Jump Bidding in the War of Attrition

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This paper investigates the effects of jump bidding between two bidders in the context of the war of attrition and its implications for market efficiency. The war of attrition is a common phenomenon where firms compete to remain in the market by investing aggressively in advertising or research. Jump bidding is a strategy that allows a firm to bypass the costly period of attrition by making a commitment to remain in the market. We first identify the dominant bidder 1 with the highest valuation for the object, and all other types will follow her strategy. This bidder faces the problems of whether to jump bid or not and when to jump bid. The shape of bidder 2's valuation distribution determines the cost of excluding bidder 2 during the war of attrition and thus answers the above two questions. We find that when bidder 2's cumulative density function is of a convex shape, bidder 1 will jump bid. Furthermore, the paper examines the inefficiencies that arise from the introduction of jump bidding in the auction.

Keywords: Jump-bidding, War of attrition

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1 Introduction

War of attrition is a common phenomenon in various aspects of our daily lives. In business, aggressive marketing and technological competition can lead to a never-ending battle between firms, with each trying to outdo the other in terms of advertising and R&D investments. One such example is the advertising campaign between Meituan and Eleme, the Chinese take-away platforms, which started in 2013. The competition between the two companies involved aggressive marketing strategies that continued until one party exhausted its resources and exited the market. However, after finding a new investor, the former loser makes a comeback and the war is still going on now. Another example is the bidding war that took place in 2011, when a group of technology companies, including Apple, Microsoft, and Sony, competed for the patent portfolio of Nortel Networks, a bankrupt Canadian telecommunications company. The bidding war lasted several weeks and involved jump bidding, with each company attempting to outbid the others to secure the patents. Similarly, in 2013, Google and Apple engaged in a bidding war to acquire the Israeli start-up Waze, which had developed a popular navigation app. The competition also involved jump bidding, with each company offering increasingly higher amounts in an attempt to outbid the other.

In this paper, we investigate the effects of jump bidding in the context of the war of attrition and examine its implications for market efficiency. The war of attrition scenario involves firms competing to remain in the market for as long as possible, often through costly investments in advertising or research. However, by making a substantial investment, a firm can signal its high type and make a credible claim that it will stay in the market for the following period. This signaling behavior allows the firm to exclude potential competitors who are perceived to be of lower type, thereby avoiding the costly war of attrition. Jump bidding, in particular, is a strategy that enables a firm to bypass the war of attrition by making a commitment to remain in the market without enduring the costly period of attrition.

By allowing one bidder to make a jump bidding and signal her commitment to staying in the game for a certain period of time, we first identify the dominant bidder with the highest valuation of the object. It is enough to discover how this dominant bidder will make the jump bidding and all the other types will follow her strategy to hide their types. The timing of the game is that two bidders initially engage in a war of attrition, and then at some time, bidder 1 can make a claim of persistence in the auction game by paying some cost.

Bidder 1 faces two critical decisions: whether to jump bid or not and when to jump bid. We find that the answers to both of the questions depend on the shape of the density function of bidder 2's valuation distribution, which determines bidder 1's cost of excluding bidder 2 during the war of attrition. Jump bidding is costly and there is only one chance, so bidder 1 wants to make the most efficient use.

If the density function of bidder 2 is flat at the beginning of the auction, this indicates that bidder 2 is less likely to have a low valuation. In such a case, it will be more beneficial for bidder 1 to exclude bidder 2 through the war of attrition, which incurs a smaller cost for bidder 1. In other words, bidder 1 sacrifices fewer of her own low types by staying in the war of attrition. Therefore, the longer the flat portion of bidder 2's density function, the longer bidder 1 would want to stay in the war of attrition and delay the jump bid.

On the other hand, a density function of this shape indicates that bidder 2 is more likely to have a high valuation, which makes the scare-off effect of the jump bidding to be even more beneficial. Consequently, bidder 1 is more likely to make the claim. In addition, we examine the inefficiencies that arise from the introduction of jump bidding in the auction. In a second-price auction, the object is typically won by the bidder with the highest valuation. However, with the inclusion of jump bidding, this may not always be the case, and the winner may be a bidder with a lower valuation. In total, the bidder who is unable to make a jump bid ends up benefiting, while the other bidder who is able to make such a claim only benefits if her valuation is high enough. Otherwise, she may end up sacrificing herself to exclude bidder 2.

1.1 Related Literature

The study of the war of attrition has its roots in biological competition, as discussed in earlier works such as Smith [1974] and Bishop et al. [1978]. In the context of complete and incomplete information, these works laid the foundation for understanding the strategic behavior of agents in a war of attrition. Further research on this topic has been carried out by Nalebuff and Riley [1985], who characterized all Bayesian equilibria in a symmetric war of attrition, which presented a challenge in terms of multiple equilibria. Fudenberg and Tirole [1986] introduced the concept of trembling hand perfection, where an extremely small probability that the competitor never quits, selects a unique equilibrium. In addition to studying the symmetric war of attrition, Myatt [2005] analyzed the asymmetric case. The introduction of the jump bidding strategy in the war of attrition is a significant contribution to the existing literature. This strategy involves bidding significantly more than the previous bid, which can help a bidder secure an item they want without giving their competitors time to react.

Jump bidding is widely studied in other auction formats: English auction (Avery [1998] and all-pay auction (Dekel et al. [2007], Leininger [1991]). There is a high potential for studying jump bidding in the war of attrition. For example, in labor negotiations, a union and employer may engage in a war of attrition to secure favorable terms. Jump bidding may occur when the union or employer offers a concession that is significantly more generous than the previous one, putting pressure on the other party to concede.

The paper is organized as follows: In section 2, we present the basic setting of our model. Section 3 discusses the classical war of attrition, which serves as a benchmark in our analysis. In section 4, we introduce our main model, which incorporates the jump bidding strategy in the war of attrition by allowing one bidder to make a claim. Finally, in section 5, we conduct welfare analyses for bidders and the auction organizer, evaluating the efficiency of the jump bidding mechanism.

2 Basic Setting

Two bidders $i = 1, 2$ are involved in a dispute for one object. Each bidder has a private valuation v_i drawn from a distribution $F_i[0, 1]$ with strictly positive and continuous density f_i.[1] Their valuations are independent of each other.

Time is modeled as a continuous variable that starts at zero and runs indefinitely. Each bidder chooses when to concede the object and their bidding strategy is a mapping from their valuation to the concession time: $v_i \rightarrow t_i(v_i)$ with $t_i \in R_+ \bigcup \{\infty\}$ for $i = 1, 2$, which may be revised at any time $t \leq t_i$. We assume that t_i as a function, is increasing in v_i, so its reverse function $y_i(t) = t_i^{-1}(v_i)$ is also an increasing function, which can be understood as the valuation of bidder i when she concedes. However, when referring to it as the bidding value, we assume that t_i follows distribution $G_i(t) = F_i(t_i^{-1}(t)) = F_i(y_i(t))$. All of the above information is common knowledge, except that bidders do not know the exact valuation of each other for the object.

We consider a second-price all-pay auction where the bidder who outlasts the other wins the object. Specifically, as proposed by Smith [1974] and Klemperer [1999], the cost of each bidder increases proportionally with the time elapsed, until one of the bidders quits. Both bidders then pay a cost equal to the loser's quitting time, and the winner obtains a utility

[1] It is possible that these two distributions have different upper bounds or lower bounds. However, considering that distributions are common knowledge, the bidder with a lower upper boundary will quit immediately because of this obvious disadvantage. Therefore, the war of attrition only occurs between well-matched bidders.

equal to the difference between their valuation and the opponent's quitting time, i.e.,

$$u_i = (v_i - t_{-i})\mathbb{1}_{t_i > t_{-i}} - t_i \mathbb{1}_{t_i < t_{-i}}$$

with $\mathbb{1}(\cdot)$ being the indicator function.

3 Benchmark: Classical War of Attrition

Under the scheme of the classical war of attrition, if we take bidder 1 as an example, her utility is given by $u_1 = (v_1 - t_2(v_2))\mathbb{1}_{t_1 > t_2(v_2)} - t_1 \mathbb{1}_{t_1 < t_2(v_2)}$. Consequently, her expected utility is

$$\begin{aligned} E[u_1(v_1, t_1)] &= \int_0^{t_1} (v_1 - t) dG_2(t) - t_1[1 - G_2(t_1)] \\ &= \int_0^{t_1} (v_1 - t) dF_2(y_2(t)) - t_1[1 - F_2(y_2(t_1))] \end{aligned} \tag{1}$$

To maximize their expected utilities, the bidders need to determine their bidding strategies. The first-order condition with respect to the bidding strategy t_1 of bidder 1 can be expressed as:

$$\begin{aligned} \frac{dE(u_1)}{dt_1} = (v_1 - t_1)F_2^{'}(y_2(t_1))y_2^{'}(t_1) - [1 - F_2(y_2(t_1))] + t_1 F_2^{'}(y_2(t_1))y_2^{'}(t_1) = 0 \\ \Longrightarrow v_1 F_2^{'}(y_2(t_1))y_2^{'}(t_1) = 1 - F_2(y_2(t_1)) \end{aligned}$$

Simplifying the above equation, we get

$$y_1(t_1)F_2^{'}(y_2(t_1))y_2^{'}(t_1) = 1 - F_2(y_2(t_1)) \tag{2}$$

where $v_1 = y_1(t_1)$ is the valuation of bidder 1 at time t_1, and $y_2(t_1)$ is the valuation of bidder 2 at the same time. Consequently, we **transfer the problem from finding the optimal**

concession time into determining the bidder's valuation when she stops bidding and quits. Similarly, we can derive the first-order condition for bidder 2. In summary, the optimal bidding strategies of two bidders (y_1, y_2) are the solutions of the following function set.

$$\begin{cases} y_1 F_2^{'}(y_2) y_2^{'} = 1 - F_2(y_2) \\ y_2 F_1^{'}(y_1) y_1^{'} = 1 - F_1(y_1) \end{cases} \tag{3}$$

This is the model of Nalebuff and Riley [1985] in which they prove the existence of a continuum of pairs of concession value functions (y_1, y_2). To simplify the question, we consider a special case where both bidders follow a uniform distribution. The cumulative density functions for both bidders are given by $F_1 = F_2 = x$ and $f_1 = f_2 = 1$. By substituting these expressions into the function set 3, we can derive the first-order conditions for the two bidders:

$$\begin{cases} y_1 y_2^{'} = y_1 \frac{dy_2}{dt} = 1 - y_2 \\ y_2 y_1^{'} = y_2 \frac{dy_1}{dt} = 1 - y_1 \end{cases} \tag{4}$$

Solving the above partial differential equations, we deduce that

$$\frac{dy_2}{y_2(1-y_2)} = \frac{dy_1}{y_1(1-y_1)} \Longrightarrow \int \frac{dy_2}{y_2(1-y_2)} = \int \frac{dy_1}{y_1(1-y_1)} \Longrightarrow ln \frac{y_2}{1-y_2} = ln \frac{y_1}{1-y_1} \cdot C \tag{5}$$

There can be several values of C. When $C = 1$, it is a symmetric case and from equation 5 we have $y_1 = y_2$. Replacing it into equation 7, it can be deduced that $y_1 \frac{dy_1}{dt} = 1 - y_1$, which by integration results in that $-y_1 - ln(1 - y_1) = t + D$. For a unique solution, boundary conditions are needed. Considering that one bidder stopping at $t = 0$ indicates her valuation $y_1(0)$ to be 0, we obtain $D = 0$ and consequently the optimal bidding strategy for bidder 1 as $t_1(v_1) = -y_1(t_1) - ln(1 - y_1(t_1)) = -v_1 - ln(1 - v_1)$. It is the same for bidder 2 under this symmetric case. Nalebuff and Riley [1985] not only focus on the symmetric case but also talk about the asymmetric cases in which they find that there is always one bidder that acts more passively in comparison with the other one, depending on whether C is larger than 1 or the reverse.

However, Fudenberg and Tirole [1986] has found a way to get rid of a continuum of equilibria by adding a tiny perturbation into the model. They assume that with positive probability one bidder might have a high enough utility that staying forever is a dominant strategy, and based on this assumption, they realize a unique equilibrium.

Following the model in this benchmark, we assume that with probability ϵ ($\epsilon > 0$ but $\epsilon \to 0$) that one bidder is irrational so she will never concede and play until the very end. This changes one bidder's expected payoff in the following way. For bidder 1, for example, her expected utility to maximize is now written as

$$E[u_1(v_1, t_1)] = \int_0^{t_1} (v_1 - t) dF_2(y_2(t))(1-\epsilon) - t_1[1 - F_2(y_2(t_1))(1-\epsilon)] \tag{6}$$

The new function set of first-order conditions with respect to two concession value functions are

$$\begin{cases} y_1 \frac{dy_2}{dt}(1-\epsilon) = 1 - y_2(1-\epsilon) \\ y_2 \frac{dy_1}{dt}(1-\epsilon) = 1 - y_1(1-\epsilon) \end{cases} \tag{7}$$

Replacing $k = 1-\epsilon$, we can get the following equation by solving these differential functions.

$$\int \frac{dy_2}{y_2(1-y_2k)} = \int \frac{dy_1}{y_1(1-y_1k)} \Longrightarrow ln \frac{y_2}{1-y_2k} = ln \frac{y_1}{1-y_1k} \cdot C_1 \tag{8}$$

Considering the restriction that as $t \to \infty$, it should hold that $y_1 = y_2 = 1$, which requires that $\frac{1}{1-k} = \frac{1}{1-k}C_1$ and consequently $C_1 = 1$. As a result, this assumption results in a unique symmetric equilibrium. Throughout the whole paper, we keep this assumption of irrational bidders so that we can focus on the **symmetric equilibrium**.

4 Jump Bidding Game

We propose a novel mechanism in the previous context of a war of attrition game, where one bidder can signal her commitment to staying in the game for a specified period of time. Specifically, bidder 1 determines to announce at time $\hat{t}$ that she will remain in the game for the subsequent Δ period, the value of which is predetermined. Therefore, only $\hat{t}$ is within bidder 1's strategic space. Bidder 2 then faces a strategic decision: either quit the game immediately at $\hat{t}$ or continue bidding until at least $\hat{t}+\Delta$. Notably, quitting between $\hat{t}$ and $(\hat{t}+\Delta)$ is not beneficial for bidder 2, as it incurs a time cost without improving his probability to win. To distinguish between two bidders, we use "she" to represent bidder 1 and "he" for bidder 2.

In this setting, the bidders' valuations can be interpreted as their types, where the quitting time of a bidder is positively related to her type, i.e. $t_i(v_i)$ is an increasing function for $i = 1, 2$. Its reverse function $y_i = t_i^{-1}$ is also increasing function. Thus, lower-value bidders will typically exit the auction earlier, whereas higher-value bidders will stay in the game longer. Within all the types of bidder 1, we focus on the **dominant bidder**, who possesses the highest possible valuation of 1 for the object. Our analysis centers on how the dominant bidder selects $\hat{t}$, given that other bidders will follow her strategy. If they do not, they effectively reveal that their valuation is less than 1. Revealing this information means they will never win against bidder 2 in this competition and it becomes better for them to quit immediately. Considering their disadvantage in valuation, the time that they can stay in the auction is shorter, so it's better to give up as soon as possible to save costs. In the following discussions, bidder 1 is always referred to as the dominant bidder who has a valuation of 1 for the object.

The bidding game commences at $t = 0$, and two bidders engage in **a classical war of attrition game**. At each subsequent time point, the two bidders have the option to either

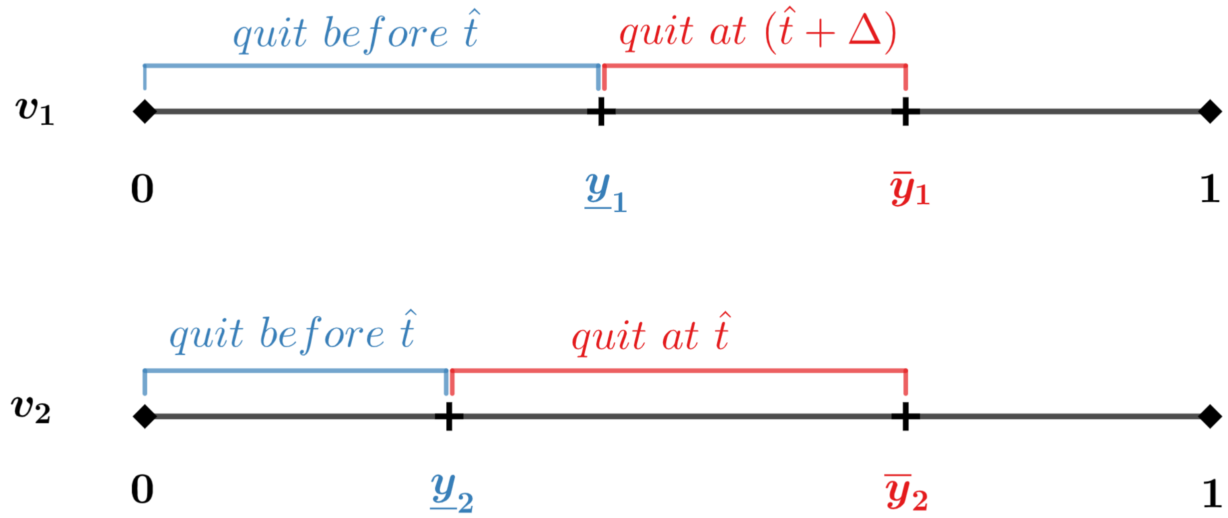

Figure 1: Bidding Strategies of two Bidders

continue or quit the game. It continues until a certain time, denoted as $\hat{t}$ when bidder 1 makes a jump bid of Δ by declaring that she will not quit for this following period of time. This time corresponds to the valuations (types) of $\underline{y}_1$ for bidder 1 and $\underline{y}_2$ for bidder 2, respectively. Therefore, we have that $y_1(\hat{t}) = \underline{y}_1$ and $y_2(\hat{t}) = \underline{y}_2$. The above two bars indicate the types of bidders who have left the game before $\hat{t}$, which are illustrated as the blue area in Figure 1.

In the next stage of the **jump bidding game** at time $\hat{t}$, bidder 1 has a strategic advantage in the auction by excluding more types of bidder 2. By selecting $\hat{t}$ strategically and making the claim, upper bars $\overline{y}_1$ and $\overline{y}_2$ are established for bidder 1 and bidder 2, respectively. This results in that bidder 2 will quit immediately if his type falls below $\overline{y}_2$. For the types of bidder 2 that are above it, they will stay longer than $(\hat{t} + \Delta)$. On the other hand, bidder 1 with a type falling between $\underline{y}_1$ and $\overline{y}_1$ will remain in the game until quitting at $(\hat{t} + \Delta)$. Only if her type is above $\overline{y}_1$ will she continue beyond $(\hat{t} + \Delta)$. This process is represented by the red section in Figure 1.

Finally, after the time of $(\hat{t} + \Delta)$, two bidders return to the war of attrition if both of them are still in the auction.

4.1 Boundary Conditions and Trade-off of Jump Bidding

Our first observation about the relationship between the upper bars of two bidders is as follows.

Proposition 1 *The upper bars of two bidders satisfy that:* $\overline{y}_1 = \overline{y}_2 = \overline{y}$.

This proposition is straightforward to prove. For example, if we assume that $\overline{y}_1 < \overline{y}_2$, then at time $\hat{t}+\Delta$, bidder 1 with a type between $\overline{y}_1$ and $\overline{y}_2$ will quit because she realizes that she cannot compete against bidder 2. Vice versa, $\overline{y}_1 > \overline{y}_2$ cannot hold in equilibrium since the disadvantaged bidder 2 will also quit, increasing $\overline{y}_2$ until it reaches $\overline{y}_1$. Consequently, when entering the next period of the war of attrition, two bidders are still of the same lower bar and upper bar for their valuation distributions. Q.E.D.

Considering bidder 1 of valuation $\underline{y}_1$, if she were to quit at $\hat{t}$, she would receive an instant utility of 0. Alternatively, if she were to make the claim and stay until $(\hat{t}+\Delta)$, she would win against bidder 2 with a valuation between $\underline{y}_2$ and $\overline{y}_2$ that quits when the claim is made. Her expected utility of claiming and staying is her value of the object $\underline{y}_1$ multiplied by the probability that bidder 2 quits, which is given by $\frac{F_2(\overline{y})-F_2(\underline{y}_2)}{1-F_2(\underline{y}_2)}$, minus the cost of claiming Δ. Since at this point, bidder 1 is indifferent between two choices, the following equation should be satisfied.

$$0 = \underline{y}_1 \cdot \frac{F_2(\overline{y}_2) - F_2(\underline{y}_2)}{1 - F_2(\underline{y}_2)} - \Delta = \underline{y}_1 \cdot \frac{F_2(\overline{y}) - F_2(\underline{y}_2)}{1 - F_2(\underline{y}_2)} - \Delta \tag{9}$$

Similarly, consider bidder 2 with valuation $\overline{y}_2$. If he were to quit at $\hat{t}$, he would receive an instant utility of 0. Alternatively, if he were to continue, he would stay in the auction until $(\hat{t}+\Delta)$ where he can win against bidder 1 with a valuation between $\overline{y}_1$ and $\underline{y}_1$ who quit before $\hat{t}$ with probability $\frac{F_1(\overline{y})-F_1(\underline{y}_1)}{1-F_1(\underline{y}_1)}$. Thus, the indifference relation is expressed as:

$$0 = \overline{y}_2 \cdot \frac{F_1(\overline{y}_1) - F_1(\underline{y}_1)}{1 - F_1(\underline{y}_1)} - \Delta = \overline{y} \cdot \frac{F_1(\overline{y}) - F_1(\underline{y}_1)}{1 - F_1(\underline{y}_1)} - \Delta \tag{10}$$

Based on these two equations, our second observation about the relationship between the lower bars is concluded as follows.

Proposition 2 *If $F_1(x) = F_2(x) = F(x)$, two bidders are symmetric and their lower bars satisfy that: $\underline{y}_1 > \underline{y}_2$.*

The above relationship can be proved by contradiction. For function $H(x) = \frac{F(\overline{y})-F(x)}{1-F(x)}$ with $x \in [0,1]$, its first-order derivative satisfies that $H'(x) = -\frac{f(x)}{(1-F(x))^2}[1 - F(\overline{y})] < 0$ since $1 - F(\overline{y}) > 0$, so $H(x)$ is a decreasing function. If we assume that $\underline{y}_1 \leq \underline{y}_2$, we will get

$$\frac{F(\overline{y}) - F(\underline{y}_1)}{1 - F(\underline{y}_1)} \leq \frac{F(\overline{y}) - F(\underline{y}_2)}{1 - F(\underline{y}_2)}$$

Given that $\underline{y}_1 < \overline{y}_1 = \overline{y}$, we have

$$\underline{y}_1 \cdot \frac{F(\overline{y}) - F(\underline{y}_1)}{1 - F(\underline{y}_1)} \leq \overline{y} \cdot \frac{F(\overline{y}) - F(\underline{y}_2)}{1 - F(\underline{y}_2)}$$

which makes equations 9 and 10 unable to hold at the same time. Therefore, it should be that $\underline{y}_1 > \underline{y}_2$. Q.E.D.

The proposition at hand embodies a trade-off for bidder 1, who must weigh the advantages and drawbacks of making a costly claim in the auction. On the one hand, by making the jump bidding, bidder 1 eliminates a larger range of opponent types from the bidding process, since up to $\overline{y}$ types of bidder 2 will immediately quit following the claim. This provides bidder 1 with a strategic advantage, as it leaves fewer bidders to compete against and raises her chances of winning the object. On the other hand, however, bidder 1 must also contend with the fact that making a claim entails sacrificing a larger portion of her own types. Specifically, the condition $\underline{y}_1 > \underline{y}_2$ indicates that more types of bidder 1 will quit the auction before the claim is made, thereby reducing the overall payoff for bidder 1.

4.2 Utility Maximization of the Dominant Bidder

Expressions for both $\underline{y}_2$ and $\overline{y}$ can be derived as functions of $\underline{y}_1$ by utilizing the two equations 9 and 10. In other words, once the concession value $\underline{y}_1$ of bidder 1, which is chosen by the dominant bidder, is known, the equilibrium solution can be identified. To calculate the profit of this dominant bidder, the following different cases are considered.

In the first case, bidder 2 quits at $\hat{t}$, so there is only the war of attrition. The dominant bidder 1 sacrifices her types below $\underline{y}_1$ and wins against bidder 2 with types below $\underline{y}_2$. Her expected utility is:

$$
\begin{aligned}
\pi_1^{WoA} &= [1 \cdot F_2(\underline{y}_2) - \int_0^{\hat{t}} t dF_2(y_2(t))] - \hat{t}[1 - F_2(y_2(\hat{t}))] \qquad (11)\\
&= F_2(\underline{y}_2) - \int_0^{\hat{t}} [1 - F_2(y_2(t))]dt \\
&= F_2(\underline{y}_2) - \int_0^{\hat{t}} y_1(t)F_2'(y_2(t))y_2'(t)dt \\
&= F_2(\underline{y}_2) - \int_0^{\underline{y}_2} y_1(y_2)dF_2(y_2)
\end{aligned}
$$

where $y_1(y_2)$ is implicitly defined by

$$
\int_{y_1(y_2)}^{\underline{y}_1} \frac{f_1(y)}{y(1 - F_1(y))}dy = \int_{y_2}^{\underline{y}_2} \frac{f_2(y)}{y(1 - F_2(y))}dy
$$

Within the expression of equation 11, the first two terms of the equation represent bidder 1's utility when she wins the object. Specifically, $1 \cdot F_2(\underline{y}_2)$ is the multiplication of her utility 1 as a dominant bidder for the object and the probability of winning, while $\int_0^{\hat{t}} t dF_2(y_2(t))$ integrates all the cost paid by bidder 1 at each time point when bidder 2 concedes and leaves the auction during the war of attrition. The last term of the equation, $-\hat{t}[1 - F_2(y_2(\hat{t}))]$, captures the cost bidder 1 pays when she fails to win the auction, which is equal to the time $\hat{t}$ multiplying the probability $[1 - F_2(y_2(\hat{t}))]$.

In the other case, bidder 2 has a high enough valuation that he does not quit at $\hat{t}$. Bidder 1 then makes a jump biding at this moment, resulting in an expected utility as follows.

$$\pi_1^{JB} = 1 \cdot \frac{F_2(\bar{y}) - F_2(\underline{y}_2)}{1 - F_2(\underline{y}_2)} - \Delta \tag{12}$$

By making such a claim, bidder 1 scares off bidder 2 with valuations between $\underline{y}_2$ and $\overline{y}$, with a probability of $\frac{F_2(\bar{y}) - F_2(\underline{y}_2)}{1 - F_2(\underline{y}_2)}$ at the cost of Δ. However, it is important to note that bidder 1 only makes a jump bid if bidder 2 is still in the auction at the time of $\hat{t}$. Therefore, in order to determine the overall utility function that bidder 1 maximizes, the utility gained from jump bidding should be multiplied by its probability $[1 - F_2(\underline{y}_2)]$.

On the other hand, if bidder 1 chooses not to make a jump bidding, she will engage in a war of attrition with bidder 2. By excluding the same range of valuations of bidder 2 as in jump bidding, bidder 1's expected utility is

$$\tilde{\pi}_1^{WoA} = F_2(\bar{y}) - \int_0^{\bar{y}} \tilde{y}_1(y_2) dF_2(y_2) \tag{13}$$

where $\tilde{y}_1(y_2)$ is implicitly defined by

$$\int_{\tilde{y}_1(y_2)}^{\bar{y}} \frac{f_1(y)}{y(1 - F_1(y))} dy = \int_{y_2}^{\bar{y}} \frac{f_2(y)}{y(1 - F_2(y))} dy$$

It is calculated following the same logic as the previous utility in the case of the war of attrition. As a result, the optimization problem for the dominant bidder 1 is summarized as follows.

$$\max_{\underline{y}_1 \in [\Delta, 1]} \quad \pi_1^{WoA} + [1 - F_2(\underline{y}_2)]\pi_1^{JB} - \tilde{\pi}_1^{WoA}$$
$$s.t. \quad \Delta = \underline{y}_1 \cdot \frac{F_2(\overline{y}) - F_2(\underline{y}_2)}{1 - F_2(\underline{y}_2)} = \overline{y} \cdot \frac{F_1(\overline{y}) - F_1(\underline{y}_1)}{1 - F_1(\underline{y}_1)}$$

Its restriction is derived from two boundary conditions with respect to the four upper and lower bars of two bidders (equations 9 and 10).

4.3 To claim or not to claim?

It is difficult to solve for the optimal valuation $\underline{y}_1$ for bidder 1 to make the jump bid. Instead, we discuss bidder 1's incentive to make such a claim by balancing its cost and benefit. To simplify the question, we focus on the symmetric equilibrium by assuming that $F_1 = F_2 = F$. The assumption of the existence of irrational bidders ensures a unique equilibrium of the game.

Taking a special case of the valuation distribution that $F(x) = x^\alpha$ with $\alpha > 0$. The cost-benefit analysis of jump bidding is contingent on the underlying distribution of valuations in the auction. The decision of whether to exercise the commitment power of jump bidding is predicated on bidder 1's belief that her opponent is more likely to hold a high valuation, which corresponds to a flatter cumulative density function at the beginning and steeper at the end. We find that bidder 1 is willing to jump bid if the following condition is satisfied.

Proposition 3 *Bidder 1 is willing to make a jump bid if the common valuation distribution of two bidders $F(x) = x^\alpha$ satisfies that $\alpha > 1$, and Δ is low enough.*

The rationale behind this strategy is that the initial period of elimination, whereby bidder 1 excludes low-valuation bidder 2, is particularly costly, especially if bidder 2 is deemed to be strong. By executing jump bidding, bidder 1 can bypass this phase by demonstrating a high valuation and committing to staying in the auction, thereby forcing low-valuation bidder 2 to exit promptly, and avoiding a protracted war of attrition.

However, jump bidding may not be advantageous if bidder 2 is likely to hold a low valuation, in which case low-valuation bidder 2 would exit the auction early on, rendering the commitment power redundant. In this scenario, committing to remain in the auction is inefficient for bidder 1, and it would be more expedient to abstain from jump bidding.

4.4 The sooner claim, the better?

The next question is about the timing of the jump bidding for bidder 1, which is determined by the value of $\hat{t}$ (or equivalently the value of $\underline{y}_1$). It reflects how long bidder 1 wants to stay in the war of attrition, during which she manages to exclude bidder 2 with a valuation below $\underline{y}$. This cost is of the size of $\int_0^{\overline{y}} y_1(y_2)dF_2(y_2)$, as calculated in equation 13. The optimal duration for bidder 1 is thus affected by the valuation distribution of bidder 2, which determines the cost of outlasting bidder 2.

If the density function $f_2(y_2)$ is flatter at the beginning, based on the correspondence of $y_1(y_2)$, bidder 1 will sacrifices fewer of her own types to exclude bidder 2 and the cost of the war of attrition for bidder 1 will also be low. In this scenario, bidder 1 may choose to postpone the jump bidding until the density function becomes steeper. Conversely, if the density function is high, indicating that the cost of the war of attrition is high, bidder 1 would prefer to make the jump bidding as soon as possible. It could even be as soon as the beginning of the auction when $\underline{y}_1 = 0$. We will use an example to illustrate this argument.

Example 1 *Assume the probability density function of the bidders' valuation follows*

$$f(x) = \begin{cases} \epsilon & x \in [0, b] \\ \frac{2(1-\epsilon b)}{(1-b)^2}(x-b) & x \in (b, 1] \end{cases}$$

As ϵ is small, the cost of war of attrition to eliminate bidder 2 with a valuation between 0 and b is low, meaning that bidder 1 sacrifices fewer of her own types. Therefore, the dominant bidder 1 is willing to stay longer in the war of attrition until bidder 2 with a valuation of b quits the auction, and then she starts the jump bidding. For instance, if $\epsilon = 0.01$, $b = 0.2$ and $\Delta = 0.0001$, we will have $\underline{y}_2 = 0.178$, $\underline{y}_1 = 0.185$ and $\bar{y} = 0.214$. □

5 Welfare Analysis

An efficient auction mechanism is one that maximizes social welfare by allocating the item to the bidder with the highest valuation. However, the jump bidding mechanism studied in our paper may introduce inefficiencies in the market. During the war of attrition period, where $t \in [0, \hat{t})$, the object may be allocated to a bidder with a lower valuation, leading to a sub-optimal allocation. Similarly, during the jump bidding period, where $t \in (\hat{t}, \hat{t} + \Delta)$, the bidder with the lower valuation may still win the auction, again leading to a less efficient outcome.

Despite these potential inefficiencies, bidders may still choose to introduce jump bidding in order to strategically manipulate their opponent's behavior and increase their own payoff. Therefore, although the jump bidding mechanism may lead to sub-optimal outcomes in terms of social welfare, it still remains a valuable tool for bidders to achieve their individual objectives.

5.1 Bidder's utility

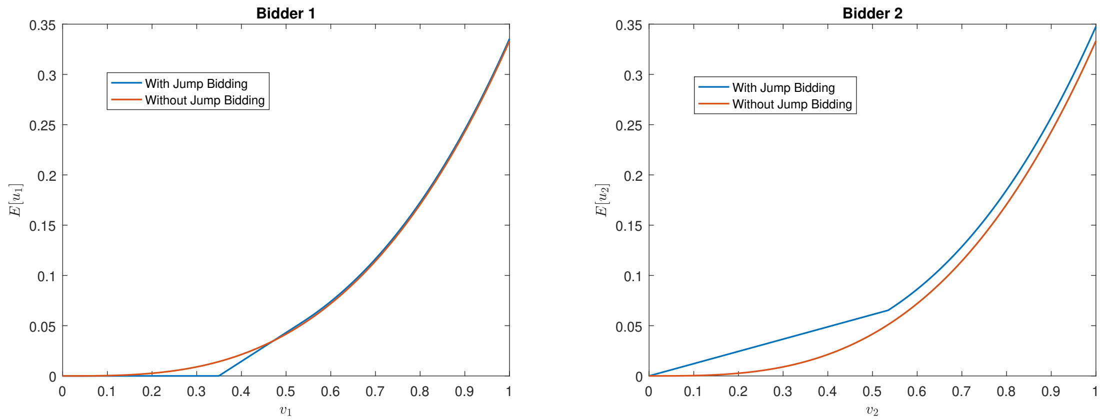

Figure 2: Bidders' expected utility with and without jump bidding.

Under the assumption of symmetry with $F_1(x) = F_2(x) = x^a$ where $a > 1$, we examine the impact of jump bidding on bidders' utility. Specifically, we analyze the scenarios for bidder 1 and bidder 2, comparing their expected utility with and without the option of jump bidding. Our findings reveal that for bidder 1, the decision to jump bid is based on their valuation. If their valuation exceeds $\overline{y}$, they jump bid and stay longer; otherwise, they quit the war of attrition early. The left panel of Figure 2 illustrates that although the expected utility of bidder 1 is lower with jump bidding, eliminating low-valuation bidder 2 becomes advantageous when bidder 1's valuation is high enough.

On the other hand, bidder 2's expected utility is always higher when bidder 1 has the right to jump bid, even when bidder 2's valuation is low. The right panel of Figure 2 highlights that this is because dominant bidder 1 assists in eliminating low-valuation bidder 1 in the early stages.

In conclusion, the commitment power of bidder 1 results in a trade-off between the benefits and drawbacks of jump bidding. Low-valuation bidder 1 is negatively impacted, while high-valuation bidder 1 and bidder 2 benefit from the presence of jump bidding. Possible extensions of this study include analyzing different auction formats, exploring the impact of asymmetry, and examining the effect of introducing multiple jump bids.

5.2 Level of attrition

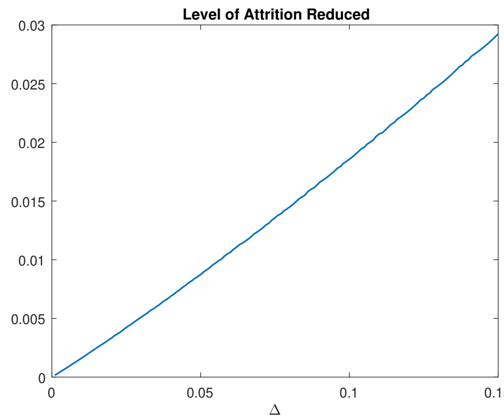

Figure 3: Level of attrition reduced with jump bidding.

The attrition level can be defined as the cumulative costs borne by both bidders, including the time spent and the claiming cost Δ incurred in the jump bidding strategy, which is only taken by bidder 1. We observe that the inclusion of jump bidding in the auction process has a significant impact in reducing the attrition level by eliminating low-valuation bidders from the competition.

To illustrate this, Figure 3 plots the reduction in the total attrition as a function of the commitment power measured by the claiming cost Δ. We find that the total attrition decreases with an increase in Δ, which reflects the bidder's commitment to the jump bidding strategy. A higher claiming cost can make it more convincing to bidder 2 that bidder 1 is of higher valuation, which makes it more efficient for bidder 1 to scare off the opponent, thus reducing the overall attrition level.

However, the societal impact of such reduction is ambiguous. While lower attrition levels may conserve resources, they can also discourage RD investment, which can ultimately harm society's long-term growth prospects. Further research could explore the trade-offs between efficiency gains and the potential long-term consequences of reduced competition.

6 Conclusions

In conclusion, our analysis has shed light on the effects of jump bidding in the context of the war of attrition and its implications for market efficiency. We have found that jump bidding can be an effective strategy for a firm to bypass the costly war of attrition and signal its high type. However, the decision to jump bid depends on the shape of the density function of the other bidder's valuation distribution, which determines the cost of excluding the other bidder during the war of attrition. In addition, the inclusion of jump bidding may lead to inefficiencies in the auction, where the winner may not be the bidder with the highest valuation.

As a possible extension of this topic, future research can examine the effects of jump bidding in more complex auction settings, such as multi-unit auctions and auctions with asymmetric information. Additionally, further research can investigate the role of other strategic behaviors, such as signaling and commitment, in the war of attrition and their implications for market efficiency. Finally, our analysis has focused on the competition between firms, but it would be interesting to extend this framework to other areas, such as political campaigns and sports competitions.

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A Proof:

Proof of proposition 3:

We prove by verifying whether dominant bidder 1 has incentive to jump bid at the very beginning of this competition when $\alpha > 1$ and $\Delta \to 0$. In this situation, $\underline{y}_2 = 0$. So, equation 9 and 10 can be rewritten as

$$\Delta = \underline{y}_1 \bar{y}^\alpha = \bar{y} \frac{\bar{y}^\alpha - \underline{y}_1^\alpha}{1 - \underline{y}_1^\alpha}$$

It is easy to see that $\underline{y}_1 \to 0$, $\bar{y} \to 0$ and $\frac{\bar{y}}{\underline{y}_1} \to k \in (0, +\infty)$, where k satisfies

$$k^\alpha - k^{\alpha-1} - 1 = 0,$$

and then

$$\Delta = \frac{1}{k} \bar{y}^{\alpha+1} = \left(1 - \frac{1}{1 + k^{\alpha-1}}\right) \bar{y}^{\alpha+1}$$

If dominant bidder 1 chooses not to jump bid, to eliminate bidder 2 with valuation from $[0, \bar{y}]$, she pays

$$\int_0^{\bar{y}} y dF(y) = \frac{\alpha}{1+\alpha} \bar{y}^{\alpha+1} = \left(1 - \frac{1}{1+\alpha}\right) \bar{y}^{\alpha+1}$$

To verify that $\Delta < \int_0^{\bar{y}} y dF(y)$, the remaining thing to prove is $k^{\alpha-1} < \alpha$ when $\alpha > 1$. To prove it, we prove that $G(\alpha) = k^{\alpha-1} - \alpha$ is decreasing in α and it is easy to see that $G(1) = 0$.

With $k^\alpha - k^{\alpha-1} - 1 = 0$ and implicit function theorem,

$$k'(\alpha) = -\frac{k^\alpha \log k - k^{\alpha-1} \log k}{\alpha k^{\alpha-1} - (\alpha - 1) k^{\alpha-2}}$$

When $\alpha > 1$, $k \in (1, 2)$. So, $k'(\alpha) < 0$. Then,

$$\begin{aligned} G'(\alpha) &= k^{\alpha-1} \log k - (\alpha - 1) k^{\alpha-2} k' - 1 \\ &= k^{\alpha-1} \log k \left(1 - (\alpha - 1) \frac{k-1}{\alpha k - \alpha + 1}\right) - 1 \\ &= \frac{\log k}{k-1} \left[1 - \frac{\alpha - 1}{\alpha} \left(1 - \frac{1}{\alpha(k-1)+1}\right)\right] - 1 \end{aligned}$$

It is easy to verify that $\frac{\log k}{k-1}$ is decreasing in $k \in (1, 2)$ and $\lim_{k \to 1} \frac{\log k}{k-1} = 1$. Also, $\frac{\alpha-1}{\alpha} \in (0, 1)$ and $1 - \frac{1}{\alpha(k-1)+1} \in (0, 1)$. Therefore, $G'(\alpha) < 0$ when $\alpha > 1$. Q.E.D.