Heterogeneous Expectations and Stock Market Cycles

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Abstract

We present a model of heterogeneous expectations. In the short run, agents learn about prices with different intensities due to their distinct levels of confidence regarding the signal-to-noise content of price news. Beliefs fluctuate around idiosyncratic means, which set agents' different views about the asset's long-run value. The model micro-founds the heterogeneous extrapolation and the persistent and procyclical disagreement present in survey data. The subjective belief system is embedded in an otherwise standard asset pricing framework, which can then quantitatively account for the dynamics of prices and trading. In the model, learning from prices leads to disagreement and trading, which reshuffles the distribution of wealth between lower- and higher-propensity-to-invest agents, affecting aggregate demand and prices. This feedback loop complements the expectations-price spiral typical of models with extrapolation, placing heterogeneity and trading as key drivers of price cycles.

Keywords: Heterogeneous Expectations, Learning, Trading, Cycles

"We need structural models of belief dynamics that can compete with RE models in explaining asset prices and empirically observed beliefs."
Brunnermeier et al. (2021)

1. Introduction

The purpose of this paper is to provide an asset pricing model with heterogeneous beliefs that can replicate basic facts about survey beliefs along with aggregate dynamics involving prices and trading. This framework allows us to shed light on the expectation formation process at the individual level and how it shapes the aggregate dynamics of the stock market.

An increasing amount of the recent asset-pricing literature has emphasized the importance of understanding how investors form beliefs and the implications for asset pricing. One of the reasons for this focus on expectation formation is the evidence coming from survey data that shows significant departures from the Rational Expectations (RE) hypothesis.[1] The opening quote is taken from the latest NBER asset pricing program agenda for future research, which clearly points out the importance of incorporating realistic belief systems in asset pricing models. We seek to contribute to this enterprise by presenting empirical facts about survey beliefs, proposing a model of expectations that replicates them and exploring their implications for asset pricing.

Two deviations from Rational Expectations have been extensively documented: people tend to extrapolate from recent events (Greenwood and Shleifer (2014)); consensus beliefs under-react to new information (Coibion and Gorodnichenko (2015)), but individual agents over-react (Bordalo et al. (2020)). Recently, Giglio et al. (2021) added a third dimension: investors' subjective expectations are characterized by persistent heterogeneity across agents ("individual fixed effects"), which cannot be explained by observables such as wealth, age, gender or past returns. Thus, the expectations coordination implied by RE is strongly rejected.

Based on this evidence, we use the cross-section of individuals from the UBS Gallup survey to build sentiment groups that replicate this persistent heterogeneity and document several facts. First, all agents extrapolate, but the optimists do it much more. Second, disagreement is always high without large variations, which we

[1] See Adam et al. (2022) for a review.

refer to as "perpetual disagreement". However, it exhibits meaningful dynamics: it comoves positively with prices and trading (as shown in early research, for instance, Vissing-Jorgensen (2003), Adam et al. (2015)) and is mostly driven by optimists.

We propose a model of expectations that is consistent with these facts. We conjecture that agents have imperfect knowledge about price formation, in line with the Internal Rationality literature (Adam and Marcet (2011)). They cope with this imperfect information by using a statistical model of prices that generalizes the RE model. They use this model to form price expectations and, as Bayesian learners, update them when new information about prices comes up.

Agents differ in two dimensions. First, they hold different views on mean price growth, which we interpret as beliefs about the long-run asset's value. Thus, beliefs fluctuate around this long-run value with the short-run dynamics arising from learning about prices. This subjective long-run growth is a micro-foundation of the statistical fixed-effect documented by Giglio et al. (2021) and the perpetual nature of the disagreement. Besides, investors differ in their speed of learning, reflecting a different way of processing public information; some are more confident, believing that that information has a high signal-to-noise ratio, and others are more skeptical about it. This heterogeneity in the processing of information can be related to two empirical observations: the different degrees of extrapolation that we document and the comovement between disagreement and prices.[2]

We embed this expectation formation process into an otherwise standard Lucas (1978) model. Apart from the price-expectations spiral typical of models of learning about prices that generates recurrent price booms and busts, the model features an additional mechanism: a feedback loop between prices and trading. Price news provoke more disagreement, as agents process the information in a heterogeneous way. This generates trading, since investors who value stocks relatively more after the price news will buy them from investors who value them less. Trading triggers a redistribution of wealth between investors with different propensities to invest, affecting aggregate demand and prices. Hence, learning connects prices to expectations; heterogeneous learning connects prices to disagreement and trading; trading changes the distribution of stocks, influencing aggregate demand and prices. Altogether, trading emerges as a key driver of asset prices, breaking with the mainstream

[2]It turns out optimists are more confident such that, when prices are high, they become even more optimistic in relation to other groups, widening the disagreement.

theory that explains asset pricing without any reference to trading dynamics.

An example of boom dynamics would be as follows. It starts with an aggregate exogenous factor (e.g. goods news, extraordinary incomes) that makes some investors more willing to invest in the stock market. This generates a rise in prices which turns all investors more optimistic, raising demand and prices further over time in a reinforcing manner. Nevertheless, not all investors react equally to the rise in prices; some are more conservative than others, interpreting the news as containing more noise, and forgoing the wave of optimism. This heterogeneous reaction implies an increase in disagreement, which leads to trading: optimists will buy from pessimists. Trading moves resources from lower- to higher-propensity-to-invest agents, which raises aggregate demand and prices, restarting the process.

Our framework also allows us to investigate the contribution of different sentiment groups to booms and busts in price cycles. Through the lens of our model, the positive correlation between disagreement and prices that we observe in booms is driven by optimists becoming more optimistic and not pessimistic agents adjusting their beliefs upward. In this regard, managing capital gain expectations for the most optimistic agents is crucial for leaning against the wind policies in reducing the inefficiencies created by belief-driven cycles.[3]

To the best of our knowledge, this is the first paper to jointly replicate quantitatively the distribution of subjective beliefs along with price and trading dynamics in the context of the stock market. The literature on belief heterogeneity and asset pricing is vast. Nevertheless, most of the literature has not provided a realistic quantitative evaluation yet. Atmaz and Basak (2018) is an example of a theoretical model of heterogeneous beliefs that is able to replicate several of the stylized facts observed in the data. In contrast to that framework in which agents possess beliefs about fundamentals (dividends), we work with expectations on expected return which allows us to compare the model directly with survey data and evaluate the quantitative performance of the model. On a similar note, WR Martin and Papadimitriou (2022) develop a model with heterogeneous beliefs about probabilities of good/bad news in which sentiment is another source of risk fully internalized by agents and which stimulates speculation and volatility. See Simsek (2021) for a comprehensive review of the literature on heterogeneous beliefs about asset prices.

[3]Belief-driven asset price cycles can impact the real economy through multiple channels: see Ifrim (2021) for demand side inefficient wealth effects, Winkler (2020) for supply side with financial frictions or Belda (2023) for supply side due to investment adjustment costs.

The rest of the paper is organized as follows. Section 2 presents several stylized facts regarding the empirical survey distribution of beliefs. Section 3 lays out a model of expectations in line with the evidence embedded in a theoretical asset pricing model. Section 4 shows the quantitative performance and the mechanism through which heterogeneous beliefs drive asset price cycles. Section 5 concludes.

2. Stylized Facts about Heterogeneous Expectations

We use the Gallup survey on future stock market return expectations of individual investors for the period 1998Q2-2007Q4. We choose this survey because it includes the most number of respondents per period (around 700), which should bring more reliability in capturing the heterogeneous dynamics of expectations.[4] We first split the distribution of beliefs into sentiment groups based on the level of optimism/pessimism of individual investors regarding future returns. Specifically, we order the distribution of beliefs across agents at each point in time in three subgroups ranked by their level of optimism and compute averages for each group. Although our data are not a panel, the evidence from Giglio et al. (2021) shows that beliefs are persistent over time, meaning that optimists remain optimists and pessimists remain pessimists without interchanging, which is robust to other surveys, as the RAND panel. Given this fact, we argue that the mean of each sentiment group captures reasonably well the heterogeneity of expectations of each group and proceed with this caveat in mind.

First, we study the features of these group-level expectations. Figure 1 presents the evolution over time of the sentiment groups, with S_1 being the most pessimistic, S_2 the average investors and S_3 representing the sentiment group of agents with the most optimistic beliefs. At the top of the dot-com bubble, optimists were expecting as high as 30% yearly returns, while pessimists only expected 7%. Sentiment groups' beliefs are highly correlated with each other (0.8-0.95).

[4] An alternative option is to work with the RAND dataset, which constitutes a true panel. One of the shortcomings is that RAND responses are coded in categories and need some assumptions to convert the answer to a continuous variable. We are currently working with the RAND dataset to test the robustness of the facts.

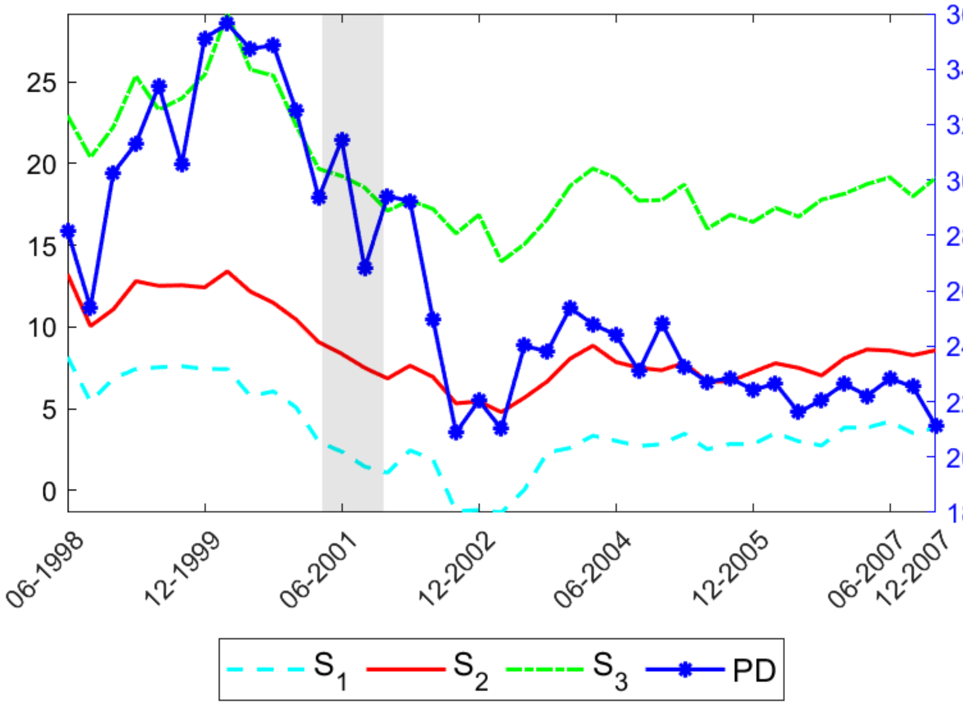

Figure 1: Return expectations by sentiment Groups. *Each sentiment group represents the average return expectation at each point in time across agents depending on the position in the distribution (eg.* S_1 *represents the average of the beliefs between 0 and* $\frac{1}{3}$ *percentiles); shaded bars denote NBER recessions.*

Extrapolation. At first sight, figure 1 suggests a positive comovement between survey expectations and prices: investors are more optimistic during the boom and more pessimistic at the bust. This eyeball test suggests the existence of extrapolation, possibly to a different degree for each group. To formally test this possibility, we run the RE test proposed in Adam et al. (2017) for each group.[5]The test implies running the following two regressions

$$\begin{aligned} \mathbb{E}^s_t[R_{t,t+n}] &= \boldsymbol{a} + \boldsymbol{c}\, PD_t + u_t + \mu_t \\ R_{t,t+n} &= a + c\, PD_t + \epsilon_t \end{aligned} \tag{1}$$

where $\mathbb{E}^s_t$ represents survey expectations regarding future returns at time t, PD_t

[5]This test is similar to the extrapolation test used by Kohlhas and Walther (2021). They collapse the two equations into a single one by subtracting the first from the second line and studying the sign of $c - \boldsymbol{c}$. We use the version with two equations as it delivers more information.

is the Price Dividend ratio and $R_{t,t+n}$ is the realized return between t and $t+n$. Moreover, u_t and ϵ_t represent variations in survey expectations and returns due to factors other than the PD ratio and μ_t captures measurement error in survey expectations, which is assumed to be uncorrelated with the previous two exogenous variations. The RE test is basically a test of equality between c and $\boldsymbol{c}$. Results from table 8 indicate that the RE hypothesis with respect to survey expectations on capital gains is rejected at the 1% significance level for each one of the three sentiment groups.[6]

	$\boldsymbol{c}$	c	*p-value* H_0: c= $\boldsymbol{c}$
p_{0-33}	0.0576***	-0.2423 ***	0.0000
p_{33-66}	0.0545***	-0.2423***	0.0000
p_{66-100}	0.0809***	-0.2423***	0.0000

Table 1: RE Tests across different sentiment groups: p_{0-33} *denotes the sentiment group whose expectations lie between the 0 and* 1/3 *percentile. The data in each group is aggregated by taking the average of that particular group at each point in time. Data used for this particular test is the Gallup UBS survey data for all individuals' expected stock market return. Estimates are based on asymptotic theory and have been adjusted for small sample bias. *** denotes significance at the 1% level.*

However, the point estimates indicate important differences, especially for optimists. To check this we run a test of equality among the coefficients, $\boldsymbol{c^s}$ with $s = 1:3$, among different sentiment groups and present the *p-value* in the following table. Results indicate that the sensitivity of expectations to the PD ratio for pessimists and moderates is statistically identical. Nevertheless, optimists exhibit a higher coefficient, $\boldsymbol{c^3}$, that is significantly different from the other two groups, suggesting a higher degree of extrapolation.

H_0:	$\boldsymbol{c^1} = \boldsymbol{c^2}$	$\boldsymbol{c^1} = \boldsymbol{c^3}$	$\boldsymbol{c^2} = \boldsymbol{c^3}$
p-value	0.363	0.016**	0.0000***

Table 2: Equality tests for coefficients, c, across sentiment groups: *See footnote for table 1 for additional details. *** denotes significance at the 1% level and ** at the 5% level*

[6]The results are unchanged if, instead of three sentiment groups, we consider two or four, see Appendix 1 for results on RE tests based on different partitions of the distribution of subjective returns.

Disagreement dynamics. Our preferred measure of disagreement/dispersion of beliefs is defined by the difference between the beliefs held by the most optimistic/pessimistic groups.[7] For three sentiment groups, this measure is defined as $DI_{33}^{33} = S_3 - S_1$. Figure 2 presents the evolution of disagreement together with the PD ratio. Disagreement about future stock returns tends to be high near the top of the price cycle and highly correlated with the PD ratio (0.7). Moreover, subjective beliefs are characterized by persistent positive disagreement with a mean of approximately 16%, in line with the evidence from Giglio et al. (2021) on the existence of individual fixed effects in the cross-section of beliefs.

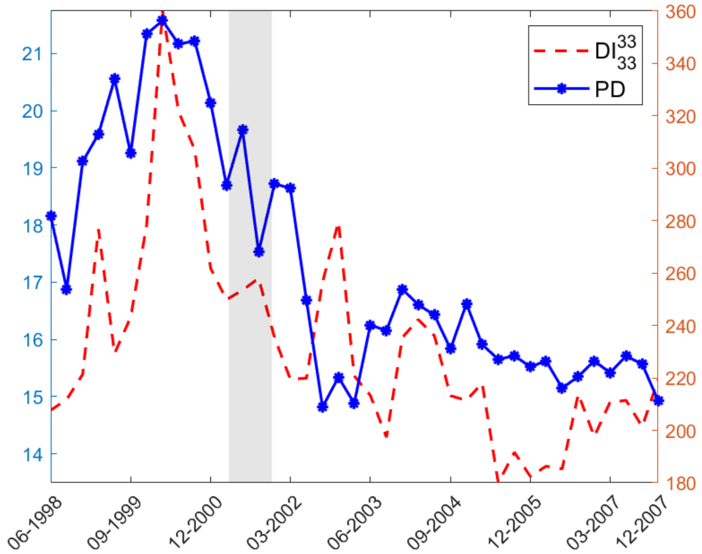

Figure 2: *Disagreement and PD ratio*

The next figure shows disagreement computed both as the inter-group standard deviation and as the difference between the 90th and 10th percentile (DI_{10}^{10}). These measures behave very similarly to our benchmark specification with correlation coefficients higher than 0.9. This suggests that the dynamics of disagreement is not sensitive on the exact measure used but instead is fundamentally rooted into the data. Table 3 collects several stylized facts about the heterogeneity of beliefs and their interaction with aggregate variables.

[7] A similar measure has been used by Giacoletti et al. (2018) to measure disagreement in bond markets.

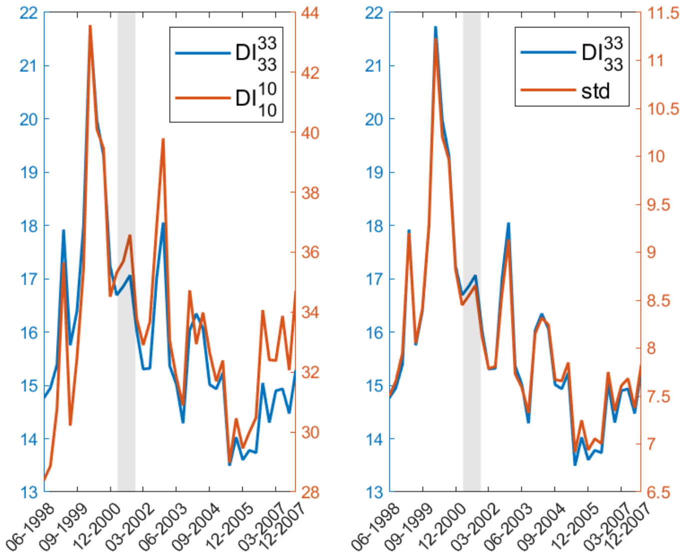

Figure 3: *Alternative measures of disagreement*

Fact	Statistic	Value
1. Persistence of expectations	ρ	0.90
2. Extrapolation of mean capital gains expectations	$corr(\beta_t, PD)$	0.82
3. Heterogeneous extrapolation	$\boldsymbol{c^1}$	0.0576
	$\boldsymbol{c^2}$	0.0545
	$\boldsymbol{c^3}$	0.0809
4. Perpetual disagreement	$\mathbb{E}(DI)$	0.04
	$\sigma(DI_t)$	0.0044
5. Disagreement led by i) optimists	$corr(DI_t, S_t^3)$	0.73
ii) pesimists	$corr(DI_t, S_t^1)$	0.36
6. Disagreement procyclicality	$corr(DI_t, PD_t)$	0.72
7. Comovement disagreement-trading	$corr(DI_t, TV_t)$	0.41
8. Correlation among sentiment groups	$corr(S^1, S^2)$	0.95
	$corr(S^1, S^3)$	0.87
	$corr(S^2, S^3)$	0.95

Table 3: Facts on Subjective Expectations.

3. A model with heterogeneous expectations

In this section, we present an asset pricing model with heterogeneous beliefs consistent with the empirical evidence from the previous section. We begin by suggesting a model of expectations that replicates the previous facts. Then, we embed this piece in an asset pricing model à la Lucas (1978), with Internal Rationality following Adam et al. (2017).

3.1. A model of expectations

In light of the previous evidence, we conjecture a model of learning about prices with different layers of heterogeneity. Investors from the sentiment group i possess the following subjective model about stock prices

$$\begin{aligned} lnP_t &= lnP_{t-1} + b_t^i + ln\varepsilon_t^{P,i} \\ b_t^i &= (1-\rho_i)\bar{\beta}_i + \rho_i b_{t-1}^i + ln\nu_t \end{aligned} \tag{2}$$

where b_t^i represents the permanent price growth component and $\varepsilon_t^{P,i}$ a transitory innovation. The permanent component, b_t^i, follows an auto-regressive process with persistence ρ_i and mean $\bar{\beta}_i$. The latter represents the perceived long-term mean of stock price return of sentiment group i. We interpret it as a subjective view of the perceived long-term growth of the asset value. Innovations $ln\varepsilon_t^P$ and $ln\nu_t$ are jointly normal but uncorrelated. The noisy price component is comprised of two independent components

$$ln\varepsilon_t^{P,i} = ln\varepsilon_{t+1}^{P1,i} + ln\varepsilon_t^{P2,i}. \tag{3}$$

where $ln\varepsilon_t^{Pj,i} \sim \mathcal{N}\left(\frac{-\sigma^{\epsilon^{Pj}}}{2}, \left(\sigma^{\epsilon^{Pj}}\right)^2\right)$ with $j = 1, 2$. We assume further that only $ln\varepsilon_t^{P1,i}$ is observed at time t. The permanent price growth component, b_t, is unobserved and is estimated optimally using the available information from price signals. Given their belief system from equation 2, the optimal posterior distribution of the permanent component of prices is

$$b_t^i \sim \mathcal{N}(\beta_t^i, (\sigma^i)^2) \tag{4}$$

where σ^2 is the steady state variance of the posterior, and β_t^i is the conditional mean. The latter is evolving according to the Kalman updating equation

$$\beta_t^i = (1-\rho_i)(1-g^i)\bar{\beta}_i + \rho_i\beta_{t-1}^i + g^i(lnP_{t-1} - lnP_{t-2} - \rho_i\beta_{t-1}^i) + g^i ln\varepsilon_t^{P1,i} \quad (5)$$

where g^i represents the steady state Kalman gain, entailing different views on the signal-to-noise ratio of the price signals. The shock $ln\varepsilon_t^{P1,i}$ will be interpreted as an information shock to the beliefs of agents from group i.

Qualitatively, equation (5) contains elements that might replicate the key observations from surveys: the heterogeneous long-run views about the fundamental value of the asset can be linked to the individual fixed-effects and the perpetual disagreement; the different views about the signal-to-noise ratio of the price signals can lead to different degrees of extrapolation; the persistence parameter can be directly linked to the persistence from the survey; the fact that all agents use the same price information would generate a high comovement between sentiment groups. To quantitatively test whether this equation is a reasonable description of the survey evidence, we estimate it for each sentiment group.[8] We estimate the parameters by NLS for each sentiment band individually and present the results in the following table.[9]

Sentiment group i	1	2	3
g^i	0.0139	0.0204	0.0301
	(0.0025)	(0.0006)	(0.007)
ρ_i	0.90	0.90	0.91
	(0.0013)	(4.4e-5)	(0.0013)
$\bar{\beta}^i$ (in %)	-0.50	1.01	4.79
	(0.14)	(0.11)	(0.5)

Table 4: Estimated Learning Parameters*: parameters have been estimated by non-linear least squares; bootstrap standard errors in parentheses calculated by a sieve bootstrap method over 1000 simulations using AR(p) innovations with order p chosen by the AIC criterion.*

[8] We transform the UBS survey return expectations into price growth using the following identity: $R_{t+1} = \frac{P_{t+1}}{P_t} + \beta^d \frac{D_t}{P_t}$ where β^d is the expected quarterly dividend growth which we set equal to 1.0048. The resulting nominal capital gain data is transformed into real series by subtracting SPF inflation forecasts.

[9] Appendix 1 presents the bootstrap distributions of these estimated parameters.

Table 4 shows that the speed of learning (g^i) is increasing with the sentiment band, with optimists (S^3) having the highest learning parameter.[10] On the other hand, the persistence is similar among these groups and the measure of long-term heterogeneity increases in optimism as expected. Figure 4 shows the fit for each sentiment band.

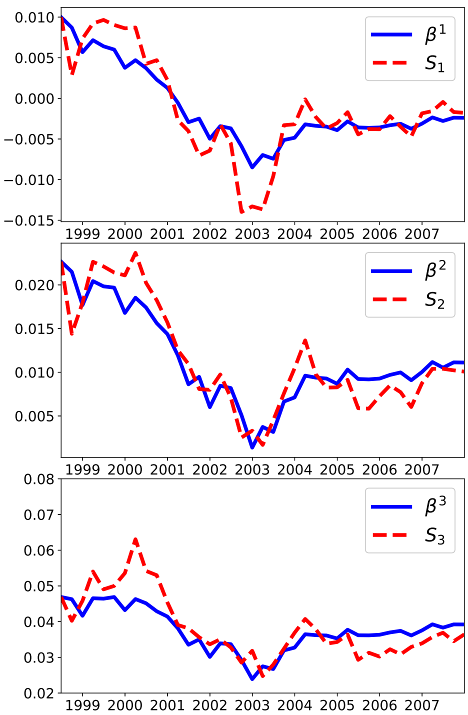

Figure 4: Model fit from equation 5 *The equation has been estimated by non-linear least squares by minimizing for each sentiment group* $\sum(S_i - \beta^i)^2$

Altogether, the different heterogeneity layers on the expectations formation process allow for capturing salient features of surveys. First, different long-run views $\overline{\beta}^i$ give rise to a perpetual disagreement: optimistic investors are always more optimistic than pessimists. This is a way of micro-found the statistical fixed-effect reported by Giglio et al. (2021), that respects the observation that this parameter is unrelated to investors' profile. Second, investors extrapolate news at different intensities g^i: some react faster, and others are more conservative. This difference

[10]Using the same survey data as us, Adam et al. (2015) show that the constant gain parameter is inversely related to investors experience of investors with low experience investors having the largest parameter. According to this evidence, the optimist investors are mostly characterized by low experience while the reverse is true for pessimists.

is in line with heterogeneous extrapolation and relates disagreement to price dynamics (see Section 4.2.). relates disagreement to price dynamics; in good times, disagreement will tend to rise, in line with the procyclicality observed in the data.

3.2. An asset pricing model

Consider an endowment economy populated by M types of agents, $i \in [1, M]$, who solve the following utility maximization problem

$$\begin{aligned} & \max_{\{C_t^i, S_t^i\}_{t=0}^{\infty}} \mathbb{E}_0^{\mathcal{P}_i} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \\ s.t. & \\ & C_t^i + P_t S_t^i \leq (P_t + D_t) S_{t-1}^i + W_t^i \\ & \underline{S} \leq S_t^i \leq \bar{S} \end{aligned} \tag{6}$$

where C denotes consumption, W income (wages) that agents receive, S the amount of stock holdings in the risky asset with price P that pays exogenous dividend D. $\mathcal{P}_i$ represents the probability measure of agents of type i. We assume that the risky asset, which we interpret as stocks, is in fixed supply $S^s > 0$. The share of each agent in the population is equal to μ_i with $\sum_{i=1}^{M} \mu_i = 1$.

Exogenous processes. Following Adam et al. (2017), we specify in a similar way the exogenous processes for dividend growth and wage-dividend ratio to obtain empirical plausible processes for dividends, consumption and consumption-to-dividend ratio.

1. Dividends: grow at a constant rate a with *iid* growth innovations $ln\varepsilon_t^D$ to be described further below

$$lnD_t = lna + lnD_{t-1} + ln\varepsilon_t^D. \tag{7}$$

2. Wage-dividend ratio: follow an AR(1) process with persistence p, mean $1+\overline{WD}$ and innovation $ln\varepsilon_t^W$

$$ln\left(1 + \frac{W_t^i}{D_t}\right) = (1-p)ln(1+\overline{WD}) + p\, ln\left(1 + \frac{W_{t-1}^i}{D_{t-1}}\right) + ln\varepsilon_t^{W,i}. \tag{8}$$

where innovations are given by the following exogenous processes

$$\begin{pmatrix} ln\varepsilon_t^D \\ ln\varepsilon_t^{W,i} \end{pmatrix} \sim \mathcal{N}\left(-\frac{1}{2}\begin{pmatrix} \sigma_D^2 \\ \sigma_W^2 \end{pmatrix}, \begin{pmatrix} \sigma_D^2 & \sigma_{DW} \\ \sigma_{DW} & \sigma_W^2 \end{pmatrix}\right), \tag{9}$$

$$\begin{pmatrix} ln\varepsilon_t^{W,i} \\ ln\varepsilon_t^{W,-i} \end{pmatrix} \sim \mathcal{N}\left(-\frac{1}{2}\begin{pmatrix} \sigma_W^2 \\ \sigma_W^2 \end{pmatrix}, \begin{pmatrix} \sigma_D^2 & \sigma_{WW} \\ \sigma_{WW} & \sigma_W^2 \end{pmatrix}\right). \tag{10}$$

Agents' Belief System. Agents are endowed with full knowledge of the law of motions for dividends and wages given by equations (7) and (8). However, we endow agents with imperfect knowledge regarding how stock prices evolve and the exact mapping from fundamentals to prices. To forecast prices, they use the price model (2) with subjective mean beliefs evolving according to (5).

Equilibrium. It consists of sequences of prices $\{P_t\}_{t=0}^{\infty}$ and allocations $\{C_t, S_t\}_{t=0}^{\infty}$ such that:

1. Given their belief system and exogenous processes, agents optimally solve their optimization problem 6.

2. Markets clear

   - Goods market: $\sum_{i=1}^{M} \mu_i C_t^i = D_t S^s + \sum_{i=1}^{M} \mu_i W_t^i$
   - Stock market: $\sum_{i=1}^{M} \mu_i S_t^i = S^s$

Recursive Solution via the Parameterized Expectations Algorithm: A recursive solution boils down to a time-invariant stock demand function $S_t = S(\boldsymbol{X}_t)$.[11] We solve the model using the PEA approach first proposed by Belda (2023) and further explored by Belda and Ifrim (2023). The idea is to numerically approximate the stock policy function via a function grounded on economic theory. Following the solution for exogenous i.i.d. returns derived in Hakansson (1970), we propose the following approximation function for the stock demand function:

$$S_t^i \approx \chi^i \beta_t^i \frac{\left(WD_t^i + (PD_t + 1)S_{t-1}^i\right)}{PD_t} = \chi^i \beta_t^i Z_t^i, \tag{11}$$

[11] Adam et al. (2017) proved the existence of a recursive equilibrium in the same model with homogeneous expectations. We assume it continues to hold in this setup.

where χ is the unique parameter of the approximating function to be estimated. This function says that stock demand is the product of two elements: $\chi^i \beta_t^i$, which can be read as a marginal propensity to invest, and Z_t^i, which are the resources of the agent i. Appendix 2 contains a detailed explanation of this approach to solving models with learning.

One of the advantages of this approach is that the stock market clearing condition

$$\sum_i \mu^i S_t^i\Big(\frac{P_t}{D_t}, \cdot\Big) = \bar{S} \tag{12}$$

can be solved for the P/D ratio in closed-form. Equilibrium prices read as

$$\frac{P_t}{D_t} = \frac{\sum_{i=1}^{M} \mu^i \chi^i \beta_t^i (\frac{W_t^i}{D_t} + S_{t-1}^i)}{\bar{S} - \sum_{i=1}^{M} \mu^i \chi^i \beta_t^i S_{t-1}^i}, \tag{13}$$

where χ^i is the only parameter of the approximation function. Thus, equilibrium prices depend on the distribution of expectations and wealth across agents. Of course, a potential cost is that the approximating function is not very flexible, as compared with arbitrary order polynomials or neural networks; however, it turns out to perform very well, with Euler Equation errors equivalent to \$1 out of a million.

Connections to demand-system asset pricing. A recent approach in quantitative asset pricing, pursued by Koijen and Yogo (2019), is to estimate characteristic demand functions for different types of investors while allowing for heterogeneity in beliefs. Specifically, the authors estimate the following equation for each type of investor

$$\delta_{i,t}(n) = exp\left(\beta_{0,i,t} ME_t(n) + \sum_{k=1}^{K-1} \beta_{k,i,t} x_{k,t}(n) + \beta_{k,i,t}\right) \epsilon_{i,t}(n) \tag{14}$$

where $\delta_{i,t}(n)$ is the demand or portfolio share of investor i in stock n, ME denotes market equity and $x_{k,t}$ is an individual characteristic of the stock among $K-1$ total characteristics (e.g. book value). The last term from the equation, $\epsilon_{i,t}(n)$, is interpreted by the authors as latent demand related to heterogeneous beliefs of each individual investor i. They show that this last term explains over 80% of the variance of stock returns.

Returning to our asset pricing framework, equation (11) can be rewritten as

$$S_t^i = exp(z_t)\beta_t^i \chi^i \tag{15}$$

where lower-case variables denote variables in logs. Two observations are in place. First, in our case, the latent demand is exactly given by the marginal propensity to invest, which is a scaled version of capital gains expectations. Secondly, the fundamental demand is determined by the wealth of each investor. One important difference between our approach and the one in Koijen and Yogo (2019) is that while the latter focuses on the portfolio choice among a universe of assets, we focus here on the aggregate stock market. Nevertheless, the aggregate demand of stocks exhibits a similar functional form in which latent demand or beliefs multiply fundamental demand.

4. Quantitative Analysis

In this section, we evaluate the quantitative performance of the model in replicating the stylized facts about the heterogeneity of beliefs and stock market cycles and then, use the model to examine the role of heterogeneity in driving the cycles.

4.1. Model performance

We start by calibrating the model parameters. We assume that there are three types of agents in our model, $M = 3$ and set their share μ_i equal to $\frac{1}{3}$. Since our model is an extension of the one from Adam et al. (2017) we approach the calibration of most of the parameters in a similar way except for the parameters concerning the dynamics of the three sentiment groups (ρ^i, g^i and $\overline{\beta}^i$), which are set according to the empirical evidence presented in the previous section. We calibrate the stock supply of stocks, S^s, such that to obtain a reasonable average price-dividend ratio while the parameter for the covariance of income shocks, σ_{WW}, implies a correlation of around 0.3 among these shocks. Table 5 gathers the calibrated parameters in our model.

Parameter	Symbol	Value
Discount factor	δ	0.995
Mean dividend growth	a	1.0048
Dividends growth standard deviation	σ_D	0.0167
Wage-dividends shocks standard deviation	σ_W	0.0167
Covariance (wage-dividend, dividend)	σ_{WD}	0.000351
Covariance wage-dividends agents	σ_{WW}	0.009
Persistence wage-dividend process	p	0.96
Average consumption-dividend ratio	$1+\overline{WD}$	23
Std of transitory component	$\sigma^{\epsilon^{P1}} = \sigma^{\epsilon^{P2}}$	0.04
Risk aversion parameter	γ	2
Stock Supply	S^s	3.3
Expectations persistence	ρ^i	Table 4
Learning speed	g^i	Table 4
Long run view on asset long-run fundamental growth	$\overline{\beta}^i$	Table 4

Table 5: Benchmark calibration. *This table reports the values of the model parameters used for the quantitative analysis.*

We introduce the quantitative performance in table 6 for three specifications of the model. The first one (column 4) represents our benchmark calibration with heterogeneous income and information shocks, in the second one (column 5) we shut off information shocks ($ln\varepsilon_t^{P1,i} = 0$), while the third specification (column 6) assumes homogeneous wages ($\varepsilon_t^{W,i} = \varepsilon_t^{W}$). On top of the statistics regarding the heterogeneity of expectations from table 3 we also present stylized facts about the trading behaviour (panel III) and aggregate stock market behaviour (panel IV).

			Model		
Fact	Statistic	US data	Benchmark	$\ln\varepsilon_t^{P1,i}=0$	$W_t^i=W_t$
I. Expectation Heterogenity					
Expectations persistence	$corr(\beta_t,\beta_{t-1})$	0.90	0.91	0.91	0.88
Correlation among sentiment groups	$corr(\beta_t^1,\beta_t^2)$	0.96	0.87	1	0.49
	$corr(\beta_t^1,\beta_t^3)$	0.87	0.86	1	0.46
	$corr(\beta_t^2,\beta_t^3)$	0.95	0.87	1	0.46
Expectations procyclicality	$corr(PD_t,\beta_t)$	0.82	0.66	0.66	0.41
	$corr(PD_t,\beta_t^3)$	0.86	0.66	0.66	0.31
	$corr(PD_t,\beta_t^1)$	0.7	0.66	0.66	0.34
II. Disagreement					
Disagreement driven by beliefs	$corr(DI_t,\beta_t^3)$	0.73	0.94	0.99	0.88
	$corr(DI_t,\beta_t^1)$	0.36	0.63	0.99	0
Perpetual disagreement	$\mathbb{E}(DI_t)$	0.04	0.04	0.04	0.04
	$\sigma(DI_t)$	0.0044	0.0047	0.0037	0.0032
Disagreement procyclicality	$corr(DI_t,PD_t)$	0.72	0.53	0.39	0.54
III. Trading					
Comovement disagreement-trading	$corr(DI_t,TV_t)$	0.41	0.24	0.26	0.36
Trading driven by beliefs	$\hat{\beta}(\lvert\Delta S_t^1\rvert,\lvert\Delta\beta_t^1\rvert)$	0.2*	0.2	0.15	0.04
	$\hat{\beta}(\lvert\Delta S_t^2\rvert,\lvert\Delta\beta_t^2\rvert)$	0.2	0.012	-0.01	0.14
	$\hat{\beta}(\lvert\Delta S_t^3\rvert,\lvert\Delta\beta_t^3\rvert)$	0.2	0.047	0.02	0.25
IV. Stock Prices					
Mean Price-Dividend	$\mathbb{E}(PD_t)$	154.86	173	173	159
Price-Dividend volatility	$\sigma(PD_t)$	64.42	55	55	13
Price-Dividend persistence	$\rho(PD_t,PD_{t-1})$	0.98	0.96	0.96	0.96
Mean returns	$\mathbb{E}(r_t)$	1.89	1.015	1.015	1.01
Returns volatility	$\sigma(r_t)$	7.70	9.2	9.1	3.8

Table 6: Model quantitative performance. *This table reports the statistics of the model together with the US data for the period 1973:I-2019:IV for prices and returns and 1998:II-2007:IV for expectations-related and trading statistics. Model implied statistics are obtained via a long simulation with T=10.000 periods;* $\hat{\beta}(Y,X)$ *denotes the OLS regression coefficient between* Y *and* X*; *estimate from Giglio et al. (2021)*

The benchmark calibration captures well all of the stylized facts, including the heterogeneity of expectations, the nature of the disagreement, trading behaviour and the excess volatility of the stock price cycles. Our model produces highly correlated beliefs among sentiment groups and positive co-movement between expectations and prices. Expectations shocks contribute to reducing the co-movement between beliefs, as can be seen when comparing with the calibration excluding sentiment shocks (column 5). The mean and volatility of disagreement match exactly those observed in the data and reproduce the positive correlation with prices.

Moreover, similarly to the data, the expectations of optimists exhibit a stronger correlation with disagreement compared to the pessimist group. As argued in the next section, the positive co-movement between prices and disagreement is driven largely by optimists becoming more optimistic, increasing trading, prices and disagreement. Panel III shows that disagreement is also positively related to trading and that changes in beliefs do not lead to trading, consistent with the empirical evidence presented in Giglio et al. (2021). Finally, panel IV documents that our model replicates closely aggregate stock market volatility and persistence.

We also run the same RE test on a simulated sample from the model under the baseline calibration. Table which is directly comparable with table 1, reveals that the model is able to replicate the stylized facts from Section 2: high prices predicts negative future returns but positive beliefs about them; this extrapolation is stronger for the most optimists ($\boldsymbol{c}^3$ is significantly different from $\boldsymbol{c}^2$ and $\boldsymbol{c}^1$).

	$\boldsymbol{c}$	c	*p-value* H_0: c= $\boldsymbol{c}$
p_{0-33}	0.0147***	-0.1726 ***	0.0000
p_{33-66}	0.0225***	-0.1726***	0.0000
p_{66-100}	0.0329***	-0.1726***	0.0000

Table 7: RE Tests on model simulated data across different sentiment groups. *The sample consists of 7000 observations simulated from the model under the baseline calibration; *** denotes significance at the 1% level.*

Figure 5 plots one simulation arising from the calibrated model. Notice that although different sentiment groups have persistently different beliefs, stock holdings vary across agents, and there is not only one group holding the largest/smallest amount of stocks. Instead, agents with the largest/smallest equity holdings alternate among sentiment groups over time.[12]

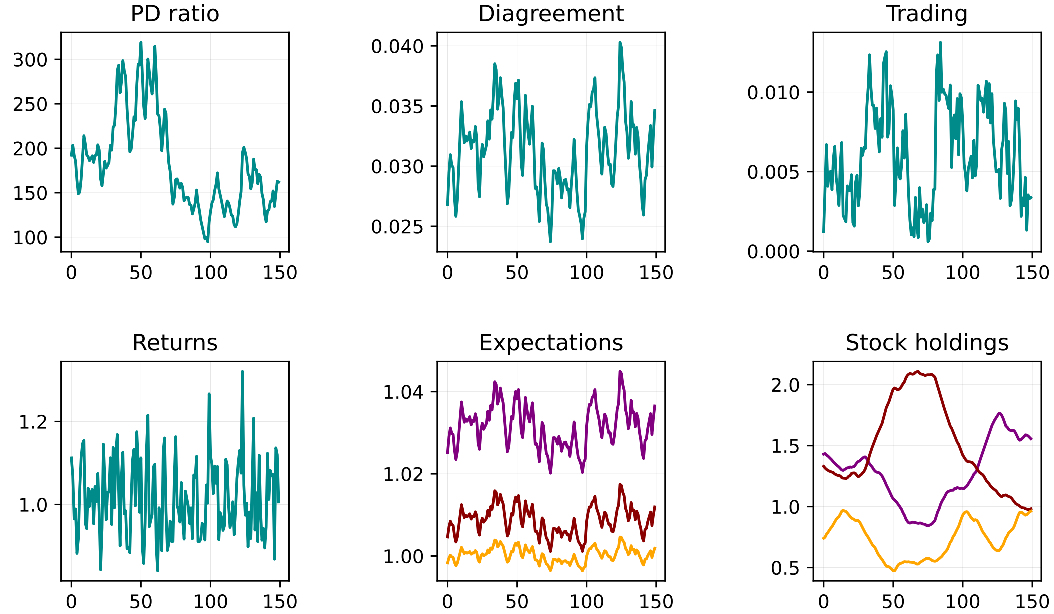

Figure 5: Simulation of 150 periods based on the benchmark model. *In the last two graphs, purple lines are for optimists, red for moderates and orange for pessimists.*

4.2. Dissecting stock market dynamics

In this section, we highlight the key mechanisms behind to joint evolution of prices, trading and expectations. The cycle starts with an exogenous factor (e.g. particular news (the "expectations shock") or extraordinary incomes (the "wage shock")) that make some investors more willing to invest in the stock market. This generates a rise in prices which turns all investors more optimistic, creating amplification over time. Nevertheless, not all investors react equally to the rise in prices due to their different expectation formation processes; some are more conservative than others, interpreting the news as containing more noise and then updating their expectations less. Thus, the heterogeneous reaction of expectations to prices increases disagreement and trading. Trading reshuffles the wealth distribution, moving resources from low to high propensity to invest agents, raising aggregate demand and prices. We first highlight the mechanisms at play and then resort to simulates to illustrate them.

[12]This is an observation from the UBS dataset that also matches Giglio et al. (2021): there is no clear mapping between the distribution of wealth and the distribution of expectations. Our model features that.

4.2.1. Mechanisms

Three mechanisms intervene in these dynamics.

Mechanism 1: expectations-price spiral. From the equilibrium P/D ratio (equation 13), it follows

$$\frac{P_{t-1}}{P_{t-2}} = f_1\Big(\{\beta^i_{t-1}, \beta^i_{t-2}\}^M_{i=1}, \cdot\Big) \tag{16}$$

and from the expectations law of motion (equation 5) it is clear that

$$\beta^i_t = f_2\Big(\frac{P_{t-1}}{P_{t-2}}, \cdot\Big). \tag{17}$$

Other things equal, these two equations constitute a feedback loop that produces endogenous price cycles as a result of self-fulfilling prophecies. An increase in optimism would raise stock demand and prices which would confirm the initial optimistic expectations (or even overcome them, rescaling the process upwards). This feedback loop is a mechanism capable of replicating the high observed volatility of stock prices. This mechanism has been exploited in the learning about prices literature, mostly focusing on the homogeneous beliefs case (see Adam et al. (2016)).

Mechanism 2: heterogeneous expectations and disagreement. Based on survey evidence, we introduce idiosyncratic long-run expectations, which are characterized by two parameters: the long-run view $\overline{\beta}^i$ and its weight on current expectations ρ^i. However, according to survey data, only $\overline{\beta}^i$ is significantly different among investors, and therefore we focus here on it. Imposing $\rho^i = \rho$ and $g^i = g$ and the same initial conditions $\beta^i_0 = \beta_0$, the expectations law of motion can be rewritten as

$$\beta^i_t = (1-\rho)(1-g)\overline{\beta^i}\sum_{j=0}^{t-1}\tilde{\rho}^j + g\sum_{j=0}^{t-1}\tilde{\rho}^j ln\frac{P_{t-j}}{P_{t-1-j}} + \tilde{\rho}^{t-1}\beta_0 \tag{18}$$

where $\tilde{\rho} = \rho(1-g\rho)$. It follows that

$$\beta^i_t - \beta^m_t = (\overline{\beta^i} - \overline{\beta^m})(1-\rho)(1-g)\frac{1-\tilde{\rho}^t}{1-\tilde{\rho}}, \tag{19}$$

where β^m_t represents the beliefs of agent $m \neq i$. Since $\tilde{\rho} < 0$, $\tilde{\rho}^t$ goes to zero relatively quickly. Thus, disagreement among investors i and m would be almost constant, reflecting their perpetual differences in long-run views up to a scale. Altogether, heterogeneous long-run expectations produce perpetual disagreement, as the one documented in surveys.

However, this idiosyncratic $\overline{\beta}^i$ does not explain the dynamics of disagreement. In particular, in the data, we observe a positive covariance between prices and disagreement. To explain these non-random movements in disagreement, we need additional heterogeneity in the expectations formation process. As in the data, consider the case of heterogeneous learning speed g^i. In this case, the disagreement between investor i and m can be written as:

$$\begin{aligned}\beta_t^i - \beta_t^m =&(1-\rho)\Big(\overline{\beta}^i \frac{(1-\rho^t(1-g^i\rho)^t)(1-g^i)}{1-\rho(1-g^i\rho)} - \overline{\beta}^m \frac{(1-\rho^t(1-g^m\rho)^t)(1-g^j)}{1-\rho(1-g^m\rho)}\Big)\\ &+\sum_{j=0}^{t-1} ln\frac{P_{t-j}}{P_{t-1-j}}\rho^j\Big(g^i(1-g^i\rho)^j - g^m(1-g^m\rho)^j\Big)\\ &+ln\beta_0(\rho^{t-1}(1-g^i\rho)^{t-1} - \rho^{t-1}(1-g^m\rho)^{t-1})\\ &\approx c(\overline{\beta}^i - \overline{\beta}^m) + \sum_{j=0}^{t-1} ln\frac{P_{t-j}}{P_{t-1-j}}\rho^j\Big(g^i(1-g^i\rho)^j - g^m(1-g^m\rho)^j\Big)\end{aligned} \tag{20}$$

where $c(\overline{\beta}^i - \overline{\beta}^m)$ is a constant, increasing on the difference of long run views. Hence, the element determining the sign of the comovement between disagreement and price growth is the parenthesis of the last line summation. It turns out that

$$\frac{\partial g^i(1-g^i\rho)^j}{\partial g^i} \begin{cases} > 0 \text{ if } j < 1/g\rho - 1 \\ \leq 0 \text{ otherwise} \end{cases}$$

In other words, for relatively recent periods (low j), the higher the learning speed, the larger the disagreement. That would reverse for higher j, but at that point, ρ^j becomes very close to zero, almost cancelling this effect. Hence,

$$g^i > g^m \Rightarrow \beta_t^i - \beta_t^m \approx f\Big(\underset{(+)}{ln\frac{P_{t-j}}{P_{t-1-j}}}, \cdot\Big).$$

Returning to the quantitative model and noting that optimistic investors have higher learning speeds than pessimistic investors ($g^3 > g^1$), an exogenous increase in the beliefs of the optimists (β^3) would imply an increase in price and, via the above equation, in disagreement producing a positive co-movement among these variables. The impulse response analysis from figure 6 illustrates this mechanism. Notice that an increase in pessimists' expectations increases prices but generates a negative co-movement with disagreement. In Appendix C, we report an equivalent shock to disagreement coming from different sources: a positive information shock to optimists and a negative shock to pessimists. In both cases, cases disagreement widens. However, the effects on aggregate prices are the opposite: when optimists become more optimistic, mean expectations and prices go up; when pessimists become more pessimist (driving up disagreement), mean expectations and prices decrease. The effect of heterogeneous ρ is similar to that of heterogeneous g.

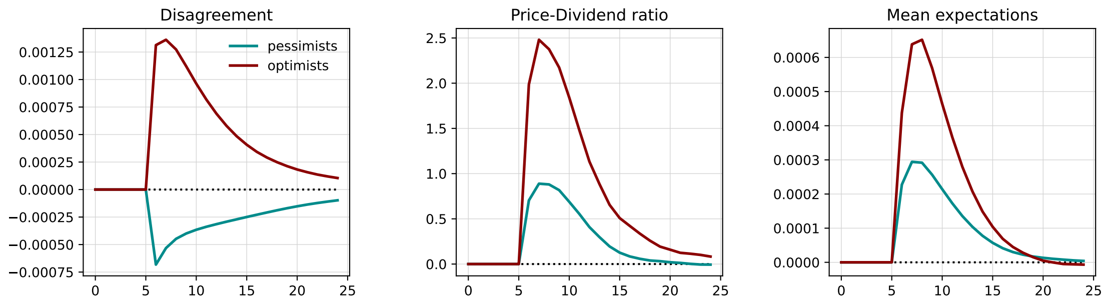

Figure 6: Responses to a positive information shock. *The graph shows the GIRFs of different variables to a positive information shock hitting either the optimists or the pessimists.*

Mechanism 3: price-trading feedback loop. Trading is an aggregate property of the model that requires a time-varying heterogeneity among agents.[13] The model includes three idiosyncratic features: wage shocks, information shocks and expectation formation parameters.[14] Thus, agents trade in the stock market to insure against income risk (fundamental motive) or because of their different views about the future evolution of stock prices (speculative motive).

Differently from income or information shocks, heterogeneity in expectation formation is a mechanism that endogenously produces disagreement and trading. Consider a price shock that surprises agents positively. Investors will tend to get more

[13]Notice that a constant heterogeneity (for instance, in terms of long run views) would generate inequality (other things equal, the most optimist would hold more stocks) but not trading.

[14]The distribution of stock holdings is time-varying, capturing nothing but the joint dynamics of the three aforementioned variables.

optimistic in general, but to a different degree; some will interpret it as truly fundamental change updating their beliefs more, while others would think of it mostly as noise, not changing their beliefs much. Due to this different processing of new information, disagreement will widen. Other things equal, investors believing the news would buy stocks from the more skeptical investors. Thus, through the heterogeneous expectation formation, a change in prices leads to disagreement and trading. On the other direction, trading implies a change in the wealth distribution; in the previous example, from pessimists to optimists. Thus, the market share of optimists is increased, which makes the market look more optimistic on average; since more optimistic agents demand more stocks, trading implies an increase in the total demand for stocks, moving prices up. Altogether, heterogeneous learning connects prices to trading, which redistributes wealth, influencing the aggregate stock demand and prices.

4.2.2. Simulated Impulse Response Functions

We resort to simulation to illustrate mechanisms 2 and 3, which emerge from the heterogeneous beliefs model. We show four experiments to explore the role of disagreement and trading.

A permanent disagreement shock. Figure 7 shows that a permanent increase in the optimist's long-run expectations implies a permanent rise in the level of their expectations. Following this burst of optimism, prices (and mean expectations) go up and, via learning, that optimism spills over the expectations of the other groups, reinforcing their effect on prices. However, the effect across groups is unequal: the impact on optimists' expectations is much larger, and their propensity to invest out of wealth increases at a faster rate compared to the ones of the other two groups. This also explains why stock holdings of pessimists and moderates decrease although their return expectations increase: since prices go up (driven by optimists' expectations), their wealth increases sufficiently rapidly to counterbalance the desire to accumulate more equity. Optimists, on the other hand, experience a rapid increase in expectations (driving up disagreement) and acquire more stocks, reducing their consumption along the way. Hence, trading increases to accommodate the stock holdings in line with the expectations distribution. Finally, the rise in prices gives rise to a temporary spike in returns due to capital gains.

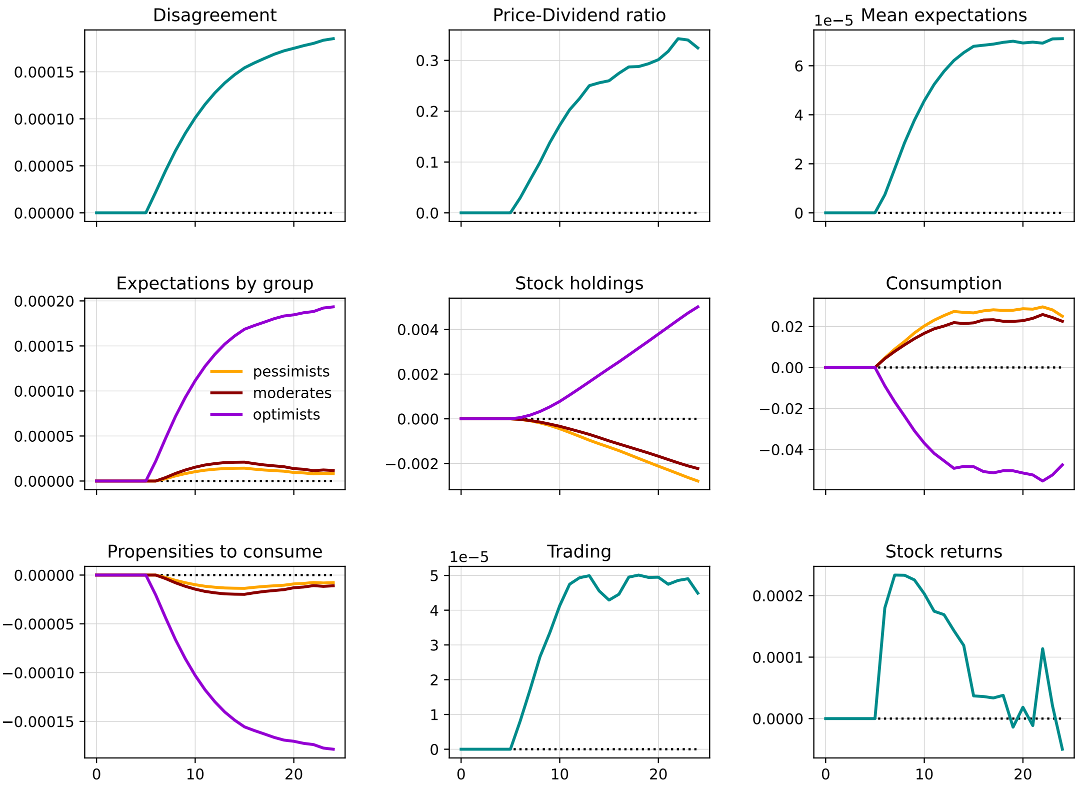

Figure 7: Responses to a long run optimism shock to optimists. *The graph show the GIRFs of different variables to a permanent increase in $\overline{\beta}^3$ in period 5. Periods are quarters. IRFs are computed following these steps: i) simulate the model T =10.000 periods across U=100 different shock realizations; ii) introduce a shock to the variable/parameter in a particular period p and compute new TxU series; iii) repeat ii) at different P=5 points, to tackle possible nonlinearities; iv) compute the differences between shocked and unshocked series at each P and U; v) average the differences across points and realizations.*

A transitory disagreement shock. Define a pure disagreement shock as a shock that increases disagreement but does not affect mean beliefs on impact. This can be implemented as a joint shock to optimists' and pessimists' beliefs of the same magnitude in absolute value but different signs. Specifically, we define an $x\%$ positive pure disagreement shock, ϵ_t^{DI}, as

$$\epsilon_t^{DI} \equiv \begin{cases} \epsilon_t^3 = \frac{x}{2} > 0, \\ \epsilon_t^2 = 0, \\ \epsilon_t^1 = -\frac{x}{2}. \end{cases} \tag{21}$$

We consider the dynamic effects of an *i.i.d* pure disagreement shock for the case in which other information shocks (ϵ^P) are absent. These results are illustrated in figure 8. It produces a sharp increase of 1% in disagreement, which exhibits high persistence over time, remaining positive even after five years. The effects on the other variables are different compared to the previously analyzed shocks: mean expectations are almost constant while the expectations of optimists and pessimists have opposite signs and manifest high persistence over time. Although average sentiment does not move significantly, the PD ratio jumps on impact and continues to increase for 3 quarters, remaining positive for the whole horizon considered. These dynamic effects generate a positive co-movement between disagreement and the PD ratio of approximately 0.8 helping in matching the high positive correlation between these two variables seen in the data.

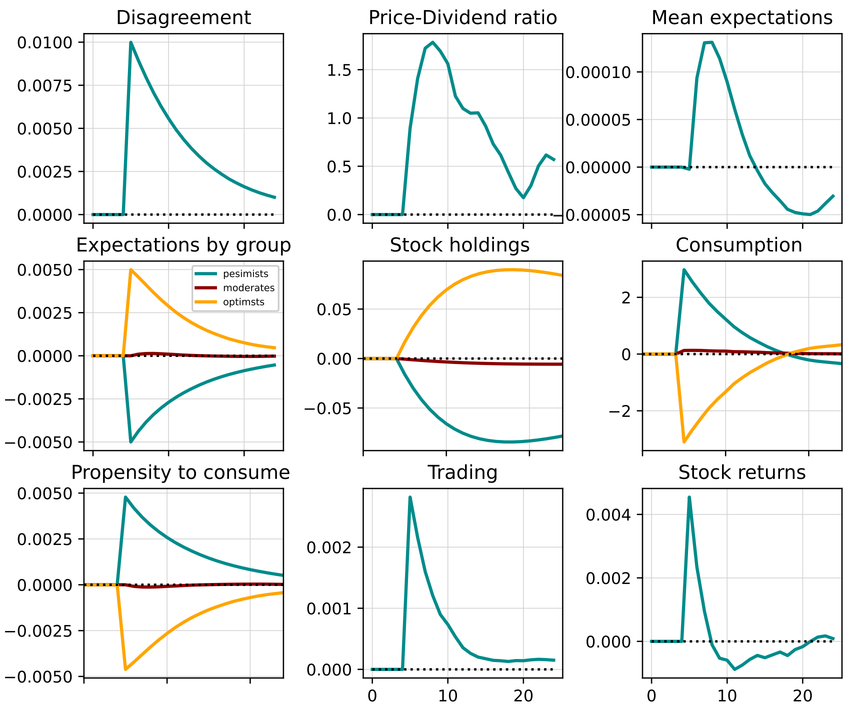

Figure 8: Responses to a pure disagreement shocks ϵ^{DI}. *Periods are quarters and the simulation does not include other information shocks that are present in the baseline calibration. IRFs are computed as in figure 7*

A trading shock. To explore the effect of trading on prices, we run the following experiment. When market-clearing prices are already set up, shock the equilibrium stock holdings:

$$S_t \equiv \begin{cases} \Delta S_t^3 = \frac{x}{2} > 0, \\ \Delta S_t^2 = 0, \\ \Delta S_t^1 = -\frac{x}{2}. \end{cases} \tag{22}$$

This is a pure redistributive shock that moves assets from pessimists to optimists. What are the effects? The weight of optimists in the market goes up, which increases mean beliefs and aggregate demand as they have a larger propensity to invest. Hence, prices go up. This is the first-round effect. Due to learning, all the agents become more optimistic, but with different intensities as they process information differently: optimists will become more optimistic than pessimists. Disagreement goes up, leading to trading; pessimists will sell assets to optimists, restarting the process. Thus, a transitory shock is propagated for a while.

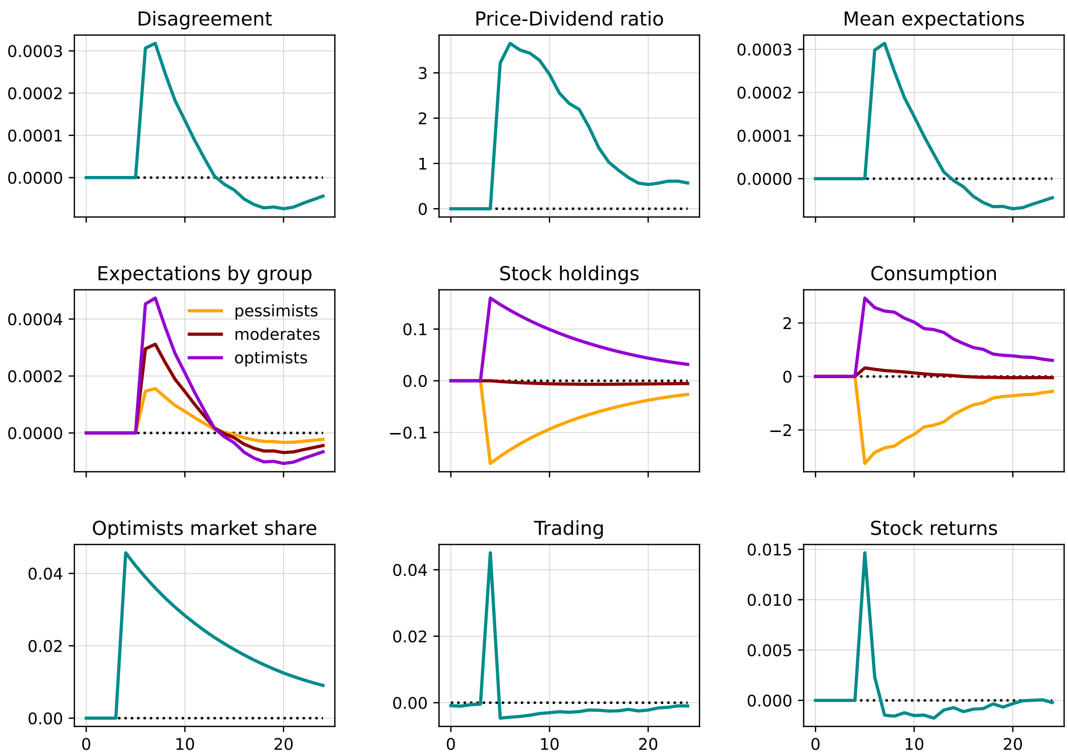

Figure 9: Responses to a trading shock that redistributes stocks from pessimists to optimists. *The graph show the GIRFs of different variables to a trading shock. Periods are quarters. IRFs are computed as in figure 7*

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An income shock. Consider now a transitory shock to optimists' wages. As with the previous redistributive shock, it represents an inflow of resources for optimists. However, now the distribution of stocks is unchanged. The dynamics resemble the ones of a wealth shock, but responses are notably less persistent. The main difference is that consumption for pessimists does not go down, as the wage shock represents a net inflow of resources into the economy while the trading shock redistributes, keeping aggregate resources unchanged. Appendix C shows the IRFs of an aggregate wage shock. The dynamics are very similar, except that, initially, that shock raises the stock market participation of pessimists and moderates who end up increasing their stock holdings by trading with optimists.

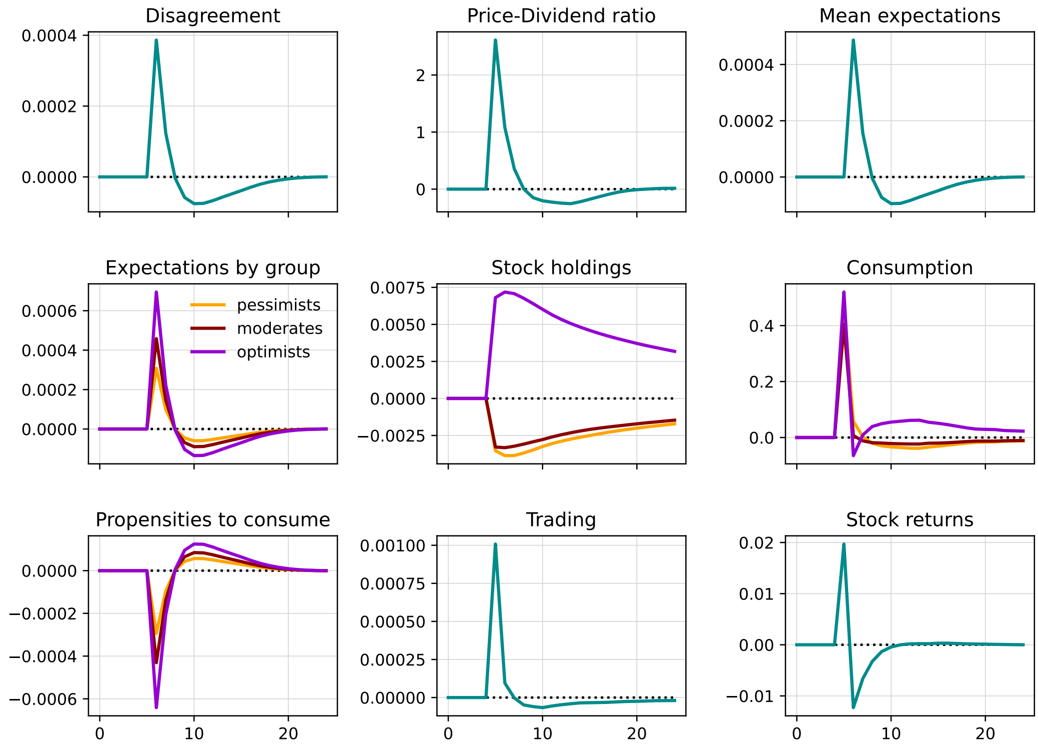

Figure 10: Responses to a shock to optimists wages $ln\varepsilon_t^{w,3}$***.*** *The graph show the GIRFs of different variables to a wage shock to group 3. Periods are quarters. IRFs are computed as in figure 7*

.

4.2.3. An agent with Rational Expectations

While the evidence on the cyclical properties individual investors' beliefs points out clear departures from Rational Expectations, the evidence for institutional investors is much less clear (Adam et al. (2022)). Despite the recent surge in retail trading, the market is still clearly dominated by institutional investors. This individual-institutional investor composition opens a question about the interaction between extrapolative and non-extrapolative agents or, in other words, whether investors who make forecast using wrong models will be kicked out of the market by agents using better models, a prediction associated with Friedman.

When the persistence of the wage process is close to 1 and all the agents hold Rational Expectations, the equilibrium PD ratio is a constant and the price growth expectations are equal to dividends growth beliefs, which boils down to a constant. However, if expectations coordination does not take place, an RE agent has to acknowledge the existence of non-RE agents. Thus, RE is the fixed point of the mapping from perceived to actual expectations

$$
\begin{aligned}
\beta_t^{RE} \equiv & \mathbb{E}_t\left(\frac{P_{t+1}}{P_t}\right) = \\
& = \mathbb{E}_t\Bigg[\frac{\bar{S} - \sum_{i=1}^{M-1} \mu^i \chi^i \beta_t^i S_{t-1}^i - \mu^{RE}\chi^{RE}\beta_t^{RE} S_{t-1}^{RE}}{\bar{S} - \sum_{i=1}^{M-1} \mu^i \chi^i \beta_{t+1}^i S_t^i - \mu^{RE}\chi^{RE}\beta_{t+1}^{RE} S_t^{RE}} \\
& \times \frac{\sum_{i=1}^{M-1} \mu^i \chi^i \beta_{t+1}^i (W_{t+1}^i/D_{t+1} + S_t^i) + \mu^{RE}\chi^{RE}\beta_{t+1}^{RE}(W_{t+1}^{RE}/D_{t+1} + S_t^{RE})}{\sum_{i=1}^{M-1} \mu^i \chi^i \beta_t^i (W_t^i/D_t + S_{t-1}^i) + \mu^{RE}\chi^{RE}\beta_t^{RE}(W_t^{RE}/D_t + S_{t-1}^{RE})} \frac{D_{t+1}}{D_t}\Bigg]
\end{aligned} \tag{23}
$$

While solving the previous equation is difficult, it is well-known in the learning literature that OLS learning converges to RE under certain conditions. Exploiting that convergence, we conjecture that the RE agents update their expectations following

$$
\hat{\beta}_t^{RE} = \hat{\beta}_{t-1}^{RE} + \frac{1}{t-1}\left(\frac{P_t}{P_{t-1}} - \hat{\beta}_{t-1}^{RE}\right) \tag{24}
$$

Figure 11 plots expectations and stock holdings of the previous 3 sentiments groups and the added RE investor after 5000 periods, implying $\hat{\beta}_t^{RE} \approx \hat{\beta}_{t-1}^{RE} \approx \beta_t^{RE}$. As expected, RE beliefs are much more stable than extrapolative beliefs. However, that does not imply they will take over the whole market. In fact, it turns out that

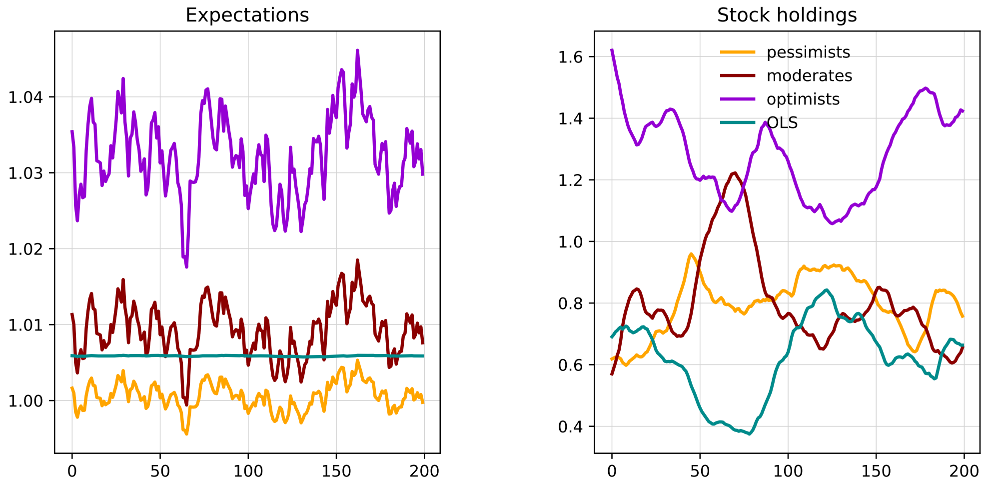

Figure 11: Model simulations for expectations and stock holdings with a Rational Expectations agent.

OLS learners are outperformed in terms of average forecast errors by the moderate extrapolators. This illustrates that RE might be the best strategy when there is belief coordination but not otherwise. In this case, in terms of Guesnerie (2011) , RE is a Nash Equilibrium but not a dominant strategy.

5. Conclusions

In this paper, we present a quantitative model that jointly replicates the empirical dynamics of stock prices, trading, and the heterogeneity of expectations. We place our emphasis on a model of expectations that allows for different layers of heterogeneity. In particular, we point out the role of heterogeneous long-run expectations and the signal-to-noise perceptions that determine the speed at which agents adapt their expectations to new information. This model captures salient features of recent survey evidence, such as high and permanent disagreement and the pro-cyclical nature of both individual expectations and disagreement, which we first document using available survey data on expected returns.

We show that an otherwise asset pricing framework endowed with this model of beliefs delivers a remarkable quantitative performance across a wide variety of stylized facts regarding the joint dynamics of prices, heterogeneous expectations, and trading patterns. The good quantitative performance legitimizes the use of the model to shed some additional light on the mechanics of stock market booms. In particular, we point out that disagreement and trading emerge as key drivers of asset price dynamics, as they shape the distribution of beliefs and wealth that determines aggregate demand and prices. This contrasts with mainstream asset pricing, where trading plays a marginal role.

Finally, we point out some shortcomings. First, the data analysis needs to be extended by using more surveys and including tests on under-reaction. Second, the model of expectations has to be compared with existing alternatives to RE, to make clear the points of continuity and divergence, with an eye on the ability to replicate the survey evidence. Finally, although the model replicates the joint movement of expectations and trading, it is completely unable to generate a level of trading similar to that of the real world. We conjecture that it is related to the type of agents we are modelling ("retail investors"), characterized by infrequent trading that accounts for a rather small fraction of total trading volume. The inclusion of institutional investors, perhaps with different mandates than households, might help in that direction.

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Appendices

Appendix A Additional Results

	c	$\boldsymbol{c}$	*p-value* H_0: c= $\boldsymbol{c}$
	2 Sentiment groups		
p_{0-50}	0.0546 ***	-0.2421***	0.0000
p_{50-100}	0.0744 ***	-0.2419 ***	0.0000
	3 Sentiment groups		
p_{0-33}	0.0576***	-0.2421 ***	0.0000
p_{33-66}	0.0545***	-0.2415***	0.0000
p_{66-100}	0.0809***	-0.2423***	0.0000
	4 Sentiment groups		
p_{0-25}	0.0591	-0.2421	0.0000
p_{25-50}	0.0501	-0.2422	0.0000
p_{50-75}	0.0621	-0.2420	0.0000
p_{75-100}	0.0867	-0.2421	0.0000

Table 8: *RE Tests across different sentiment groups:* p_{0-50} *denotes the sentiment group which expectations lies between between the 0 and 50th percentile. The data in each group is aggregated by taking the average of that particular group. Data used for this particular test is the Gallup UBS survey data for expected stock market return of all individuals. Estimates have are based on asymptotic theory and have been adjusted for small sample bias*

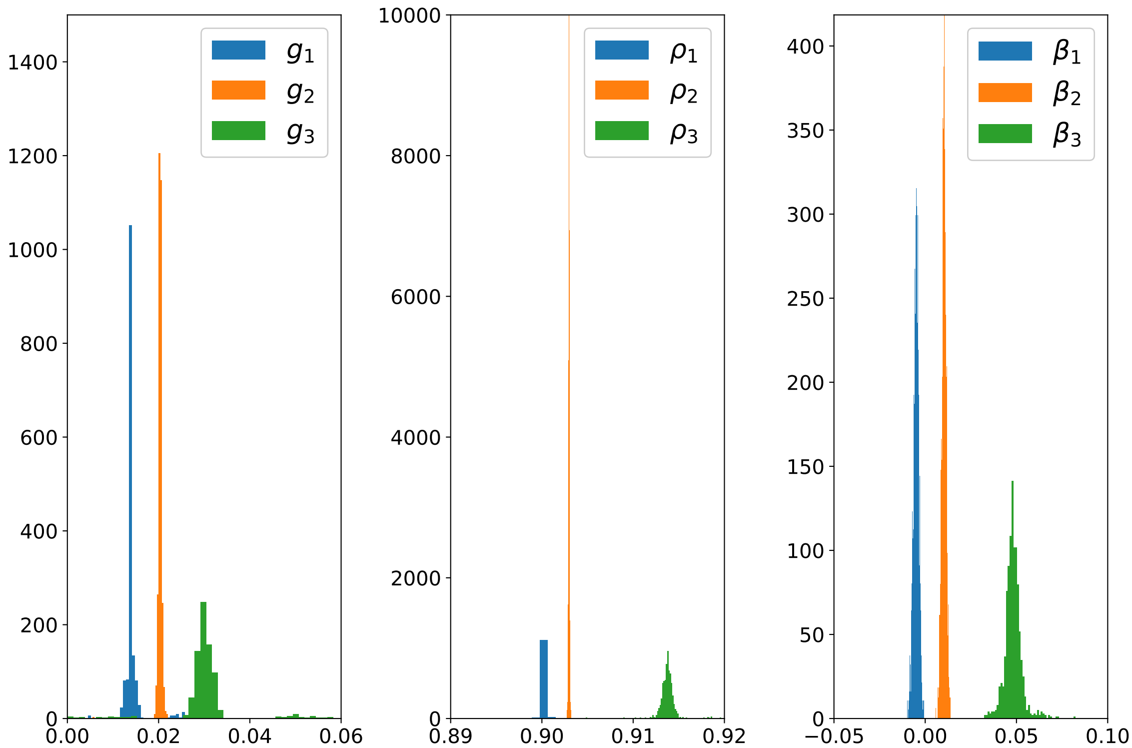

Figure 12: *Bootstrap densities of estimated parameters from equation* **??**

Appendix B Model Solution Strategy

The concavity of the objective function and the convexity set guarantee the sufficiency of the first order conditions for an interior optimal plan. The optimal condition for the household plan is given by the Euler equation:

$$(CD_t^i)^{-\gamma} = \delta \mathbb{E}_t^{\mathcal{P}} \left(\left(\frac{D_{t+1}}{D_t} \right)^{1-\gamma} \frac{(PD_{t+1}+1)}{PD_t} (CD_{t+1}^i)^{-\gamma} \right) = \delta \mathcal{E}(\boldsymbol{X}_t^i) \tag{25}$$

where $\boldsymbol{X}_t^i$ are the state variables. The problem is that this Euler Equation includes an unknown conditional expectation. To solve the model, it must be computed somehow. The Parameterized Expectations Algorithm (PEA) is one of the alternatives. PEA consists of replacing the conditional expectation $\mathcal{E}(\boldsymbol{X}_t^i)$ by some parametric function $\boldsymbol{\psi}$ (Marcet (1988)). The choice of the approximating functions $\boldsymbol{\psi}$ is not obvious and not unique. Popular possibilities are polynomials, splines, neural networks, etc. We follow the approach outlined by **?**: use approximating functions rooted in economic theory. Among the advantages of that approach is the possibility of getting closed-form solutions. Altogether, we follow the next steps

1. Approximate the consumption policy using a

$$CD_t^i = CD(S_{t-1}^i, PD_t, WD_t, \beta_t^i) = B(\beta_t^i)\Big(WD_t + (PD_t + 1)S_{t-1}^i\Big) \tag{26}$$

$$B_t^i = B(\beta_t^i) = 1 - \chi^i \beta_t^i \tag{27}$$

   where χ^i is an unknown parameter which will be estimated via PEA to be discussed below. The consumption policy function is linear in wealth and the propensity to consume depends negatively on expectations.

2. Obtain the stock holdings policy function by plugging the consumption policy in the budget constraint:

$$S_t^i = (1 - B_t^i)\frac{\Big(WD_t^i + (PD_t + 1)S_{t-1}^i\Big)}{PD_t}. \tag{28}$$

3. Determine market-clearing prices by adding individual demands, equating them to the aggregate supply and solving for prices. In this case,

$$\frac{P_t}{D_t} = \frac{\sum_{i=1}^{M} \mu_i (1 - B_t^i)(S_{t-1}^i + \frac{W_t^i}{D_t})}{S^s - \mu_i \sum_{i=1}^{M}(1 - B_t^i) S_{t-1}^i}. \tag{29}$$

The only unknown at this point is the parameter χ^i from equation 27. To obtain this parameter we make use of PEA on the first order condition of the agent which we rewrite as

$$(CD_t^i)^{-\gamma}\delta^{-1} = \mathbb{E}_t^{\mathcal{P}}\left(\left(\frac{D_{t+1}}{D_t}\right)^{1-\gamma}\frac{(PD_{t+1}+1)}{PD_t}(CD_{t+1}^i)^{-\gamma}\right). \tag{30}$$

The PEA algorithm involves the following steps:

1. Draw a series of the exogenous processes for a large T.
2. For a given $\boldsymbol{\chi} \in \mathbb{R}^n$, recursively compute the series of the endogenous variables.
3. Minimize the Euler Equation square residuals

$$G(\chi) = \underset{\chi}{\text{argmin}} \left[\left(\left(\frac{D^{\mathcal{P}}_{t+1}}{D_t} \right)^{1-\gamma} \frac{(PD^{\mathcal{P}}_{t+1} + 1)}{PD_t} (CD(\chi)^{i,\mathcal{P}}_{t+1})^{-\gamma} \right) - \frac{(CD(\chi)^{i}_{t})^{-\gamma}}{\delta} \right]^2 \tag{31}$$

Note the interior of the expectation must be computed according to investor's beliefs. Since investors know the process for dividends and wage-dividends, the only problematic objects are PD_{t+1} and CD_{t+1}. Using agents subjective price model

$$\beta^{i,\mathcal{P}}_{t+1} = \beta^i_t \nu_{t+1} \Rightarrow \left(\frac{P_{t+1}}{P_t} \right)^{\mathcal{P}} = \beta^i_t \nu_{t+1} \varepsilon^p_{t+1} \Rightarrow \left(\frac{P_{t+1}}{D_{t+1}} \right)^{\mathcal{P}} = \left(\frac{P_{t+1}}{P_t} \right)^{\mathcal{P}} \frac{D_t}{D_{t+1}} \frac{P_t}{D_t}$$

In turn, expected consumption reads

$$CD^{\rangle,\mathcal{P}}_{t+1} = (1 - \chi \beta^{i,\mathcal{P}}_{t+1}) \left[WD^i_{t+1} + \left(\left(\frac{P_{t+1}}{D_{t+1}} \right)^{\mathcal{P}} + 1 \right) S_t \right]$$

4. Find a fixed point $\chi = G(\chi)$. For that, update χ following

$$\chi^{j+1} = \chi^j + d(G(\chi^j) - \chi^j) \tag{32}$$

where j iteration number and d the dampening parameter.

Appendix C Responses to aggregate shocks

In this section we report the responses of the model main variables to simultaneous equivalent shocks on investors wages (figure 13) and transitory information (figure 14).

Figure 13: Responses to a general wage shock. *The graph show the GIRFs of different variables to an equivalent wage shock enjoyed by all investors. group 3. Periods are quarters.*

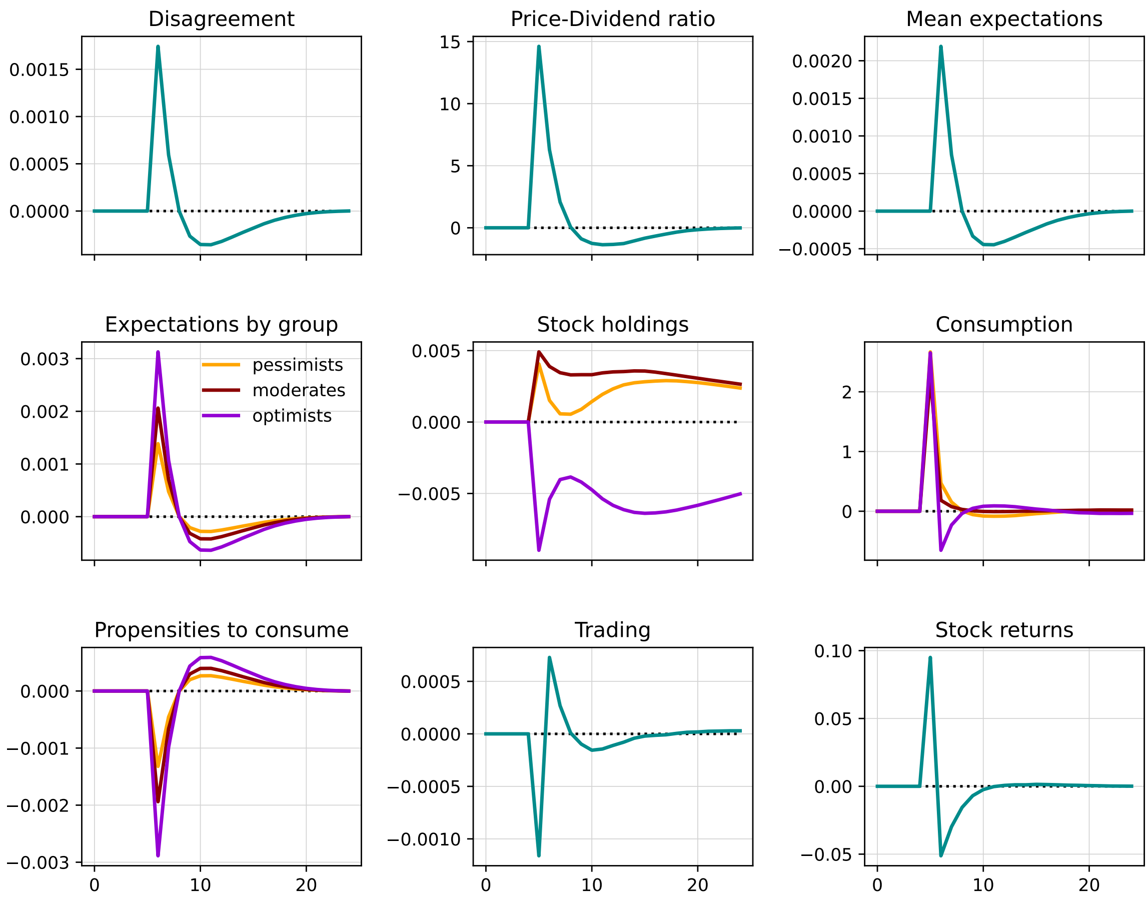

__Figure 14: Responses to a general information shock.__ The graph show the GIRFs of different variables to an information equally received by all investors. Periods are quarters.

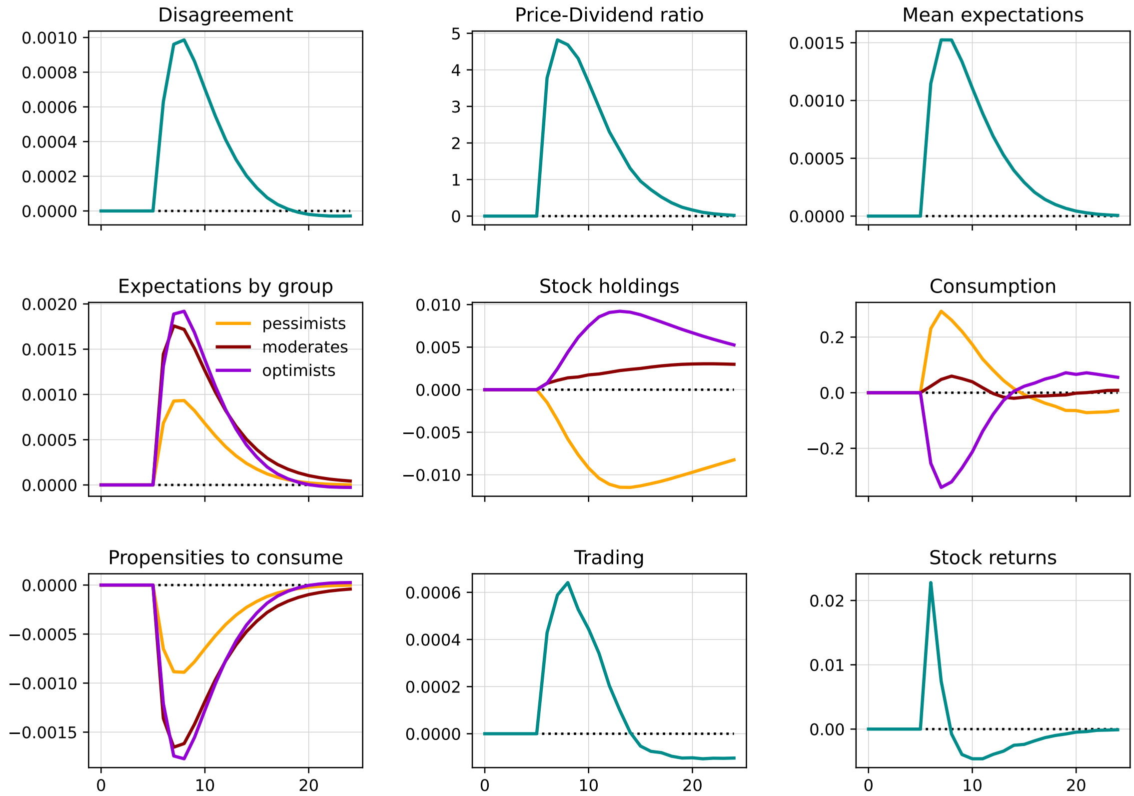

__Figure 15: Responses to a disagreement shock.__ The graph show the GIRFs of different variables to a positive information shock hitting the optimists and a negative shock hitting the pessimists.