Scoring and Favoritism in Optimal Procurement Design

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Abstract

In the pioneering work by Che (1993), a scoring auction was shown to be the optimal mechanism for procurement where firms compete in both price and quality of the future contract. Many procurement contracts, however, are awarded based on either observed characteristics of the firm or quality recorded in past contracts — these are quality characteristics paid for regardless of winning today.

We show that, when some costs are all-pay, that is, borne regardless of winning, scoring auction remains optimal among *symmetric* mechanisms, but with a different scoring rule, which may be either steeper or flatter than that in Che (1993) depending on the relative elasticities of winner-pay and all-pay costs. However, the symmetry of the mechanism may be restrictive, when the informational asymmetry over all-pay costs is relatively low.

For two firms, we identify two new families of optimal *asymmetric* mechanisms. When marginal all-pay costs are sufficiently convex, it is a scoring auction with individual *reserve scores* and a side payment. When they are sufficiently concave, it is a scoring auction with individual score *ceilings* and a side payment. Both mechanisms exhibit ex-ante favoritism while partially retaining auction-style competition.

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1 Introduction

In recent decades, procurement agencies across the world have come under increasing pressure to improve performance and deliver projects faster. The reason is that the traditional approach, when the contract is awarded to the lowest bidder, fails to capture the trade-off between costs and quality of procurement. At the same time, quality may represent a large portion of the buyer's utility.[1] As a result, numerous alternative auction designs have emerged, see Molenaar et al. (2007) for an overview.

One such design is the scoring auction, where the contract is awarded to the firm with the best combination of price and quality. It was shown to be superior to the traditional approach theoretically (see Che (1993)) when quality is contractible, and the associated costs are costs winner-pay. A special case when quality is represented by the speed of delivery is known as A+B auctions in road construction (see Lewis and Bajari (2011)). On the other hand, when quality is not contractible, the average bid auction (see Albano et al. (2006); Decarolis (2014), Decarolis (2018)) and the low-ball lottery auction (see Lopomo et al. (2022)) have been proposed, to combat adverse selection.

Our main goal is to find an optimal mechanism when quality is contractible, but the associated costs can be all-pay. The firm's experience and past performance, among others, have this feature. This is in stark contrast with most of the theoretical literature on procurement design, where quality is either adjusted as part of the contract design (see Che (1993); Asker and Cantillon (2008, 2010)) or exogenously given (see Lopomo et al. (2022)). In the tradition of the mechanism design literature, we model the firms as having a single latent type $\theta \in [0, 1]$, which represents her inefficiency.

We begin with the analysis of optimal symmetric mechanisms, which are relatively easy to characterize, see Proposition 1. We show that, under mild conditions, it is implemented by a scoring auction with a quasi-linear scoring rule, see Proposition 2. The equilibrium quality, however, is not independent of the number of bidders, as in Che (1993), but is decreasing, see Proposition 3. The intuition is that, as the market share of a firm shrinks, it can not afford to put the same level of investment upfront. Similarly, the scoring rule is not independent of the number of bidders, but the comparative statics is more complicated and depends on the relative elasticities of winner-pay and all-pay costs, see Proposition 4.

We proceed with several arguments showing that symmetric mechanisms can be sub-optimal, see Propositions 5 to 7. One such argument is simple: if there is no private information, sole sourcing (i.e., procurement from a single supplier) is strictly optimal. But

[1] Decarolis et al. (2016) document long-lasting blackouts associated with traditional price-only procurement auctions for electricity works in Italy.

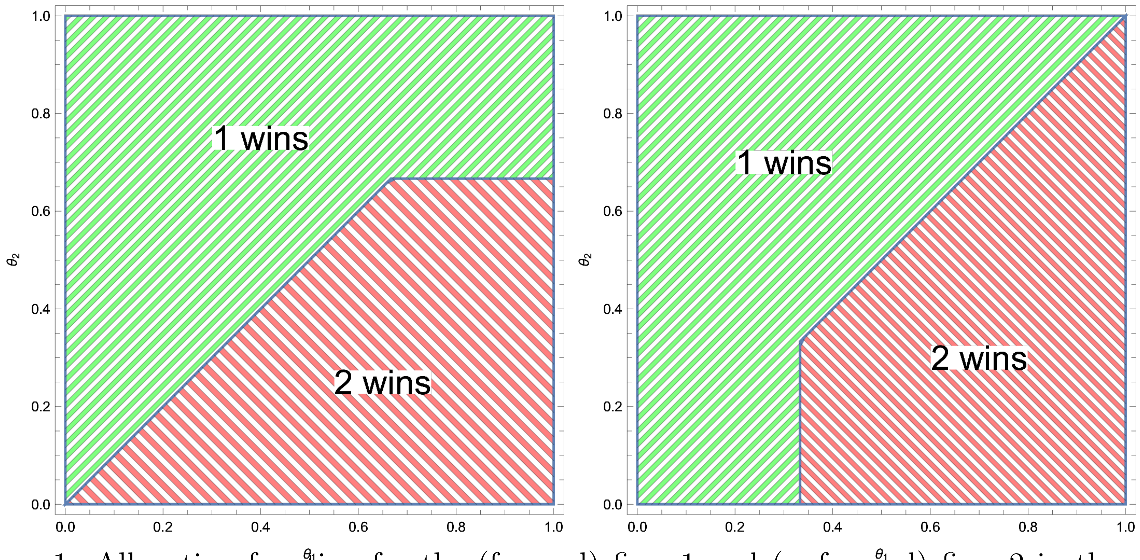

Figure 1: Allocation function for the (favored) firm 1 and (unfavored) firm 2 in the score floor (left) and score ceiling (right) mechanisms.

even when private information is present, the trade-off between the all-pay costs and price competition is sometimes resolved in asymmetric mechanisms. We show that an optimal symmetric mechanism can be sometimes dominated by a member of an ad-hoc family of "threshold" mechanisms. Finally, we show that even when the number of firms grows, the limiting utility of the optimal symmetric mechanism falls short of that of sole sourcing.

The most intriguing (to our minds) part the paper is the analysis of optimal asymmetric mechanisms, in the presence of all-pay costs, see Theorem 1. To be precise, we identify four families of mechanisms that could be optimal, when there are two firms. The first two are the standard scoring auction and sole sourcing. The other two are new and represent a surprising mixture of symmetric and asymmetric design (i.e., scoring and favoritism), see Propositions 8 to 11. Notably, only the classical scoring auction is a fully symmetric mechanism. The key factors determining the shape of the optimal mechanism appear to be the informational asymmetry, as well as the curvature of the marginal all-pay costs.

Our first new mechanism can be though of as an optimal symmetric mechanism with the efficiency parameter $(1-\theta)$ of the favored firms censored from below. We illustrate it with two ex-ante symmetric firms in Figure 1 (left) and Figure 2 (left). When the efficiency of both firms is above the threshold, the mechanism proceeds as usual. However, the unfavored firm 2, whose efficiency is below the threshold, can not win against the favored firm of equal type, because the latter firm's efficiency is reported at the threshold. As a result, the unfavored firm does not invest, if her type is below the threshold. This mechanism alleviates the duplication of effort among the most inefficient firms, and it is optimal when the marginal costs of investment is relatively convex.

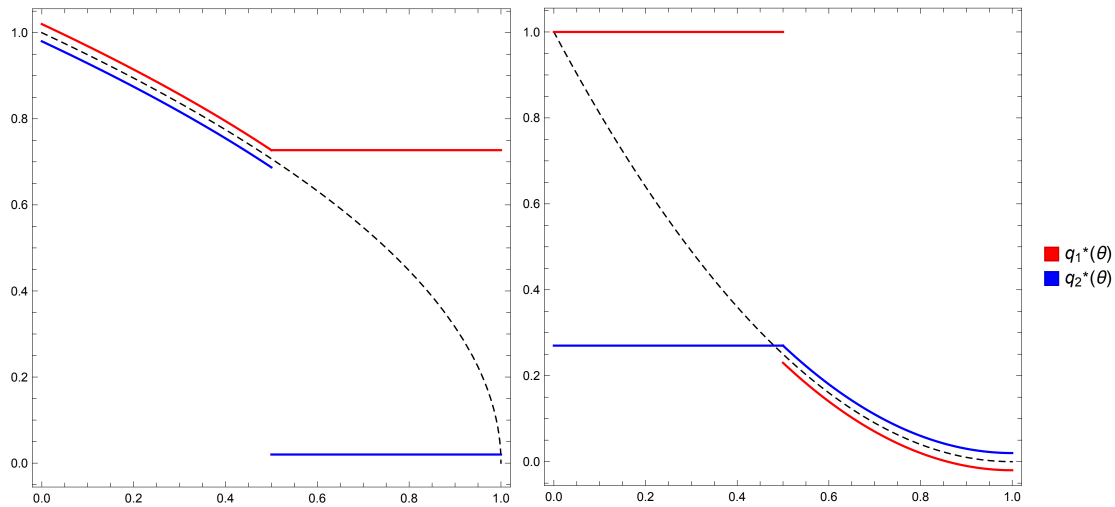

Figure 2: Quality for the (favored) firm 1 and (unfavored) firm 2 in the score floor (left) and score ceiling (right) mechanisms.

Our second new mechanism can be though of as an optimal symmetric mechanism with the efficiency parameter $(1-\theta)$ of the unfavored firms censored from above. We illustrate it with two ex-ante symmetric firms in Figure 1 (right) and Figure 2 (right). When the efficiency of both firms is below the threshold, the mechanism proceeds as usual. However, the favored firm 1, whose efficiency is above the threshold, always wins against the unfavored firm of equal type, because the latter firm's efficiency is reported at the threshold. As a result, the unfavored firm invests as if her type was at the threshold. This mechanism alleviates the duplication of effort among the most efficient firms, and it is optimal when the marginal costs of investment is relatively concave.

Both mechanisms can be implemented by a modified scoring auction, when quality is contractible. We will refer to the first mechanism as a score *floor*, applied to the favored firms, see Proposition 12. That is, for these firms, the score is censored from below. We will refer to the second mechanism as a score *ceiling*, applied to the unfavored firms, see Proposition 13. That is, for these firms, the score is censored from above. Additionally, side payments have to be made, to make firms indifferent between symmetric and asymmetric behavior, when they are exactly at the threshold (floor or ceiling).

To conclude, sole sourcing and favoritism are often viewed as signs of collusion and inefficiency. We show that, when the nature of quality is all-pay, some kind of favoritism could be efficient. This contradicts the conventional wisdom that the designer should always try to level the playing field, for example, by giving bid preferences to small, or otherwise disadvantaged firms. To the contrary, in the presence of all-pay costs, the designer might want to make the playing field uneven.

1.1 Related literature

Our paper contributes to the literature on scoring auctions, where quality is endogenously chosen as part of the contract design, see Che (1993); Branco (1997); Asker and Cantillon (2008, 2010); Nishimura (2015) for important theoretical contributions, and Adani (2018); Lewis and Bajari (2011) for empirical.

Two more strands of literature work with quality which is exogenous but not contractible, see Manelli and Vincent (1995) and Lopomo et al. (2022), and quality which is chosen before learning one's type Tan (1992); Piccione and Tan (1996); Arozamena and Cantillon (2004).

But the closest to us is the literature that deals with investment decisions (or entry) made after learning ones type. Notable theoretical contributions to this study were made in Celik et al. (2009); Zhang (2017); Gershkov et al. (2021). There are two important economic differences between our model and the models in this sub-literature: (1) in contrast to our paper, in Celik et al. (2009); Zhang (2017); Gershkov et al. (2021), an agent's action is not contractible which precludes the use of scores which are a focus of the present paper; and (2) in this literature an agent's action does not directly benefit the principal whereas this (variable) benefit a key concern it procurement with endogenous quality. Nevertheless, we use the mathematical techniques of Zhang (2017) to prove our result in section 5.

Finally, a rare empirical study of scoring auctions in an environment where costs associated with past performance (i.e., quality) are clearly all-pay, was done in Decarolis et al. (2016).

1.2 Organization of the paper

In Section 2 we set the environment. In Section 3 we derive the optimal symmetric mechanism and study it's comparative statics. In Section 4 we show suboptimality of symmetric mechanisms among all mechanisms. In Section 5 we derive the optimal asymmetric mechanisms. Finally, in Section 6, we consider ex-ante exclusion of firms.

2 Setup

Consider a single buyer (principal) who wishes to procure a contract, for which there are n potential suppliers (agents). The quality of the good to be procured is endogenous. Upon privately learning her cost-efficacy parameter (type) $\theta_i \in [0, 1]$, an agent chooses quality $q_i \in \mathbb{R}_+$, which is perfectly observed by the principal and can be verified by a court. Following Che (1993), we assume that quality is one-dimensional although some of our results are valid in the case of multidimensional quality as well. The contract can be allocated among the agents in shares $z_i \geq 0$, $\sum_{i=1}^{n} z_i = 1$. Thus, also following Che (1993), we posit that the good must be procured in any case: the utility of the buyer from the good, relative to the outside option, is sufficiently high.

Quality is costly to produce and each supplier will incur convex non-decreasing per-unit production costs $c^P(q_i, \theta_i)$, if he is selected for the contract. The novel feature is that in addition to the production (winner-pay) costs, each supplier i also incurs convex non-decreasing investment costs $c_I(q_i, \theta_i)$, which are sunk before the auction.[2] Thus, the investment costs are *all-pay* costs that are borne regardless of winning the contract. Types θ_i are independently distributed with cdf $F(\theta_i)$ and strictly positive density $f(\theta_i)$. Finally, the principal's payoff is captured by $\sum_{i=1}^{n} z_i v(q_i)$ for some concave non-decreasing function $v(q_i)$.

We impose the following natural assumptions on the cost functions.

Assumption 1. *$c_q^P > 0$, $c_\theta^P > 0$, $c_{q\theta}^P > 0$, $c_q^I > 0$, $c_\theta^I > 0$, $c_{q\theta}^I > 0$.*

We seek allocations, implementable in a Bayes-Nash equilibrium (BNE). Denote the whole profile of types by θ. By the Revelation Principle, it is without loss of generality to restrict our search to direct mechanisms, that is, mappings $(z(\theta), t(\theta), \{q_i(\theta_i)\}_{i=1}^{n})$, where t is a vector of transfers to the agents. The buyer's ex-post utility u_b and sellers' ex-post utility u_{is} are

$$u_b(z,t,q) = \sum_{i=1}^{n} \left(v(q_i)z_i - t_i\right), \quad u_{is}(z,t,q_i,\theta_i) = t_i - c^P(q_i,\theta_i)z_i - c^I(q_i,\theta_i).$$

Crucially, while the ex-post allocation z_i and transfer t_i depend on the whole profile of types θ, *quality q_i depends only on private type θ_i, because quality is chosen beforehand.* That is, we assume that the decisions about investment in quality are *independent* — a bidder cannot condition her quality on her competitor's types. This is a common assumption in the literature on auctions with endogenous valuations (Gershkov et al. (2021), Celik et al. (2009)) which we think is a reasonable approximation to reality.

We aim at buyer's expected utility:

$$\text{(P1)} \quad U = \max_{z,t,q} \mathbb{E}\left(u_b(z(\theta), t(\theta), q(\theta))\right),$$

[2] Decarolis et al. (2016) documents quality considerations in procurement, such as documentation, equipment and machinery, works execution, safety and regularity. One could speculate, that for some of them, the associated investment costs are more sunk than for the others. However, since these characteristics were measured in past contracts, from the viewpoint of the current contract, they all of the associated costs are effectively sunk.

subject to $z_i(\theta) \geq 0$, $\sum_{i=1}^n z_i(\theta) = 1$ and the standard Bayesian IC and IR constraints

$$\theta_i \in \arg\max_{\theta'} \mathbb{E}_{\theta_{-i}} \left(u_{is}(z(\theta', \theta_{-i}), t(\theta', \theta_{-i}), q_i(\theta'), \theta_i) \right),$$
$$\mathbb{E}_{\theta_{-i}} \left(u_{is}(z(\theta_i, \theta_{-i}), t(\theta_i, \theta_{-i}), q_i(\theta_i), \theta_i) \right) \geqslant 0,$$

for all θ_i in the support.

2.1 Relaxed problem

By the standard envelope argument, the derivative of expected interim transfers can be related to the expected derivative of interim utility, w.r.t. the true type, almost everywhere. In other words, interim expected transfers are uniquely defined, up to a constant. Therefore, we can formulate a relaxed problem:

$$\text{(P2)} \quad U = \max_{z,q} \mathbb{E} \sum_{i=1}^{n} \left(v(q_i) z_i - \tilde{c}_P(\theta_i, q_i) z_i - \tilde{c}_I(\theta_i, q_i) \right), \quad \text{s.t.} \quad z_i \geq 0, \quad \sum_{i=1}^{n} z_i(\theta) = 1,$$
$$\tilde{c}_P = c_P + \frac{\partial c_P}{\partial \theta_i} \frac{F(\theta_i)}{f(\theta_i)}, \quad \tilde{c}_I = c_I + \frac{\partial c_I}{\partial \theta_i} \frac{F(\theta_i)}{f(\theta_i)},$$

ignoring a large portion of IC and IR constraints.

In the tradition of mechanism design literature, we will first solve the relaxed problem, and, upon observing the monotonicity of interim expected allocation in type, argue that it is, indeed, the solution to the full problem. To do this, we shall show (in Proposition 2 below) that under the appropriate regularity conditions, the quality schedule $q^*(\theta)$ solving (P2) can be indeed achieved in an equilibrium of a mechanism, namely, a first-score scoring auction. The appropriate regularity conditions are formulated in assumption 2.

Assumption 2. *The regularity conditions are:*

1. $\tilde{c}^P_\theta > 0$, $\tilde{c}^P_{qq} > 0$, $\tilde{c}^P_{q\theta} > 0$, $\tilde{c}^i_\theta > 0$, $\tilde{c}^I_{qq} > 0$, $\tilde{c}^I_{q\theta} > 0$.
2. *The function* $(v(q) - \tilde{c}^P(q, \theta))(1 - F(\theta))^{n-1} - \tilde{c}^I(q, \theta)$ *is submodular in* (q, θ).

We will assume assumptions 1 and 2 by default.

3 Optimal symmetric mechanisms

We start with characterizing optimal symmetric mechanisms, similar in spirit to the optimal mechanism in Che (1993). However, in the presence of investment (all-pay) costs the actual answer is different and is somewhat more difficult to characterize.

3.1 The optimal quality

The buyer's payoff is equal to

$$U = \mathbb{E} \sum_{i=1}^{n} \left(\left(v(q_i) - \tilde{c}^P(q_i, \theta_i) \right) z_i - \tilde{c}^I(q_i, \theta_i) \right) \tag{1}$$

The buyer will award the contract to a firm for which the virtual production surplus

$$x(q(\theta_i), \theta_i) := v(q_i(\theta_i)) - \tilde{c}^P(q_i(\theta_i), \theta_i)$$

is maximal. Under appropriate regularity conditions (assumption 2, part 1), this will be the firm with the lowest θ_i.

Without the investment costs, as in Che (1993), the optimal symmetric quality schedule $q(\theta)$ would be the one maximizing $x(q, \theta)$ pointwise. With investment costs, it will be different.

Denoting by $\theta_{(1)}$ the lowest type, the buyer's payoff may be rewritten as

$$U = \mathbb{E} \left(x(q(\theta_{(1)}), \theta_{(1)}) - \sum_{i=1}^{n} \tilde{c}^I(q_i, \theta_i) \right)$$

Given $F(\cdot)$, the pdf of $\theta_{(1)}$ is given by $n(1 - F(\theta))^{n-1} f(\theta)$. Due to symmetry, the buyer's payoff is finally

$$U = n \int \left(x(q(\theta), \theta))(1 - F(\theta))^{n-1} f(\theta) - \tilde{c}^I(q(\theta), \theta) f(\theta) \right) d\theta. \tag{2}$$

This representation allows to characterize the optimal quality schedule.

Proposition 1. *The quality schedule $q^*(\theta)$ solving the relaxed problem (P2) under the symmetry constraint $q_i(\theta) \equiv q_j(\theta)$ for all i, j is determined by maximizing (2) pointwise. That is, $q^*(\theta)$ maximizes*

$$(v(q) - \tilde{c}^P(q, \theta))(1 - F(\theta))^{n-1} - \tilde{c}^I(q, \theta)$$

over q for each θ. Moreover, $q^*(\theta)$ is decreasing.

We also denote the optimal symmetric quality schedule $q^*(\theta)$ by $q_{symm}(\theta)$, that is, $q^*(\theta) \equiv q_{symm}(\theta)$.

3.2 Implementation

In this section, we show that the quality schedule $q^*(\theta)$ identified in proposition 1 in fact solves the full mechanism design problem. We do this by showing that $q^*(\theta)$ can arise in an equilibrium of a mechanism, namely a first-score scoring auction with an appropriate score function.

Scoring auctions are widely used in practice and have been studied extensively in the procurement literature (see section 1.1), but, to the best of our knowledge, not in the presence of an all-pay cost component.

A *first-score scoring auction* is the following mechanism. (1) The bidders submit multi-dimensional bids (q_i, p_i) where q_i is the offered quality and p_i is the offered price. (2) a score $S_i = S(q_i, p_i)$ is computed for every bidder; (3) the bidder with the highest score (call her i^*) is awarded the contract in which it is stipulated that she must supply a good of quality q_{i^*} for a price of p_{i^*}. That is, the bidder supplies her own submitted quality for her own submitted price[3].

In fact we will show that for *any* decreasing quality schedule $q(\theta)$ there exists an appropriate scoring function $S(q, p)$ such that $q(\theta)$ is played by every bidder in an equilibrium of the first-score scoring auction with the scoring function $S(q, p)$.

As is usual in the quasi-linear environments, a quasi-linear score $S(q, p) = s(q) - p)$ suffices. It is not hard to show that the scoring function $s(q)$ implementing $q(\theta)$ must, due to the every firm's first-order condition, satisfy

$$s'(q) \equiv C_q^P(q, \theta(q)) + \frac{C_q^I(q, \theta(q))}{(1 - \theta(q))^{n-1}}, \tag{3}$$

where $\theta(q)$ is the inverse of $q(\theta)$.

However, it is not clear that second-order conditions will hold. Recall that in a first-score scoring auction, the bid is a quality-price pair, so it is two-dimensional. The second-order condition for a two-dimensional problem is involved since it must exclude joint quality-price deviations, among others. Without the investment costs, as in Che (1993), it is relatively easy to show unprofitability of such deviations, as quality bid optimization can be effectively decoupled from the price bid optimization. However, with investment costs such decoupling is not possible: for a bidder, the optimal price bid depends quality and at the same time optimal quality bid depends on price since the price-dependent marginal benefits of investing in quality must be weighed against marginal costs of investment. In the proof of proposition 2, we show that despite this potential complication, no condition beyond the already assumed

[3] In this specification, the contract is always awarded, as we assume throughout. If a contract does not have to be always awarded, one may amend the auction with a reserve score

assumption 1 is needed to guarantee the implementation in a FSA.

Proposition 2. *Suppose assumption 1 holds but not necessarily assumption 2. For any decreasing quality schedule $q(\theta)$ with inverse $\theta(q)$,including the quality schedule $q^*(\theta)$ identified in Proposition 1, the first-score auction with quasi-linear score $S(q,p) = s(q) - p$, where $s(q)$ satisfies (3), has a BNE in which quality strategy of every firm is $q(\theta)$.*

3.3 Comparative statics

Having characterized the optimal symmetric quality schedule to implement and the optimal score function, we are now in a position to answer qualitative questions regarding how the solution would change upon a change in parameters. In particular, we are interested in how the solution in our model differs from that without investment costs, in Che (1993).

We consider the comparative statics with respect to two parameters — the size of the investment costs and the number of bidders. To parameterize the size of the investment costs, we simply let the investment costs equal $\beta C^I(q,\theta)$, $\beta \geq 0$. Note that $\beta = 0$ corresponds to the model of Che (1993). For convenience, we denote the dependence on parameters explicitly by writing $q^*(\theta, \beta, n)$ and $s^*(q, \beta, n)$.

3.3.1 Comparative statics of the optimal quality

Proposition 3. *The optimal symmetric quality schedule $q^*(\theta, \beta, n)$ is:*

1. *Decreasing in the size of the investment costs β;*
2. *Decreasing in the number of bidders n.*

Proof. By the analysis above, the optimal quality $q^*(\theta, \beta, n)$ maximizes

$$\psi(q,\theta)(1-F(\theta))^{n-1} - \beta C^I(q,\theta) - \beta C^I_\theta(q,\theta)\frac{F(\theta)}{f(\theta)} \tag{4}$$

pointwise.

As $C^I_q > 0$, $C^I_{\theta q} > 0$, the function (4) is submodular in (q, β). Thus, the optimal $q^*(\theta, \beta, n)$ is decreasing in β. Furthermore, maximizing (4) is the same as maximizing

$$\psi(q,\theta) - \frac{\beta}{(1-F(\theta))^{n-1}} C^I(q,\theta) - \frac{\beta}{(1-F(\theta))^{n-1}} C^I_\theta(q,\theta)\frac{F(\theta)}{f(\theta)}.$$

As $1 - F(\theta) < 1$ almost everywhere, increasing n has the same effect on the objective as increasing β. Thus, the optimal $q^*(\theta, \beta, n)$ is decreasing in n as well. □

The intuition behind proposition 3 is straightforward. A higher scale of investment costs implies higher marginal virtual investment costs; this leads to a lower optimal quality. Note that this implies that the optimal quality to implement is always lower than without the investment costs, in the model of Che (1993). A higher number of bidders, in its turn, lowers the probability that any particular bidder's quality will be consumed; this lowers the principal's marginal benefits of inducing a higher quality and thus reduces the optimal one. Note that this result is in contrast to Che (1993)'s model where the optimal quality is *independent* of the number of bidders. There, the marginal benefits of increasing quality are not weighed against the marginal investment costs. Even though with $C^I \equiv 0$ the marginal benefits do decrease as n grows, the point at which the marginal benefits are equal to 0 does not depend on n; hence, the optimal quality stays constant in n.

3.3.2 Comparative statics of the optimal score function

How does the size of investment costs and the number of bidders affect the optimal incentives, i.e., the score function $s^*(q)$? Given that, as we just showed, the optimal quality is decreasing in the size of investment costs β, one may conjecture that the higher β, the flatter quality incentives $s(q)$ should be. This intuition is wrong, because upon an increase in β the bidders will themselves take the increased marginal investment costs into account, which may render flattening the score unnecessary. The question of how the optimal score slope changes turns out to be subtler, with the answer depending on the relative elasticities of production and investment costs w.r.t. private information θ.

First, we pinpoint a special case where the size of investment costs β does not matter at all.

Example 1. *Suppose $c^P(q,\theta) = \theta \cdot g_1(q)$ and $c^I(q,\theta) = \theta \cdot g_2(q)$ where g_1 and g_2 are any weakly increasing functions. $F(\theta) = \theta^{\frac{1}{d}}$ for some $d > 0$, $\theta \in [0,1]$.*

Then, the buyer wants the quality to satisfy

$$(1 - F(\theta))^{n-1}(v'(q) - \theta(1+d)g_1'(q)) - \theta(1+d)g_2'(q) = 0.$$

At the same time, in a first-score scoring auction with score $S = s(q) - p$, a supplier with type θ sets q so that

$$(1 - F(\theta))^{n-1}(s'(q) - \theta g_1'(q)) - \theta g_2'(q) = 0.$$

Comparing the two conditions, we immediately see that the score $s^*(q) = \frac{v(q)}{1+d}$ implements the optimal mechanism regardless of g_1, g_2 and n. *Thus, investment costs do not affect the* *slope of optimal score whatsoever (in particular, this score is also optimal in the model of Che (1993)). Note also that the quality incentives are depressed relative to the truthful score but this distortion tends to vanish when $d \to 0$, i.e., when the distribution of types approaches a constant $\theta = 1$ and asymmetry of information disappears.*

Special in the setting of Example 1 is the fact that production and investment costs exhibit the same linear dependence on the private information θ. Inspired by Example 1, we surmise that when one cost function exhibits a stronger dependence on θ than the other cost function, the slope of the optimal score becomes responsive to β. It might increase or decrease in β depending on which costs depend stronger on the private information.

To gain traction, we restrict attention to environments in which the elasticity of costs and $F(\theta)$ w.r.t θ is constant, as in Example 1.

Proposition 4. *Suppose $c^P(q,\theta) = \theta^{E_1} g_1(q)$ and $c^I(q,\theta) = \beta \cdot \theta^{E_2} g_2(q)$ where $E_1, E_2 > 0$ and $g_1(q)$, $g_2(q)$ are some well-behaved functions. Suppose $F(\theta) = \theta^{\frac{1}{d}}$ for some $d > 0$. Denote by $s_q^*(q)$ the slope of optimal score. Then:*

1. *If $E_1 > E_2$ then $s_q^*(q)$ increases in β and n at every q;*
2. *If $E_1 < E_2$ then $s_q^*(q)$ decreases in β and n at every q.*

Proof. From (3), we have

$$s^{*\prime}(q) \equiv \theta^{E_1}(q) g_1'(q) + \frac{\beta \theta^{E_2}(q) g_2'(q)}{(1 - F(q))^{n-1}}, \tag{5}$$

where $\theta(q)$ is the inverse of the optimal $q^*(\theta, n, \beta)$. We suppress the dependence of $\theta(q)$ on β and n for brevity.

At the same time, from the optimality of $\theta(q)$,

$$(1 - \theta(q))^{n-1}(v'(q) - (1 + E_1 d)\theta^{E_1}(q)) - \beta(1 + E_2 d)\theta^{E_2}(q) g_2'(q) = 0. \tag{6}$$

Solving (5) for $\beta\theta^{E_2}(q) g_2'(q)$ and plugging this in (5), we get

$$s^{*\prime}(q) \equiv \theta^{E_1}(q) g_1'(q) + \frac{v'(q) - (1 + E_1 d)\theta^{E_1}(q) g_1'(q)}{1 + E_2 d} = \frac{v'(q) + d(E_2 - E_1)\theta^{E_1}(q) g_1'(q)}{1 + E_2 d}.$$

It follows from proposition 3 that $\theta(q)$ is decreasing in both β and n for a fixed q. From this, the result is immediate. □

To understand the intuition behind proposition 4, recall from Che (1993) that the optimal score without investment costs is less high-powered than the truthful one, and the size of the discrepancy between the two increases in the severity of the asymmetric information problem, i.e. size of info rents. Then one adds investment costs. When they are less elastic in private information than production costs, adding them reduces "overall" information rents and and this should make the optimal score closer to the truthful one – but this means more-high powered. When investment costs are more elastic in private information than production costs, adding them increases overall information rents, moving the optimal score farther from the truthful one – and thus making it less high-powered. This is exactly what proposition 4 says.

4 Suboptimality of symmetric mechanisms

In this subsection, we show that a solution to our mechanism design problem is in general not a symmetric mechanism. This is again unlike Che (1993) the fact that the optimal mechanism is symmetric follows easily from pointwise maximizaion.

Intuitively, the desirability of a symmetric treatment depends on the degree of importance of private information. If private information is not important at all, it is almost obvious that an optimal thing to do is to always award the contract to one specific firm (so that others do not incur investment costs), rather than to have a symmetric contest. On the other hand, if private information is very important, it becomes important to reveal the identity of the most efficient firm — and this can be most effectively done by a symmetric mechanism. This may make a symmetric mechanism optimal in spite of the desire to avoid duplication of costs.

The first result establishes that this intuition is correct: when importance of private information is sufficiently low, the optimal symmetric mechanism can in fact be improved upon under almost no structural assumptions. The second result shows that a symmetric mechanism is necessarily suboptimal *regardless* of the degree of importance of private information if the elasticity of investment costs with respect to quality at zero is sufficiently high.

We introduce "the importance of private information" in a simple way. Suppose the production and investment costs are parametrized by a parameter $\alpha \geq 0$ such that $C^P(q, \theta|\alpha)$ and $C^I(q, \theta|\alpha)$ do not depend on θ if $\alpha = 0$ and do depend on θ if $\alpha > 0$. The functions are continuous in α. For concreteness, one may think of the parametrizations $C^P(q, \alpha\theta)$ and $C^I(q, \alpha\theta)$.

For our results, we need to introduce a reduction of the problem under consideration to an equivalent problem with investment costs only. Namely, denote by x the virtual production surplus, that is, $x = v(q) - \tilde{C}^P(q,\theta)$. Denote by $C(x,\theta)$ the minimal virtual investment costs possible when the virtual production surplus is x. Namely, $C(x,\theta) = \min\{\tilde{C}^I(q,\theta) : v(q) - \tilde{C}^P(q,\theta) = x\}$. Given that quality is one-dimensional, $C(x,\theta)$ is achieved simply at the lowest root q of the equation $v(q) - \tilde{C}^P(q,\theta) = x$. It is not hard to show that given our assumptions that $\tilde{C}^I(q,\theta)$ and $\tilde{C}^P(q,\theta)$ are supermodular, $C(x,\theta)$ is increasing in both arguments and supermodular as well. We call the function $C(x,\theta)$ the *indirect investment costs* function.

After this cost minimization step, the principal's utility may be written simply as

$$U = \mathbb{E}\max_i x_i(\theta_i) - \mathbb{E}\sum_{i=1}^n C(x_i(\theta_i),\theta_i), \tag{7}$$

where $x_i(\theta_i)$ is the virtual production surplus of bidder i with type θ_i. Any tuple of functions $x_i(\theta_i)$ induces a mechanism, so we might think about choosing directly the functions $x_i(\theta_i)$ to maximize (7). This (equivalent) problem looks like the original one, but without any production costs.

4.1 A small-alpha result

We show that when the importance of private information is sufficiently low, any optimal mechanism must be asymmetric.

Proposition 5. *There exists $\overline{\alpha} > 0$ such that for all $\alpha \in [0,\overline{\alpha}]$ the optimal symmetric mechanism is not an optimal mechanism.*

Proof. We will show the suboptimality of the optimal symmetric mechanism for $\alpha = 0$. Namely, we will show that for $\alpha = 0$ the completely asymmetric mechanism (only one bidder is left) dominates the optimal symmetric mechanism. Then the result will follow from continuity (the objective at the completely asymmetric mechanism and the objective at the optimal symmetric mechanism are continuous in α).

With $\alpha = 0$, the indirect investment costs $C(x,\theta)$ is just a function of x. Abusing notation, we write $C(x)$. Denote by $H_i(x)$ the cdf of $x_i(\theta_i)$. We can write the objective as a functional of $H_i(x)$ and optimize over $H_i(x)$. That is,

$$\mathbb{E}U = \mathbb{E}\max_i x_i(\theta_i) - \mathbb{E}\sum_{i=1}^{n} C(x_i(\theta_i)) =$$
$$= \int_0^{+\infty}\left(1 - \prod_{i=1}^{n} H_i(x)\right)dx - \int_0^{+\infty}\sum_{i=1}^{n} C'(x)(1 - H_i(x))dx =$$
$$\int_0^{+\infty}\left(1 - \prod_{i=1}^{n} H_i(x) - \sum_{i=1}^{n} C'(x)(1 - H_i(x))\right)dx. \quad (8)$$

Ignoring the monotonicity constraint for $H_i(x)$, let's optimize (8) pointwise. The integrand is linear in each of H_i, so without loss of optimality at every x $H_i(x) \in \{0, 1\}$. It is easy to see that at optimum at every x at most one of $H_i(x)$ is 0; otherwise the objective can be increased since $C'(x) \geq 0$ for $x > 0$. It remains to compare two cases: all of H_i are 1 and all but one of H_i is 1, and the remaining one is zero. In the first case, the integrand is 0; in the second case, it is $1 - C'(x)$. Thus, the optimal mechanism (up to permutation) is as follows:

$$M^* = \begin{cases} H_1^*(x) = 0,\ H_i^*(x) = 1 \text{ for } i = 2, \dots, n & \text{if } C'(x) < 1; \\ H_i^*(x) = 1 \text{ for } i = 1, \dots, n & \text{if } C'(x) \geq 1. \end{cases} \quad (9)$$

The functions H_i^* are nondecreasing, hence this is indeed the optimal mechanism. This is the completely asymmetric mechanism in which only one bidder is allowed to participate and this bidder produces the monopoly surplus x_m that satisfies $C'(x_m) = 1$.

The optimal symmetric mechanism M^*_{symm} maximizes (8) pointwise under the constraint that $H_i(x) = H_j(x)$ for all i, j. Taking FOC, one gets $H^*_{symm}(x) = \sqrt[n-1]{C'(x)}$. After changing $H_1(x)$ from $H^*_{symm}(x)$ to $H_1^*(x)$ given by (9) the objective does not change but if after that $H_2(x)$ is changed from $H^*_{symm}(x)$ to $H_1^*(x)$ given by (9), the objective strictly increases; hence M^*_{symm} is not optimal and is strictly dominated by M^*.

Write $M^*(\alpha)$ and $M^*_{symm}(\alpha)$ for the optimal mechanism and the optimal symmetric mechanism as functions of α. Write $\mathbb{E}U(M, \alpha)$ as the principal's expected utility as the function of mechanism and α. We have shown above that

$$\mathbb{E}U(M^*(0), 0) > \mathbb{E}U(M^*_{symm}(0), 0).$$

By continuity, this implies that

$$\mathbb{E}U(M^*(0), \alpha) > \mathbb{E}U(M^*_{symm}(\alpha), \alpha)$$

for all sufficiently small α. □ □

Remark: It may seem obvious that the one-bidder mechanism M^* is optimal — after all, why do we need several costly-to-have bidders if the goal is to maximize $\mathbb{E}\max_i x_i$? The subtlety is that if $x_i(\theta_i)$ are not constant in θ_i, they are stochastic, and hence, $\mathbb{E}\max_i x_i(\theta_i) > \max_i Ex_i(\theta_i)$. That is, by having two active bidders the principal can achieve on average more than by having the best of the two bidders only. This is because with two bidders, the principal has two draws from theta distribution rather than one. Our result in the proof of proposition 5 is that the single-bidder mechanism is optimal *in spite* of this effect.

4.2 An all-alpha result

Now we provide a result that holds for all values of α, not only small ones. The condition that we use instead is that the elasticity of the indirect investment costs is sufficiently high (or n is sufficiently low).

Suppose indirect investment costs are given by the function $C(x,\theta)$ (here, we suppress dependence on α). Denote by $x^*(z,\theta)$ the solution to the problem

$$\max_x \left(zx - C(x,\theta)\right),$$

and by $\psi(z,\theta)$ the maximum value of this problem. Recall that the optimal symmetric mechanism $x^*_{symm}(\theta)$ satisfies $x^*_{symm}(\theta) = x^*((1-F(\theta))^{n-1},\theta)$.

Consider the following family of (asymmetric) threshold mechanisms $\tilde{x}(\cdot|\theta_0)$, parametrized by a threshold $\theta_0 \in [0,1]$:

- For bidders $i = 2,3,\ldots,n$, $\tilde{x}_i(\theta_i|\theta_0) = \begin{cases} x^*_{symm}(\theta_i), & \theta_i \le \theta_0; \\ x^*(0,\theta_i), & \theta_i > \theta_0. \end{cases}$
- For bidder 1, $\tilde{x}_1(\theta_1|\theta_0) = \begin{cases} x^*_{symm}(\theta_1|\theta_0), & \theta_1 \le \theta_0; \\ x^*((1-F(\theta_0))^{n-1},\theta_1), & \theta_1 > \theta_0. \end{cases}$

That is, before the threshold θ_0, the mechanism $\tilde{x}(\cdot|\theta_0)$ follows the optimal symmetric mechanism; after the threshold θ_0 all bidders but bidder 1 are "excluded" so that for them, the minimal possible x is chosen (corresponding to zero quality), and the production surplus for bidder 1 is chosen optimally given the exclusion of other bidders. For $n = 2$, this is the mechanism analyzed in detail in section 5.

Denote by $\underline{x}$ the production surplus that satisfies $0 = C_x(\underline{x},1)$. That is, $\underline{x}$ is the lowest production surplus that is possible at an optimum (symmetric or otherwise). (Under our assumptions, $\underline{x}$ also solves the equation $0 = C(\underline{x}, 1)$ and is given by $\underline{x} = v(0) - c^P(0, 1)$.) For the suboptimality of symmetric mechanisms, we need the following condition:

$$\text{There exists } \gamma > n \text{ such that } \lim_{x \to \underline{x}^+} \frac{C(x, 1)}{(x - \underline{x})^\gamma} \text{ is positive and finite.} \tag{10}$$

Condition 10 says that around $x = \underline{x}$, the indirect investment costs behave a like a function with a sufficiently high elasticity γ (the precise threshold for which happens to be n, the number of bidders). In an environment where $v(q) = q$, $C^P(\theta) = b(\theta)$ and $C^I(q, \theta) = g(q)$ this condition is equivalent to saying that the elasticity of direct investment costs $g(q)$ at zero is higher than n. Another reading of the condition 10 is that the number of bidders, n, is relatively small (smaller than the investment cost elasticity).

Proposition 6. *Suppose condition 10 holds, i.e. the indirect investment costs are sufficiently elastic at minimal production surplus. Then, there exists* $\theta_0 < 1$ *such that the mechanism* $\tilde{x}(\cdot|\theta_0)$ *yields strictly higher value of principal's objective than the optimal symmetric mechanism* $x^*_{symm}(\cdot)$.

The result is intuitive: the symmetric mechanism should be suboptimal when the problem of duplication of costs is severe, but it is so exactly when the elasticity of investment costs is high.

Note that when $\theta_0 = 1$, the mechanism $\tilde{x}(\cdot|\theta_0)$ is just the optimal symmetric mechanism. So we prove Proposition 6 by investigating the derivative of the principal's objective $U(\theta_0)$ with respect to θ_0 in a neighborhood of $\theta_0 = 1$. We show that even though this derivative is zero at $\theta_0 = 1$, it is strictly negative in a neighborhood of $\theta_0 = 1$ which shows that $U(\theta_0) > U(1)$ for some $\theta_0 < 1$.

4.3 An asymptotic result

The last two results were for a fixed number of bidders n. However, one may conjecture that when n grows without bound, the symmetric mechanism should perform well as the more draws from the type distribution we have, the better the most efficient draw becomes and thus it becomes crucial to treat the bidders symmetrically to not lose the benefit of that most efficient draw. In this subsection, we show that this intuition is wrong, in the sense that the optimal symmetric mechanism is not even *asymptotically* optimal for small α.

Let n be the number of bidders (firms), $U_n(M)$ be the principal's objective as a function of mechanism M when there are n bidders, and M^*_n be an optimal mechanism with n bidders.

Definition 1. *A sequence of mechanisms M_n is* asymptotically optimal *iff*

$$\lim_{n\to\infty}(U_n(M_n) - U_n(M_n^*)) = 0.$$

We parametrize the indirect cost function as $C = C(x, \alpha\theta)$ where α is the "importance of private information".

Recall that $\psi(z, \theta)$ is given by

$$\psi(z,\theta) := \max_x (zx - C(x, \alpha\theta).)$$

Note that $\psi(\cdot, \theta)$ is convex (and, under mild conditions, strictly so). Also, as $C(x, \alpha\theta) \geq 0$ and for all $\alpha\theta$ there exists an x_0 such that $C(x_0, \alpha\theta) = 0$, we get $\psi(0, \theta) = 0$.

In section 4.1, we have shown that with a sufficiently small α, the optimal symmetric mechanism is not optimal. The proposition below gives a stronger result: with a sufficiently small α, the utility that the principal wins by using an optimal (asymmetric) mechanism rather than the optimal symmetric mechanism does not vanish when the number of bidders tends to infinity.

Proposition 7. *There exists $\overline{\alpha} > 0$ such that for all $\alpha \in [0, \overline{\alpha})$, the sequence of optimal symmetric mechanisms M_n^{symm} is not asymptotically optimal.*

Proof. We prove the result by showing that with a sufficiently small α, simply leaving one bidder provides asymptotically higher utility than using the optimal symmetric mechanism.

First, we derive the limiting utility of the optimal symmetric mechanism. The principal's objective at the optimal symmetric mechanism M_n^{symm} can be written as

$$U_n(M_n^{symm}) = n\int_0^1 \psi((1-\theta)^{n-1}, \alpha\theta)d\theta.$$

After the substitution $y = (1-\theta)^{n-1}$, we get for $n > 1$:

$$U_n(M_n^{symm}) = \frac{n}{n-1}\int_0^1 \psi(y, \alpha(1-y^{\frac{1}{n-1}}))y^{\frac{1}{n-1}-1}dy.$$

By continuity of ψ, it follows that

$$\lim_{n\to\infty} U_n(M_n^{symm}) = \int_0^1 \frac{\psi(y,0)}{y}dy.$$

Now consider the principal's utility from leaving only one bidder. It is given by

$$U_1(M_1^{symm}) = \int_0^1 \psi(1, \alpha\theta) d\theta.$$

At $\alpha = 0$, this is just

$$\int_0^1 \psi(1, 0) d\theta = \psi(1, 0).$$

If we prove that

$$\int_0^1 \frac{\psi(y, 0)}{y} dy < \psi(1, 0), \tag{11}$$

the result will follow by continuity of $U_1(M_1^{symm}) = \int_0^1 \psi(1, \alpha\theta) d\theta$ in α. But note that because $\psi(\cdot, \theta)$ is strictly convex and $\psi(0, \theta) = 0$, the function $y \to \frac{\psi(y,0)}{y}$ is strictly increasing, thus for all $y \in (0, 1)$ we have

$$\frac{\psi(y, 0)}{y} < \psi(1, 0)$$

and thus (11) holds. □

The crux of the proof above is that the limiting utility from the optimal symmetric mechanism is strictly less than the utility of just always procuring from the best type of one bidder (even though the law of large numbers would suggest otherwise). This is precisely because when using the symmetric mechanism, the principal has to pay for *all* the bidders' investment costs.

5 Optimal asymmetric mechanisms for $n = 2$

5.1 The set-up

Now we provide partial results about optimal mechanisms without the *a priori* restriction to the symmetric ones. In general, the analysis is hard; to the best of our knowledge, only partial results are available in the literature for the class of problems to which our problem belongs[4]. The mathematical difficulty comes from the assumption of *independent* investments, i.e., that fact that a bidder's quality is allowed to depend only on her own type but not on competitor's type. This precludes the use of pointwise integral maximization which is typical in mechanism design. As a result, one has to use perturbation techniques to characterize an optimum.

[4] Zhang (2017) provides the result for $n = 2$ and additively separable environment with quadratic costs; Gershkov et al. (2021) provide a general condition under which the optimal mechanism is symmetric; Celik et al. (2009) study only binary investment decisions.

Here, we provide a new set of partial results which to a certain extent generalize what is currently known for $n = 2$. The economic novelty lies in the characterization of the dependence of the optimal mechanism on the curvature of marginal all-pay costs. Our proof is based on Zhang (2017), Gershkov et al. (2021). However, we would like to stress again that even though our settings are related mathematically, they represent qualitatively different situations economically as in Zhang (2017), Gershkov et al. (2021) an agent's action is not contractible whereas in our model it is.

In this section, we restrict attention to situations with two bidders ($n = 2$) and in which $C^P(q, \theta) = \alpha\theta$, $C^I(q, \theta) = g(q)$ with $g'(q) > 0$, $g''(q) > 0$ and $v(q) = q$. That is, the production costs depend only on private information, with α being a parameter measuring its importance or the degree of information asymmetry. At the same time, the investment costs depend only on the quality. Given that quality is one-dimensional, setting $v(q) = q$ is without loss of generality. This is a setting generalizing the one in Zhang (2017) who considers only quadratic investment costs $g(q) = Kq^2/2$.

We start by defining, for the set-up just described, the notion of convexity that, as we show, is relevant for the shape of the optimal mechanism. Recall that $F(\theta)$ is the cdf of the type θ. Denote by $J(\theta) = \theta + \frac{F(\theta)}{f(\theta)}$ the standard virtual type.

Define $\xi(z) := 1 - J(F^{-1}(1 - z))$. Note that under our assumption that $J(\theta)$ is increasing, $\xi(z)$ is also increasing. When θ is uniform on [0,1], $\xi(z) = 2z - 1$.

Definition 2. *We say that* marginal investment costs are sufficiently convex *iff the function $q \to \alpha\xi(g'(q)) - q$ is strictly quasi-convex. We say that* marginal investment costs are sufficiently concave *iff the function $q \to \alpha\xi(g'(q)) - q$ is strictly quasi-concave.*

Note that the above definition depends not only on the marginal investment costs $g'(q)$ themselves but on the other primitives as well. This just means that the meaning of the qualifier "sufficiently" is context-dependent. In the simple setting where the type θ is distributed uniformly on [0,1], $\alpha\xi(g'(q)) - q = 2\alpha g'(q) - q - 1$, so any convex marginal costs $g'(q)$ would be sufficiently convex while any concave marginal costs $g'(q)$ would be sufficiently concave according to definition 2.

5.2 The main result

We are now in a position to state the main result of this section. Recall that $q_{symm}(\theta)$ is the optimal symmetric quality identified in proposition 1.

Theorem 1. *Suppose $n = 2$, $C^P(q, \theta) = \alpha\theta$, $C^I(q, \theta) = g(q)$ and $v(q) = q$. Then:*

1. *If the marginal costs are sufficiently convex, there exists a* $\theta_0 \in [0,1]$ *such that an optimal pair of quality schedules is*

$$q_1^*(\theta) = \begin{cases} q_{symm}(\theta), & \theta < \theta_0; \\ q_{symm}(\theta_0), & \theta > \theta_0; \end{cases} \quad q_2^*(\theta) = \begin{cases} q_{symm}(\theta), & \theta < \theta_0; \\ 0, & \theta > \theta_0. \end{cases} \tag{12}$$

2. *If the marginal costs are sufficiently concave, there exists a* $\theta_0 \in [0,1]$ *such that an optimal pair of quality schedules is*

$$q_1^*(\theta) = \begin{cases} q_{symm}(0), & \theta < \theta_0; \\ q_{symm}(\theta), & \theta > \theta_0; \end{cases} \quad q_2^*(\theta) = \begin{cases} q_{symm}(\theta_0), & \theta < \theta_0; \\ q_{symm}(\theta), & \theta > \theta_0. \end{cases} \tag{13}$$

In both cases, bidder 1 is the "favored" bidder while bidder 2 is the "unfavored" one.

Thus, in an optimal mechanism discrimination takes a relatively simple form: one should take the optimal symmetric quality schedule and truncate it either from the left (if marginal investment costs are sufficiently convex) or from the right (if marginal investment costs are sufficiently concave). In the region where quality schedules are asymmetric (so that one bidder is "favored"), one quality stays at the level the optimal symmetric quality reached at the threshold θ_0 while the other jumps to an extreme value.

Typical pairs of quality schedules (12) and (13) are depicted in Figure 2, left and right correspondingly. (These are for an example with $g(q) = q^\gamma/\gamma$, analyzed in detail in section 5.3.)

Both asymmetric mechanisms (12) and (13) can be implemented as appropriate modifications of a first-score scoring auction. The quality schedules given by (12) can be implemented via a scoring auction with discriminatory reserve scores (score floors) while quality schedules given by (13) can be implemented via a scoring auction in which the unfavored bidder faces a score ceiling. For this reason, we say that quality schedules (12) represent a "score floors" mechanism while the quality schedules (13) represent a "score ceilings" mechanism.

Importantly, and somewhat unexpectedly, in both cases the implementation includes a side-payment for the favored bidder — it is either a bonus she gets from passing the (higher) unfavored bidder's reserve score or a kickback she pays for winning without competition when the unfavored bidder's score hits its ceiling. The role of the side-payments is to ensure the correct incentives of the "intermediate" types θ located in the vicinity of the threshold θ_0 that are supposed to choose the symmetric quality. We describe the proposed implementation in detail in subsection 5.5.

The result of Theorem 1 is especially clean when the production costs are distributed uniformly, that is, $F(\theta) = \theta$. The following corollary stems from the fact that when the type θ is distributed uniformly on [0,1], $\alpha\xi(g'(q)) - q = 2\alpha g'(q) - q - 1$.

Corollary 1. *Suppose that in the setting of Theorem 1 it additionally holds that θ_i is distributed uniformly on [0,1]. Then:*

1. *If $g'''(q) > 0$, an optimal mechanism is a "score floors" mechanism given by (12).*

2. *If $g'''(q) < 0$, an optimal mechanism is a "score ceilings" mechanism given by (13).*

Thus, that θ_i is distributed uniformly, all that matters is whether *marginal* investment costs are convex or concave. Also note that Corollary 1 implies that quadratic (total) costs, that are often assumed, are a knife-edge case with $g''' = 0$ and thus are not representative.

5.3 Example: constant-elasticity investment costs

Now we illustrate Theorem 1 and Corollary 1 with an example. Suppose that $g(q) = q^\gamma/\gamma$ for some $\gamma > 1$. In this case, Corollary 1 identifies an optimal mechanism for (almost) all $\gamma > 1$: it is a "score floors" mechanism for $\gamma > 2$ and a "score ceilings" mechanism for $\gamma < 2$. However, for some values of α the optimal threshold θ_0 can be 0 or 1, implying that the optimal mechanism is in fact symmetric, coinciding with the one identified in section 3, or the *sole-sourcing* — a completely asymmetric mechanism in which one bidder is totally excluded *a priori*[5]. We provide a complete description of the solution as a function of parameters γ and α in Proposition 8 and illustrate it in Figure 3.

Proposition 8. *Suppose $n = 2$, $F(\theta) = \theta$, $C^P = \alpha\theta/2$, $C^I = q^\gamma/\gamma$ where $\alpha \geq 0$, $\gamma > 1$. Then, an optimal mechanism is:*

- *sole-sourcing if $\alpha < \min\left\{2 - \frac{2}{\gamma}, \frac{2}{\gamma}\right\}$;*

- *a scoring auction with discriminatory score ceilings if $\gamma \leq 2$ and $2 - \frac{2}{\gamma} < \alpha < \frac{1}{\gamma - 1}$;*

- *a non-discriminatory scoring auction (the optimal symmetric mechanism) if $\gamma < 2$ and $\frac{1}{\gamma-1} < \alpha$;*

- *a scoring auction with discriminatory score floors (reserve scores) if $\gamma > 2$ and $\frac{2}{\gamma} < \alpha$.*

[5]In fact, one may show that it is always the case for $\gamma = 2$: the optimal mechanism can be represented by both (12) and (13) and thus involves a boundary value of θ_0.

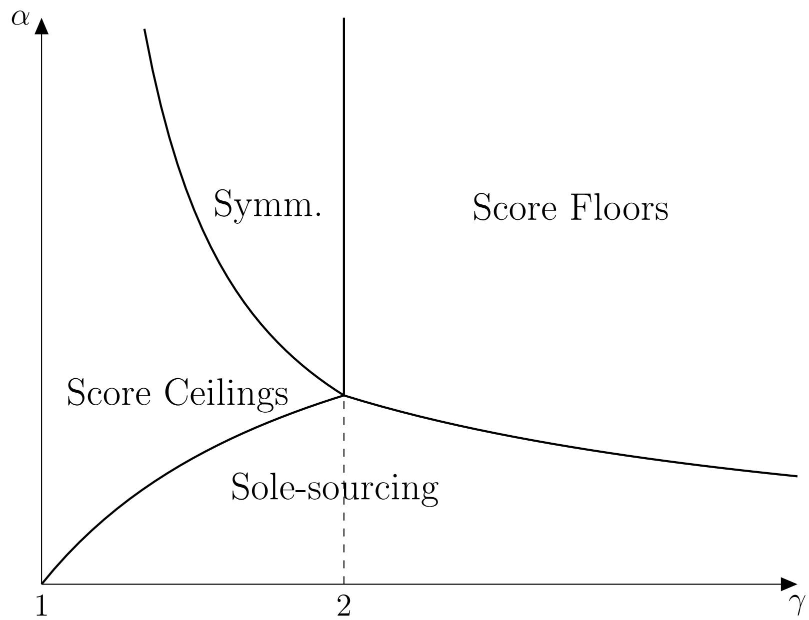

Figure 3: Type of optimal mechanism with $n = 2$ as a function of (γ, α). Costs are given by $C^P = \alpha\theta/2$, $C^I = q^\gamma/\gamma$.

When the optimal mechanism is "score floors", the optimal threshold θ_0^* is given by

$$\theta_0^* = 1 - \left(\frac{2}{\alpha\gamma}\right)^{\frac{\gamma-1}{\gamma-2}}.$$

When the optimal mechanism is "score ceilings", the optimal threshold θ_0^* satisfies

$$\frac{\gamma-1}{\gamma}\left(1-(1-\theta_0^*)^{\frac{\gamma}{\gamma-1}}\right) - \alpha\theta_0^{*2}/2 = \theta_0^*(1-\theta_0^*)^{\frac{1}{\gamma-1}}.$$

Proposition 8 is a testimony to the richness of the problem even under a relatively simple parametrization. Note that for every γ the optimal mechanism tends to become more symmetric as α grows — this is not a coincidence and is explained in the next subsection. However, while the mechanism becomes completely symmetric for high α when $\gamma < 2$, it maintains some (slight but positive) degree of asymmetry even for very high α when $\gamma > 2$.

5.4 Comparative statics

5.4.1 Dependence on the importance of private information α

How does the optimal mechanism change when the importance of private information (the degree of information asymmetry) α changes? In subsection 5.3 we saw that as α grows, the mechanism becomes *more symmetric*. In fact, this result holds in general in the class of settings we consider in section 5.

We state this observation in the next two propositions.

Proposition 9. *Suppose $n = 2$, $C^P = \alpha\theta$, $C^I = g(q)$ and θ_i is not necessarily uniformly distributed. Then, for the optimal score floors mechanism (including boundary cases), the optimal threshold θ_0^* is weakly increasing in the importance of private information α. Thus, the optimal score floors mechanism becomes more symmetric as α grows.*

Proposition 10. *Suppose $n = 2$, $C^P = \alpha\theta$, $C^I = g(q)$ and θ_i is not necessarily uniformly distributed. Then, for the optimal score ceilings mechanism (including boundary cases), the optimal threshold θ_0^* is weakly decreasing in the importance of private information α. Thus, the optimal score ceilings mechanism becomes more symmetric as α grows.*

The intuition behind propositions 9 and 10 is related to the main economic trade-off faced by the principal when deciding on the level of mechanism asymmetry. This is the trade-off between ex-post efficiency and avoidance of duplication of investment costs. On the one hand, the principal would like to award the contract to the most efficient producer to obtain the best possible price-quality combination. However, to find out which one is the most efficient best, one has to treat the bidders symmetrically — and this will inevitably lead to both bidders incurring investment costs. By bidders' participation constraints, these costs must ultimately be paid by the principal for both bidders. On the other hand, if the principal constrains herself to a sole supplier *a priori*, she will have to compensate only one firm for the investment costs; however, it might not turn out to be the most efficient supplier. In general the principal's dilemma is resolved at some intermediate level of mechanism symmetry. When the importance of private information α grows, the ex-post efficiency motive becomes stronger and the optimal solution moves towards the optimal symmetric mechanism.

Because we can also interpret α as the degree of information asymmetry between the buyer and the suppliers (the higher α, the higher is the degree of the principal's uncertainty about a supplier's production costs), the message of Propositions 9 and 10 can also be formulated as follows: *Information asymmetry is associated with an optimal mechanism's symmetry. Information symmetry is associated with an optimal mechanism's asymmetry.*

5.4.2 Optimal vs. Efficient Mechanisms

Even though in this paper we analyze mostly buyer-optimal mechanisms, an interesting question is how the degree of asymmetry compares between the buyer-optimal and the society-optimal (efficient) mechanism. We can easily answer this question when θ is uniform.

Somewhat unexpectedly, we find that the efficient mechanism exhibits *more* favoritism than the buyer-optimal mechanism, not less.

Proposition 11. *Suppose $n = 2$, $C^P = \alpha\theta$, $C^I = g(q)$ with either $g'''(q) > 0$ for all q or $g'''(q) < 0$ for all q, and $F(\theta) = \theta$. Then, the efficient mechanism is weakly more asymmetric (exhibits weakly more favoritism) than the buyer-optimal mechanism.*

The intuition behind Proposition 11 is that while the buyer takes into account virtual costs, the social planner takes into account just costs when determining the optimal degree of mechanism asymmetry. Since virtual costs are more responsive to θ than just costs (because they also include type-dependent information rents) the buyer is more concerned about ex-post efficiency than the social planner. Hence, by the logic of Propositions 9 and 10 the buyer should choose a more symmetric mechanism than the social planner. This is exactly what Proposition 11 says.

5.5 Implementation

Now we describe the implementation of the optimal quality schedules 12 and 13 in more detail. Even though they may be implemented in a direct mechanism, we seek an implementation via an appropriate modification of a first-score scoring auction which will presumably be more practical.

5.5.1 "Score floors" mechanisms

As noted above, when the optimal quality schedules are given by (12), they can be implemented with a modified first-score scoring auction in which the bidders face different score floors (reserve scores) and also the favored bidder gets a bonus in case she passes the un-favored bidder's (higher) reserve score.

We describe the implementation formally in a more general setting than the one considered in section 5 so far. Recall from Proposition 2 that the optimal symmetric quality $q^*_{symm}(\theta)$ can be implemented by a score function $S(q,p) = s(q) - p$ where $s(q)$ is defined by (3). Given a score function $s(q)$ defined by (3) for $q^*_{symm}(\theta)$, define

$$\psi_s(e,\theta) := \max_q \left(e \cdot (s(q) - c^P(q,\theta)) - c^I(q,\theta)\right).$$

Let $q^*(e,\theta)$ be the quality that solves this problem.

Also, denote $t \wedge \theta_0 = \min\{\theta_0, t\}$ and

$$IR(a,b) := \int_a^b \left((1 - F(t\wedge\theta_0))c^P_\theta(q^*(1 - F(t\wedge\theta_0),t),t) + c^I_\theta(q^*(1 - F(t\wedge\theta_0),t),t)\right) dt.$$

(The notation “IR” stands for “information rents”).

Proposition 12. *Suppose an optimal mechanism is a “score floors” mechanism with a threshold $\theta_0 \in [0,1)$. Then, the optimal quality schedules given by (12) can be implemented in a modified first-score scoring auction in which:*

- *the quasi-linear score $S(q,p) = s(q) - p$ is used;*
- *the favored bidder faces a reserve score (score floor) of $S_1^r = \frac{\psi_s(1-F(\theta_0),1)}{1-F(\theta_0)}$. That is, her score S_1 counts only if $S_1 \geq S_1^r$.*
- *the unfavored bidder faces a (higher) reserve score (score floor) of $S_2^r = \frac{\psi_s(1-F(\theta_0),\theta_0)}{1-F(\theta_0)}$. That is, her score S_2 counts only if $S_2 \geq S_2^r$.*
- *the favored bidder gets a bonus $B_1 = IR(\theta_0, 1)$ if her score exceeds the unfavored bidder's reserve score S_2^r, regardless of whether the favored bidder wins or not.*
- *the equilibrium score bid of the favored bidder is*

$$S_1^*(\theta_1) = \begin{cases} \frac{\psi_s(1-F(\theta_1),\theta_1) - IR(\theta_1,\theta_0)}{1-F(\theta_1)}, & \theta_1 \leq \theta_0; \\ S_1^r, & \theta_1 > \theta_0. \end{cases}.$$

  The score bid $S_1^*(\theta_1)$ jumps down at $\theta_1 = \theta_0$.

- *the equilibrium score bid of the unfavored bidder is*

$$S_2^*(\theta_2) = \begin{cases} \frac{\psi_s(1-F(\theta_2),\theta_2) - IR(\theta_2,\theta_0)}{1-F(\theta_2)}, & \theta_2 \leq \theta_0; \\ \textit{no bid}, & \theta_2 > \theta_0. \end{cases}$$

Two comments are in order about the structure of the mechanism described in Proposition 12.

- **The role of the bonus.** The bonus serves to create the correct incentives for the types of the favored bidder that are immediately to the left of and including θ_0. The score strategy of bidder 1 jumps down at θ_0, so the type θ_0 is exactly indifferent between bidding the two scores. Note that her quality strategy is continuous at θ_0 meaning that the type θ_0 should be indifferent between bidding two different prices. To achieve the indifference, she must be compensated. Without the bonus, θ_0 and some more efficient types would opt out of the “fierce competition” high-scores, choosing to win with score S_1^r and with comfortable probability of $1 - F(\theta_0)$ instead. Thus, the bonus is needed to achieve the efficient symmetric competition among types $\theta < \theta_0$.

- **Why two reserve scores?** Here, the reserve scores are only needed to create asymmetry, not to exclude inefficient types, as the contract should be always awarded. So one might surmise that having only the reserve score for the unfavored bidder would suffice. So why are there two reserve scores but not one? The answer is that when we impose a binding reserve score for the unfavored bidder, there emerges a region of types of the favored bidder which win with a probability *that does not depend on their bid*. (This probability is equal to the probability that the unfavored bidder does not pass her reserve score.) Without a reserve score for the favored bidder, this would incentivize such types of her to choose the minimal quality and infinite price which would not be efficient/optimal. Thus, the reserve score is needed for the favored bidder as well. It should be low enough so that in equilibrium, the favored bidder always passes her reserve score, but it will be binding for all types $\theta_1 > \theta_0$.

5.5.2 "Score ceilings" mechanisms

The quality schedules given by (13) are implemented by an even less standard modification of the first-score scoring auction than the one described for the case of "score floors". Namely, now the score *ceiling* is introduced, ties happen in equilibrium and are resolved in favor of the favored bidder, and the favored bidder pays a side-payment upon winning which we call *a kickback*. For the purposes of the proposition to follow, redefine information rents by

$$IR(a,b) := \int_a^b \big((1-F(t))c_\theta^P(q^*(1-F(t),t),t) + c_\theta^I(q^*(1-F(t),t),t)\big)\,dt.$$

Proposition 13. *Suppose an optimal mechanism is a "score ceilings" mechanism with a threshold* $\theta_0 \in [0,1)$*. Then, the optimal quality schedules* (13) *can be implemented in a modified first-score scoring auction in which:*

- *the quasi-linear score* $S(q,p) = s(q) - p$ *is used;*
- *the unfavored bidder faces a score ceiling of* $\tilde{S} = \frac{\psi_s(1-F(\theta_0),\theta_0) - IR(\theta_0,1)}{1-F(\theta_0)}$*. That is, the unfavored bidder's score* S_2 *does not count if* $S_2 > \tilde{S}$*.*
- *the favored bidder faces any score ceiling weakly higher than* $\tilde{S}$*. In particular, the favored bidder may face no score ceiling.*
- *ties are resolved in favor of the favored bidder;*

- *the bidders use the same continuous equilibrium score strategy given by*

$$S_i^*(\theta) = \begin{cases} \tilde{S}, & \theta < \theta_0; \\ \frac{\psi_s(1-F(\theta),\theta) - IR(\theta,1)}{1-F(\theta)}, & \theta \geq \theta_0. \end{cases}$$

- *Upon winning with score $\tilde{S}$, the favored bidder must pay the buyer a kickback of*

$$T = \psi_s(1,\theta_0) - \frac{\psi_s(1-F(\theta_0),\theta_0)}{1-F(\theta_0)} + \frac{F(\theta_0)}{1-F(\theta_0)} IR(\theta_0, 1). \tag{14}$$

From (14), it may not be immediate that the kickback is nonnegative (and thus the name is warranted). We establish that it indeed is in the following proposition:

Proposition 14. *In the modified first-score auction implementation described above, the kickback T is nonnegative (and positive if $\theta_0 > 0$).*

We again make several comments.

- **The role of the kickback.** We again stress that the role of the side-payment, in this case kickback, is to provide correct incentives; it is in no way evidence of corruption which is absent from our model as the buyer and the auctioneer are one and the same. The kickback is needed to ensure that the types $\theta_1 > \theta_0$ of the favored bidder do not rush to win for sure with the score $\tilde{S}$ and instead maintain efficient symmetric competition with the types $\theta_2 > \theta_0$ bidder 2 in the low-score range.

- **Alternative description**. Also, this mechanism may be described as follows. After the bidders submit bids (q_i, p_i), the buyer first looks if the favored bidder's bid is *satisfactory*, in the sense that $s(q_i) - p_i \geq \tilde{S}$. If it is, it is taken and the game ends. If it is not, the bid is compared with that of bidder 2 and then the best bid in terms of score wins.

6 Allow-k-bidders mechanisms

One interesting family of asymmetric mechanisms is mechanisms where the principal allows only $k \leq n$ bidders to enter, and employs the optimal symmetric mechanism for these k bidders. Such mechanisms may be more practical than arbitrary asymmetric mechanisms since this particular kind of asymmetry may be less salient, and thus on the surface such mechanisms may look more "fair".

Given that the principal only chooses $k \leq n$, which k would be optimal for him? One reasonable guess is that it may be often optimal to set $k = 2$: this choice saves a lot of investment costs while still preserving some competition. We show that in a large class of settings, including the example in subsection 5.3, this guess is wrong. Namely, we show that often the optimal solution is "one-or-all": depending on the importance of private information, it is always optimal for the principal to either allow only 1 bidder or *all* of the bidders to enter. In this sizable set of situations, no intermediate $k \in \{2, 3, \ldots, n-1\}$ can ever be optimal.

Proposition 15. *Suppose $C^P = \alpha\theta$, $C^I = g(q)$ where $g(0) = 0$, $g'(q)$ is strictly increasing and $\frac{g(q)}{\sqrt{g'(q)}}$ is also strictly increasing. Suppose also that $F(\theta) = \theta$. Then, the buyer's utility $U(k)$ is a quasi-convex function of the number of bidders k allowed to enter. Thus, it is optimal for the principal to allow either one or all bidder to enter, that is, $k^* \in \{1, n\}$.*

The condition that $\frac{g(q)}{\sqrt{g'(q)}}$ is strictly increasing means that, in a certain sense, investment costs are not too convex. However, this condition is admittedly mild, as it is satisfied for $g(q) = q^\gamma$ and all $\gamma > 1$ and even for exponential costs $g(q) = \exp(q) - 1$.

Example: Suppose $g(q) = q^2/2$. Then, the buyer's utility from the optimal symmetric mechanism with k bidder is

$$U(k) = \frac{1}{2}\frac{k}{2k-1} - \frac{\alpha}{k+1},$$

which is a quasi-convex function. Thus, there exists an α_0 such that for

$$k^*(\alpha) = \begin{cases} 1, & \alpha < \alpha_0; \\ n, & \alpha > \alpha_0. \end{cases}$$

Proposition 15 also explains the quasi-convex behavior of utility depicted in Figure 2 in Gershkov et al. (2021) for a related additively separable setting.

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Appendix

A Omitted proofs

A.1 Proof of Proposition 2

Proof. Choosing quality q and price p is equivalent to choosing quality q and score $S = s(q) - p$. Given score strategy $S(\theta)$, price strategy is recovered by $p(\theta) = s(q(\theta)) - S(\theta)$. So it is sufficient to establish the existence of equilibrium in quality-score pairs.

Given quality schedule $q(\theta)$ and $s(q)$ given by (3), define the *pseudotype* by $k(\theta) := s(q(\theta)) - C^P(q(\theta), \theta)$. Imagine the firms already chose qualities $q(\theta)$ and now choose which scores S_i to bid. The problem of firm i is

$$\max_S \; (k(\theta_i) - S)\mathbb{P}(S > \max_{j \neq i} S_j) - C^I(q(\theta_i), \theta_i).$$

Since C^I is constant (already sunk), we have a textbook first-price auction problem in which the valuation of each bidder is $k_i = k(\theta)$ and the bid is S. This auction has the textbook symmetric equilibrium $S_i = S^*(k_i)$.

We claim that $(q_i, S_i) = (q(\theta_i), S^*(k(\theta_i)))$, $i = 1, \dots, n$ is in fact a BNE of the scoring auction.

First, we show that the score strategy $S^*(k(\theta_i))$ is decreasing in θ_i. Indeed, $S^*(k)$ is increasing being the textbook equilibrium while $k(\theta_i)$ is decreasing. To show the latter, it is sufficient to show that $k(\theta(q))$ is increasing. But $k(\theta(q)) \equiv s(q) - C^P(q, \theta(q))$ so

$$\frac{dk(\theta(q))}{dq} = s'(q) - C^P_q(q, \theta(q)) - C^P_\theta(q, \theta(q))\theta'(q) = \frac{C^I_q(q, \theta(q))}{(1 - \theta(q))^{n-1}} - C^P_\theta(q, \theta(q))\theta'(q) > 0,$$

where we used (3) for the second equality and known signs of C^I_q, C^P_θ, $\theta'(q)$ for the inequality.

Abusing notation, we write $S^*(\theta) \equiv S^*(k(\theta))$. We proved that $S^*(\theta)$ is decreasing.

Now suppose every bidder but the bidder i plays according to the alleged BNE strategy $(q, S) = (q(\theta), S^*(\theta))$. We shall show that it is indeed optimal for bidder i to respond with exactly the same strategy. To this end, consider bidder's i two-dimensional optimization problem:

$$\max_{q,S} \Pi(q, S|\theta_i) = \max_{q,S} \; \left(s(q) - C^P(q, \theta_i) - S\right) \mathbb{P}(S > \max_{j \neq i} S^*(\theta_j)) - C^I(q, \theta_i).$$

Because $S^*(\theta)$ is decreasing and all θ_i are iid uniform on $[0, 1]$, the optimization problem can be rewritten as

$$\max_{q,S} \ \left(s(q) - C^P(q,\theta_i) - S\right)\left(1 - S^{*-1}(S)\right)^{n-1} - C^I(q,\theta_i).$$

Denote $\Pi_{\max}(q|\theta_i) := \max_S \Pi(q,S|\theta_i)$. This is the maximum profit bidder i can get by bidding some score if her quality is fixed at some (possibly suboptimal) level q. Denote by $\sigma^*(q,\theta_i)$ the optimal score bid given that quality is fixed at level q and type of bidder i is θ_i. Thus, $\Pi_{\max}(q|\theta_i) \equiv \Pi(q,\sigma^*(q,\theta_i)|\theta_i)$. It follows from the above construction that $\sigma^*(q(\theta_i),\theta_i) \equiv S^*(\theta_i)$. Thus, for $(q,S) = (q(\theta_i), S^*(\theta_i))$ to be bidder's i best response, it is sufficient that $q(\theta_i)$ is a maximizer of $\Pi_{\max}(q|\theta_i)$.

It follows from the structure of the objective function $\Pi(q,S|\theta)$ that the optimal score bid given that quality $\sigma^*(q,\theta_i)$ depends only on the expression $s(q) - C^P(q,\theta_i)$ (pseudotype at arbitrary q), so we can write $\sigma^*(s(q) - C^P(q,\theta_i))$. Note that because $S^*(\cdot)$ is decreasing, the profit function Π is supermodular in $(S, s(q) - C^P(q,\theta_i))$; thus, the function $\sigma^*(\cdot)$ is increasing.

By Envelope Theorem, the first-order condition for the maximization of $\Pi_{\max}(q|\theta_i)$ over q is

$$\left(s'(q) - C^P_q(q,\theta)\right)\left(1 - S^{*-1}(\sigma^*(s(q) - C^P(q,\theta)))\right)^{n-1} - C^I_q(q,\theta) = 0. \tag{15}$$

Set $q = q(w)$, where $w \in [0,1]$ is the "misreported type", in (15):

$$\left(s'(q(w)) - C^P_q(q(w),\theta)\right)\left(1 - S^{*-1}(\sigma^*(s(q(w)) - C^P(q(w),\theta)))\right)^{n-1} - C^I_q(q(w),\theta) = 0. \tag{16}$$

Because $S^*(\theta) \equiv \sigma^*(s(q(\theta)) - C^P(q(\theta),\theta_i))$ and equation (3), the equation (16) is always satisfied at $w = \theta$, so the first-order condition (15) is always satisfied at $q = q(\theta)$. To prove that $q = q(\theta)$ indeed maximizes $\Pi_{\max}(q|\theta)$, we need to check some second-order conditions of some form. We use the following two-step argument: first, we show that (16) is satisfied *only* at $w = \theta$; second, we show that in a neighborhood $w = \theta$, the LHS of (16) is increasing in w. This implies that the LHS of (15) is decreasing in q in a neighborhood of $q(\theta)$, meaning that $q = q(\theta)$ is a local maximum of $\Pi_{\max}(q|\theta)$. Thus, we show that $\Pi_{\max}(q|\theta)$ has only one critical point, $q = q(\theta)$, and it is a local maximum. It follows that it must be the global maximum as well, proving the result.

Denote the LHS of (16) by $g(w,\theta)$.

Lemma 1. *For all w and θ, $g(w,\theta) = 0$ implies $w = \theta$.*

Proof. One (straightforward) approach would be to try to show that for every θ, the equation (16) has only one solution $w = \theta$. Instead, we show that for every w, the equation (16) has only one solution $\theta = w$. This is an equivalent statement which is easier to show.

To this end, note that:

1. $s'(q(w)) - C^P_q(q(w),\theta)$ is decreasing in θ by supermodularity of $C^P(q,\theta)$.

2. $\left(1 - S^{*-1}(\sigma^*(s(q(w)) - C^P(q(w),\theta)))\right)^{n-1}$ is decreasing in θ because (1) $C^P_\theta > 0$; (2) $\sigma^*(\cdot)$ is increasing by the discussion above; (3) $S^{*-1}(\cdot)$ is decreasing as $S^*(\cdot)$ is decreasing.

3. $C^I_q(q(w),\theta)$ is increasing in θ by supermodularity of $C^I(q,\theta)$.

These three observations imply that whenever $s'(q(w)) - C^P_q(q(w),\theta) > 0$ (for θ lower than some threshold), $g(w,\theta)$ is decreasing in θ, and whenever $s'(q(w)) - C^P_q(q(w),\theta) \leq 0$ (for θ higher than this threshold), $g(w,\theta) < 0$. Thus, for every w there can be at most one θ such that $g(w,\theta) = 0$. Since this is true for $\theta = w$, the result follows. □

Lemma 2. *For all θ, $g_w(w,\theta)|_{w=\theta} > 0$.*

Proof. By (3), $s'(q(w)) \equiv C^P_q(q(w),w) + \frac{C^I_q(q(w),w)}{(1-w)^{n-1}}$. Plugging this into (16), one gets

$$g(w,\theta) = \left(C^P_q(q(w),w) - C^P_q(q(w),\theta) + \frac{C^I_q(q(w),w)}{(1-w)^{n-1}}\right)(1 - S^{*-1}(\sigma^*(\cdot)))^{n-1} - C^I_q(q(w),\theta).$$

Rearranging, one gets

$$\begin{aligned} g(w,\theta) = &\int_\theta^w C^P_{q\theta}(q(w),y)dy \cdot (1 - S^{*-1}(\sigma^*(\cdot)))^{n-1} \\ &+ \int_\theta^w C^I_{q\theta}(q(w),y)dy \\ &- \left(1 - \left(\frac{1 - S^{*-1}(\sigma^*(\cdot))}{1-w}\right)^{n-1}\right) C^I_q(q(w),w). \end{aligned}$$

As $S^{*-1}(\sigma^*(s(q(\theta)) - C^P(q(\theta),\theta)) \equiv \theta$, one obtains

$$\begin{aligned} g_w(w,\theta)|_{w=\theta} = &C^P_{q\theta}(q(\theta),\theta) \cdot (1-\theta)^{n-1} && (17) \\ &+ C^I_{q\theta}(q(\theta),\theta) && (18) \\ &+ (n-1)1^{n-2} C^I_q(q(\theta),\theta) \frac{\partial}{\partial w}\left\{\frac{1 - S^{*-1}(\sigma^*(\cdot))}{1-w}\right\}|_{w=\theta}. && (19) \end{aligned}$$

The first two terms are positive by supermodularity of costs. The last term is

$$\frac{\partial}{\partial w}\left\{\frac{1 - S^{*-1}(\sigma^*(s(q(w)) - C^P(q(w),\theta)))}{1-w}\right\}|_{w=\theta} = -\frac{\sigma^{*\prime}(\cdot)[s'(q) - C^P_q]q'(\theta)}{S^{*\prime}(S^{*-1}(\cdot))(1-\theta)} + \frac{1 - S^{*-1}(\cdot)}{(1-\theta)^2}.$$

Again using $S^{*-1}(\sigma^*(s(q(\theta)) - C^P(q(\theta), \theta)) \equiv \theta$, we get that the last expression is

$$-\frac{\sigma^{*\prime}(\cdot)[s'(q) - C_q^P]q'(\theta)}{S^{*\prime}(\theta)(1-\theta)} + \frac{1}{1-\theta} = \frac{1}{1-\theta}\left(1 - \frac{\sigma^{*\prime}(\cdot)[s'(q) - C_q^P]q'(\theta)}{\sigma^{*\prime}(\cdot)([s'(q) - C_q^P]q'(\theta) - C_\theta^P)}\right) = -\frac{C_\theta^P}{k'(\theta)(1-\theta)}.$$

As $C_\theta^P > 0$ and $k'(\theta) < 0$ (we proved earlier that the pseudotype is decreasing), $C_q^I > 0$, this implies that the term (19) is positive, yielding the desired result. □

The above two lemmas together show show that $\Pi_{\max}(q|\theta)$ has only one critical point, $q = q(\theta)$, and it is a local maximum. It follows that it must be the global maximum as well; thus $(q(\theta), S^*(\theta))$ is a best response to itself, and hence a strategy forming a symmetric BNE. □

A.2 Proof of Proposition 6

Proof. Principal's utility as a function of threshold θ_0 can be written as

$$U(\theta_0) = n \int_0^{\theta_0} \max_x \left[(1-\theta)^{n-1}x - C(x,\theta)\right] d\theta + \int_{\theta_0}^1 \max_x \left[(1-\theta_0)^{n-1}x - C(x,\theta)\right] d\theta.$$

Denote by $x^*(\theta, \theta_0)$ is the solution to the problem $\max_x [(1-\theta_0)^{n-1}x - C(x,\theta)]$. Note that $x^*_{symm}(\theta) \equiv x^*(\theta,\theta)$. Differentiating $U(\theta_0)$, we get

$$U'(\theta_0) = (n-1)\max_x \left[(1-\theta_0)^{n-1}x - C(x,\theta_0)\right] - (n-1)(1-\theta_0)^{n-2}\int_{\theta_0}^1 x^*(\theta,\theta_0)d\theta,$$

where we used Envelope theorem to evaluate the derivative of the second addend in $U(\theta_0)$. This may be further rewritten as

$$U'(\theta_0)/(n-1) = (1-\theta_0)^{n-2}\left((1-\theta_0)x^*(\theta_0,\theta_0) - \int_{\theta_0}^1 x^*(\theta,\theta_0)d\theta\right) - C(x^*(\theta_0,\theta_0),\theta_0). \quad (20)$$

In what follows, we prove that $\lim_{\theta_0\to 1} \frac{U'(\theta_0)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}} < 0$ which implies the result.

To evaluate the limit

$$L_1 := \lim_{\theta_0\to 1} \frac{(1-\theta_0)^{n-2}\left((1-\theta_0)x^*(\theta_0,\theta_0) - \int_{\theta_0}^1 x^*(\theta,\theta_0)d\theta\right)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}},$$

use the first-order Taylor expansion of the function $t \to x^*(t,\theta_0)$ at $t = 1$ and the second-order Taylor expansion of the function $t \to \int_t^1 x^*(\theta,\theta_0)d\theta$ at $t = 1$. (The expansions are valid because $x^*(\theta, \theta_0)$ is differentiable in the first argument by Implicit Function theorem.) After simplifications, we get

$$L_1 = \lim_{\theta_0 \to 1} \frac{(1-\theta_0)^{n-2}\left((1-\theta_0)x^*_\theta(1,\theta_0)(\theta_0 - 1) + x^*_\theta(1,\theta_0)(1-\theta_0)^2/2 + o((1-\theta_0)^2)\right)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}} =$$

$$= \lim_{\theta_0 \to 1} \frac{-x^*_\theta(1,\theta_0)(1-\theta_0)^n/2 + o((1-\theta_0)^n)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}} = 0,$$

because $x^*_\theta(1,1)$ is finite and $n > \frac{\gamma(n-1)}{\gamma-1}$ since $\gamma > n$.

Now consider $L_2 := \lim_{\theta_0 \to 1} \frac{C(x^*(\theta_0,\theta_0),\theta_0)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}}$. We prove that $L_2 > 0$. First, denoting $\lim_{x \to \underline{x}^+} \frac{C(x,1)}{(x-\underline{x})^\gamma} := D > 0$, by L'Hopital's rule we get that $\lim_{x \to \underline{x}^+} \frac{C_x(x,1)}{(x-\underline{x})^{\gamma-1}} = \gamma D > 0$. Thus,

$$\lim_{x \to \underline{x}^+} \frac{C(x,1)}{[C_x(x,1)]^{\frac{\gamma}{\gamma-1}}} = \left[\lim_{x \to \underline{x}^+} \frac{[C(x,1)]^{\gamma-1}}{[C_x(x,1)]^\gamma}\right]^{\frac{1}{\gamma-1}} = \left[\lim_{x \to \underline{x}^+} \frac{D^{\gamma-1}(x-\underline{x})^{\gamma(\gamma-1)}}{(\gamma D)^\gamma (x-\underline{x})^{(\gamma-1)\gamma}}\right]^{\frac{1}{\gamma-1}} = \frac{D}{(\gamma D)^{\frac{\gamma}{\gamma-1}}} > 0.$$

Now substitute $x = x^*(1,\theta_0)$ with $\theta_0 \to 1$ to the above limit (this is valid since $\lim_{\theta_0 \to 1} x^*(1,\theta_0) = \underline{x}$). Note that by FOC $C_x(x^*(1,\theta_0),1) \equiv (1-\theta_0)^{n-1}$. Thus,

$$0 < \frac{D}{(\gamma D)^{\frac{\gamma}{\gamma-1}}} = \lim_{x \to \underline{x}^+} \frac{C(x,1)}{[C_x(x,1)]^{\frac{\gamma}{\gamma-1}}} = \lim_{\theta_0 \to 1} \frac{C(x^*(1,\theta_0),1)}{[C_x(x^*(1,\theta_0),1)]^{\frac{\gamma}{\gamma-1}}} = \lim_{\theta_0 \to 1} \frac{C(x^*(1,\theta_0),1)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}}.$$

Finally, note that

$$0 < \lim_{\theta_0 \to 1} \frac{C(x^*(1,\theta_0),1)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}} = \lim_{\theta_0 \to 1} \lim_{\theta \to 1} \frac{C(x^*(\theta,\theta_0),\theta)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}} = \lim_{\substack{\theta \to 1 \\ \theta_0 \to 1}} \frac{C(x^*(\theta,\theta_0),\theta)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}} = L_2.$$

To sum up,

$$\lim_{\theta_0 \to 1} \frac{U'(\theta_0)}{(1-\theta_0)^{\frac{\gamma(n-1)}{\gamma-1}}} = (n-1)(L_1 - L_2) = (n-1)(0 - L_2) < 0.$$

□

A.3 Proof of Theorem 1

Proof. Part 1. If marginal costs are sufficiently convex, by definition 2 the function $q \to \alpha\xi(g'(q))) - q = \alpha(1 - J(F^{-1}(1 - g'(q)))) - q$ is strictly quasi-convex. Quasi-convexity is preserved under any monotone transformation of the argument, so plugging $q = g'^{-1}(1 - F(\theta))$, we get that the function $\alpha(1 - J(\theta)) - g'^{-1}(1 - F(\theta))$ is strictly quasi-convex as well.

The environment considered in the theorem is additively separable. It may be shown that in an additively separable environment the optimal mechanism is the same in two cases: (1) quality choice (agent's action) is contractible and $v(q)$ enters principal's utility (our case); (2) quality choice (agent's action) is non-contractible and $v(q)$ enters agent's utility (the case in Zhang (2017)). Thus, we can use the analysis in Zhang (2017).

By following the same steps as in Zhang (2017), we reach the conclusion that the function $\alpha(1 - J(\theta)) - g'^{-1}(1 - F(\theta))$ plays in our model exactly the same role as the function $J(\theta) - KF(\theta)$ in Zhang (2017), with the proviso that θ is a cost type here and utility type in Zhang (2017), so the notions of F and J should be modified appropriately. We will invoke the appropriate generalizations of the results in Zhang (2017) with the expressions "strictly increasing" and "strictly decreasing" interchanged (again as θ increases costs in this paper and decreases costs in Zhang (2017)). As the function $\alpha(1 - J(\theta)) - g'^{-1}(1 - F(\theta))$ is strictly quasi-convex, it either (1) strictly decreasing, (2) strictly increasing, or (3) first strictly decreasing and then strictly increasing. In the case (1) by Corollary 1, part I in Zhang (2017), the optimal mechanism is symmetric so the quality schedules (12) with $\theta_0 = 1$ are optimal. In the case (2) by Theorem 1, part I in Zhang (2017), the optimal mechanism is completely asymmetric so that one bidder gets the contract with probability 1. This corresponds to quality schedules (12) with $\theta_0 = 0$. Finally, in the case (3) by Theorem 1 in Zhang (2017) there exists $\theta_0 \in [0, 1]$ such that an optimal mechanism is symmetric for $\theta < \theta_0$ while having $[\theta_0, 1]$ is a "favored bidder interval". That is, for $\theta > \theta_0$ one bidder is "favored" and gets the contract with probability $1 - F(\theta_0)$; and the other bidder is "un-favored" and gets the contract with probability $1 - F(1) = 0$. This implies the quality schedules (12) with that particular θ_0.

Part 2 is proved analogously. □

A.4 Proofs of remaining results in section 5

Proof of proposition 8:

Proof. It follows from Corollary 1 that if $\gamma > 2$ an optimal mechanism is either score floors, sole-sourcing, or symmetric while if $\gamma < 2$ an optimal mechanism is either score ceilings, sole-sourcing, or symmetric.

For a score floors mechanism, the FOC for the optimal threshold θ_0 is

$$\psi(1 - \theta_0, \theta_0) = \int_{\theta_0}^{1} x^*(1 - \theta_0, \theta) d\theta. \tag{21}$$

Plugging in $\psi(e, \theta) = \frac{\gamma-1}{\gamma} e^{\frac{\gamma}{\gamma-1}} - \alpha\theta e$ and $x^*(e, \theta) = \psi'_e = e^{\frac{1}{\gamma-1}} - \alpha\theta$, we get an equation which is under $\theta_0 < 1$ simplified to

$$(1-\theta_0)^{\frac{\gamma-2}{\gamma-1}} = \frac{2}{\alpha\gamma}.$$

sole-sourcing corresponds to $\theta_0 = 0$ and it obtains if the derivative of the objective at $\theta_0 = 0$ is non-positive. Thus, it will obtain if $1 - \frac{2}{\alpha\gamma} \leq 0$, $\alpha \leq 2/\gamma$, otherwise there will be a $\theta_0 \in (0,1)$ satisfying FOC and giving a score floors mechanism.

For a score ceiling mechanism, the FOC for the optimal threshold θ_0 is

$$\psi(1,\theta_0) - \psi(1-\theta_0,\theta_0) = \int_0^{\theta_0} x^*(1-\theta_0,\theta)d\theta. \tag{22}$$

Plugging in ψ and x^*, we get

$$\frac{\gamma-1}{\gamma}\left(1-(1-\theta_0)^{\frac{\gamma}{\gamma-1}}\right) - \alpha\theta_0^2/2 = \theta_0(1-\theta_0)^{\frac{1}{\gamma-1}}. \tag{23}$$

In general, there is no closed-form solution to this equation. sole-sourcing now corresponds to $\theta_0 = 1$ while the symmetric solution to $\theta_0 = 0$. To obtain an α threshold separating the sole-sourcing region from the score ceiling region, plug in $\theta_0 = 1$ to (23). To obtain an α threshold separating the symmetric solution region region from the score ceiling region, solve (23) for α and then take the limit when $\theta_0 \to 0$. □

Proof of proposition 9:

Proof. For this costs specification, $\psi(e,\theta) = H(e) - \alpha l(\theta)e$ for some function $H(e)$. Rewriting (21), we get that

$$U'_{\theta_0} = \psi(1-\theta_0,\theta_0) - \int_{\theta_0}^1 x^*(1-\theta_0,\theta)d\theta = H(1-\theta_0) - (\theta_0)(1-\theta_0) - \int_{\theta_0}^1 (H'(1-\theta_0) - \alpha l(\theta))d\theta =$$
$$\alpha\left[\int_{\theta_0}^1 l(\theta)d\theta - l(\theta_0)(1-\theta_0)\right] + H(1-\theta_0) - (1-\theta_0)H'(1-\theta_0).$$

Thus,

$$U''_{\theta_0,\alpha} = \int_{\theta_0}^1 l(\theta)d\theta - l(\theta_0)(1-\theta_0) > 0,$$

where the inequality follows from the fact that $l(\theta)$ is strictly increasing. By the standard monotone comparative statics theorem, the optimal $\theta_0^*(\alpha)$ must be a weakly increasing function. □

Proof of proposition 10 is analogous to that of proposition 9 and is omitted.

Proof of proposition 11:

Proof. The efficient mechanism is characterized by a statement analogous to Theorem 1 in which in the definition of the function $\xi(\cdot)$ one replaces $J(\theta)$ with just θ. With $F(\theta) = \theta$ $J(\theta)$ is proportional to θ and thus Corollary 1 applies fully to the efficient mechanisms as well. Hence, an optimal mechanism is either a "score floors" mechanism or a "score ceilings" mechanism. Now note that the efficient mechanism in a given setting 1 is the same as a buyer-optimal mechanism in another setting 2 such that *virtual* costs in setting 2 are equal to costs in setting 1. If $C^P = \alpha\theta/2$, the *virtual* production costs $C^P + C^P_\theta F/f$ are exactly $\alpha\theta$. Thus, the efficient mechanism for $C^P = \alpha\theta$ is equal to the buyer-optimal mechanism for $C^P = \alpha\theta/2$. So we need to compare the buyer-optimal mechanism for $C^P = \alpha\theta/2$ with the buyer-optimal mechanism for $C^P = \alpha\theta$. Since α halves and thus decreases, by Propositions 9 and 10 the optimal threshold moves in a way that makes the mechanism less symmetric. Thus, the efficient mechanism is less symmetric than the buyer-optimal mechanism. □

Proof of proposition 12:

Proof. Denote by θ_0 the optimal threshold. Denote by $x^*(e, \theta)$ the solution to the problem $\max_x(ex - C(x, \theta))$.

Bidder 1 is the favored bidder. The x bids are just the optimal x'es to implement:

$$x_1^*(\theta_1) = x^*(1 - \min\{\theta_1, \theta_0\}, \theta_1);$$

$$x_2^*(\theta_2) = \begin{cases} x^*(1 - \theta_2, \theta_2), & \theta_2 \leq \theta_0; \\ x^*(0, \theta_2), & \theta_2 > \theta_0. \end{cases}$$

The equilibrium expected transfers can be recovered using the envelope formula:

$$t_i^*(\theta_i) = C(x_i^*(\theta_i), \theta_i) + \int_{\theta_i}^1 C_\theta(x^*(t), t)dt,$$

so

$$t_1^*(\theta_1) = \begin{cases} C(x^*(1 - \theta_1, \theta_1), \theta_1) + \int_{\theta_1}^{\theta_0} C_\theta(x^*(1 - t, t), t)dt + \int_{\theta_0}^1 C_\theta(x^*(1 - \theta_0), t), t)dt, & \theta_1 \leq \theta_0; \\ C(x^*(1 - \theta_0, \theta_1), \theta_1) + \int_{\theta_1}^1 C_\theta(x^*(1 - \theta_0), t), t)dt, & \theta_1 > \theta_0; \end{cases}$$

and

$$t_2^*(\theta_2) = \begin{cases} C(x^*(1-\theta_2,\theta_2),\theta_2) + \int_{\theta_2}^{\theta_0} C_\theta(x^*(1-t,t),t)dt + \int_{\theta_0}^{1} C_\theta(x^*(0,t),t)dt, & \theta_2 \leq \theta_0; \\ C(x^*(0,\theta_2),\theta_2) + \int_{\theta_2}^{1} C_\theta(x^*(0,t),t)dt, & \theta_2 > \theta_0; \end{cases}$$

The bonuses to be given are naturally the constants in the expressions above for $\theta_i < \theta_0$:

$$B_1 := \int_{\theta_0}^{1} C_\theta(x^*(1-\theta_0),t),t)dt$$

$$B_2 := \int_{\theta_0}^{1} C_\theta(x^*(0,t),t)dt$$

For many specifications (including all examples in section ...), $B_2 = 0$, so only the favored bidder gets the bonus.

The bonuses are necessary to render the equilibrium (and allocation) symmetric for $\theta_i < \theta_0$.

For $\theta_i < \theta_0$, the equilibrium (symmetric) price bid can be recovered as

$$p_i^*(\theta_i) = \frac{t_i^*(\theta_i) - B_i}{Prob.(\text{i wins})} = \frac{t_i^*(\theta_i) - B_i}{1-\theta_i}.$$

For $\theta_1 > \theta_0$, the price bid of bidder 1 has to satisfy

$$p_1^*(\theta_1) = \frac{t_1^*(\theta_1)}{Prob.(\text{1 wins})} = \frac{t_1^*(\theta_1)}{1-\theta_0} = \frac{C(x^*(1-\theta_0,\theta_1),\theta_1) + \int_{\theta_1}^{1} C_\theta(x^*(1-\theta_0),t),t)dt}{1-\theta_0}.$$

On the other hand, for $\theta_1 > \theta_0$, in our construction bidder 1 must not try to outbid the good types of bidder 2 (those with $\theta_2 < \theta_0$), opting instead for the guaranteed victory with prob. $1-\theta_0$ provided she passes her reserve score. Given this, she will optimally set the price at the maximum level that ensures barely passing her reserve score, so

$$x_1^*(\theta_1) - p^*(\theta_1) \equiv S_1^r \text{ for all } \theta_1 > \theta_0$$

$$p^*(\theta_1) \equiv x^*(1-\theta_0,\theta_1) - S_1^r \text{ for all } \theta_1 > \theta_0$$

Thus, for the implementation to be valid it must be that

$$x^*(1-\theta_0,\theta_1) - S_1^r \equiv \frac{C(x^*(1-\theta_0,\theta_1),\theta_1) + \int_{\theta_1}^{1} C_\theta(x^*(1-\theta_0),t),t)dt}{1-\theta_0},$$

so

$$x^*(1-\theta_0,\theta_1) - \frac{C(x^*(1-\theta_0,\theta_1),\theta_1) + \int_{\theta_1}^{1} C_\theta(x^*(1-\theta_0),t),t)dt}{1-\theta_0} \equiv S_1^r = \text{const in } \theta_1. \quad (24)$$

So the LHS of (24) must not depend on θ_1. In fact, this is so; this is a consequence of the envelope theorem applied to the maximization problem $\max_x(ex - C(x,\theta))$ and parameter θ.

As the LHS of (24) is a constant in θ_1, we can recover S_1^r by evaluating the RHS at any $\theta_1 > \theta_0$, in particular, at $\theta_1 = 1$. Thus, the principal has to set

$$S_1^r := x^*(1-\theta_0,1) - \frac{C(x^*(1-\theta_0,1),1)}{1-\theta_0} = \frac{\psi(1-\theta_0,1)}{1-\theta_0}.$$

To recover the unfavored bidder's reserve score, note that it must be equal to the score bid of the marginal non-excluded type θ_0. That is,

$$S_2^r := x_2^*(\theta_0) - p_2^*(\theta_0) = x^*(1-\theta_0,\theta_0) - \frac{C(x^*(1-\theta_0,\theta_0)}{1-\theta_0} = \frac{\psi(1-\theta_0,\theta_0)}{1-\theta_0}.$$

Now it is easy to see that the reserve for the unfavored bidder is higher, i.e. $S_2^r > S_1^r$. Indeed, this just follows from the fact that $\psi(e,\theta)$ is decreasing in the second argument.

A.4.1 Checking monotonicity

The envelope formulas take care of only the FOC of the firms' problems. For the whole construction to be valid (i.e., for the strategies described indeed be optimal for the firms), one must also check that the score bid strategies $S^*(\theta) = x^*(\theta) - p^*(\theta)$ are monotone (non-increasing).

The favored firm's score bid jumps down at θ_0, from S_2^r to S_1^r and stays constant for $\theta_1 > \theta_0$. The unfavored firm does not bid anything for $\theta_2 > \theta_0$. So it is sufficient to check that the (symmetric) score bids are nonincreasing for $\theta < \theta_0$.

For example, if $C = (x+\alpha\theta)^\gamma/\gamma$, for $\theta < \theta_0$ we have $S^*(\theta) = \frac{\gamma-1}{\gamma}(1-\theta)^{\frac{1}{\gamma-1}} - \alpha\frac{\theta_0-\theta_0^2/2-\theta^2/2}{1-\theta}$ which is a decreasing function for $\theta < \theta_0$ although it is not obvious.

In general, the equilibrium score bid for $\theta < \theta_0$ is

$$S^*(\theta) = x^*(1-\theta,\theta) - \frac{C(x^*(1-\theta,\theta),\theta) + \int_\theta^{\theta_0} C_\theta(x^*(1-t,t),t)dt}{1-\theta} = \frac{\psi(1-\theta,\theta) - \int_\theta^{\theta_0} C_\theta(x^*(1-t,t),t)dt}{1-\theta}. \quad (25)$$

The next lemma shows that it is indeed a nonincreasing function in general.

Lemma 3. *The score bid strategy defined by* (25) *is nonincreasing for in* θ *for* $\theta < \theta_0$*.*

Proof. Note that by envelope theorem, $\psi_e = x^*$ and $\psi_\theta = -C_\theta$ where we omit the arguments for brevity.

Differentiating, we obtain

$$(1-\theta)^2 S^{*\prime}(\theta) = (-x^* - C_\theta + C_\theta)(1-\theta) + \psi(1-\theta,\theta) - \int_\theta^{\theta_0} C_\theta(x^*(1-t,t),t)dt =$$
$$- x^* \cdot (1-\theta) + \psi(1-\theta,\theta) - \int_\theta^{\theta_0} C_\theta(x^*(1-t,t),t)dt =$$
$$- C(x^*(1-\theta,\theta),\theta) - \int_\theta^{\theta_0} C_\theta(x^*(1-t,t),t)dt < 0,$$

where we used the facts that $\psi(1-\theta,\theta) = (1-\theta)x^*(1-\theta,\theta) - C(x^*(1-\theta,\theta),\theta)$ and that $C(\cdot,\cdot) > 0$, $C_\theta(\cdot,\cdot) > 0$, $\theta < \theta_0$. □ □

□

Proof of proposition 14:

Proof. Denote by $x_m(\theta_1)$ the "monopolistic" production surplus that the favored bidder offers when she wins with prob.1 and by $x_s(\theta)$ the optimal symmetric surplus that each bidder offers when $\theta_i > \theta_0$. In general, the derivations similar to the above lead to the following formula for the kickback T:

$$T(\theta_1,\theta_0) = x_m(\theta_1) - x_s(\theta_0) + \frac{C(x_s(\theta_0),\theta_0) + \int_{\theta_0}^1 C_\theta(x_s(t),t)dt}{1-\theta_0}$$
$$- C(x_m(\theta_1),\theta_1) - \int_{\theta_1}^{\theta_0} C_\theta(x_m(t),t)dt - \int_{\theta_0}^1 C_\theta(x_s(t),t)dt.$$

It is easy to see that $T_1(\theta_1,\theta_0) = 0$, so the kickback does not depend on the reported type not only in the example above but in general. Thus,

$$T(\theta_1,\theta_0) = T(\theta_0,\theta_0) = x_m(\theta_0) - x_s(\theta_0) + \frac{C(x_s(\theta_0),\theta_0) + \int_{\theta_0}^1 C_\theta(x_s(t),t)dt}{1-\theta_0}$$
$$- C(x_m(\theta_0),\theta_0) - \int_{\theta_0}^1 C_\theta(x_s(t),t)dt.$$

As $\theta_0 \geq 0$,

$$T(\theta_0,\theta_0) \geq x_m(\theta_0) - x_s(\theta_0) + \frac{C(x_s(\theta_0),\theta_0) + \int_{\theta_0}^1 C_\theta(x_s(t),t)dt}{1} - C(x_m(\theta_0),\theta_0) - \int_{\theta_0}^1 C_\theta(x_s(t),t)dt$$
$$= [x_m(\theta_0) - C(x_m(\theta_0),\theta_0)] - [x_s(\theta_0) - C(x_s(\theta_0),\theta_0)] \geq 0,$$

where the last inequality is due to the fact that $x_m(\theta)$ maximizes the expression $x - C(x,\theta)$.

If $\theta_0 > 0$, the last inequality is strict, so $T > 0$. □

A.5 Proof of proposition 15

Proof. Recall that $\psi(e,\theta) \equiv \max_x (ex - C(x,\theta))$.

For $C(x,\theta) = g(x + \alpha\theta)$, $\psi(e,\theta) = H(e) - \alpha\theta e$ for some function $H(e)$. Denote $h(e) := H(e)/e$. First, we show that the condition that $\frac{g(t)}{\sqrt{g'(t)}}$ is strictly increasing implies that $h'(e)e^{3/2}$ is increasing.

Indeed, given that $g(0) = 0$ it is easy to show that

$$h(e) = \begin{cases} 0, & e < g'(0); \\ g'^{-1}(e) - \frac{g(g'^{-1}(e))}{e}, & e \geq g'(0). \end{cases}$$

and thus after simplifications

$$h'(e)e^{3/2} = \begin{cases} 0, & e < g'(0); \\ \frac{g(g'^{-1}(e))}{\sqrt{e}}, & e \geq g'(0). \end{cases}$$

As $g'^{-1}(e)$ is increasing and $\frac{g(t)}{\sqrt{g'(t)}}$ is increasing by assumption, $h'(e)e^{3/2}$ is increasing.

Now, the principal's payoff from playing the optimal symmetric mechanism among k remaining bidders is

$$U(k) = \int_0^1 k\left(H((1-\theta)^{k-1}) - \alpha\theta(1-\theta)^{k-1}\right) d\theta. \tag{26}$$

We shall show that $U(k)$ is quasi-convex when k is treated as a continuous variable[6] This implies the result.

[6]The integrand in (26) is *quasi-concave* in k. However, a sum (integral) of quasi-concave functions can be quasi-convex and not quasi-concave.

Using the substitution $y = (1-\theta)^{k-1}$, $U(k)$ may be rewritten as

$$U(k) = \frac{k}{k-1}\int_0^1 \left(H(y) - \alpha(1 - y^{\frac{1}{k-1}})y\right) y^{\frac{2-k}{k-1}} dy = \frac{k}{k-1}\int_0^1 \left(h(y) - \alpha(1 - y^{\frac{1}{k-1}})\right) y^{\frac{1}{k-1}} dy.$$

Now substitute $\delta := \frac{1}{k-1}$. Because this is a monotone transformation, it is sufficient to show that $U(k(\delta))$ is quasi-convex in δ. When $k = 1$, we set $\delta = +\infty$.

$$U = (1+\delta)\int_0^1 \left(h(y) - \alpha(1-y^\delta)\right) y^\delta dy = (1+\delta)\int_0^1 h(y)y^\delta dy - \alpha(1+\delta)\int_0^1 (1-y^\delta)y^\delta dy.$$

We now compute U'_δ. Integrating the first term by parts we get

$$(1+\delta)\int_0^1 h(y)y^\delta dy = \int_0^1 h(y) dy^{1+\delta} = h(1) - \int_0^1 h'(y)y^{1+\delta}dy.$$

Thus,

$$\left((1+\delta)\int_0^1 h(y)y^\delta dy\right)'_\delta = \int_0^1 h'(y)y^{1+\delta}\ln(1/y)dy.$$

The second integral can be computed explicitly:

$$(1+\delta)\int_0^1 (1-y^\delta)y^\delta dy = \frac{\delta}{2\delta+1}.$$

Thus,

$$\left((1+\delta)\int_0^1 (1-y^\delta)y^\delta dy\right)'_\delta = \frac{1}{(2\delta+1)^2} = \frac{1}{4}\frac{1}{(\delta+\frac{1}{2})^2} = \frac{1}{4}\int_0^1 y^{\delta-\frac{1}{2}}\ln(1/y)dy,$$

where the last equality may be verified by integration by parts.

Tacking stock,

$$\begin{aligned} U'_\delta = \int_0^1 h'(y)y^{1+\delta}\ln(1/y)dy - \frac{\alpha}{4}\int_0^1 y^{\delta-\frac{1}{2}}\ln(1/y)dy \\ = \left(\frac{\int_0^1 h'(y)y^{1+\delta}\ln(1/y)dy}{\int_0^1 y^{\delta-\frac{1}{2}}\ln(1/y)dy} - \frac{\alpha}{4}\right)\int_0^1 y^{\delta-\frac{1}{2}}\ln(1/y)dy. \end{aligned}$$

Now, we shall show that the expression in parentheses above is increasing in δ. This will imply that $U'_\delta(\delta)$ is of increasing sign and hence $U(\delta)$ is quasi-convex.

Rewrite

$$\frac{\int_0^1 h'(y)y^{1+\delta}\ln(1/y)dy}{\int_0^1 y^{\delta-\frac{1}{2}}\ln(1/y)dy} = \frac{\int_0^1 \left[h'(y)y^{\frac{3}{2}}\right] y^{\delta-\frac{1}{2}}\ln(1/y)dy}{\int_0^1 y^{\delta-\frac{1}{2}}\ln(1/y)dy}$$

and consider the following family of density functions on $(0,1)$ parametrized by δ:

$$f(y|\delta) = \frac{y^{\delta-\frac{1}{2}}\ln(1/y)}{\int_0^1 y^{\delta-\frac{1}{2}}\ln(1/y)dy}.$$

It is not hard to show that for any $\delta_1 < \delta_2$ $f(y|\delta_2)$ first-order stochastically dominates $f(y|\delta_1)$.[7] Thus, for any increasing function $b(y)$

$$\mathbb{E}_{y\sim f(\cdot|\delta)}(b(y)) = \frac{\int_0^1 [b(y)]\, y^{\delta-\frac{1}{2}}\ln(1/y)dy}{\int_0^1 y^{\delta-\frac{1}{2}}\ln(1/y)dy}$$

is increasing in δ. However, we have shown above that the function $h'(y)y^{3/2}$ is increasing indeed. □

[7]For this, note that $f(y|\delta_2)/f(y|\delta_1)$ is increasing in y and thus apart from endpoints the two densities cross only once, with $f(y|\delta_2)$ crossing $f(y|\delta_1)$ from below.