition because firms react to their reputations more sensitively in price competition. Given realized reputations, one firm has an advantage over the other. Such an advantage affects the equilibrium quantities more severely in price competition than in quantity competition. Thus, the equilibrium quantities fluctuate more in price competition. Second, the reputations themselves fluctuate more in price competition than in quantity competition because the number of fake reviews is correlated with the product's quality and the consumer reacts to the ratings more in price competition.

In addition to the effects on the consumer surplus, Singh and Vives (1984) showed that the firms face lower profits in price competition if and only if the products are substitutes. With fake reviews, however, the profit depends on the fluctuations of equilibrium quantities. Therefore, the profit can be higher in price competition.

[21] Klemperer and Meyer (1986) and Reisinger and Ressner (2009) analyzed the role of uncertainty in choice of price or quantity as a strategic variable. They assume that firms cannot observe any information on the underlying uncertainty when they choose their actions. In this article, by contrast, the consumer and firms face the same information once the ratings are realized. Therefore, the resulting formula for the surpluses and the mechanism behind the results are totally different.

5 Extension to n-Firms

In this section, given the results from the two-firm oligopoly models with fake reviews, we now extend our models to n-firm oligopoly competition settings. Our n-firm extensions are based on three simplifications. First, we focus on linear demand-intercept-asymmetric firms. In both Cournot and Bertrand competition environments, n-firms share the same demand slope and demand substitution parameters. On the other hand, they are asymmetric in their linear (inverse) demand function intercepts and marginal costs. Second, given the linear demand setting, the profit function for each firm remains quadratic with respect to the fake-review action. Thus, the analytic framework explored for two-firm situations in the previous section(s) can straightforwardly be extended to n-firm oligopoly competition settings. Third, we use a linear fake-review strategy.

That is, we assume a representative consumer with the following *ex-post* utility function:

$$U = \theta'\mathbf{q} + \frac{1}{2}\mathbf{q}'\Sigma\mathbf{q} - \mathbf{p}'\mathbf{q}, \tag{11}$$

where $\theta = (\theta_1, ..., \theta_n)'$, $\mathbf{q} = (q_1, ..., q_n)'$, $\mathbf{p} = (p_1, ..., p_n)'$, and Σ is n-by-n matrix with b's in its diagonal elements and s's in its off-diagonal elements.

Then, given the ratings $\mathbf{R} = (R_1, ..., R_n)'$, the consumer faces the *interim* expected utility:

$$E[U|\mathbf{R}] = \mathbf{Y}'\mathbf{q} + \frac{1}{2}\mathbf{q}'\Sigma\mathbf{q} - \mathbf{p}'\mathbf{q}, \tag{12}$$

and the inverse demand function for product i is derived by the first-order condition with respect to q_i:

$$p_i = Y_i - bq_i - s\sum_{j\neq i} q_j \tag{13}$$

Given the demand function, the price (or quantity) competition results in a profit function quadratic in its own and others' reputation levels:

$$\pi_i = J_i\left(Y_i - c_i + K_i\sum_{j\neq i}(Y_j - c_j)\right)^2 - \frac{\phi_i}{2}F_i^2,$$

where K and L are positive constants. (See the appendix for the derivation.) Then, by replacing $K(Y_2 - c_2)$ in the main part to $K_i\sum_{j\neq i}(Y_j - c_j)$, we can generalize all the results so far to the n-firms oligopoly setting. Furthermore, we can analyze how the fake review strategy and the surplus change as the market becomes more competitive.

6 Extensions to Behavioral Consumers

In this section, we report the analyses of fake review oligopoly models with naive and partially naive consumers, who cannot fully recognize the fake review writing activities strategically conducted by firms. Given the existence of such (partially) naive consumers, firms adjust their fake-review-writing strategies, and resulting market outcomes could be different from those with rational consumers.

This section is organized as follows. First, we define a naive consumer, who has a lack of understanding of firm-side fake review actions. We then characterize the market equilibrium with a representative naive consumer. Second, we define a partially naive representative consumer, who has a belief of the convex combination of a rational belief (analyzed in previous sections) and a naïve belief. We then analyze the firm-side conduct and characterize equilibrium properties with this partially naive representative consumer.

As a summary, given the existence of the partially naive representative consumer, a firm faces a subtle trade-off with three channels: First, a firm has more incentive to write fake reviews to exploit a (partially) naive belief, who cannot discount the inflated ratings. Second, a firm has less incentive to write fake reviews as a (partially) naïve consumer cannot discount the opponents' inflated ratings, diminishing the firm's relative reputation advantage. As a consequence, a strategic fake review substitute property arises with a (partially) naïve consumer. Third, a firm faces less incentive to write fake reviews as a partially naïve consumer does not fully understand the signaling role of fake reviews. Thus, the equilibrium amount of generated fake reviews is determined by accounting these three channels of fake reviews. As such, throughout this section, we emphasize the channels and tradeoff with marginal benefit functions.

6.1 Naive/Credulous Consumers

As in Yasui (2020), we consider consumers who cannot take the existence of fake reviews into account while understanding that the reviews are noisy signals of underlying qualities. Such a consumer's naive belief in the products' qualities corresponds to a belief on θ_i, given $\alpha_i = 0$ and $\gamma_i = 0$. That is written as follows:

$$Y_i^N \equiv E^N\left[\theta_i | R_1,\, R_2, s\right] = \mu + \underbrace{\frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\epsilon^2}}_{\equiv \lambda_i^N}(R_i - \underbrace{\mu}_{E^N[R_i]})$$

where E^N denotes a naive consumer's belief. The naive consumer cannot discount the boosted ratings. The naive consumer maximizes their *interim* utility with a belief described above, given ratings' realization. Such a maximization results in the following first-order conditions:

$$
\begin{aligned}
0 &= Y_1^N - b_1 q_1 - s q_2 - p_1 \\
0 &= Y_2^N - b_2 q_2 - s q_1 - p_2
\end{aligned}
$$

which is rearranged as

$$
\begin{aligned}
q_1 &= \frac{b_2 Y_1^N - s Y_2^N}{b_1 b_2 - s^2} - \frac{b_2}{b_1 b_2 - s^2} p_1 + \frac{s}{b_1 b_2 - s^2} p_2 \equiv q_1^N \\
q_2 &= \frac{b_1 Y_2^N - s Y_1^N}{b_2 b_1 - s^2} - \frac{b_1}{b_2 b_1 - s^2} p_2 + \frac{s}{b_2 b_1 - s^2} p_1 \equiv q_2^N
\end{aligned}
$$

Then, given the above demand functions, firms choose their optimal price (or quantities) given the opponent's strategies. Again, this results in profit functions quadratic in $Y_i^N - c_i$:

$$
\pi_1 = J_1 \times \left(Y_1^N - c_1 + K_1(Y_2^N - c_2)\right)^2 - \frac{\phi_1}{2} F_1^2.
$$

Firm i maximizes the profit function described above with respect to F_i, given

$$
R_i = \theta_i + F_i + \epsilon_i, \tag{14}
$$

$$
Y_i^N = \mu + \lambda_i^N (R_i - \mu)
$$

instead of

$$
Y_i = \mu + \lambda_i (R_i - (1 + \alpha_i)\mu).
$$

Proposition 10 (Best response with the naive consumer)**.** *If the consumer is naive, then*

1. α_i *is positive and uniquely determined, independent of the other firm's fake review strategy;*

2. γ_i *is decreasing in* γ_j $(j \neq i)$

3. $E[F_i]$ *is decreasing in* $E[F_j]$ $(j \neq i)$

Proposition 11. *If the consumer is naive, the equilibrium exists and it is unique.*

Proposition 12. *If the consumer is naive,*

1. α_i *is smaller than one with the rational consumer;*

Because α_i is still positive, fake reviews enhance the correlation between its rating and the underlying quality of the product. However, the naive consumer suffers from bias when they evaluate the rating. Such a trade-off is summarized in the following expression of the expected consumer surplus.[22]

$$\begin{aligned} E[U] = E&\left[\frac{b_1}{2}q_1^2 + \frac{b_2}{2}q_2^2 + \delta q_1 q_2\right] \\ &- E\left[\left(Y_1^N - Y_1^R\right) q_1 + \left(Y_2^N - Y_2^R\right) q_2\right] \end{aligned}$$

The first line is the same as the rational consumer. This line implies that even the naive consumer somewhat benefits from the equilibrium patterns of fake reviews.

In the second line, $Y_i^N - Y_i^R$ corresponds to the bias caused by fake reviews. The naive consumer has higher expectations than the rational consumer. The harm to the naive consumer is greater if the consumer purchases a larger volume of the product with biased expectations.

Proposition 13. *If the consumer is naive, the consumer surplus is lower/higher with fake reviews than without fake reviews.*

6.1.1 Partially Naive Consumers

If the consumer is partially rational, the product i's reputation can be defined as follows:

$$\begin{aligned} Y_i^\eta &= \eta Y_i^R + (1-\eta) Y_i^N \\ &= \eta \left\{\mu + \lambda_i^R \left(R_i - \{(1+\alpha_i)\mu + \gamma_i\}\right)\right\} + (1-\eta)\left\{\mu + \lambda_i^N (R_i - \mu)\right\} \\ &= \mu + \lambda_i^\eta (R_i - \mu) - \eta\lambda_i^R \underbrace{(\alpha_i \mu + \gamma_i)}_{=E[F_i]} \end{aligned}$$

where Y_i^R is the reputation formed by a rational consumer and Y_i^N is the reputation formed by a naive consumer, and $\lambda_i^\eta = \eta\lambda_i^R + (1-\eta)\lambda_i^N$. Such a convex combination of beliefs can be interpreted that either (i) the representative consumer (each consumer in a mass) is partially rational, or (ii) there are a mass η of rational consumers and a mass $(1-\eta)$ of naive consumers. See the Appendix **??** for the second interpretation. Given this belief, the consumer's interim utility maximization leads to the following demand function

$$q_i = \frac{b_j Y_i^\eta - s Y_j^\eta}{b_i b_j - s^2} - \frac{b_j}{b_i b_j - s^2} p_i + \frac{s}{b_i b_j - s^2} p_j$$

[22] See the Appendix for the derivation.

Then, firm 1's optimal behavior is characterized by maximizing the following objective function

$$
\begin{aligned}
\widetilde{\pi}_1 &= E_1\left[\left(\left(Y_1^\eta - c_1\right) + K_1\left(Y_2^\eta - c_2\right)\right)^2 - \frac{\phi_1}{2J_1}F_1^2|\theta_1\right] \\
&= E_1\left[\Big(\big(\mu + \lambda_1^\eta\left(\theta_1 + F_1 + \epsilon_1 - \mu\right) - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big) + K_1\left(Y_2^\eta - c_2\right)\Big)^2 - \frac{\phi_1}{2J_1}F_1^2|\theta_1\right] \\
&= E_1\left[\Big(\big(\mu + \lambda_1^\eta\left(\theta_1 + F_1 - \mu\right) - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big) + \lambda_1^\eta\epsilon_1 + K_1\left(Y_2^\eta - c_2\right)\Big)^2|\theta_1\right] - \frac{\phi_1}{2J_1}F_1^2 \\
&= \big(\mu + \lambda_1^\eta\left(\theta_1 + F_1 - \mu\right) - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big)^2 \\
&+ 2\big(\mu + \lambda_1^\eta\left(\theta_1 + F_1 - \mu\right) - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big) E_1\left[\lambda_1^\eta\epsilon_1 + K_1\left(Y_2^\eta - c_2\right)|\theta_1\right] \\
&+ E_1\left[\left(\lambda_1^\eta\epsilon_1 + K_1\left(Y_2^\eta - c_2\right)\right)^2|\theta_1\right] \\
&- \frac{\phi_1}{2J_1}F_1^2 \\
&= \big(\mu + \lambda_1^\eta\left(\theta_1 + F_1 - \mu\right) - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big)^2 \\
&+ 2\big(\mu + \lambda_1^\eta\left(\theta_1 + F_1 - \mu\right) - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big) K_1E_1\left[\left(Y_2^\eta - c_2\right)|\theta_1\right] \\
&+ E_1\left[\left(\lambda_1^\eta\epsilon_1 + K_1\left(Y_2^\eta - c_2\right)\right)^2|\theta_1\right] \\
&- \frac{\phi_1}{2J_1}F_1^2
\end{aligned}
$$

The first-order condition with respect to F_1 is written as follows:

$$
\begin{aligned}
0 &= 2\lambda_1^\eta\big(\mu + \lambda_1^\eta\left(\theta_1 + F_1 - \mu\right) - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big) + 2\lambda_1^\eta K_1E_1\left[\left(Y_2^\eta - c_2\right)|\theta_1\right] - \frac{\phi_1}{J_1}F \\
\Leftrightarrow \left(\frac{\phi_1}{J_1} - 2\left(\lambda_1^\eta\right)^2\right)F_1 &= 2\lambda_1^\eta\big(\mu + \lambda_1^\eta\left(\theta_1 - \mu\right) - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big) + 2\lambda_1^\eta K_1E_1\left[\left(Y_2^\eta - c_2\right)|\theta_1\right] \\
\Leftrightarrow F_1 &= \frac{2\left(\lambda_1^\eta\right)^2\theta_1 + 2\lambda_1^\eta\big(\left(1 - \lambda_1^\eta\right)\mu - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big) + 2\lambda_1^\eta K_1E_1\left[\left(Y_2^\eta - c_2\right)|\theta_1\right]}{\frac{\phi_1}{J_1} - 2\left(\lambda_1^\eta\right)^2}
\end{aligned}
$$

Then. we have the following restrictions on equilibrium coefficients:

$$
\begin{aligned}
\alpha_1 &= \frac{2\left(\lambda_1^\eta\right)^2}{\frac{\phi_1}{J_1} - 2\left(\lambda_1^\eta\right)^2} \\
\lambda_1^\eta &= \eta\frac{\left(1 + \alpha_1\right)}{\left(1 + \alpha_1\right)^2 + \left(\sigma_\epsilon/\sigma_\theta\right)^2} + \left(1 - \eta\right)\frac{1}{1 + \left(\sigma_\epsilon/\sigma_\theta\right)^2} \\
\gamma_1 &= \frac{2\lambda_1^\eta\big(\left(1 - \lambda_1^\eta\right)\mu - \eta\lambda_1^R\left(\alpha_1\mu + \gamma_1\right) - c_1\big) + 2\lambda_1^\eta K_1E_1\left[\left(Y_2^\eta - c_2\right)|\theta_1\right]}{\frac{\phi_1}{J_1} - 2\left(\lambda_1^\eta\right)^2}
\end{aligned}
$$

Note that

$$\begin{aligned}
E_1\left[Y_2^{\eta}|\theta_1\right] &= E_1\left[\mu + \lambda_2^{\eta}\left(\theta_2 + \alpha_2\theta_2 + \gamma_2 - \mu\right) - \eta\lambda_2^{R}\left(\alpha_2\mu + \gamma_2\right)|\theta_1\right] \\
&= \mu + \lambda_2^{\eta}\left(\alpha_2\theta_2 + \gamma_2\right) - \eta\lambda_2^{R}\left(\alpha_2\mu + \gamma_2\right) \\
&= \mu + \left(\eta\lambda_2^{R} + (1-\eta)\lambda_2^{N}\right)\left(\alpha_2\mu + \gamma_2\right) - \eta\lambda_2^{R}\left(\alpha_2\mu + \gamma_2\right) \\
&= \mu + (1-\eta)\lambda_2^{N}\left(\alpha_2\mu + \gamma_2\right)
\end{aligned}$$

Thus, the restriction for γ_1 is rewritten as

$$\begin{aligned}
\gamma_1 &= \frac{2\lambda_1^{\eta}\left(\left(1-\lambda_1^{\eta}\right)\mu - \eta\lambda_1^{R}\left(\alpha_1\mu + \gamma_1\right) - c_1\right) + 2\lambda_1^{\eta}K_1\left(\mu + (1-\eta)\lambda_2^{N}\left(\alpha_2\mu + \gamma_2\right) - c_2\right)}{\frac{\phi_1}{J_1} - 2\left(\lambda_1^{\eta}\right)^2} \\
&= 2\lambda_1^{\eta}\frac{\left(1-\lambda_1^{\eta}\right)\mu - c_1 + K_1\left(\mu - c_2\right) - \eta\lambda_1^{R}\left(\alpha_1\mu + \gamma_1\right) + (1-\eta)\lambda_2^{N}K_1\left(\alpha_2\mu + \gamma_2\right)}{\frac{\phi_1}{J_1} - 2\left(\lambda_1^{\eta}\right)^2}
\end{aligned}$$

Thus, unless η is equal to one, the equilibrium coefficient γ_i is determined by the best responses between firms 1 and 2. This determines the amount of fake reviews at equilibrium. The relationship between the consumer's rationality can be explained more intuitively by taking the expectation of firm 1's first-order condition:

$$\phi_1 E\left[F_1\right] = 2J_1\lambda_1^{\eta}\left(\mu - c_1 + K_1\left(\mu - c_2\right) + \lambda_1^{\eta}E\left[F_1\right] - \eta\lambda_1^{R}E\left[F_1\right] + (1-\eta)K_1\lambda_2^{N}E\left[F_2\right]\right)$$

The left-hand side is the expected marginal cost of making a fake review and the right-hand side is the expected marginal revenue of making a fake review. In the 4-th term of the marginal revenue, $\lambda_1^{\eta}E\left[F_1\right]$ appears because, if the reputation is shifted upward, then, so is the marginal effect of fake reviews. (Note that the revenue is quadratically increasing in the reputation.) However, such a reputational advantage is diminished by (i) expectations of the firm i's fake review strategy or (ii) the other firm's fake review strategy, depending on how much the consumer is rational. (i) If the consumer is rational, the consumer discounts the inflated ratings, and the reputation of firm i does not change on average. (ii) If the consumer is naive, the consumer is deceived by the other firm in the same way as being deceived by firm i. Thus, the firm i again loses its reputational advantage on average. The rationality parameter η smoothly connects these two extremes.

On top of that, λ_1^{η} becomes larger as the consumer becomes rational. The rational consumer can take positive α_i into account. Therefore, the rational consumer believes more that a high rating implies high quality. Thus, the rational consumer reacts to the ratings more, and the marginal effect of fake reviews is higher for rational consumers.

7 Conclusion

We model fake reviews in oligopoly market quantity and price competitions, which have not been reported in the literature. The presence of fake reviews could alter the oligopoly asymmetric information market competitions and their outcomes. Resulting market consequences are of the interest among researchers, as well as informative to market regulators and competition authority policymakers. In our models, each firm engages in a costly fake review writing action, inflates its review rating, and attempts to make its product look more attractive than others. On the other side of the market, given available rating information, consumers infer the quality of the product offered by each firm. Specifically, with an inflated rating, a firm could raise consumers' expected willingness to pay for its product, potentially upwardly shifting its demand function. Given such an inflated demand system, firms then engage in a standard static oligopoly market competition.

Analytically, we propose a linear rating specification with an idiosyncratic review noise (see Holmström (1999)): A product rating consists of the linear combination of (1) a true product-quality type, (2) fake reviews, and (3) an idiosyncratic review shock. (1) and (3) are from independent normal distributions, while (2) is the subject of strategic manipulation by a firm. Then, given a linear demand system, and by focusing on a linear strategy in private information of product quality type, we report that linear equilibrium fake-review writing strategies exist in Cournot and Bertrand competitions, respectively. Moreover, we apply the model to the market with both rational and (partially) naive consumers. Notably, regardless of price or quantity competition, and regardless of the consumers' rationality, there exists a positive monotone strategy for each firm: the equilibrium amount of fake reviews generated by an oligopoly firm increases with respect to a firm's quality type.

With rational consumers, we report our benchmark result of the fake review market outcome partial equivalence: expected prices and quantities remain unchanged with fake reviews, while the second moments of prices and quantities and expected surpluses ¬ are altered. With rational consumers, we report the following four findings.

First, regarding the linear fake-review equilibrium strategies, we report the unresponsive-best-response property, indicating that a firm's equilibrium fake review strategy does not depend on other firms' fake review strategies, and the signaling property, indicating that high-quality firm writes more fake reviews. In equilibrium, after rational consumers observe a firm's rating, they rationally conjecture its product quality type, forming their willingness to pay for this firm's product. The rational consumers correctly discount the inflated review ratings. Given this conjecturing process, each firm chooses its fake review writing action based on the ex-ante expected types of opponent firms because the opponent's fake reviews affect only via consumers' expectation on the opponent's quality, which is correctly discounted. Accordingly, in equilibrium, a firm does not need to condition its fake review writing strategy on other firms' fake reviews, as rational consumers eventually discount inflated ratings. In contrast, the firm monotonically increases the fake-review writing effort according to its own quality-type because high quality-type leads to large sales even without fake reviews, which then implies large marginal impact of fake reviews.

Second, stemming from the signaling role of fake reviews described above, we report that expected market quantities and prices remain the same with and without fake reviews, regardless of the Cournot and Bertrand competitions. Third, counter-intuitively, fake reviews improve rational consumers' surplus, as rational consumers could recognize observed ratings as signals and could infer the process for inflating ratings with costly fake-review writing activities. In other words, ratings inflated by fake reviews could better signal product quality than ratings based only on authentic reviews. This surplus improvement is related to the consumer surplus (expected utility) with variance terms. The existence of fake reviews results in a higher variance of review scores, which is more informative to rational consumers for conjecturing product quality, compared to a no-fake-review environment Fourth, with rational consumers, we report some competition policy implications, such as exogenously increased fake review writing costs, as well as considering the special case of the market with some honest firms.

Next, given the benchmark results with rational consumers, we extend our model to (partially) naive consumers, who cannot fully comprehend fake review writing actions done by oligopoly firms. Note that partially naive consumers nest both rational and naive consumers as extreme cases. With (partially) naive consumers, this study reports the fake review market outcome non-equivalence: the existence of fake reviews results in large distortions in oligopoly market outcomes. Specifically, we report a takeaway.

With (partially) naive consumers, a strategic substitution property emerges in firms' fake-review writing actions, which is a drastic contrast to the unresponsive-best-response property with rational consumers. When rival firms write a large amount of fake reviews, (partially) naive consumers believe that fake reviews are genuine and are attracted by those rival firms' products. As a result, the firm in question faces relatively diminished demand function due to the product substitutability. Then, the marginal return from a fake-review writing action for this firm is now relatively smaller (than the one with rational consumers), resulting in a relatively smaller amount of fake-review writing effort. Similar logic applies to the vice-versa case, consisting of strategic fake-review substitutability.

Lastly, it is worth mentioning that the equilibrium oligopoly market outcomes crucially depend on consumers' rationality in conjecturing the costly fake-review generating process.

Given this, we would like to report that the scope of consumer rationality regarding the inflated ratings formation is the subject of experimental and empirical investigations (for example, see Akesson et al. (2023)), and we leave such applied topics for future studies, which are currently and actively being researched.

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Appendix A Proofs

A.1 Derivation of the Consumer's Belief

The representative consumer rationally believes that the sellers are taking the linear strategy $F_i = \alpha_i\theta_i + \gamma_i$. That is, the consumer believes that the true quality θ_i and the rating R_i are interacting as follows:

$$\begin{aligned}
\begin{bmatrix} \theta_i \\ R_i \end{bmatrix} &= \begin{bmatrix} \theta_i \\ \theta_i + F_i + \epsilon_i \end{bmatrix} = \begin{bmatrix} \theta_i \\ (1+\alpha_i)\,\theta_i + \gamma_i + \epsilon_i \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \gamma_i \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ (1+\alpha_i) & 1 \end{bmatrix} \begin{bmatrix} \theta_i \\ \epsilon_i \end{bmatrix} \\
&\sim N\left(\begin{bmatrix} \mu \\ (1+\alpha_i)\,\mu + \gamma_i \end{bmatrix}, \Omega \right),
\end{aligned}$$

where

$$\begin{aligned}
\Omega &= \begin{bmatrix} 1 & 0 \\ (1+\alpha_i) & 1 \end{bmatrix} \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & \sigma_\epsilon^2 \end{bmatrix} \begin{bmatrix} 1 & (1+\alpha_i) \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ (1+\alpha_i) & 1 \end{bmatrix} \begin{bmatrix} \sigma_\theta^2 & (1+\alpha_i)\,\sigma_\theta^2 \\ 0 & \sigma_\epsilon^2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_\theta^2 & (1+\alpha_i)\,\sigma_\theta^2 \\ (1+\alpha_i)\,\sigma_\theta^2 & (1+\alpha_i)^2\,\sigma_\theta^2 + \sigma_\epsilon^2 \end{bmatrix}.
\end{aligned}$$

Thus, the required property is derived as the conditional expectation of multivariate normal distribution. Given this multivariate normal distribution, the reputation of firm i's product (Y_i) is written as

$$\underbrace{E\left[\theta_i | R_1,\ R_2\right]}_{=Y_i} = \mu + \underbrace{\frac{(1+\alpha_i)\,\sigma_\theta^2}{(1+\alpha_i)^2\,\sigma_\theta^2 + \sigma_\epsilon^2}}_{=\lambda_i}(R_i - \underbrace{\{(1+\alpha_i)\,\mu + \gamma_i\}}_{E[R_i]}),$$

where λ_i is a discounting parameter.

A.2 Proof of Proposition 2

In eq. (6), λ changes from 0 to $\sqrt{\phi_1/(2J_1)}$ as α changes from 0 to ∞. In eq. (7), λ changes from $1/(1+(\sigma_\epsilon/\sigma_\theta))$ to zero as α changes from 0 to ∞. Thus, those equations intersect each other at some point. The corresponding γ_1 is determined by eq. (8). Thus, there exist α

and γ which satisfy the equilibrium condition. The uniqueness is obtained from the unique intersection of eqs. (6) and (7) when $\sigma_\theta > \sigma_\epsilon$. With this parameter requirement, λ_1 from eq. (7) is decreasing in $\alpha > 0$ while λ_1 from eq.(6) is always increasing in α.

Appendix B Reduced Form Profit Functions

In this Appendix section, we define the coefficients of reduced form profit functions used throughout this study. The section consists of two subsections: First, we report two-firm Cournot and Bertrand models, respectively. Second, we extend the modeling framework to n-firm Cournot and Bertrand competition, respectively. Within each model, we initially define the payoff function. We then derive the first-order condition, as well as defining the coefficients in the reduced form profit function. A caveat is required that, although all profit functions have the common quadratic form in terms of the firm's own fake review action, the coefficients vary across different models. Throughout this study, we exploit such a commonality in the quadratic form.

B.1 Two-Firm Bertrand Competition

In this appendix subsection, we derive the reduced-form Bertrand oligopoly equilibrium profit functions. Given the utility function of the representative consumer, we have the demand system of

$$\begin{aligned} q_1 &= \dot{Y}_1 - \dot{b}_1 p_1 + \dot{s} p_2, \\ q_2 &= \dot{Y}_2 - \dot{b}_2 p_2 + \dot{s} p_1, \end{aligned}$$

where

$$\dot{Y}_i = \frac{b_i}{b_1 b_2 - s^2} Y_i - \frac{s}{b_1 b_2 - s^2} Y_j, \qquad \dot{b}_i = \frac{b_j}{b_1 b_2 - s^2}, \qquad \dot{s} = \frac{s}{b_1 b_2 - s^2},$$

for $i = 1, 2$ and $j \neq i$. The profit function for firm i is $\pi_i = (p_i - c_i)\, q_i - \frac{\phi_i}{2} F_i^2$. The first-order condition yields the Bertrand competition equilibrium price, quantity, markup, and profit for firm i

$$\begin{aligned} p_i &= \frac{2\dot{b}_j}{4\dot{b}_i\dot{b}_j - \dot{s}^2}\left(\dot{Y}_i + \dot{b}_i c_i\right) + \frac{\dot{s}}{4\dot{b}_i\dot{b}_j - \dot{s}^2}\left(\dot{Y}_j + \dot{b}_j c_j\right), \\ q_i &= \frac{2\dot{b}_i\dot{b}_j}{4\dot{b}_i\dot{b}_j - \dot{s}^2}\dot{Y}_i + \frac{\left(\dot{s}^2 - 2\dot{b}_i\dot{b}_j\right)\dot{b}_i}{4\dot{b}_i\dot{b}_j - \dot{s}^2} c_i + \frac{\dot{b}_i\dot{s}}{4\dot{b}_i\dot{b}_j - \dot{s}^2}\left(\dot{Y}_j + \dot{b}_j c_j\right), \\ p_i - c_i &= \frac{1}{\dot{b}_i} q_i \end{aligned}$$

Then, the Bertrand competition equilibrium profit function is written by the equation of

$$\pi_i = J_i \left(Y_i - c_i + K_i(Y_j - c_j)\right)^2 - \frac{\phi_i}{2} F_i^2,$$

where

$$\begin{aligned}
J_i &= \frac{\dot{b}_i \left(2\dot{b}_i\dot{b}_j - \dot{s}^2\right)^2}{\left(4\dot{b}_i\dot{b}_j - \dot{s}^2\right)^2} = \frac{b_j \left(2b_ib_j - s^2\right)^2}{\left(b_ib_j - s^2\right)\left(4b_ib_j - s^2\right)^2}, \\
K_i &= -\frac{\dot{b}_j \dot{s}}{2\dot{b}_i\dot{b}_j - \dot{s}^2} = -\frac{b_i s}{2b_ib_j - s^2}.
\end{aligned}$$

B.2 Two-Firm Cournot Competition

In this appendix subsection, we derive the reduced-form Cournot oligopoly equilibrium profit functions and define shorthand coefficient notations. Throughout this subsection, for simplicity, we treat the reputation variables Y_1 and Y_2 as given and constant terms. Given the utility function of the representative consumer, we have the inverse demand system of

$$\begin{aligned}
p_1 &= Y_1 - b_1 q_1 - s q_2, \\
p_2 &= Y_2 - b_2 q_2 - s q_1.
\end{aligned} \tag{15}$$

We use the notation of $i, j \in \{1, 2\}$ for a relevant profit-maximizing firm and an opponent firm. The profit function for firm i is $\pi_i = (p_i - c_i)\, q_i - \frac{\phi_i}{2} F_i^2$. The first-order condition yields the Cournot competition equilibrium quantity, price, markup, and profit for firm i

$$\begin{aligned}
q_i &= \frac{2b_j}{4b_ib_j - s^2}\left(Y_i - c_i\right) - \frac{s}{4b_ib_j - s^2}\left(Y_j - c_j\right), \\
p_i &= \frac{2b_ib_j}{4b_ib_j - s^2} Y_i - \frac{s^2 - 2b_ib_j}{4b_ib_j - s^2} c_i - \frac{b_i s}{4b_ib_j - s^2}\left(Y_j - c_j\right), \\
p_i - c_i &= b_i q_i, \qquad \pi_i = b_i q_i^2 - \frac{\phi_i}{2} F_i^2.
\end{aligned}$$

Then, the Cournot competition equilibrium profit function is written as

$$\pi_i = J_i \left(Y_i - c_i + K_i(Y_j - c_j)\right)^2 - \frac{\phi_i}{2} F_i^2,$$

where we use the shorthand coefficient notations of

$$J_i = \frac{4b_i b_j^2}{\left(4b_i b_j - s^2\right)^2}, \quad K_i = -\frac{s}{2b_j}.$$

B.3 Derivation of Ex Ante Surpluses at Equilibrium

The *ex post* consumer surplus is

$$U = \theta_1 q_1 + \theta_2 q_2 - \frac{b_1}{2} q_1^2 - \frac{b_2}{2} q_2^2 - s q_1 q_2 - p_1 q_1 - p_2 q_2$$

The consumer (with passive belief) tries to maximize the following *interim* consumer surpluls:

$$E\left[U | R_1,\, R_2\right] = Y_1 q_1 + Y_2 q_2 - \frac{b_1}{2} q_1^2 - \frac{b_2}{2} q_2^2 - s q_1 q_2 - p_1 q_1 - p_2 q_2$$

as a result, the optimal quantities to purchase (demand) are characterized by the following first-order equations with respect to q_1 and q_2:

$$\begin{aligned} 0 &= Y_1 - b_1 q_1 - s q_2 - p_1 \\ 0 &= Y_2 - b_2 q_2 - s q_1 - p_2 \end{aligned}$$

Therefore, the *ex post* consumer surplus *at equilibrium* is

$$\begin{aligned} U &= \theta_1 q_1 + \theta_2 q_2 - \frac{b_1}{2} q_1^2 - \frac{b_2}{2} q_2^2 - s q_1 q_2 - p_1 q_1 - p_2 q_2 \\ &= \left(\theta_1 - p_1 - \frac{b}{2} q_1 - \frac{s}{2} q_2\right) q_1 + \left(\theta_2 - p_2 - \frac{b}{2} q_2 - \frac{s}{2} q_1\right) q_2 \\ &= \left(\theta_1 - (Y_1 - b_1 q_1 - s q_2) - \frac{b}{2} q_1 - \frac{s}{2} q_2\right) q_1 + \left(\theta_2 - (Y_2 - b_2 q_2 - s q_1) - \frac{b}{2} q_2 - \frac{s}{2} q_1\right) q_2 \\ &= \left(\theta_1 - Y_1 + \frac{b_1}{2} q_1 + \frac{s}{2} q_2\right) q_1 + \left(\theta_2 - Y_2 + \frac{b_2}{2} q_2 + \frac{s}{2} q_1\right) q_2 \end{aligned}$$

Then, the *ex ante* consumer surplus *at equilibrium* is

$$
\begin{aligned}
E[U] &= E[E[U|R_1,\, R_2]] \\
&= E\left[E\left[\left(\theta_1 - Y_1 + \frac{b_1}{2}q_1 + \frac{s}{2}q_2\right)q_1 + \left(\theta_2 - Y_2 + \frac{b_2}{2}q_2 + \frac{s}{2}q_1\right)q_2|R_1,\, R_2\right]\right] \\
&= E[E[(\theta_1 - E[\theta_1|R_1,\, R_2])|R_1,\, R_2]\, q_1 + E[(\theta_2 - E[\theta_2|R_1,\, R_2])|R_1,\, R_2]\, q_2] \\
&\quad + E\left[\left(\frac{b_1}{2}q_1 + \frac{s}{2}q_2\right)q_1 + \left(\frac{b_2}{2}q_2 + \frac{s}{2}q_1\right)q_2\right] \\
&= E\left[\frac{b_1}{2}q_1^2 + \frac{b_2}{2}q_2^2 + sq_1q_2\right] \\
&= \frac{b_1}{2}Var(q_1) + \frac{b_2}{2}Var(q_2) + sCov(q_1,\, q_2) + \frac{b_1}{2}E[q_1]^2 + \frac{b_2}{2}E[q_2]^2 + sE[q_1]\,E[q_2]
\end{aligned}
$$

Appendix C Derivations Related to the Naive Consumer

C.1 Derivation of the Consumer Surplus for the Naive Consumer

The *ex post* consumer surplus is

$$U = \theta_1 q_1 + \theta_2 q_2 - \frac{b_1}{2} q_1^2 - \frac{b_2}{2} q_2^2 - \delta q_1 q_2 - p_1 q_1 - p_2 q_2$$

The consumer with *partially rational* and *passive* belief tries to maximize the following *interim* consumer surpluls:

$$E^{\eta}\left[U | R_1,\, R_2\right] = Y_1^{\eta} q_1 + Y_2^{\eta} q_2 - \frac{b_1}{2} q_1^2 - \frac{b_2}{2} q_2^2 - \delta q_1 q_2 - p_1 q_1 - p_2 q_2$$

as a result, the optimal quantities to puchase (demand) is characterized by the following first-order equations with respect to q_1 and q_2:

$$\begin{aligned} 0 &= Y_1^{\eta} - b_1 q_1 - \delta q_2 - p_1 \\ 0 &= Y_2^{\eta} - b_2 q_2 - \delta q_1 - p_2 \end{aligned}$$

Therefore, the *ex post* consumer surplus *at equilibrium* is

$$\begin{aligned} U &= \theta_1 q_1 + \theta_2 q_2 - \frac{b_1}{2} q_1^2 - \frac{b_2}{2} q_2^2 - \delta q_1 q_2 - p_1 q_1 - p_2 q_2 \\ &= \left(\theta_1 - p_1 - \frac{b}{2} q_1 - \frac{\delta}{2} q_2\right) q_1 + \left(\theta_2 - p_2 - \frac{b}{2} q_2 - \frac{\delta}{2} q_1\right) q_2 \\ &= \left(\theta_1 - (Y_1^{\eta} - b_1 q_1 - \delta q_2) - \frac{b}{2} q_1 - \frac{\delta}{2} q_2\right) q_1 + \left(\theta_2 - (Y_2^{\eta} - b_2 q_2 - \delta q_1) - \frac{b}{2} q_2 - \frac{\delta}{2} q_1\right) q_2 \\ &= \left(\theta_1 - Y_1^{\eta} + \frac{b_1}{2} q_1 + \frac{\delta}{2} q_2\right) q_1 + \left(\theta_2 - Y_2^{\eta} + \frac{b_2}{2} q_2 + \frac{\delta}{2} q_1\right) q_2 \end{aligned}$$

Then, the *ex ante* consumer surplus *at equilibrium* is

$$
\begin{aligned}
E[U] &= E[E[U|R_1,\, R_2]] \\
&= E\left[E\left[\left(\theta_1 - Y_1^{\eta} + \frac{b_1}{2}q_1 + \frac{\delta}{2}q_2\right)q_1 + \left(\theta_2 - Y_2^{\eta} + \frac{b_2}{2}q_2 + \frac{\delta}{2}q_1\right)q_2|R_1,\, R_2\right]\right] \\
&= E\left[E\left[\left(\theta_1 - Y_1^{\eta}\right)|R_1,\, R_2\right]q_1 + E\left[\left(\theta_2 - Y_2^{\eta}\right)|R_1,\, R_2\right]q_2 + \left(\frac{b_1}{2}q_1 + \frac{\delta}{2}q_2\right)q_1 + \left(\frac{b_2}{2}q_2 + \frac{\delta}{2}q_1\right)q_2\right] \\
&= E\left[\left(Y_1^R - Y_1^{\eta}\right)q_1 + \left(Y_2^R - Y_2^{\eta}\right)q_2 + \left(\frac{b_1}{2}q_1 + \frac{\delta}{2}q_2\right)q_1 + \left(\frac{b_2}{2}q_2 + \frac{\delta}{2}q_1\right)q_2\right] \\
&= E\left[(1-\eta)\left(Y_1^R - Y_1^N\right)q_1 + (1-\eta)\left(Y_2^R - Y_2^N\right)q_2 + \left(\frac{b_1}{2}q_1 + \frac{\delta}{2}q_2\right)q_1 + \left(\frac{b_2}{2}q_2 + \frac{\delta}{2}q_1\right)q_2\right] \\
&= E\left[\frac{b_1}{2}q_1^2 + \frac{b_2}{2}q_2^2 + \delta q_1 q_2\right] \\
&- (1-\eta)\,E\left[\left(Y_1^N - Y_1^R\right)q_1 + \left(Y_2^N - Y_2^R\right)q_2\right] \\
&= \frac{b_1}{2}Var(q_1) + \frac{b_2}{2}Var(q_2) + \delta Cov(q_1,\, q_2) + \frac{b_1}{2}E[q_1]^2 + \frac{b_2}{2}E[q_2]^2 + \delta E[q_1]\,E[q_2] \\
&- (1-\eta)\,E\left[\left(Y_1^N - Y_1^R\right)q_1 + \left(Y_2^N - Y_2^R\right)q_2\right]
\end{aligned}
$$

C.2 Interpretation of the Partially Rational Consumer

If the market is filled with a mass η of rational consumers and a mass $(1-\eta)$ of naive consumers, the market for product i is defined as $q_i = \eta q_i^R + (1-\eta)\,q_i^N$ where q_i^R is the demand from a rational consumer and q_i^N is the demand from a naive consumer, each of which is defined so far. Then, the market demand for product i is

$$
\begin{aligned}
q_i &= \eta q_i^R + (1-\eta)\,q_i^N \\
&= \eta\left\{\frac{b_j Y_i^R - s Y_j^R}{b_i b_j - s^2} - \frac{b_j}{b_i b_j - s^2}p_i + \frac{s}{b_i b_j - s^2}p_j\right\} \\
&+ (1-\eta)\left\{\frac{b_j Y_i^N - s Y_j^N}{b_i b_j - s^2} - \frac{b_j}{b_i b_j - s^2}p_i + \frac{s}{b_i b_j - s^2}p_j\right\} \\
&= \frac{b_j Y_i^{\eta} - s Y_j^{\eta}}{b_i b_j - s^2} - \frac{b_j}{b_i b_j - s^2}p_i + \frac{s}{b_i b_j - s^2}p_j
\end{aligned}
$$

where $Y_i^{\eta} = \eta Y_i^R + (1-\eta)\,Y_i^N$. Thus, the demand function is equivalent to the one from the partially rational representative consumer.

Appendix D Model with a Joint Elliptical Distribution

Note that we actively use the normality assumption when we calculate the rating as a linear combination of quality and noise, and we calculate the conditional expectation of quality given the rating's realization. Those properties of the jointly normal distribution are robust in the general elliptical distribution in a sense explained below. For the normative analysis in Subsection 4.1, the surplus functions, which is characterized by second moments, are just multiplied by a scalar. Therefore, the ordinal property of the surplus function is robust with the general elliptical distribution as well. Thus, the results in the main part is robust with a non-negative elliptical distirbution whose tail is truncated.

Suppose that $x_i = \begin{bmatrix} \theta_i \\ \epsilon_i \end{bmatrix}$ follows elliptical distribution with a location parameter $\bar{\mu} = \begin{bmatrix} \mu_i \\ 0 \end{bmatrix}$, scale parameter $\Sigma = \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & \sigma_\epsilon^2 \end{bmatrix}$, and a density function $g(\cdot)$ (Cambanis et al., 1981; Gómez et al., 2003). That is, the probability density function of $\begin{bmatrix} \theta_i \\ \epsilon_i \end{bmatrix}$ is characterized as $f_i(x_i) = g_i(x_i' \Sigma x_i)$. More precisely, we suppose that x_i and x_j are jointly follow an elliptial distribution with Σ's as block diagonal elements and 0's off-diagonal elements of scaling parameters. Therefore, (i) x_i and x_j are not correlated, (ii) an affine transformation of $[x_i, x_j]'$ follows an elliptical distribution, whose location and scaling parameters are characterized as in a joint normal distribution, but (iii) x_i and x_j are not necessarily independent each other (e.g., If the support of (θ_1, θ_2) is an area enclosed by a circle with a finite radius, the support of $(\theta_1 | \theta_2)$ depends on the value of θ_2).

Let a rating on Seller i as a noisy and potentially biased signal of the underlying quality, $R_i = \theta_i + F_i + \epsilon_i$. Suppose that Seller i uses a linear strategy in fake reviews, that is,

$$F_i = \alpha_i \theta_i + \gamma_i$$

Then, a rating on Seller i's product is written as

$$R_i = \theta_i + \alpha_i \theta_i + \gamma_i + \epsilon_i$$

Now, $\begin{bmatrix} \theta_i \\ R_i \end{bmatrix}$ is written as a linear transformation of $\begin{bmatrix} \theta_i \\ \epsilon_i \end{bmatrix}$:

$$\begin{bmatrix} \theta_i \\ R_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 + \alpha_i & 1 \end{bmatrix} \begin{bmatrix} \theta_i \\ \epsilon_i \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma_i \end{bmatrix}$$

Then, by Theorem 5 of Gomez et al (2003), $\begin{bmatrix} \theta_i \\ R_i \end{bmatrix}$ follows an elliptical distribution with a location parameter, $\begin{bmatrix} 1 & 0 \\ 1+\alpha_i & 1 \end{bmatrix} \begin{bmatrix} \mu_i \\ \epsilon_i \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma_i \end{bmatrix} = \begin{bmatrix} \mu_i \\ (1+\alpha_i)\,\mu_i + \gamma_i \end{bmatrix}$, scaling parameter

$$\begin{aligned} & \begin{bmatrix} 1 & 0 \\ (1+\alpha_i) & 1 \end{bmatrix} \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & \sigma_\epsilon^2 \end{bmatrix} \begin{bmatrix} 1 & (1+\alpha_i) \\ 0 & 1 \end{bmatrix} \\ = & \begin{bmatrix} 1 & 0 \\ (1+\alpha_i) & 1 \end{bmatrix} \begin{bmatrix} \sigma_\theta^2 & \sigma_\theta^2\,(1+\alpha_i) \\ 0 & \sigma_\epsilon^2 \end{bmatrix} \\ = & \begin{bmatrix} \sigma_\theta^2 & \sigma_\theta^2\,(1+\alpha_i) \\ \sigma_\theta^2\,(1+\alpha_i) & \sigma_\theta^2\,(1+\alpha_i)^2 + \sigma_\epsilon^2 \end{bmatrix} \end{aligned}$$

with some density function $\tilde{g}_i\,(\cdot)$.

Then, by Theorem 8 of Gómez et al. (2003), $(\theta_i|R_i)$ follows an elliptical distribution with a location parameter

$$\mu_i + \frac{\sigma_\theta^2\,(1+\alpha_i)}{\sigma_\theta^2\,(1+\alpha_i)^2 + \sigma_\epsilon^2}\,(R_i - ((1+\alpha_i)\,\mu_i + \gamma_i))$$

scaling parameter $\sigma_\theta^2 - \frac{\sigma_\theta^4 (1+\alpha_i)^2}{\sigma_\theta^2 (1+\alpha_i)^2 + \sigma_\epsilon^2}$ with some density function $\hat{g}\,(\cdot)$.

Appendix E Notation Table

Throughout this study, we use many notations for strategic choice variables, primitive parameters, and others, such as shorthand notations. In this Appendix section, we list such notations. In the table below, we first list the notations. We then report the label and short descriptions for each notation.

Notation	Label	Descriptions and Notes
i and j	Firm index	Generic indices of firms. We often use $i = 1$ and $j = 2$ specifications in a two-firm oligopoly competition setting.
q_i	Quantity supplied	Choice variable of firm i in a Cournot competition.
p_i	Price of firm i's product	Choice variable of firm i in a Bertrand competition.
c_i	Marginal production cost	Marginal production costs are public information among all market participants. Also, marginal costs could be different across firms in this study. c_is could be heterogeneous among firms
ϕ_i	Coefficient of quadratic fake-review-writing cost of firm i	ϕ_is are public information among all market participants. ϕ_is could be heterogeneous among firms.
F_i	Fake-review-writing effort	Firm i's fake-review-writing strategy. In this study, we focus on linear fake-review-writing strategies.
π_i or $\dot{\pi}_i$	Profit of firm i	A profit function of firm i in a Cournot competition. The dot notation is used for a profit funcition in a Bertrand competition. The profit of firm i is quadratic in its fake review effort (F_i).
$\tilde{\pi}_i$ or $\dot{\tilde{\pi}}_i$	Reduced-form profit of firm i	A reduced-form profit function of firm i in a Cournot competition, which is used for shortening mathematical notations. The dot notation is used for a reduced-form profit function in a Bertrand competition. The reduced-form profit of firm i is quadratic in its fake review effort (F_i).
α_i	Slope coefficient of linear fake-review-writing strategy	The linear fake-review-writing strategy is $F_i = \alpha_i\theta_i + \gamma_i$. This slope could be heterogeneous among firms.
γ_i	Intercept of linear fake-review-writing strategy	The linear fake-review-writing strategy is $F_i = \alpha_i\theta_i + \gamma_i$. This intercept could be heterogeneous among firms.
θ_i	True quality of firm i's product	The true quality of firm i's product is private information of firm i. This random variable is distributed as a normal distribution of $\theta_i \sim \mathcal{N}(\mu, \sigma_\theta^2)$.
μ	Expectation of true quality (θ_i)	The expectation of the true product quality of firm i, which is common among all firms.
σ_θ^2	Variance of true quality (θ_i)	The variance of the true product quality of firm i, which is common among all firms.
b_i or $\dot{b}_i$	(Inverse) Demand function slope coefficient	The slope coefficient of firm i's inverse demand function. The dot notation is used for the slope coefficient of a demand function.
s or $\dot{s}$	(Inverse) Demand substitution parameter	The substitution parameter in an inverse demand function, common across all firms. The dot notation is used for the substitution parameter in a demand function.

R_i	Rating of firm i's product	The rating of firm i's product, publicly observed. $R_i = \theta_i + F_i + \epsilon_i$.
ϵ_i	Review shock on firm i's rating	The review shock in the rating of firm i's product, distributed as a normal distribution of $\epsilon_i \sim \mathcal{N}(0, \sigma_\epsilon^2)$. This review shock normal distribution is common across all firms.
σ_ϵ^2	Variance of review shock (ϵ_i)	The variance of the review shock, common across all firms.
a_i	Action in an oligopoly market competition	An action for firm i at the market competition stage. In a Cournot competition, $a_i = q_i$. In a Bertrand competition, $a_i = p_i$.
$E_c[\cdot]$ and $E_c[\cdot\|\cdot]$	Expectation of the representative consumer	Expectation and conditional expectation from the viewpoint of the representative consumer.
$E_i[\cdot]$ and $E_i[\cdot\|\cdot]$	Expectation of firm i	Expectation and conditional expectation from the viewpoint of firm i, who has the private information of its product type (θ_i). Under the two-firm duopoly competition setting, we often use the specification of $i = 1$, as well as $i = 2$, for emphasizing the nature of duopolistic interactions.
$E[\cdot]$	Ex-ante expectation	Ex-ante expectation from the view of a market designer (e.g., market competition and regulatory authority). Here, ex-ante means before each firm draws its product quality type.
Y_i or $\dot{Y}_i$	Reputation of firm i	The intercept of firm i's inverse demand function, which is $Y_i = E_c[\theta_i\|R_i, R_j]$ in a two-firm duopoly or $Y_i = E_c[\theta_i\|R_1, \cdots, R_n]$ in an n-firm oligopoly. This variable is an expected quality of firm i's product, which the representative consumer conjectures, conditional on publicly observed ratings. In this study, we broadly refer to this variable as reputation, attributing the representative consumer's willingness to pay for firm i's product.The dot notation is used for the intercept of a demand function.
λ_i	Sensitivity parameter	This parameter indicated the representative consumer's sensitivity in her/his evaluation of the observed rating of firm i's product (R_i). The sensitivity parameter stems from the representative consumer's conjecturing process for evaluating the true quality of firm i's product (θ_i), in response to the observed rating of firm i's product (R_i). The higher this parameter, the higher the marginal conjectured expectation with respect to an observed rating (R_i). $\lambda_i = [(1+\alpha_i)\sigma_\theta^2]/[(1+\alpha_i)^2\sigma_\theta^2 + \sigma_\epsilon^2]$.
J_i or $\dot{J}_i$	Outer coefficient of quadratic profit function (of firm i)	A shorthand notation for the outer coefficient of the quadratic profit function (of firm i). Note that in our study, at a market competition stage, each firm has a quadratic profit function with respect to its fake-review-writing effort (F_i). The dot notation is used for a coefficient in a profit function in a Bertrand competition.
K_i or $\dot{K}_i$	Inner coefficient of quadratic profit function (of firm i)	A shorthand notation for the inner coefficient of the quadratic profit function (of firm i). Note that in our study, at a market competition stage, each firm has a quadratic profit function with respect to its fake-review-writing effort (F_i). The dot notation is used for a coefficient in a profit function in a Bertrand competition.

n	Number of firms	Number of firms in an n-firm oligopoly competition market.
$\mathbf{Y}$ or $\dot{\mathbf{Y}}$	Vector of reputations	A reputation vector in an n-firm oligopoly competition market, which stacks the reputation variable of all firms, $Y_i = E[\theta_i \vert R_1, \ldots, R_n]$. The dot notation is used for the vector of demand function intercepts.
$\mathbf{q}$	Vector of supplied quantities	A supply quantity vector in an n-firm oligopoly competition market, which stacks the supply quantities of all firms (q_is).
$\mathbf{p}$	Vector of prices	A price vector in an n-firm oligopoly competition market, which stacks the prices ($p_i s$).
b or $\dot{b}$	Common (inverse) demand function slope parameter	A common (across all firms) slope coefficient of the inverse demand function in an n-firm oligopoly competition. The dot notation is used for the common slope coefficient of a demand function in an n-firm oligopoly competition.
s or $\dot{s}$	Common (inverse) demand function substitution parameter	A common (across all firms) substitution coefficient of the inverse demand function in an n-firm oligopoly competition. The dot notation is used for the common substitution coefficient of a demand function in an n-firm oligopoly competition.
Σ or $\dot{\Sigma}$	Matrix of demand slope and substitution parameters	Representative consumer's utility function coefficient matrix, which consists of a common inverse demand function slope (b) and substitution (s) parameters. The dot notation is used for the common slope ($\dot{b}$) and common substitution ($\dot{s}$) parameters of the demand function.
$\mathbf{R}$	Vector of ratings	A rating vector in an n-firm oligopoly competition market, which stacks the (public information of) ratings of all firms' products, R_is.