Optimal procurement auctions with audit

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Abstract

One solution to limit the contractor's default risk in first-price procurement auctions is to audit the winning firm in order to assess the reliability of its bid, and, in case of an unsatisfactory response, moving to the next firm. We provide conditions under which a standard mechanism with audit – a mechanism in which allocation and audit proceed monotonically, starting from the lowest cost firm – is indeed optimal. These conditions involve the cost of the auditing procedure, the direct benefit of audit (the reduction in the risk of having the project unfinished), and, crucially, the relationship between firms' costs and default risks. We also provide conditions such that the procurer would rather use a mechanism in which allocation and audit are done randomly, at least at some point of the awarding process.

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1 Introduction

A well-known problem in the procurement of works is the risk of non-performance: the firm that is awarded the contract, in the (often long) time span between the assignment and the completion of the work, may encounter unforeseen contingencies that may hinder, partially or totally, the realization of the object of the contract. Such unforeseen contingencies can be of several types: some of them could be specific to the project to be realized (technical difficulties, a sudden increase in the cost of the inputs), others could affect the general economic conditions of the firm (financial distress, a drop in the demand for other firm's products). A lack of performance by the firm may be extremely costly, especially when the object of the procurement is a public good or service. These costs may consist in delays, litigations, extra-costs to complete the work if the appointed firm defaults, political costs for the procurer and, of course, welfare losses for the public who misses the benefits from the good/service.

Public contracts are often awarded via competitive bidding, usually through first-price auctions. Limiting corruption of public officials and fostering an efficient contract allocation are some of the well known advantages of competitive bidding. However, when the risk of the contractor's non performance is an issue, competitive bidding may even exacerbate this risk, as it econourages the bidder most likely to default to bid low and hence win.

This perverse effect may occur for two reasons. On the one hand, a riskier firm may bid lower because it has less to lose in case of default. This may occur when firms have heterogenous assets. Being protected by limited liability, the value of a firm's assets actually constitutes an upper bound on its potential losses, as the firm will rather go bankrupt and lose its assets when the expected loss from going on with the project exceeds this value. Hence, a firm with little assets is more likely to go bankrupt if a negative shock occurs ex-post, and, at the same time, having less to lose in case of a negative shock, can also afford to bid lower than a firm with larger assets.

On the other hand, a riskier firm may bid lower because it has more to gain in case of no default. This may occur when firms adopt different solutions to realize the tendered project and these solutions are characterized different cost-risk balances: some solutions may be more cost-effective if everything goes well, but more difficult to adapt to negative contingencies; others are more costly on average, but more flexible and secure. In this case, upon bidding in the auction, a firm with a riskier technology only takes into account its (low) production cost in the (low probability) good states, as it is protected by limited liability in the (high probability) bad states of the world.

An approach to alleviate the contractor's default problem is to rule out bids that are perceived as abnormally low. This approach has been and still is adopted in several European and non-European countries. In general, the mechanism is formulated as a first-price auction in which, however, bids that are below some threshold defined as a function of all bids (often their average) are considered unreasonable and automatically eliminated. Hence, the winning bid turns out to be the one that is closest and above to a certain average of all submitted bids, which actually makes this mechanism an instance of average bid auction. An average bid auction has an equilibrium (possibly the unique) in which all firms make the same bid: in other words, it is essentially equivalent to a lottery. Therefore, an average bid auction may effectively reduce the risk that the winning firm defaults with respect to a genuine first-price auction.

Despite its potential in reducing default risk, the average bid auction mechanism has been viewed with suspect for its anticompetitive flavor.[1] In the European Union, such mechanism is prohibited for contracts above a certain value (see Directive 2004/18, art. 55), but is still used in practice for contracts below that value. In particular, the European Law dictates that bids that appear abnormally low can be eliminated but only after auditing the firms that made it. In this sense, an average bid method can possibly be used, but only to identify those bids that are suspect. Implicitly, the European provision introduces a different awarding mechanism, that can be called a first-price auction with audit: the contract is provisionally awarded to the firm that

[1]Another shorthcoming of average bid auctions is that they are weak to collusion, as the members of a cartel, by placing coordinated bids, may pilot the average (see Conley and Decarolis, 2016).

makes the lowest bid, but the procurer, before coming to a final decision, can decide to audit this firm. Then, depending on the information collected from this audit, the procurer can then either definitively assign the contract to this firm or, instead, moving to the firm that made the second-lowest bid, which, in turn, can be audited; and so on. The rationale behind a first-price auction with audit is then to alleviate the risk of awarding the contract to a risky firm without giving up the benefits of competition.

The goal of this paper is to study whether and when auditing firms does indeed represent an optimal compromise between the two competing goals of reducing the awarding price and minimizing the risk of non-performance. This potential remedy has been greatly overlooked by the literature, which has mainly focussed on other solutions (e.g. penalties, perfomance bonds). To tackle this question, we adopt a mechanism design approach. We assume that firms are characterized by an idyosincratic risk of non-performing ex-post, which is correlated with their expected cost for completing the project. Although our model allows for any possible relation between default risk and (expected) production cost, what we have in mind is a situation in which this relation is negative: a firm that plans to adopt a cheaper technology to realize the project is more likely to be unable to adapt to unexpected negative shocks and thereby to default. This assumption captures a situation in which the perverse effect of competitive bidding originates from the fact that firms have technologies characterized by different riskiness rather than heterogeneous assets' value. This situation seems more in line with the scope of the provision of the European Union. In fact, Directive 2004/18 specifies that the audit should be concerned with the "details of the constituent elements of the tender"(art. 55). Hence, its goal is not to assess the general financial conditions of the firm, but, rather, to evaluate the reliability of the firm's bid and, in particular, to appraise how likely it is that, given the technical solutions the firm declares it will adopt to realize the project, the firm will be able to effectively complete the project. We also assume that the firm that is awarded the contract receives the payment only upon performance, and that, if the project is assigned to a firm but not completed, the procurer suffers a loss, which captures any delay, social, or political cost. We finally pose that auditing firms is costly and, by simplicity, that audit fully reveals whether a firm will be able to fulfill the contract or not.

As a benchmark, we begin the analysis with the case in which the contract is allocated at one go, without auditing any firm. We provide the condition such that the optimal mechanism is equivalent to a first-price auction and the condition such that the optimal mechanism is a lottery (i.e. it is implemented by an average bid auction). These conditions are generalizations of the standard result of Myerson (1981), and involve the shape of the virtual surplus functions, which, in the present context, also depend on the relationship between firms' expected production cost and risk of default and on the magnitude of the loss suffered by the procurer in case of default.

We then introduce audit in the awarding mechanism. We define a *mechanism with audit* as an incentive compatible and individually rational direct mechanism that allows the procurer to audit a firm that has been (provisionally) awarded the contract, and, depending on the outcome of this audit, to definitively assign the contract to this firm or to move to another firm, which, in turn, can be audited; and so on. We also define a *standard* mechanism with audit as a mechanism with audit where the (provisional) allocation follows the ranking of firms' costs; and a *random* mechanism with audit as a mechanism with audit where the (provisional) allocation is made randomly, at least in some round of the assignment process.

We first consider the case in which any firm, upon assignment, must be audited (with some fixed probability). We provide a sufficient condition that guarantees that a standard mechanism with audit is indeed optimal. This condition involves the shape of a different virtual surplus function that takes also into account the option value of assigning the contract to a certain firm. Such option value captures the fact that, if a firm is audited, the procurer still has the possibility to have the project done by another firm. We also characterize the first-price (and the second-price) auction with audit that implements such a mechanism.

We finally move to the case in which the procurer can optimally decide whether to audit a certain firm or not. In this case, the trade-off between firms' costs and default risks interacts with a second trade-off, the one between the cost and the benefits of audit. We provide sufficient conditions under which a standard mechanism with audit is optimal. In particular, these conditions imply that, in those situations in which, in the absence of audit, the procurer should adopt a lottery (i.e. an average bid auction) to maximize her expected payoff, a standard auction with audit can indeed be optimal, but only if the relationship between firms' expected cost and default risk is not too steep, the probability of firms' default is neither too high nor too low, and, in addition, the ratio between auditing cost and procurer's loss in case of default is sufficiently low. When these conditions fail to hold, instead, a standard mechanism with audit will in general be suboptimal. This does not mean that auditing is not beneficial at all, but, rather, that, at least at some round of the allocation process, the procurer should assign the project randomly to one of the eligible firms, and then audit this firm. In particular, we characterize random mechanisms with audit that, under certain conditions, dominate the best standard mechanism with audit.

Our paper is related to the literature, especially theoretical, on the problem of contractor's default in procurement auctions and its remedies.

Spulber (1990) was the first to note that auctions may provide incentives for contractors to default when there are cost overruns. Zheng (2001) highlights how this problem is exacerbated when bidders have heterogenous limited liabilities. He shows that, in a first-price selling auction, it is possible that, in equilibrium, low-budget bidders bid high and high-budget bidders bid low, so that the winner is the most budget-constrained bidder, who is most likely to declare bankruptcy. Burguet et al. (2012) consider a procurement setting which is equivalent to that of Zheng (2001), and show that a standard auction can haradly be optimal. Waehrer (1995) and Board (2007) investigate the implications on the auctioneer's payoff of different post-default situations.

Several papers then explored possible remedies to the risk of default's problem. Calveras et al. (2004) and Birulin (2020) consider surety (or performance) bonds.[2] The latter article, in particular, shows that, under the (crucial) hypothesis that the possible ex-post cost overrun is realized after the contractor has already made a significant sunk investment, a first-price auction together with an optimally designed surety bond can be the optimal mechanism. Chillemi and Mezzetti (2014) considers a different setting in which the procurer can impose penalties (stipulated damage payments) in case of contract's breach. They show that the optimal mechanism may lead to an inefficient lock-in effect (the contractor completes the project even if it is not efficient to do so), and, interestingly, that that, if the cost types of all agents are above a threshold, then the project is assigned by lottery. Birulin and Izmalkov (2022) study the problem of optimally balancing up-front and final payments to the contractor. They highlights how an increase in the up-front payments may alleviate opportunistic behavior at the bidding stage; however, this comes at the cost of increasing the risk of opportunistic behavior ex-post.

Among the variuos remedies to the contractor's default problem in competitive auctions, the most relevant to us is the authomatic elimination of abnormally low tenders, which actually yields to an average bid auction. There is not a single definition of this type of auction, as there are various actual examples around the world that differ in the details (especially the way in which the average is defined).[3] For the average bid methods in use in Italy, Galavotti et al. (2018) and Decarolis (2018) show that there is a continuum of equilibria, in all of which all bidders make the same (sufficiently high) bid,[4] making an average bid auction essentially equivalent to a lottery. However, most of the studies on average bid auctions, including the cited article by Decarolis (2018), are empirical and aimed at comparing the relative performance of this method with respect to the first-price auction (see also Decarolis, 2014 and Chang et al., 2015). In this respect, one of the contribution of our paper is to provide conditions under which an average bid auction (i.e. a lottery) is indeed the optimal mechanism (at least when the contract is assigned without audit).[5]

[2]A surety bond is a financial instrument by which a specialized firm, the surety company, guarantees to the procurer that the contractor will perform the procurement contract. In case of difficulties by the contractor, the surety company can help the contractor completing the job. In case of default, the surety company refunds the buyer.

[3]See, e.g., Decarolis (2018) for a list of countries that adopt average bid auctions with different averages.

[4]This equilibrium can be unique or not, depending on the details, but even if it is not unique, the other equilibria still display a large degree of pooling.

[5]It should be noted, nonetheless, that both Burguet et al. (2012) and Chillemi and Mezzetti (2014), in their

Even more scarce is the theoretical literature that deals with bid audit as a potential solution to the default risk. This lack of research could be due to the fact that, while the literature has almost exclusively focussed on heterogeneous limited liability as the main driver of the adverse effect of competitive auctions, the European law dictates that audit should not be concerned with evaluating a firm's financial status but only the reliability of its bid. In this respect, our model shows how the perverse effect of first-price auctions may also arise as a consequence of different technologies or technical solutions adopted by the firms, a situation that justifies auditing firms' bids. The only article that deals explicitely with bid audit in auctions is, again, Decarolis (2018). Although the interest of this paper is mainly empirical, it also presents a theoretical analysis of first-price auctions with audit. However, it does not address the optimality of such a method: in fact, in his model, the decision to audit a bid is not optimized over and totally exogenous, and only the lowest bid can be audited (although it is assumed that all firms that can possibly default are immediately eliminated).

Finally, the results of our paper are reminiscent of those arising from situations in which a buyer is interested both in the price paid and in the quality of the supplier. In particular, Manelli and Vincent (1995), in a setting where there is a trade-off between suppliers' costs and qualities, show that, depending on the primitives of the problem, the buyer, rather than exploiting competition among suppliers, may prefer to adopt a sequential offer mechanism, in which the buyer makes a take-it-or-leave-it offer to a randomly selected supplier, and, if this offer is rejected, then the buyer makes another take-it-or-leave-it offer to a different supplier chosen randomly, and so on. This mechanism has the same structure as what, in our paper, we call a random mechanism mith audit. In Wan and Beil (2009), instead, the buyer is interested in contracting with a supplier that satisfies a minimum level of quality, and screens suppliers to assess their qualities. In a setting in which there is no trade-off between suppliers' costs and qualities (cost and qualities are independent), the question they address is how to optimally balance ex-ante and ex-post screening, where the ex-post screening is done sequentially, one firm at a time, until a firm that satisfies the minimum quality requirement is found.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 develops a preliminary analysis of the buyer's problem. Section 4 collects the results on the optimal mechanisms, which are then discussed in Section 5. Section 6 briefly concludes.

2 Model

A procurer wants to assign the realization of a project to one out of n risk neutral firms. Each firm $i \in N = \{1, 2, \ldots, n\}$ is characterized by its type $c_i \in C = [\underline{c}, \overline{c}]$, which we refer to as firm i's *cost.* c_i is private information to the firm, but it is commonly known that c_i is drawn from the cumulative distribution function G_i, with strictly positive density g_i, and that, for all i and j, c_i and c_j are stochastically independent. It is not guaranteed that firms will indeed be able to finalize the project. In particular, we assume that a firm with cost c, if assigned the project, will complete it with probability $\phi(c) \in (0, 1]$, where ϕ is a commonly known function.

The simple relationship between cost and probability of completing the project, summarized by the function ϕ, can be interpreted as the reduced form of the following richer model. Suppose there is a continuum of possible states of the world S, with generic element s, and corresponding density $f(s)$. On the other hand, there is a (potentially infinite) set of possible technologies to realize the project. Each technology j is summarized by a function $\kappa_j(s)$ that associates, to each state of the world, the *actual* cost borne by the firm that adopts that technology if state s realizes. For each technology j, there is a subset of states of the world $S_D(j) \in S$ such that, if any of these states realizes, the actual cost would be so high that the firm is not able to complete the project (i.e. the firm defaults). On the other hand, if the realized state of the world does not belong to

mechanism design exercises, show that, under some conditions, the optimal mechanism may involve a random allocation.

$S_D(j)$, the firm will complete the project. We can thus define the probability that a firm that with technology j defaults as $\text{Prob}(s \in S_D(j))$, and the expected cost associated to the technology j conditional on completing the project as $c(j) = \mathbb{E}[\kappa_i(s)|s \notin S_D(j)]$. Of course, it perfectly possible that two different technologies have the same expected cost conditional on completing the project, but, different default sets and different probabilities of default. If this is the case, knowing a firm's expected cost does not allow to know the actual technology and the actual probability of default. Hence, our model can be interpreted as follows: c_i – the cost of firm i – is the expected cost to realize the project associated to the (privately known) technology adopted by firm i, conditional on the event that a state of the world realizes such that, with that technology, firm i will be able to complete the project. Clearly, firm i knows her own technology and thus her actual probability of default. But the procurer (and the other firms), even knowing c_i, cannot ascertain what the actual technology is: hence, from their point of view, $\phi(c_i)$ is the probability that a firm with expected cost c_i is indeed able to complete the project, which is the result of the expected probability of default associated with all the technologies with expected cost c_i

Only if the appointed firm completes the work, the firm will receive the agreed payment p, and the procurer will enjoy the benefits from the project, equal to v. In the opposite case, the firm's payoff will be equal to zero, while the procurer will suffer a loss equal to d.[6]

To face this risk, the procurer, before awarding the project, has the option to audit firms in order to ascertain their ability to perform the task and use the information obtained from this auditing process to reach a final decision. We assume that auditing a firm has a cost equal to a, but fully reveals whether the firm is able to complete the project or not. In other words, if a firm with cost c is audited, with probability $\phi(c)$, the response will be positive and the firm would perform with probability 1, whereas, with probability $1 - \phi(c)$, the response will be negative and the firm would certainly default.[7]

To sum up, the ex-post payoff of the procurer, *gross of the auditing cost* (which is given by a times the number of firms that have been audited) is equal to $v - p$ if the project is assigned to a firm and completed, it is equal to $-d$ if the project is assigned to a firm but not completed (the appointed firm has defaulted), it is equal to 0 if the project is not assigned. On the other hand, the ex-post payoff of the firm that has been assigned the project is equal to $p - c$ if the firm completes the project, it is equal to 0 if it defaults. The payoff of all other firms is equal to zero.

3 Preliminaries

3.1 Mechanism with audit

Our goal is to determine an optimal, individually rational mechanism, i.e. a mechanism such that all firms would like to take part in it and that, in equilibrium, maximizes the expected payoff of the procurer. From the revelation principle, we can confine ourselves to incentive compatible direct revelation mechanisms, namely mechanisms in which firms' message spage is C and truthtelling (i.e. all firms reveal their types) is a Bayes-Nash equilibrium of the game induced by the mechanism.

Now, if audit is not possible, the mechanism has simply to determine to which firm, if any, the project is assigned and the payment to this firm conditional on completing the project. Instead, when auditing is allowed, the decision is more complicated: the procurer has also to decide whether to audit or not a certain firm and, depending on the response of this audit, whether to assign the project to that firm, to another firm or to none. Moreover, if the project is re-assigned to another firm, the procurer can decide to audit this firm as well. And so on.

Therefore, when audit is envisaged, we can think that the allocation of the project is decided by means of a sequential process, where, in each round, three things can happen:

[6] d can be interpreted as a political and/or a delay cost (the procurer will have to postpone the realization of the project).

[7] The assumption that the audit yields a perfect response is clearly extreme, but is made for the sake of simplicity as it reduces the complexity of the procurer's decision problem.

1. the project is not assigned to any firm;
2. the project is *provisionally* assigned to a certain firm, this firm is not audited or is audited and the response is positive: in this case, the project is *definitively* assigned to that firm (and the firm is then paid but only if it actually concludes the work);
3. the project is *provisionally* assigned to a certain firm, this firm is audited and the response is negative: in this case, the project is not assigned to that firm.

We will restrict attention to mechanisms in which the transition to one round to the next occurs only in case 3. above. In other words, if, in a certain round, the contract is not assigned to any bidder, then the process is stopped (no other round takes place). Also, we will consider only mechanisms in which, if a firm who has been provisionally assigned the project is audited and the audit gives a negative response, then this firm is no longer eligible in the following rounds of allocation.[8] In symbols, letting b_i be the firm that is provisionally assigned the project in the i-th round and letting $\mathbf{b}^{[t-1]} = (b_1, b_2, \ldots, b_{t-1})$ denote the sequence of firms that are provisionally assigned the project in the first $(t-1)$ rounds, the mechanisms we consider are such that $\mathbf{b}^{[t-1]}$ is a sequence of $(t-1)$ *different* firms and, in round t, only firms in the set $N \setminus \mathbf{b}^{[t-1]}$ are eligible.[9] For conciseness, we can refer to $\mathbf{b}^{[t-1]}$ as a *history.* Notice, finally that this implies that there can be at most n rounds of allocations.

Hence, letting $\mathbf{c} = (c_1, c_2, \ldots, c_n) = (c_i, \mathbf{c}_{-i})$ denote a generic vector of reported costs by firms ($\mathbf{c}_{-i}$ is the vector of types of all firms except i), a *mechanism with audit* is given by the following functions:

- $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$: probability that firm i is provisionally assigned the project in the t-th round when firms report $\mathbf{c}$, conditional on the history $\mathbf{b}^{[t-1]}$. Notice that, for any $1 \leq t \leq n$ and for any $\mathbf{b}^{[t-1]}$, $\sum_i \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \leq 1$. Moreover, to guarantee that firms that were assigned the project in the previous rounds (but received a negative audit) are not eligible any more, we impose $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 0$ for $i \in \mathbf{b}^{[t-1]}$. When $t = 1$, $\mathbf{b}^{[t-1]}$ is an empty sequence: in this case, we simply write $\pi_i^{(1)}(\mathbf{c})$, which is the probability that firm i is provisionally assigned the project in the first round. We can refer to $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ as the *t-round allocation rule*, conditional on history $\mathbf{b}^{[t-1]}$;
- $\theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$: probability that firm i is audited when firms report $\mathbf{c}$, conditional on firm i being assigned the project in round t and on the history $\mathbf{b}^{[t-1]}$;
- $p_i(\mathbf{c})$: payment to firm i if firm i is assigned the project and completes it, when firms report $\mathbf{c}$.[10]

It is also useful to define $\rho^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ as the probability of reaching round t of assignment after the sequence of provisional assignments $\mathbf{b}^{[t-1]}$ in the previous rounds, and to define $\pi_i(\mathbf{c})$ as the probability that firm i is provisionally assigned the project throughout the assignment process. Specifically, omitting the dependence from $\mathbf{c}$ and writing ϕ_j for $\phi(c_j)$, it is

$$\begin{aligned}\rho^{(t|\mathbf{b}^{[t-1]})} \quad &\equiv \quad \pi_{b_1}^{(1)}\theta_{b_1}^{(1)}\left(1-\phi_{b_1}\right) \times \pi_{b_2}^{(2|[b_1])}\theta_{b_2}^{(2|[b_1])}\left(1-\phi_{b_2}\right) \times \\ &\qquad \ldots \times \pi_{b_{t-1}}^{(t-1|[b_1,\ldots,b_{t-2}])}\theta_{b_{t-1}}^{(t-1|[b_1,\ldots,b_{t-2}])}\left(1-\phi_{b_{t-1}}\right),\end{aligned}$$

[8]The assumptions that auditing is costly and fully reveals whether the firm is able to complete the project or not ensures that the allocation process described above is without loss of generality. In fact, it is clearly preferable to audit firms one at a time than more than one simultaneously. Moreover, it is clearly optimal not to assign the project to a firm when the response of the audit is negative, and to definitively assign it to her when the response of the audit is positive (provided that the firm to be audited were selected optimally). Notice, finally, that, for any mechanism in which the transition from one round to the next occurs in case 1. and case 3. there is an equivalent (in the sense that produces the same allocation and gives the same expected payoff to the procurer) mechanism in which the transition from one round to the next occurs only in case 3. By the way, the latter mechanism would be strictly preferable when there was even a small cost for each transition.

[9]$N \setminus \mathbf{b}^{[t-1]}$ denotes the difference between the set N and the set of firms contained in $\mathbf{b}^{[t-1]}$.

[10]We are thus assuming that the payment to firm i depends neither on the round in which firm i is assigned the project nor on the sequence of firms that were assigned the project previously.

and

$$\pi_i \equiv \pi_i^{(1)} + \sum_{t=2}^{n} \sum_{\mathbf{B}^{[t-1]}} \rho^{(t|\mathbf{b}^{[t-1]})} \pi_i^{(t|\mathbf{b}^{[t-1]})},$$

where the second summation above is made over the set $\mathbf{B}^{[t-1]}$ of every possible sequence of $(t-1)$ different firms. We will refer to $\pi_i(\mathbf{c})$ as the *allocation rule.* Notice that $\pi_i(\mathbf{c})$ depends on all the t-round allocation rules and on the probabilities of auditing. Notice also that $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ becomes relevant only if $\rho^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) > 0$ (round t can possibly be reached); and that $\theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ becomes relevant only if $\rho^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) > 0$ and $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) > 0$ (round t can possibly be reached and firm i can possibly be assigned the project).

In the sequel, we will use uppercase letters to denote (interim) expected values computed with respect to the other firms' types distribution. For example, the (interim) expected value of $\pi_i(\mathbf{c})$ is

$$\Pi_i(c_i) \equiv \mathbb{E}_{-i}\left[\pi_i(c_i, \mathbf{c}_{-i})\right] = \int_{C^{n-1}} \pi_i(c_i, \mathbf{c}_{-i}) g_{-i}(\mathbf{c}_{-i}) d\mathbf{c}_{-i},$$

where $g_{-i}(\mathbf{c}_{-i}) = \prod_{j \neq i} g_j(c_j)$ and $d\mathbf{c}_{-i} = \prod_{j \neq i} dc_j$.

3.2 Incentive Compatibility and Individual Rationality

The expected profit of firm i with cost c_i when she reports r_i (while the other firms report truthfully) is

$$\begin{aligned} U_i(r_i; c_i) &= \int_{C^{n-1}} \pi_i(r_i, \mathbf{c}_{-i}) \phi(c_i) \left[p_i(r_i, \mathbf{c}_{-i}) - c_i\right] g_{-i}(\mathbf{c}_{-i}) d\mathbf{c}_{-i} \\ &= \phi(c_i) \left[Q_i(r_i) - \Pi_i(r_i) \times c_i\right], \end{aligned}$$

where $Q_i(c_i) \equiv \mathbb{E}_{-i}\left[\pi_i(c_i, \mathbf{c}_{-i}) \times p_i(c_i, \mathbf{c}_{-i})\right]$.

Incentive Compatibility (IC) amounts to the requirement that for all i, $U_i(c_i; c_i) \geq U_i(r_i; c_i)$ for all c_i, r_i. Since, by assumption, $\phi(c_i) > 0$ for all c_i, this is equivalent to require that, for all i, and for all c_i, r_i,

$$Q_i(c_i) - \Pi_i(c_i) \times c_i \geq Q_i(r_i) - \Pi_i(r_i) \times c_i,$$

an expression that is analogous to that of a standard problem.[11] We can thus immediately obtain the following lemma.

Lemma 1 [Incentive Compatibility]. *A direct mechanism with audit is incentive compatible if and only if, for all i, the expected allocation rule* Π_i *is weakly decreasing on* C*. Moreover, in an incentive compatible direct mechanism with audit, for all i, the expected payment function* $Q_i(c_i) \equiv \mathbb{E}_{-i}\left[\pi_i(c_i, \mathbf{c}_{-i}) \times p_i(c_i, \mathbf{c}_{-i})\right]$ *is such that*

$$Q_i(c_i) = Q_i(\overline{c}) - \Pi_i(\overline{c}) \times \overline{c} + \Pi_i(c_i) \times c_i + \int_{c_i}^{\overline{c}} \Pi_i(y) dy. \tag{1}$$

Finally, Individual Rationality (IR) amounts to the requirement that $U_i(c_i; c_i) \geq 0$ for all i and all c_i, which, using (1), reduces to $Q_i(\overline{c}) - \Pi_i(\overline{c}) \times \overline{c} \geq 0$.

[11] See, e.g., Krishna (2009).

3.3 Setup of the procurer's optimization problem

The expected payoff of the procurer is the following (omitting the dependence from $\mathbf{c}$ and writing ϕ_j for $\phi(c_j)$):

$$
\begin{aligned}
V \quad = \quad & \int_{C^n} \sum_{i=1}^{n} \left\{ [\phi_i(v - p_i) - (1-\phi_i)d] \cdot \left[\pi_i^{(1)} + \sum_{t=2}^{n} \sum_{\mathbf{B}^{[t-1]}} \rho^{(t|\mathbf{b}^{[t-1]})} \pi_i^{(t|\mathbf{b}^{[t-1]})} \right] + \right. \\
& \left. [(1-\phi_i)d - a] \cdot \left[\pi_i^{(1)} \theta_i^{(1)} + \sum_{t=2}^{n} \sum_{\mathbf{B}^{[t-1]}} \rho^{(t|\mathbf{b}^{[t-1]})} \pi_i^{(t|\mathbf{b}^{[t-1]})} \theta_i^{(t|\mathbf{b}^{[t-1]})} \right] \right\} g(\mathbf{c}) d\mathbf{c}.
\end{aligned}
$$

The first term in square brackets is the payoff to the procurer when the project is assigned to firm i. This term is multiplied by the probability that this occurs during the assignment process (which is simply π_i). The third term in square brackets is the additional payoff (which can be negative) to the procurer when firm i is audited: auditing firm i allows the procurer to avoid the delay cost d when the response is negative (in this case the procurer will withdraw the project from firm i) but costs a. This term is multiplied by

$$
\theta_i \equiv \pi_i^{(1)} \theta_i^{(1)} + \sum_{t=2}^{n} \sum_{\mathbf{B}^{[t-1]}} \rho^{(t|\mathbf{b}^{[t-1]})} \pi_i^{(t|\mathbf{b}^{[t-1]})} \theta_i^{(t|\mathbf{b}^{[t-1]})}, \tag{2}
$$

which is the probability of auditing firm i during the assignment process. The expected payoff of the procurer can be written as

$$
V = \int_{C^n} \sum_{i=1}^{n} \{ [\phi_i v - (1-\phi_i)d] \cdot \pi_i + [(1-\phi_i)d - a] \cdot \theta_i \} \, g(\mathbf{c}) d\mathbf{c} - \int_{C^n} \sum_{i=1}^{n} \phi_i p_i \pi_i g(\mathbf{c}) d\mathbf{c}. \tag{3}
$$

Consider the second integral in (3). It is:

$$
\int_{C^n} \sum_{i=1}^{n} \phi(c_i) p_i(\mathbf{c}) \pi_i(\mathbf{c}) g(\mathbf{c}) d\mathbf{c} = \sum_{i=1}^{n} \int_C \phi(c_i) Q_i(c_i) g_i(c_i) dc_i,
$$

where the equality above follows from computing the integral with respect to $\mathbf{c}_{-i}$. Using (1), and letting $Z_i(\bar{c}) = Q_i(\bar{c}) - \Pi_i(\bar{c}) \times \bar{c}$, the above integral reduces to

$$
\begin{aligned}
\int_C \phi(c_i) Q_i(c_i) g_i(c_i) dc_i \quad &= \quad \int_C \phi(c_i) \left[\Pi_i(c_i) c_i + Z_i(\bar{c}) + \int_{c_i}^{\bar{c}} \Pi_i(t) dt \right] g_i(c_i) dc_i \\
&= \quad \int_C \phi(c_i) \left[\Pi_i(c_i) c_i + Z_i(\bar{c}) \right] g_i(c_i) dc_i + \int_C \int_{\underline{c}}^{c_i} \phi(t) g_i(t) dt \; \Pi_i(c_i) dc_i \\
&= \quad \int_{C^n} \pi_i(c_i) \left[\phi(c_i) c_i + \int_{\underline{c}}^{c_i} \frac{\phi(t) g_i(t)}{g_i(c_i)} dt \right] g(\mathbf{c}) d\mathbf{c} + \int_C \phi(c_i) Z_i(\bar{c}) g_i(c_i) dc_i.
\end{aligned}
$$

Substituting this expression into (3), we obtain that the expected payoff of the procurer is

$$
\begin{aligned}
V \quad = \quad & \int_{C^n} \sum_{i=1}^{n} \left[\phi(c_i)(v - c_i) - (1 - \phi(c_i))d - \int_{\underline{c}}^{c_i} \frac{\phi(t) g_i(t)}{g_i(c_i)} dt \right] \cdot \pi_i(\mathbf{c}) g(\mathbf{c}) d\mathbf{c} \\
& + \int_{C^n} \sum_{i=1}^{n} [(1 - \phi(c_i))d - a] \cdot \theta_i(\mathbf{c}) g(\mathbf{c}) d\mathbf{c} - \int_C \sum_{i=1}^{n} \phi(c_i) Z_i(\bar{c}) g_i(c_i) dc_i \\
= \quad & \int_{C^n} \sum_{i=1}^{n} \left\{ \gamma_i^0(c_i) \pi_i(\mathbf{c}) + [(1 - \phi(c_i))d - a] \, \theta_i(\mathbf{c}) \right\} g(\mathbf{c}) d\mathbf{c} - \int_C \sum_{i=1}^{n} \phi(c_i) Z_i(\bar{c}) g_i(c_i) dc_i,
\end{aligned} \tag{4}
$$

where, in the above expression,

$$
\gamma_i^0(c_i) \equiv \phi(c_i)(v - c_i) - (1 - \phi(c_i))d - \int_{\underline{c}}^{c_i} \frac{\phi(t) g_i(t)}{g_i(c_i)} dt. \tag{5}
$$

Thus, the procurer's problem is that of maximizing (4) under the IR constraint ($Z_i(\overline{c}) \geq 0$ for all i), the IC constraint ($\Pi_i(c_i)$ must be weakly decreasing for all i), and the feasibility constraints

$$0 \leq \theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \leq 1, \quad 0 \leq \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \leq 1, \quad \sum_{i=1}^{n} \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \leq 1, \tag{6}$$

which must hold for all $i = 1, 2, \ldots, n$, for all $\mathbf{c} \in C^n$, for all $t = 1, 2, \ldots, n$, and for all $\mathbf{b}^{[t-1]} \in \mathbf{B}^{[t-1]}$.

From (4), we can immediately obtain the revenue equivalence principle in our setting (for which we omit the proof).

Lemma 2 [Revenue Equivalence]. *Any two direct mechanisms with the same $\pi_i(\mathbf{c})$ and $\theta_i(\mathbf{c})$, $i \in N$, $\mathbf{c} \in C^n$, and with the same value of $Z_i(\overline{c})$, $i \in N$, yield the same expected payoff to the procurer.*

Before moving to the derivation of the optimal mechanism, it is instructive to highlight the recursive structure of the procurer's objective function (4). For $t = 1, \ldots, n$, let

$$\begin{aligned} V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \quad = \quad & \sum_{i \in N \setminus \mathbf{b}^{[t-1]}} \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \left\{ \gamma_i^0(c_i) + \theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \left[(1-\phi(c_i))\, d - a \right] + \right. \\ & \left. + \, \theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \, (1-\phi(c_i)) \, V^{(t+1|\mathbf{b}^{[t-1]} \cup \{i\})}(\mathbf{c}) \right\}, \end{aligned} \tag{7}$$

where it is intended that $V^{(n+1|\mathbf{b}^{[n]})} \equiv 0$ for all $\mathbf{b}^{[n]}$, and, for $t = 1$, $\mathbf{b}^{[t-1]}$ is an empty sequence and we write $V^{(1)}(\mathbf{c})$. It is immediate to see that the expected payoff of the procurer is

$$V = \int_{C^n} V^{(1)}(\mathbf{c}) g(\mathbf{c}) \; d\mathbf{c} - \int_C \sum_{i=1}^{n} \phi(c_i) Z_i(\overline{c}) g_i(c_i) dc_i.$$

Looking at (7), we see that $V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ can be interpreted as the procurer's t-round continuation payoff: it is the additional value accruing to the procurer's payoff if round t is reached after the sequence $\mathbf{b}^{[t-1]}$ of assignments in the previous $(t-1)$ rounds (when firms' vector of costs is $\mathbf{c}$). In particular, the expression within curly brackets in (7) is the benefit to the procurer from allocating the project to firm i in round t. This is made of four terms: *(i)* the first term, $\gamma_i^0(c_i)$ (see (5)), is the direct contribution to the procurer's payoff from assigning the project to firm i. The other three terms accrue to the procurer only if firm i is audited; in particular,: *(ii)* if firm i is audited, the procurer has to bear the auditing cost a; *(iii)* if firm i is audited and the response of the audit is negative (an event that occurs with probability $1-\phi(c_i)$), the procurer saves on the delay cost d (the project is withdrawn from firm i); *(iv)* if firm i is audited and the response of the audit is negative (an event that occurs with probability $1-\phi(c_i)$), the procurer can opt for moving to round of assignment $(t+1)$, where she will obtain the continuation value $V^{(t+1|\mathbf{b}^{[t-1]} \cup \{i\})}(\mathbf{c})$. This last term captures the *option value* of auditing firm i.

4 Optimal mechanism

Observe first that the most convenient way (to the procurer) to satisfy the IR constraint is to have $Z_i(\overline{c}) = 0$ for all i. This can easily be achieved by setting $p_i(\overline{c}, \mathbf{c}_{-i}) = \overline{c}$ for all $\mathbf{c}_{-i}$. Hence, in the procurer's objective function (4), the second integral is equal to zero and we can concentrate on the first term only.

4.1 No audit

We begin with the benchmark case in which audit is not allowed, i.e. $\theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 0$ for all i, for all $\mathbf{c}$, for all t, and for all $\mathbf{b}^{[t-1]}$. Notice that, in this case, $\pi_i(\mathbf{c})$ reduces to $\pi_i^{(1)}(\mathbf{c})$ and the procurer's optimization problem simplifies to:

$$\max_{\pi_i^{(1)}} V = \int_{C^n} \left\{ \sum_{i=1}^{n} \gamma_i^0(c_i) \cdot \pi_i^{(1)}(\mathbf{c}) \right\} g(\mathbf{c}) d\mathbf{c},$$

under the IC constraint ($\Pi_i(c_i) = \int_{C^{n-1}} \pi_i^{(1)}(c_i, \mathbf{c}_{-i}) g_{-i}(\mathbf{c}_{-i}) d\mathbf{c}_{-i}$ must be weakly decreasing), and under the feasibility constraints

$$0 \leq \pi_i^{(1)}(\mathbf{c}) \leq 1, \qquad \sum_i \pi_i^{(1)}(\mathbf{c}) \leq 1.$$

This expression is akin to the one of a standard optimal auction problem, with $\gamma_i^0(c_i)$ represent the virtual surplus of firm i in this context. Thus, we can apply directly Myerson (1981) to obtain the following:

Proposition 1 [Optimal mechanism without audit]. *Suppose that, for all i, $\gamma_i^0(c_i)$ is decreasing in the subset of C in which $\gamma_i^0(c_i) > 0$, and let $w = \arg\max_i \gamma_i^0(c_i)$. Then the optimal mechanism assigns the project to firm w if $\gamma_w^0(c_w)$ is strictly positive, does not assign the project otherwise. Suppose that, for all i, γ_i^0 is increasing over C: if, for all i, $\mathbb{E}[\gamma_i^0(c_i)] = k > 0$, the optimal mechanism assigns the project with equal probability to each firm; if, instead, for all i, $\mathbb{E}[\gamma_i^0(c_i)] < 0$, the project is not assigned.*

Hence, as in a standard problem, it is the shape of the virtual surplus functions γ_i^0 that determines how the procurer should allocate the project. Notice however that, in our setting, the virtual surpluses do not only depend on the cost distributions, but also, and crucially, on the function ϕ. In fact, consider the derivative of firm i's virtual surplus

$$\left[\gamma_i^0(c_i)\right]' = \phi'(c_i)(v - c_i + d) - 2\phi(c_i) + \frac{g_i'(c_i)}{g_i^2(c_i)} \int_{\underline{c}}^{c_i} \phi(t) g_i(t) dt.$$

When there is a trade-off between a firm's production cost and her probability of performing ($\phi' > 0$), but this trade-off is moderate (ϕ' positive but sufficiently low), the procurer will typically still find it optimal to award the project to the firm with the lowest cost. When, instead, the trade-off is severe (ϕ' positive and sufficiently high), the procurer would rather award the project to the firm with the lowest risk (i.e. the one with the highest cost). However, doing this would violate incentive compatibility, according to which a high cost firm should have a (weakly) lower probability of being assigned the project than a low cost firm. As a result, the procurer optimally decides to assign the project randomly (provided that, in so doing, her expected payoff is positive).

Notice, finally, that Proposition 1 implies that, in the symmetric case – $g_i = g$ and, thereby, $\gamma_i^0 = \gamma^0$ for all i –, when γ^0 is decreasing, a first-price auction (possibly with reserve price) is optimal. If, instead, γ^0 is increasing and $\mathbb{E}[\gamma^0(c_i)] > 0$, then any auction mechanism such that, in equilibrium, all firms make the same bid, the winner is chosen randomly and, upon completion of the work, is paid $\bar{c}$, is optimal.

4.2 Fixed audit probability

As a second case, let's consider a situation in which, whenever a firm is provisionally assigned the project, it is audited with some exogenously fixed probability, i.e. $\theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = \bar{\theta} \in (0, 1]$ for all

i, for all $\mathbf{c}$, for all t, and for all $\mathbf{b}^{[t-1]}$. Notice that, in this case, θ_i reduces to $\overline{\theta} \cdot \pi_i$ (see (2)), and the procurer's optimization problem simplifies to:

$$\max_{\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})} V = \int_{C^n} \sum_{i=1}^{n} \left\{ \gamma_i^0(c_i) + \overline{\theta} \left[(1-\phi(c_i))d - a \right] \right\} \pi_i(\mathbf{c}) g(\mathbf{c}) d\mathbf{c}$$

$$= \int_{C^n} \sum_{i=1}^{n} \gamma_i^{\overline{\theta}}(c_i) \pi_i(\mathbf{c}) g(\mathbf{c}) d\mathbf{c}, \tag{8}$$

subject to the IC constraint ($\Pi_i(c_i)$ must be weakly decreasing), and the feasibility constraints:

$$0 \leq \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \leq 1, \qquad \sum_{i=1}^{n} \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \leq 1.$$

The expression for the objective function (8) follows from defining

$$\gamma_i^{\overline{\theta}}(c_i) \equiv \gamma_i^0(c_i) + \overline{\theta} \left[(1-\phi(c_i))d - a \right]. \tag{9}$$

The function $\gamma_i^{\overline{\theta}}(c_i)$ can be interpreted as firm i's virtual surplus in this context: it is the sum of $\gamma_i^0(c_i)$ – the direct contribution to the procurer's payoff from assigning the project to firm i – plus the additional net payoff accruing from auditing firm i (something that occurs with probability $\overline{\theta}$). Notice, however, that, differently from a standard problem, $\sum_{i=1}^{n} \gamma_i^{\overline{\theta}}(c_i)\pi_i(\mathbf{c})$ is not a weighted average because $\pi_i(\mathbf{c})$ is not a probability distribution, but a function of all the probability distributions $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ (one for each t and for each $\mathbf{b}^{[t-1]}$), and of the exogenous function ϕ. Notice, in particular, that $\sum_i \pi_i(\mathbf{c})$ can well exceed 1. Hence, the results in Myerson (1981) cannot be applied directly. The next Proposition provides a sufficient condition under which it is optimal for the procurer to assign the project following the ranking of firms in terms of costs, starting from the lowest cost firm and, if this firm is audited and the response is negative, moving to the firm with the second lowest cost, and so on.

Proposition 2 [Optimal mechanism with fixed audit probability]. *Suppose that, for all i, $\Gamma_i^{\overline{\theta}}(c_i) \equiv \gamma_i^{\overline{\theta}}(c_i) / \left[1 - \overline{\theta}\,(1-\phi(c_i))\right]$ is decreasing in the subset of C in which $\Gamma_i^{\overline{\theta}} > 0$. Then the optimal mechanism involves at most n rounds, where the t-th round of assignment, $t = 1, \ldots, n$, works as follows:*

- *if $\max_{i \in N \setminus \mathbf{b}^{[t-1]}} \Gamma_i^{\overline{\theta}}(c_i) < 0$, the project is not assigned;*
- *if $\max_{i \in N \setminus \mathbf{b}^{[t-1]}} \Gamma_i^{\overline{\theta}}(c_i) \geq 0$, the project is provisionally assigned to any of the firms in $M(\mathbf{b}^{[t-1]}) = \arg\max_{i \in N \setminus \mathbf{b}^{[t-1]}} \Gamma_i^{\overline{\theta}}(c_i)$. The assignment becomes definitive if this firm is not audited or is audited and the response is positive. If, instead, this firm is audited and the response is negative, then we move to the next round $t+1$.*[12]

Proof. See the Appendix.

Proposition 2 holds that, when it is established that every firm that is provisionally assigned the project will be audited with some fixed probability $\overline{\theta} > 0$, the first-best mechanism allocates the project according to the ranking of firms in terms of the values $\Gamma_i^{\overline{\theta}}(c_i)$, assigning the project in the first round to the first firm in this ranking, and, if this firm is actually audited and the response is negative, to the second firm, and so on. Hence, the optimal allocation is driven by the functions

$$\Gamma_i^{\overline{\theta}}(c_i) \equiv \frac{\gamma_i^{\overline{\theta}}(c_i)}{1 - \overline{\theta}\,(1-\phi(c_i))}.$$

[12]In the statement of Proposition 2, $N \setminus \mathbf{b}^{[t-1]}$ denotes the difference between the set N and the set of firms contained in the sequence $\mathbf{b}^{[t-1]}$.

Observe that Γ_i is the ratio between firm i's virtual surplus $\gamma_i^{\overline{\theta}}$ and the probability that firm i is *definitively* assigned the project (either because it is not audited, or because it is audited and the response is positive). The reason why, for the procurer, it is optimal to do so is intuitive. When the project is assigned to firm i, the effect on the procurer's expected payoff is twofold. The first effect is the direct benefit, measured by firm i's virtual surplus $\gamma_i^{\overline{\theta}}(c_i)$. The second effect is indirect, and is related to the option value of assigning the project to firm i: in fact, if firm i is audited and the response is negative, the procurer can reallocate the project to the next firm in the ranking. Clearly, this second effect is the higher, the lower the probability that the allocation to firm i becomes definitive. It is perfectly possible that, between two firms i and j, firm i has a higher virtual surplus than firm j ($\gamma_i^{\overline{\theta}}(c_i) > \gamma_j^{\overline{\theta}}(c_j)$), but a lower option value $(1 - \overline{\theta}\,(1 - \phi(c_i)) > 1 - \overline{\theta}\,(1 - \phi(c_j))$, i.e. $\phi(c_i) > \phi(c_j)$), so that the procurer prefers to assign the project earlier to firm j ($\Gamma_i^{\overline{\theta}}(c_i) < \Gamma_j^{\overline{\theta}}(c_j)$). Clearly, while allocating the project according to the ranking of firms in terms of $\Gamma_i^{\overline{\theta}}(c_i)$ would certainly be the first-best for the procurer, this is incentive compatible (and thereby optimal) only if this ranking respects the ranking of firms in terms of cost efficiency, i.e. only if the functions $\Gamma_i^{\overline{\theta}}(c_i)$ are decreasing.

The next Corollary shows that, when firms are ex-ante symmetric, the optimal mechanism of Proposition 2 is implemented by means of auction mechanisms that are generalizations of the first-price and the second-price auction to this environment with (exogenous) audit.

Corollary 1 [Optimal auction with fixed audit probability]. *Suppose that all firms are ex-ante homogeneous, i.e., $g_i = g$ and, thereby, $\gamma_i^{\overline{\theta}} = \gamma^{\overline{\theta}}$ for all $i \in N$, and that $\Gamma^{\overline{\theta}}(c) \equiv \gamma^{\overline{\theta}}(c)/\left[1 - \overline{\theta}\,(1 - \phi(c))\right]$ is decreasing over C. Then the following auctions with fixed audit probability $\overline{\theta}$ are optimal.*

1. Vickrey Auction. *Firms submit sealed bids. Let $\left(b^{(1)}, b^{(2)}, \ldots, b^{(n)}\right)$ be the vector of submitted bids in increasing order. If there exists $\hat{c} \in C$ such that $\gamma^{\overline{\theta}}(\hat{c}) = 0$, let $r = \hat{c}$ be the reserve price, otherwise let $r = \overline{c}$. Finally let $p^{(n)} = r$ and, for $t = 1, \ldots, n-1$, let $p^{(t)} = \left[1 - \overline{\theta}\left(1 - \phi(b^{(t+1)})\right)\right] \times \min\left\{b^{(t+1)}, r\right\} + \overline{\theta}\left(1 - \phi(b^{(t+1)})\right) \times p^{(t+1)}$.*
   - Assignment. *The assignment involves at most n rounds. In round t, $t = 1, \ldots, n$, if $b^{(t)} > r$, the project is not assigned; if $b^{(t)} \leq r$, the project is provisionally assigned to the firm who bid $b^{(t)}$: if this firm is not audited or it is audited and the response is positive, the assignment becomes definitive, otherwise, if $t < n$, we move to round $t+1$ where the firm who bid $b^{(t)}$ is not eligible any more, whereas, if $t = n$, the project is not assigned.*
   - Payments. *If the project is definitively assigned in round t, the firm who is assigned it is payed $p^{(t)}$ conditionally on performing.*
2. Pay-your-bid Auction. *Firms submit sealed bids. Let $\left(b^{(1)}, b^{(2)}, \ldots, b^{(n)}\right)$ be the vector of submitted bids in increasing order. If there exists $\hat{c} \in C$ such that $\gamma^{\overline{\theta}}(\hat{c}) = 0$, let $r = \hat{c}$ be the reserve price, otherwise let $r = \overline{c}$.*
   - Assignment. *The assignment rule is the same as in the previous auction.*
   - Payments. *If the project is definitively assigned in round t, the firm who is assigned the project is payed her own bid $b^{(t)}$ conditionally on performing.*

Proof. See the Appendix.

4.3 Endogenous audit probability

Finally, let's consider the most general case in which the procurer can choose whether or not to audit a certain firm i, i.e. the audit probabilities $\theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$, $i = 1, 2, \ldots, n$, $t = 1, 2, \ldots, n$, $\mathbf{b}^{[t-1]} \in \mathbf{B}^{[t-1]}$, are endogenous. For simplicity, in this section we focus on the case in which firms are ex-ante identical in all respects, i.e. $g_i(c) = g(c)$ for all i, which implies that $\gamma_i^0(c) = \gamma^0(c)$ for all i. We also define

$$\gamma^1(c) \equiv \gamma^0(c) + (1 - \phi(c))d - a.$$

Notice that $\gamma^1(c)$ is the analogue of (9) with $\overline{\theta} = 1$ (and $g_i(c) = g(c)$ for all i): hence, it can be interpreted as the virtual surplus of a firm with cost c when this firm is audited for sure.

Our question is whether the European law on procurement – according to which the firm that makes the lowest bid cannot be excluded unless it is audited – is the best compromise between the competing goals of reducing procurement cost and minimizing the risk of default of the winning firm. To this end, let's define a *standard mechanism with audit* as a direct mechanism in which:

- the contract is provisionally assigned monotonically, starting from the firm that reports the lowest cost and then possibly moving to the second-lowest cost firm, to the third-lowest cost firm, and so on;
- a provisional assignment to a certain firm can be undone only after that firm is audited (and the outcome of the audit is negative).

Notice that, in a standard mechanism with audit, it is perfectly possible to definitively assign the contract without audit. Hence, a provisional assignment can become definitive not only when a firm is audited and the outcome is positive, but also when a firm is not audited at all. In other words, a standard mechanism with audit can envisage a threshold such that firms whose cost is below the threshold are audited (if provisionally assigned the contract), whereas firms whose cost is above the threshold are not. The next propositions identify situations in which a standard mechanism with audit is indeed optimal.

Proposition 3.1 [Optimal standard mechanism with audit (1)]. *Suppose that there exists $\hat{c} \in (\underline{c}, \overline{c}]$ such that: (i) $\max\{\gamma^1(c), \gamma^0(c), 0\} = \gamma^1(c)$ for $c \leq \hat{c}$, (ii) $\max\{\gamma^1(c), \gamma^0(c), 0\} = 0$ for $c > \hat{c}$, and (iii) $\Gamma^1(c) \equiv \gamma^1(c)/\phi(c)$ is decreasing in $(\underline{c}, \hat{c})$. Then a standard mechanism with audit is optimal. In this case, any firm that is provisionally assigned the project is audited.*

Proof. See the Appendix.

Proposition 3.2 [Optimal standard mechanism with audit (2)]. *Suppose that there exist $\hat{c}_1$ and $\hat{c}_2$, with $\underline{c} \leq \hat{c}_1 < \hat{c}_2 \leq \overline{c}$ such that: (i) $\max\{\gamma^1(c), \gamma^0(c), 0\} = \gamma^1(c)$ for $c < \hat{c}_1$, (ii) $\max\{\gamma^1(c), \gamma^0(c), 0\} = \gamma^0(c)$ for $\hat{c}_1 \leq c \leq \hat{c}_2$, (iii) $\max\{\gamma^1(c), \gamma^0(c), 0\} = 0$ for $c > \hat{c}_2$, (iv) $\Gamma^1(c) \equiv \gamma^1(c)/\phi(c)$ is decreasing in $(\underline{c}, \hat{c}_2)$, (v) $\gamma^0(c)$ is strictly decreasing in $(\hat{c}_1, c'')$, and (vi) $\phi(c)$ is increasing in $(\hat{c}_1, \hat{c}_2)$. Then a standard mechanism with audit is optimal. In this case, the mechanism involves a threshold, which is a function of the vector of reported costs: only firms that report a cost that is below the threshold are audited.*

Proof. See the Appendix.

The next propositions, instead, identifies situations in which a standard mechanism with audit is certainly suboptimal. In particular, we show that, in such situations, there exists a *random mechanism with audit* that yields a higher expected payoff to the procurer than the best standard mechanism with audit. A random mechanism with audit is a mechanism with audit in which, in some round of the assigment process, the allocation is done randomly (and the selected firm may or may not be audited). Notice that Propositions 4.1 and 4.2 below are the counterparts of Proposition 3.1 and 3.2, respectively. Before stating the Propositions, we need to define the following quantities: let

$$W^0(c) = \gamma^0(c),$$

and, for $s \geq 1$, let

$$W^s(c_1, c_2 \ldots, c_{s+1}) = W^0(c_1) - a + (1 - \phi(c_1)) \left(d + W^{s-1}(c_2, \ldots, c_{s+1})\right).$$

W^s can be interpreted as the continuation payoff if the procurer decides to assign the project to the firm with cost c_1 in the current round and audit this firm, to the firm with cost c_2 in the next round and audit this firm, ..., to the firm with cost c_{s+1} in s rounds from the current one and *not* audit this firm. Moreover, for all s, let

$$\overline{W}^s(c_1) = \mathbb{E}[W^s(c_1, c_2 \ldots, c_{s+1})|c_1].$$

$\overline{W}^s(c_1)$ is the continuation payoff the procurer expects to gain if the procurer decides to go on assigning the project for other s rounds (without auditing the firm that is assigned the project in s rounds from the current one), knowing only that the cost of the firm that is assigned the project in the current period is c_1.

PROPOSITION 4.1 [RANDOM MECHANISM WITH AUDIT (1)]. *Suppose that there exists $\hat{c} > \underline{c}$ such that (i) $\max\{\gamma^1(c), \gamma^0(c), 0\} = \gamma^1(c)$ for $c \leq \hat{c}$, and (ii) $\Gamma^1(c) \equiv \gamma^1(c)/\phi(c)$ is strictly increasing in $(\underline{c}, \hat{c})$. For any vector of firms' costs $\mathbf{c}$, define $\hat{N}(\mathbf{c})$ as the set of firms whose cost is below $\hat{c}$, and let $\hat{n}(\mathbf{c})$ be the cardinality of $\hat{N}(\mathbf{c})$. Let M_S be the incentive compatible standard mechanism with audit that maximizes the procurer's expected payoff and let M_R be the following random mechanism with audit:*

- *when $\hat{n}(\mathbf{c}) \leq 1$, M_R allocates as M_S;*
- *when $\hat{n}(\mathbf{c}) \geq 2$, in all rounds $t > \hat{n}(\mathbf{c})$, M_R allocates as M_S, whereas in any round $1 \leq t \leq \hat{n}(\mathbf{c})$, M_R allocates as follows:*

$$\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = \frac{1}{\hat{n}(\mathbf{c}) - t + 1} \quad \text{and} \quad \theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 1$$

if $\mathbf{b}^{[t-1]}$ is a sequence of $(t-1)$ (different) firms in $\hat{N}(\mathbf{c})$, and $i \in \hat{N}(\mathbf{c}) \backslash \mathbf{b}^{[t-1]}$; $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 0$ otherwise.

Then M_R is incentive compatible and yields a strictly higher expected payoff to the procurer than M_A.

Proof. See the Appendix.

PROPOSITION 4.2 [RANDOM MECHANISM WITH AUDIT (2)]. *Suppose that there exist $\hat{c} < \overline{c}$ such that (i) $\max\{\gamma^1(c), \gamma^0(c), 0\} = \gamma^1(c)$ for $c < \hat{c}$, (ii) $\max\{\gamma^1(c), \gamma^0(c), 0\} = \gamma^0(c)$ for $c \geq \hat{c}$, (iii) $\gamma^0(c)$ is strictly increasing in $(\hat{c}, \overline{c})$, and (iv) for all $\hat{n} \leq n-2$,*

$$\mathbb{E}\left[\max_{s \in \{0,\ldots,n-\hat{n}-1\}} W^s\left(c^{(1)}, c^{(2)} \ldots, c^{(s+1)}\right)\right] < \sum_{s=0}^{n-\hat{n}-1} \mathbb{E}\left[W^s\left(c_1, c_2, \ldots, c_{s+1}\right) \right|$$
$$\left. \arg\max_{s' \in \{0,\ldots,n-\hat{n}-s-1\}} \overline{W}^{s'}(c_{s+1}) = 0 \wedge \arg\max_{s' \in \{0,\ldots,n-\hat{n}-r-1\}} \overline{W}^{s'}(c_{r+1}) \neq 0, \quad r < s\right] \times$$
$$\mathit{Prob}\left[\arg\max_{s' \in \{0,\ldots,n-\hat{n}-s-1\}} \overline{W}^{s'}(c_{s+1}) = 0 \wedge \arg\max_{s' \in \{0,\ldots,n-\hat{n}-r-1\}} \overline{W}^{s'}(c_{r+1}) \neq 0, \quad r < s\right],$$

where $(c_1, c_2 \ldots, c_{s+1})$ are $(s+1)$ independent draws from the distribution $G(c|c > \hat{c})$, and $c^{(t)}$ denotes the t-th lowest out of $n - \hat{n}$ draws from that distribution. For any vector of firms' costs $\mathbf{c}$, define $\hat{N}(\mathbf{c})$ as the set of firms whose cost is below $\hat{c}$, and let $\hat{n}(\mathbf{c})$ be the cardinality of $\hat{N}(\mathbf{c})$. Let M_S be the (not necessarily incentive compatible) standard mechanism with audit that maximizes the procurer's expected payoff and let M_R be the following random mechanism with audit:

- *when $\hat{n}(\mathbf{c}) \geq n-1$, M_R allocates as M_S;*
- *when $\hat{n}(\mathbf{c}) \leq n-2$, in all rounds $t \leq \hat{n}(\mathbf{c})$, M_R allocates as M_S, whereas, in any round $\hat{n}(\mathbf{c}) + 1 \leq t \leq n$, M_R allocates as follows:*

$$\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = \frac{1}{n-t+1},$$

$$\theta_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = \begin{cases} 1 & \textit{if} \quad \arg\max_{s=0,\dots,n-t} \left\{ \overline{W}^s(c_i) \right\} \neq 0 \\ 0 & \textit{if} \quad \arg\max_{s=0,\dots,n-t} \left\{ \overline{W}^s(c_i) \right\} = 0 \end{cases},$$

if $\mathbf{b}^{[t-1]}$ *is a sequence of* $(t-1)$ *(different) firms, where the first* $\hat{n}(\mathbf{c})$ *elements belong to* $\hat{N}(\mathbf{c})$ *and the remaining* $(t-1-\hat{n}(\mathbf{c}))$ *elements belong to* $N \setminus \hat{N}(\mathbf{c})$*, and* $i \in N \setminus \mathbf{b}^{[t-1]}$*;* $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 0$ *otherwise.*

Then M_R *is incentive compatible and yields a strictly higher expected payoff to the procurer than* M_A*.*

Proof. See the Appendix.

5 Discussion

The previous results allow us to draw some general comments on the effectiveness of a standard mechanism with audit as an optimal solution to the trade-off between procurement's cost and risk of non-performance. To this end, let's focus a situation in which the trade-off between firms' cost efficiency and risk of non-performance is indeed present, i.e. ϕ is strictly increasing over the entire support of the cost distribution C. For simplicity, it is useful to restrict attention to a situation in which both γ^0 and γ^1 are positive on C: in words, this means that, whenever a firm is audited and the response of the audit is negative, it is always in the interest of the procurer to reassign the project to another firm.

Propositions 3.1 and 3.2 state that a standard mechanism with audit is optimal if the function Γ^1 is decreasing and, at the same time, either of the following two conditions is satisfied:

(a) the auditing cost a is always lower than the expected default cost $(1-\phi(c))d$;

(b) γ^0 is decreasing whenever the auditing cost is above the expected default cost.

The decreasingness of Γ^1 guarantees that, as long as the procurer goes on auditing firms throughout the allocation process, it is indeed optimal to allocate the project monotonically, starting from the lowest cost firm. To assess when this condition is satisfied, let's look at the derivative of Γ^1, which is

$$\begin{aligned} \left[\Gamma^1(c)\right]' &= \frac{\left[\gamma^0(c)\right]' \phi(c) - \left(\gamma^0(c) + d - a\right) \phi'(c)}{\left[\phi(c)\right]^2} \\ &= \frac{\phi'(c)}{\left[\phi(c)\right]^2} \left[a + \int_{\underline{c}}^{c} \frac{\phi(t)g(t)}{g(c)} dt \right] + \frac{g'(c)}{\phi(c)g(c)} \int_{\underline{c}}^{c} \frac{\phi(t)g(t)}{g(c)} dt - 2. \end{aligned}$$

Notice, first, that, with ϕ increasing, it is possible to have $\gamma^0(c)$ increasing and Γ^1 decreasing: this corresponds to a situation in which, without audit, the procurer would find it optimal to allocate the project randomly, but, if a mechanism with audit is adopted, then the procurer finds it optimal to allocate it monotonically, starting from the firm with the lowest cost. Second, the shape of Γ^1 depends on the auditing cost a but is unaffected by the value of the project v. Finally, observe that Γ^1 will typically be decreasing when ϕ is relatively flat and bounded away from zero, so that the ratio ϕ'/ϕ is never too high.

However, the decreasingness of Γ^1 is not enough, because it is not guaranteed that procurer really wants to go on auditing all the firms. This is certainly the case when $a < (1-\phi(c))d$ for all c, i.e. when the cost of auditing a certain firm i is always lower than the direct benefit from auditing $(1-\phi(c_i))d$ (the procurer avoids the default cost if it discovers that the firm will not perform), regardless of the potential indirect benefit that can possibly accrue from reassigning the project to another firm. Notice that, since ϕ is assumed to be everywhere increasing, this amounts to the requirement that $a/d < (1-\phi(\overline{c}))$, a condition that can possibly be satisfied only if $\phi(\overline{c}) < 1$, i.e.

only if no firm, even one with very high cost, is guaranteed to finalize the project. On the other hand, if $a > (1 - \phi(c))d$ for some c, then, at some round t of the assignment process, the procurer may find it optimal to stop auditing firms. At this point, a standard mechanism (with audit) assigns the project definitively to the firm with the lowest cost among the eligible ones, something that is indeed optimal only $\gamma^0(c_t)$ – the virtual surplus from definitively assigning the project to firm t – is higher than the virtual surplus of all other remaining firms, i.e. only if γ_0 is decreasing.

Hence, we can conclude that the sufficient conditions of Propositions 3.1 and 3.2 are lielky to be satisfied when

(a) either ϕ is relatively flat and bounded away from zero (so that the ratio ϕ'/ϕ is never too high), and, at the same time, the cost of auditing is small (per se and with the respect to the default cost);

(b) or ϕ is relatively flat and bounded away from zero (so that the ratio ϕ'/ϕ is never too high), and, at the same time, the value of the project and the default cost are not too large (so that γ_0 is decreasing).

Propositions 4.1 and 4.2 deals with the opposite situations considerd in Propositions 3.1 and 3.2. In such circumstances, a standard mechanism with audit will typically be suboptimal, and the procurer would be better off by using a mechanism in which firms may still be audited but in which, at least at some round of assignment, the project is allocated randomly. For example, if Γ^1 is decreasing but, for some c, $a > (1 - \phi(c))d$ and γ_0 is increasing, then the procurer would find it optimal to start assigning the project monotonically to the firms and auditing them, but only up to same point; at that point, the project should definitively be assigned randomly to one of the remaining firms. If, instead, Γ^1 is increasing, this does not imply that the procurer should never audit firms. Rather, the project should be assigned randomly from the very first round, but still the procurer may find it convenient to audit the firm that has been selected and, if the outcome of the audit is negative, possibly reassign the project, again randomly, to another firm.

6 Conclusion

Among the various remedies suggested to limit the risk of contractor's default in procurement, this paper provides a formal analysis of mechanisms with audit, i.e. mechanisms in which, before determining the final allocation of the tendered project, the procurer can sequentially audit firms, allocating the project to the firm that, on the basis of the auditing, appears to guarantee the completion of the work. Our interest was motivated by the fact that, in the European Union, the public procurement law dictates that, for contracts above a certain value, abnormally low bids in first-price auctions can be disregarded only after auditing the firms that made those bids. On the other hand, there are many Contracting Authorities, both in European (for contracts below the EU threshold) and non-European countries, that still adopt an average bid awarding rule, according to which abnormally low bids, i.e. bids that are below a certain average of all submitted bids, are authomatically eliminated, without any further inquiry. The co-existence of these two different approaches to deal with abnormally low tenders, led us to question the effectiveness of both.

Using a model in which firms are heterogeneous with respect to their technologies, and the risk of default is correlated with the technology adopted, we first showed that, when no audit is envisaged, an average bid auction (i.e. a lottery) can indeed be optimal. We then introduced audit and provided sufficient conditions for a standard mechanism with audit (the mechanism design equivalent of the first-price auction with audit suggested by the European Union) to be the best compromise between procurement cost and risk of contractor's default.

These conditions are useful to assess whether the European provision of imposing audit only for contracts of high value is justifiable. Our results clearly indicate that the optimality of a first-price auction with audit is unaffected by the value of the tendered project. However, not surprisingly, it crucially depends on the costs of the auditing process (which include the cost of the people – engineers, accountants, etc. – in charge of the audit, plus administrative and legal costs) and on its benefits, given by the (social) loss associated with the contractor's default, which audit allows to avoid (or at least to limit). It seems reasonable to believe that the auditing costs should be lower for larger CAs, which are likely to have skilled and dedicated personnel to conduct the audit and more resources to face potential legal disputes. Moreover, it is also likely to be the case that the auditing conducted by a large CA is more precise and effective than the auditing conducted by a small one in identifying whether the firm will indeed be able to complete the project: in this sense, also the expected benefit of audit should be larger for large CAs. These considerations seem to support the EU approach to impose audit only for large procurement projects.

However, our results show that the optimality of a standard mechanisms with audit not only depends on the direct cost-benefit balance of audit, but also, crucially, on the relation between firms' costs and default risk. In particular, a first-price auction with audit can be optimal only if this relation is not too steep (i.e. firms are not too heterogeneous in terms of default risk) and, moreover, the probability of default is not too high. Hence, any other instrument that can affect such a relation in this direction should ideally be adopted together with this mechanism. In the opposite case, the procurer would be better off by using a mechanism that involves some randomness in the allocation process, still possibly auditing the firm that was (randomly) selected in the first place.

Appendix

A Proof of Proposition 2

We proceed as follows: we disregard the IC constraint (namely that $\Pi_i(c_i)$ must be weakly decreasing) and we show that the stated allocation *pointwise* maximizes the procurer's objective function, i.e. it maximizes

$$V(\mathbf{c}) = \sum_{i=1}^{n} \gamma_i^{\overline{\theta}}(c_i) \cdot \pi_i(\mathbf{c}) \tag{10}$$

for every profile of costs $\mathbf{c}$, under the feasibility constraints

$$0 \leq \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \leq 1, \qquad \sum_{i=1}^{n} \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \leq 1.$$

We will check ex-post that the IC constraint is indeed satisfied. Again, it is useful to rewrite (10) in a recursive manner: in this framework, (7) becomes

$$V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = \sum_{i \in N \setminus \mathbf{b}^{[t-1]}} \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \left[\gamma_i^{\overline{\theta}}(c_i) + \overline{\theta}\,(1-\phi(c_i))\, V^{(t+1|\mathbf{b}^{[t-1]}\cup\{i\})}(\mathbf{c})\right], \tag{11}$$

and the maximand (10) is simply $V(\mathbf{c}) = V^{(1)}(\mathbf{c})$. Notice that the mechanism that maximizes (10) must be such that, whenever $\rho^{(t|\mathbf{b}^{[t-1]})} > 0$ (i.e. there is a strictly positive probability of reaching round t of assignment after a sequence of provisional assignments $\mathbf{b}^{[t-1]}$), (11) must be maximized.

The mechanism in Proposition 2 can be written as follows: for all $t = 1, \ldots, n$, and for every sequence $\mathbf{b}^{[t-1]}$ of $(t-1)$ different firms,

$$\begin{cases} \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 0 & \text{if} \quad i \notin M(\mathbf{b}^{[t-1]}) \quad \text{or} \quad \Gamma_i(c_i) < 0 \\ \sum_{i \in M(\mathbf{b}^{[t-1]})} \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 1 & \text{if} \quad \Gamma_i(c_i) \geq 0 \quad \text{for} \quad i \in M(\mathbf{b}^{[t-1]}) \end{cases}. \tag{12}$$

To show that the above mechanism is indeed optimal, we will proceed by induction, i.e. we will show that, if, in all rounds $t' > t$, with $t < n$, and for all $\mathbf{b}^{[t'-1]}$, the allocation corresponds to (12), then the optimal allocation in round t also corresponds to (12).

Now, consider a certain round $t < n$ and a certain sequence $\mathbf{b}^{[t-1]}$ of $(t-1)$ different firms. The set of eligible firms in round t is $N \setminus \mathbf{b}^{[t-1]}$. It is useful to rename such firms according to their ranking with respect to the value of the function $\Gamma_i^{\overline{\theta}}$: let $N \setminus \mathbf{b}^{[t-1]} = \{1, 2, \ldots, n-t+1\}$, where $\Gamma_i(c_i) \geq \Gamma_j(c_j)$ for all

$i, j \in N \setminus \mathbf{b}^{[t-1]}$. Finally, let $0 \le k \le n - t + 1$ be the number of firms in $N \setminus \mathbf{b}^{[t-1]}$ for which $\gamma_i^{\overline{\theta}}(c_i) \ge 0$ (i.e. the first k firms in $N \setminus \mathbf{b}^{[t-1]}$). Notice finally that $\gamma_i^{\overline{\theta}}(c_i)$ and $\Gamma_i^{\overline{\theta}}(c_i)$ have always the same sign.

Clearly, if $\rho^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 0$, then any allocation $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ is optimal, including the allocation corresponding to (12). So, consider the case $\rho^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) > 0$. Notice, first, that, if $\gamma_i^{\overline{\theta}}(c_i) < 0$ for all $i \in N \setminus \mathbf{b}^{[t-1]}$ (i.e. $k = 0$), then $V^{(t+1|\mathbf{b}^{[t-1]} \cup \{i\})} = 0$ for all $i \in N \setminus \mathbf{b}^{[t-1]}$, and it is clearly optimal to set $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 0$ for all $i \in N \setminus \mathbf{b}^{[t-1]}$, as stated in (12). Suppose, instead, that there is some $i \in N \setminus \mathbf{b}^{[t-1]}$ such that $\gamma_i^{\overline{\theta}}(c_i) \ge 0$ (i.e. $k > 0$), and that, contrary to (12), $\pi_j^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) > 0$ for some $j \notin M(\mathbf{b}^{[t-1]})$. We show that this cannot be optimal, in the sense that a decrease in $\pi_j^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ by ε and a corresponding increase in $\pi_1^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ by ε would increase the value of $V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$. In fact, this would change the value of $V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ by (again, we omit the dependence from $\mathbf{c}$ and we write ϕ_j for $\phi(c_j)$)

$$\varepsilon \cdot \left[\gamma_1^{\overline{\theta}} - \gamma_j^{\overline{\theta}} + \overline{\theta}\,(1-\phi_1)\, V^{(t+1|\mathbf{b}^{[t-1]} \cup \{1\})} - \overline{\theta}\,(1-\phi_j)\, V^{(t+1|\mathbf{b}^{[t-1]} \cup \{j\})}\right]. \tag{13}$$

Now, if $j > k$ (i.e. $\gamma_j^{\overline{\theta}} < 0$), (13) reduces to

$$\varepsilon \cdot \left[\left(1 - \overline{\theta}\,(1-\phi_j)\right)\left(\gamma_1^{\overline{\theta}} + \sum_{i=2}^{k} \overline{\theta}^{\,i-1}\,(1-\phi_1)\cdot\ldots\cdot(1-\phi_{i-1})\,\gamma_i^{\overline{\theta}}\right) - \gamma_j^{\overline{\theta}}\right],$$

which is strictly positive (because $\gamma_j^{\overline{\theta}} < 0$ whereas all $\gamma_i^{\overline{\theta}}$, $i = 1, \ldots, k$ are positive). If, instead, $j > k$ (i.e. $\gamma_j^{\overline{\theta}} < 0$), (13) reduces to

$$\varepsilon \cdot \left[\left(1 - \overline{\theta}\,(1-\phi_j)\right)\left(\gamma_1^{\overline{\theta}} + \sum_{i=2}^{j-1} \overline{\theta}^{\,i-1}\,(1-\phi_1)\cdot\ldots\cdot(1-\phi_{i-1})\,\gamma_i^{\overline{\theta}}\right) + \left(\overline{\theta}^{\,j-1}\,(1-\phi_1)\cdot\ldots\cdot(1-\phi_{j-1}) - 1\right)\gamma_j^{\overline{\theta}}\right].$$

The above expression is positive if

$$\left(1 - \overline{\theta}\,(1-\phi_j)\right)\left(\gamma_1^{\overline{\theta}} + \sum_{i=2}^{j-1} \overline{\theta}^{\,i-1}\,(1-\phi_1)\cdot\ldots\cdot(1-\phi_{i-1})\,\gamma_i^{\overline{\theta}}\right) \ge \left(1 - \overline{\theta}^{\,j-1}\,(1-\phi_1)\cdot\ldots\cdot(1-\phi_{j-1})\right)\gamma_j^{\overline{\theta}}.$$

Observe that, since firms' indexes are ordered according to the value of $\Gamma_i^{\overline{\theta}}$, it is $\gamma_i^{\overline{\theta}}\,(1-\theta(1-\phi_j)) \ge \gamma_j\,(1-\theta(1-\phi_i))$ for all $i < j$. Therefore, the left-hand-side in the above inequality is certainly greater than

$$\left(\left(1-\overline{\theta}\,(1-\phi_1)\right) + \sum_{i=2}^{j-1}\left(1-\overline{\theta}\,(1-\phi_i)\right)\overline{\theta}^{\,i-1}\,(1-\phi_1)\cdot\ldots\cdot(1-\phi_{i-1})\right)\gamma_j^{\overline{\theta}} = \left(1 - \overline{\theta}^{\,j-1}\,(1-\phi_1)\ldots(1-\phi_{j-1})\right)\gamma_j^{\overline{\theta}},$$

which is exactly the right-hand-side of the same inequality. We thus conclude that a decrease in $\pi_j^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ by ε and a corresponding increase in $\pi_1^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ by ε does always increase the value of $V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$, which implies that it is optimal to set $\pi_j^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 0$ for all $j \notin M(\mathbf{b}^{[t-1]})$. Finally, after noticing that, whenever $\gamma_i^{\overline{\theta}}(c_i) \ge 0$, an increase in $\pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$ is always profitable, we conclude that, whenever $\gamma_i^{\overline{\theta}}(c_i) \ge 0$ for $i \in M(\mathbf{b}^{[t-1]})$, it must be $\sum_{i \in M(\mathbf{b}^{[t-1]})} \pi_i^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 1$.

We have thus shown that, if, in all rounds $t' > t$ and for all $\mathbf{b}^{[t-1]}$, the allocation corresponds to (12), then the optimal allocation in round $t < n$ also corresponds to (12). To show that (12) is indeed optimal, we just need to make sure that the allocation corresponding to (12) is indeed optimal in the last round, i.e. when only one eligible firm is left, which is quite obvious.

Finally, the assumption that $\Gamma_i^{\overline{\theta}}$ is decreasing guarantees that, for all i and for all $\mathbf{c}_{-i}$, $\pi_i(c_i, \mathbf{c}_{-i})$ – the probability that firm i is assigned the project throughout the assignment process, is weakly decreasing: hence, (12) satisfies IC in the sense that it generates a weakly decreasing function $\Pi_i(c_i)$. This completes the proof. □

B Proof of Corollary 1

1. Vickrey Auction. Notice that a firm that is assigned the project in round t is payed, upon completion of the work, an amount that is a weighted average among r and the bids that are above that firm's bid and below r. First, we show that truthtelling is an (ex post) equilibrium of this auction. Now, consider a firm with cost c and denote this firm's bid by b. Suppose the other firms bid truthfully (each firm's bid is equal to its cost), and let $(c^{(1)}, c^{(2)}, \ldots, c^{(n-1)})$ be the vector of the other $n-1$ firms' costs/bids in increasing order.

Consider, first, the case $c > r$.[13] If the firm bids truthfully ($b = c$), the firm will be assigned the project with probability zero, so its profit will be equal to zero, as it will be with any other bid greater than r. To have a positive probability of being assigned the project, the firm should bid $b < c$, but in this case the payment received (upon completion) will be no greater than r, leading to a negative profit.

Consider now the case $c \leq r$. If this firm bids truthfully, its profit is positive: this firm is assigned the project with positive probability and is payed (upon completion) an amount that is no smaller than c. Let's verify that no deviation is profitable. Notice, first, that bidding $b > r$ is obviously non-profitable (the firm's profit would be equal to zero). Notice, also, that any deviation that does not alter the position of this firm in the ranking of bids, does not affect the firm's profit (the probability of being assigned the project is unchanged, as it is the payment, which only depends on the bids of the firms that follow in the ranking). Hence, we have to focus on deviations that change the position of the firm in the ranking of bids. Now, suppose, that $c^{(t-1)} < c < c^{(t)}$, i.e. if this firm bids truthfully it is ranked t-th in the ranking of bids. Its profit under truthtelling is

$$u(b=c) = \overline{\theta}^{t-1} \prod_{i=1}^{t-1} (1-\phi(c^{(i)}))\phi(c) \left\{ \left[1 - \overline{\theta}\left(1-\phi(c^{(t)})\right)\right] \min\left\{c^{(t)}, r\right\} + \overline{\theta}\left(1-\phi(c^{(t)})\right) p^{(t)} - c \right\}$$

A downward deviation to $b' \in (c^{(t-2)}, c^{(t-1)})$[14] would make this firm rank $(t-1)$-th and would change its profit to

$$\begin{aligned} u(b') &= \overline{\theta}^{t-2} \prod_{i=1}^{t-2} (1-\phi(c^{(i)}))\phi(c) \left\{ \left[1 - \overline{\theta}\left(1-\phi(c^{(t-1)})\right)\right] c^{(t-1)} + \overline{\theta}\left(1-\phi(c^{(t-1)})\right) \times \right. \\ &\quad \left. \times \left\{ \left[1 - \overline{\theta}\left(1-\phi(c^{(t)})\right)\right] \min\left\{c^{(t)}, r\right\} + \overline{\theta}\left(1-\phi(c^{(t)})\right) p^{(t)} \right\} - c \right\}. \end{aligned}$$

It is then immediate to verify that

$$u(b=c) - u(b') = \overline{\theta}^{t-2} \prod_{i=1}^{t-2} (1-\phi(c^{(i)}))\phi(c) \left\{ \left[1 - \overline{\theta}\left(1-\phi(c^{(t-1)})\right)\right] \left[c - c^{(t-1)}\right] \right\} \geq 0,$$

i.e. a downward deviation to $b' \in (c^{(t-2)}, c^{(t-1)})$ is not profitable. One could then recursively apply this argument to show that any other downward deviation (e.g. a deviation to $b' \in (c^{(t-3)}, c^{(t-2)})$) is not profitable either.

An upward deviation to $b'' \in (c^{(t)}, \min\{c^{(t+1)}, r\})$[15] would make this firm rank $(t+1)$-th and would change its profit to

$$u(b'') = \overline{\theta}^{t} \prod_{i=1}^{t} (1-\phi(c^{(i)}))\phi(c) \left\{ \left[1 - \overline{\theta}\left(1-\phi(c^{(t+1)})\right)\right] \min\left\{c^{(t+1)}, r\right\} + \overline{\theta}\left(1-\phi(c^{(t+1)})\right) \times p^{(t+1)} - c \right\}.$$

It is then immediate to verify that

$$u(b=c) - u(b'') = \overline{\theta}^{t-1} \prod_{i=1}^{t-1} (1-\phi(c^{(i)}))\phi(c) \left\{ \left[1 - \overline{\theta}\left(1-\phi(c^{(t)})\right)\right] \left[c^{(t)} - c\right] \right\} \geq 0,$$

i.e. an upward deviation to $b'' \in (c^{(t)}, \min\{c^{(t+1)}, r\})$ is not profitable. One could then recursively apply this argument to show that any other upward deviation (e.g. a deviation to $b'' \in (c^{(t+1)}, \min\{c^{(t+2)}, r\})$) is not profitable either.

[13] Clearly, this case is possible only if there is a reserve price, i.e. only if there exists $\hat{c} \in [\underline{c}, \overline{c}]$ such that $\gamma^{\overline{\theta}}(\hat{c}) = 0$ (in which case the reserve price is $r = \hat{c}$).

[14] Clearly, such a deviation is feasible only for $c > c^{(1)}$.

[15] Clearly, such a deviation is feasible only for $c^{(t)} < r$.

We have thus shown that the Vickrey Auction is an incentive compatible direct mechanism. After noticing that the Vickrey Auction has the same allocation rule as the optimal mechanism in Proposition 2, and that, in both mechanism, $Z_i(\overline{c}) = 0$ for all i, revenue equivalence guarantees that the Vickrey Auction is optimal.

2. Pay-your-bid Auction. We claim that the following is the (symmetric) equilibrium bidding function of this auction: firms with $c > r$ bid $\beta(c) = c$; firms with $c \leq r$ bid according to

$$\beta(c) = \frac{\sum_{t=1}^{n-1} \mathbb{E}\left[(1-\nu^{(t)}) \min\{c^{(t)}, r\} \prod_{j=0}^{t-1} \nu^{(j)} \middle| c^{(t)} > c\right] \cdot \Pr(c^{(t)} > c) + \mathbb{E}\left[r \prod_{j=1}^{n-1} \nu^{(j)} (1-\nu^{(t)})\right]}{\sum_{t=1}^{n-1} \mathbb{E}\left[(1-\nu^{(t)}) \cdot \prod_{j=0}^{t-1} \nu^{(j)} \middle| c^{(t)} > c\right] \cdot \Pr(c^{(t)} > c) + \mathbb{E}\left[\prod_{j=1}^{n-1} \nu^{(j)} (1-\nu^{(t)})\right]}, \tag{14}$$

where, for $t = 1, \ldots, n-1$, $c^{(t)}$ is the t-th lowest out of $n-1$ draws from the distribution G, $\nu^{(t)} = \overline{\theta}(1 - \phi(c^{(t)}))$, and $\nu^{(0)} = 1$. Notice that $\beta(r) = r$ and that, for $c < r$, $\beta(c) > c$. Notice also that, when there is no audit, i.e. $\overline{\theta} = 0$, (14) reduces to $\mathbb{E}[\min\{c^{(1)}, r\} | c^{(1)} > c]$, which is indeed the equilibrium of a standard first-price auction (with reserve price r).

It is trivial to see that, for firms with $c > r$, it is optimal to bid $\beta(c) = c$: doing so, their probability of being assigned the project (and hence their expected profit) is zero, whereas a bid below r would yield a strictly negative expected profit. Similarly, for a firm with $c = r$ it is optimal to bid $\beta(r) = r$: doing so, its expected profit is zero, whereas a bid below r would yield a strictly negative expected profit.

To see that, for firms with $c < r$, (14) is an equilibrium, observe preliminarily that (14) is strictly increasing in c (this can be shown formally, although we know that, given the rules of the auction, the equilibrium must necessarily be strictly increasing). Hence, if all firms follow the strategy $\beta(\cdot)$, the auction allocates the project in the various rounds according to the ranking in terms of costs, starting from the lowest cost firm. Given this, one can easily see that the denominator in (14) is the probability that a firm that bids $\beta(c)$ $(c < r)$ is assigned the project throughout the auction, when all other firms follow $\beta(\cdot)$.

Now, consider a certain firm with cost $c < r$, and suppose that all other firms bid according to $\beta(\cdot)$. Clearly, for this firms making a bid above r cannot be optimal (its expected profit would be zero). On the other hand, suppose this firm bids $\beta(z) < r$ (i.e. it mimics type $z < r$): its expected profit is

$$\begin{aligned}
U(z; c) &= \mathrm{PW}(z) \times (\beta(z) - c) \\
&= \sum_{t=1}^{n-1} \mathbb{E}\left[(1-\nu^{(t)}) \min\{c^{(t)}, r\} \prod_{j=0}^{t-1} \nu^{(j)} \middle| c^{(t)} > z\right] \cdot \Pr(c^{(t)} > z) + \mathbb{E}\left[r \prod_{j=1}^{n-1} \nu^{(j)} (1-\nu^{(t)})\right] \\
&\quad -c \left\{\sum_{t=1}^{n-1} \mathbb{E}\left[(1-\nu^{(t)}) \cdot \prod_{j=0}^{t-1} \nu^{(j)} \middle| c^{(t)} > z\right] \cdot \Pr(c^{(t)} > z) + \mathbb{E}\left[\prod_{j=1}^{n-1} \nu^{(j)} (1-\nu^{(t)})\right]\right\} \\
&= \sum_{t=1}^{n-1} \mathbb{E}\left[(1-\nu^{(t)}) \left(\min\{c^{(t)}, r\} - c\right) \prod_{j=0}^{t-1} \nu^{(j)} \middle| c^{(t)} > z\right] \cdot \Pr(c^{(t)} > z) \\
&\quad +\mathbb{E}\left[(r-c) \prod_{j=1}^{n-1} \nu^{(j)} (1-\nu^{(t)})\right]
\end{aligned}$$

It follows that

$$\begin{aligned}
U(c; c) - U(z; c) &= \sum_{t=1}^{n-1} \mathbb{E}\left[(1-\nu^{(t)}) \left(\min\{c^{(t)}, r\} - c\right) \prod_{j=0}^{t-1} \nu^{(j)} \middle| c^{(t)} > c\right] \cdot \Pr(c^{(t)} > c) \\
&\quad - \sum_{t=1}^{n-1} \mathbb{E}\left[(1-\nu^{(t)}) \left(\min\{c^{(t)}, r\} - c\right) \prod_{j=0}^{t-1} \nu^{(j)} \middle| c^{(t)} > z\right] \cdot \Pr(c^{(t)} > z),
\end{aligned}$$

which, for $c < z < r$, equals

$$\sum_{t=1}^{n-1} \mathbb{E}\left[(1-\nu^{(t)}) \left(\min\{c^{(t)}, r\} - c\right) \prod_{j=0}^{t-1} \nu^{(j)} \middle| c < c^{(t)} < z\right] \cdot \Pr(c < c^{(t)} < z) \geq 0,$$

while, for $z < c$, equals

$$-\sum_{t=1}^{n-1} \mathbb{E}\left[(1-\nu^{(t)}) \left(\min\{c^{(t)}, r\} - c\right) \prod_{j=0}^{t-1} \nu^{(j)} \middle| z < c^{(t)} < c\right] \cdot \Pr(z < c^{(t)} < c) \geq 0.$$

We have thus shown that $\beta(\cdot)$ is an equilibrium of the Pay-your-bid Auction. The fact that (14) is strictly increasing implies that the Pay-your-bid Auction is equivalent to a direct mechanism with the same allocation rule as the optimal mechanism in Proposition 2. Moreover, in both mechanism, $Z_i(\bar{c}) = 0$ for all i. Then revenue equivalence guarantees that the Pay-your-bid Auction is optimal. □

C Proof of Proposition 3.1

We proceed like in the proof of Proposition 2: we disregard the IC constraint and, under the assumptions stated in this Proposition, we characterize the mechanism with audit that pointwise maximizes the procurer's objective function (4) under the feasibility constraints (6), showing that such a mechanism is indeed a standard mechanism with audit. We then check ex-post that the optimal mechanism just characterized does also satisfy IC.

Notice, preliminarily, that, thanks to the linear structure of the objective function, the above maximization problem has one solution in which, in any round of assignment t that can be reached with strictly positive probability:

(i) the project is either not assigned at all (i.e. $\pi_i^{(t|\mathbf{b}^{[t-1]})} = 0$ for all $i \in N \setminus \mathbf{b}^{[t-1]}$), or it is assigned with certainty to one of the eligible firms (i.e. $\pi_i^{(t|\mathbf{b}^{[t-1]})} = 1$ for some $i \in N \setminus \mathbf{b}^{[t-1]}$), and

(ii) the firm that is provisionally assigned the project (i.e. the only firm i such that $\pi_i^{(t|\mathbf{b}^{[t-1]})} = 1$) is either audited for sure ($\theta_i^{(t|\mathbf{b}^{[t-1]})} = 1$) or it is not audited ($\theta_i^{(t|\mathbf{b}^{[t-1]})} = 0$).

We will concentrate on a solution with such structure.

To see that there is one solution that satisfies conditions *(i)* and *(ii)* above, observe, first, that, in any round t, for all audit probabilities $\theta_i^{(t|\mathbf{b}^{[t-1]})}$, it is optimal to provisionally assign the project with probability one to the eligible firm for which the expression within curly brackets in (7) is maximal, provided this quantity is positive, and not to assign the contract if this expression is negative; and, if there is more than one firm for which the expression within curly brackets in (7) is maximal and positive, it is optimal to assign the contract with certainty to any of such firms. This means that, for given $\mathbf{c}$, the sequence of firms that can be assigned the project throughout the allocation process is uniquely determined, so that any round t can be reached uniquely after a specific history $\mathbf{b}^{[t-1]}$; moreover, in that round t, a specific firm i is assigned the project with probability one, and, conversely, firm i can be assigned the project only in that specific round t (if reached). This implies that the probability that firm i is audited is relevant only in that unique round, whereas it is immaterial in all other rounds. Hence, among all the audit probabilities $\theta_i^{(t|\mathbf{b}^{[t-1]})}$, we can concentrate only on the one associated with $\pi_i^{(t|\mathbf{b}^{[t-1]})} = 1$. We can thus neglect the dependence of $\theta_i^{(t|\mathbf{b}^{[t-1]})}$ from t and $\mathbf{b}^{[t-1]}$, and simply write $\tilde{\theta}_i$ to denote the probability that firm i is audited in the unique round in which firm i is assigned the project. From (7), it is immediate to see that, if the allocation prescribes that firm i is assigned the project (only) in period t (i.e. $\pi_i^{(t|\mathbf{b}^{[t-1]})} = 1$), then it is optimal to set $\tilde{\theta}_i = 1$ if

$$(1 - \phi(c_i)) \left(d + V^{(t+1|\mathbf{b}^{[t-1]} \cup \{i\})}(\mathbf{c}) \right) - a \geq 0, \tag{15}$$

in which case (7) reduces to

$$\begin{aligned} V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) &= \gamma^0(c_i) + (1 - \phi(c_i))\, d - a + (1 - \phi(c_i))\, V^{(t+1|\mathbf{b}^{[t-1]} \cup \{i\})}(\mathbf{c}) \\ &= \gamma^1(c_i) + (1 - \phi(c_i))\, V^{(t+1|\mathbf{b}^{[t-1]} \cup \{i\})}(\mathbf{c}). \end{aligned}$$

If, on the contrary, (15) is not satisfied, it is optimal to set $\tilde{\theta}_i = 0$, in which case (7) simply reduces to $V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = \gamma^0(c_i)$.

Notice that, in the solution, it must necessarily be that $V^{(t|\mathbf{b}^{[t-1]}} \geq 0$ for all t (the procurer can always set this quantity to zero by simply not assigning the project in that period). Hence, if $\gamma^1(c_i) - \gamma^0(c_i) = (1 - \phi(c_i))\, d - a \geq 0$, condition (15) is certainly satisfied, i.e. it is certainly optimal to audit firm i.

Now, it is immediate to observe that a firm with cost below $\hat{c}$ must be audited when provisionally assigned the contract (for such firm $\gamma^1 > \gamma^0$). It is also quite intuitive to see that, if there are firms with cost above $\hat{c}$, these firms should never be assigned the project during the assignment process. Suppose, to the contrary, that there is one firm i with cost $c_i > \hat{c}$ that is assigned the project at some round t (that can be reached with strictly positive probability). If t is the last round in which the project is assigned to a firm, then the continuation payoff is $V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = \max\left\{\gamma^1(c), \gamma^0(c)\right\}$, which is strictly negative, so it would be better stop assigning the project in round $(t-1)$. If round t is not the last round in which the project is assigned to a firm, this means that $\tilde{\theta}_i = 1$ and that, in the next round, the project is assigned (with audit) to a firm j with cost $c_j < \hat{c} < c_i$. But this cannot be optimal, because it would certainly be profitable for the procurer to switch the order, assigning the project first to firm j and the to firm i: this would postpone (i.e. reduce the probability of getting) a negative contribution to the procurer's payoff ($\gamma^1(c_i) \leq 0$) and anticipate (i.e. increase the probability of getting) a positive contribution to the procurer's payoff ($\gamma^1(c_j) \geq 0$). Hence, only firms with cost below $\hat{c}$ are assigned the project during the allocation process, and since it is optimal to always audit these firms, we can directly invoke Proposition 2 (with $\overline{\theta} = 1$) to conclude that a standard mechanism with audit (with all firms that are provisionally assigned the project being audited) does indeed pointwise maximize the procurer's objective function and it is incentive compatible (i.e. the resulting function $\Pi_i(c_i)$ is decreasing). This completes the proof. □

D Proof of Proposition 3.2

It is immediate to see that, if there are firms with cost above $\hat{c}_2$, these firms should never be assigned the project during the assignment process (the argument is the same as above). Hence, in what follows, we just need to focus on the firms with cost below $\hat{c}_2$. We have to consider two cases.

Case 1: there is no firm with cost $c \in (\hat{c}_1, \hat{c}_2)$. In this case, we are exactly in the same situation as in statement (a) above, so the same argument applies.

Case 2: there is at least one firm with cost $c \in (\hat{c}_1, \hat{c}_2)$. For illustration, consider the vector of firms' costs $\mathbf{c} = [c_1, c_2, \ldots, c_n]$ and suppose without loss of generality that the elements of $\mathbf{c}$ are in increasing order. This case corresponds to a situation in which the first k firms have costs $c \leq \hat{c}_1$, firms from $k+1$ to $k+j$ $(1 \leq j \leq n-k)$ have cost $c \in (\hat{c}_1, \hat{c}_2)$, the last $n-k-j$ firms have cost $c \geq \hat{c}_2$. First, it is immediate to see that the first k firms must be audited when provisionally assigned the project. Second, a mechanism in which all firms that are provisionally assigned the project are audited cannot be optimal. To see this, let s be the last round of assignment (i.e. round s can be reached with strictly positive probability, in round s one eligible firm is assigned the project, and, if round $(s+1)$ is reached, no firm is assigned the project anymore), and suppose, to the contrary, that, also in round s, the firm that is provisionally assigned the project is audited. If $s \leq k$, then there is still at least one eligible firm with $c \in (\hat{c}_1, \hat{c}_2)$ (i.e. with $\gamma^0(c) > 0$), and it would be better to move to round $(s+1)$, assigning the project in that round to such firm with no audit; if, instead, $k+1 \leq s \leq k+j$, then Proposition 2 implies that, in these s rounds with audit, the order of assignment should reflect the order of firms' costs: hence, the firm that is assigned the project in round s is certainly one with $c \in (\hat{c}_1, \hat{c}_2)$, for which it holds that $\gamma^0(c) > \gamma^1(c)$, so it would be better not to audit such firm. We conclude that the optimal mechanism: *(i)* has $(s-1)$ rounds of assignment with audit (with $s-1 \geq k$) and a last round (round s) in which a firm with $c \in (\hat{c}_1, \hat{c}_2)$ is assigned the project and not audited; *(ii)* in the first $(s-1)$ rounds (with audit), the order of assignment reflects the order of firms' costs; *(iii)* in round s, given that γ^0 is strictly decreasing, the project is assigned to the firm with the lowest cost among the eligible ones. It remains to show that the firm that is assigned the project in round s has a higher cost than any of the firms that are assigned the project in the first $(s-1)$ rounds. If $s = k+1$, this is obvious: the project is assigned to the k firms with $c \leq \hat{c}_1$ in the first k rounds, and to firm $s = k+1$, whose cost is $c \in (\hat{c}_1, \hat{c}_2)$, in the last round. If $s > k+1$, we just need to show that it is unprofitable to switch firm s and firm $(s-1)$ (notice that these two firms necessarily have both costs in $(\hat{c}_1, \hat{c}_2)$). In fact, if the project is assigned to firm $(s-1)$ in round $(s-1)$ and to firm s in round s, we have

$$V^{(s-1|\mathbf{b}^{[s-2]})}(\mathbf{c}) = \gamma^0(c_{s-1}) + (1-\phi(c_{s-1}))\, d - a + (1-\phi(c_{s-1}))\, \gamma^0(c_s),$$

whereas, if the project is assigned to firm s in round $(s-1)$ and to firm $(s-1)$ in round s, we have

$$V^{(s-1|\mathbf{b}^{[s-2]})}(\mathbf{c}) = \gamma^0(c_s) + (1-\phi(c_s))\, d - a + (1-\phi(c_s))\, \gamma^0(c_s).$$

The difference is

$$\phi(c_s)\left[\gamma^0(c_{s-1}) + d\right] - \phi(c_{s-1})\left[\gamma^0(c_s) + d\right] = \phi(c_s)\phi(c_{s-1})\left[\frac{\gamma^0(c_{s-1}) + d}{\phi(c_{s-1})} - \frac{\gamma^0(c_s) + d}{\phi(c_s)}\right],$$

which is strictly positive (this follows easily from the fact that, on $(\hat{c}_1, \hat{c}_2)$, γ^0 is strictly decreasing and strictly positive and ϕ is increasing). We thus conclude that, also in this case, a standard mechanism with audit does indeed pointwise maximize the procurer's payoff.

As a last step, we characterize the threshold, i.e. the value $\tilde{c}$ such that firms with cost lower than $\tilde{c}$ are audited (when provisionally assigned the project), while firms with cost higher than $\tilde{c}$ (and lower than $\hat{c}_2$) are not (i.e. they are definitively assigned the project without audit). This is equivalent to determine the last round of assignment. In case 1 considered above, the threshold is exactly $\hat{c}_1$. Now consider case 2: as before, let's say that the first k firms have costs $c \leq \hat{c}_1$, the next $j \geq 1$ firms have $c \in (\hat{c}_1, \hat{c}_2)$, the last $n-k-j$ firms have $c \geq \hat{c}_2$. The previous analysis has made it clear that, if $j = 1$, there will necessarily be $k+1$ rounds (and firm $k+1$ is not audited if assigned the project): in this case, again, $\tilde{c} = \hat{c}_1$. Consider, instead, the case of $j \geq 2$. Exploting the fact that, on $(\hat{c}_1, \hat{c}_2)$, ϕ is increasing and γ^0 is strictly decreasing, it is quite easy to verify that the last round of assignment is:

- $k+1$ if $a \geq (1-\phi(c_{k+1}))(\gamma^0(c_{k+2}) + d)$: in this case, $\tilde{c} = \hat{c}_1$;
- $k+l$, $1 < l < j$, if $(1-\phi(c_{k+l}))(\gamma^0(c_{k+l+1}) + d) \leq a < (1-\phi(c_{k+l-1}))(\gamma^0(c_{k+l}) + d)$: in this case, $\tilde{c} = c_{k+l-1}$;
- $k+j$ if $a < (1-\phi(c_{k+j-1}))(\gamma^0(c_{k+j}) + d)$: in this case, $\tilde{c} = c_{k+j-1}$.

Hence, the threshold depends on the number of firms with $c \in (\hat{c}_1, \hat{c}_2)$ and, given the number of such firms, on the actual realization of their costs.

Finally, it remains to check whether the mechanism is incentive compatible (i.e. the resulting function $\Pi(c)$ is decreasing). It is straightforward to see that, for given vector of costs of the other firms $\mathbf{c}_{-i}$ and *for fixed threshold* $\tilde{c}$, an increase in c_i (the cost of firm i) lowers the probability that firm i is assigned the project throughout the assignment process. But we have to take into account that the threshold is not fixed, i.e. an increase in c_i may change the threshold, thus affecting the probability that firm i is assigned the project. Notice, however, that the fact that γ^0 is strictly decreasing and ϕ increasing on $(\hat{c}_1, \hat{c}_2)$, implies that an increase in c_i can change the threshold only in a direction that is unfavorable to firm i (in the sense that, if anything, it reduces the probability that firm i is assigned the project). To see this, suppose that, given the realization of $\mathbf{c}$, the optimal threshold is $\tilde{c} = c_{k+l-1}$ (i.e. the last rouns is $k+l$). This means that

$$(1-\phi(c_{k+l}))(\gamma^0(c_{k+l+1}) + d) \leq a < (1-\phi(c_{k+l-1}))(\gamma^0(c_{k+l}) + d).$$

Consider firm $k+l$, whose cost is c_{k+l}: this firm is assigned the project in the last round. An increase in c_{k+l} reduces the right-hand-side above, and, when sufficiently high, this reduction may upset the inequality, leading to a reduction in threshold to $\tilde{c} = c_{k+l-2}$, so that firm $k+l$ is no longer assigned the project with strictly positive probability. On the other hand, consider firm $k+l+1$, whose cost is c_{k+l+1}: this firm is certainly not assigned the project, and to have the project assigned with strictly positive probability, it should be that $a < 1-\phi(c_{k+l}))(\gamma^0(c_{k+l+1}) + d)$. But this firm has nothing to gain from increasing her reported cost, as this would reduce, not increase, the left-hand-side above. This completes the proof. □

E Proof of Proposition 4.1

Let's first characterize M_S, the mechanism that maximizes the procurer's payoff among the incentive compatible mechanisms that allocate the project according to the (increasing) order of firms' costs. It is quite immediate to see that such a mechanism, whenever $\hat{n}(\mathbf{c}) \geq 1$, in each round $t \leq \hat{n}(\mathbf{c})$, allocates the project with probability one to the firm with the t-th lowest cost and audits this firm with probability one. To see this, let $c^{(t)}$ denote the t-th lowest element of the vector of firms' costs $\mathbf{c}$, and suppose we are in round $t \leq \hat{n}(\mathbf{c})$. The procurer's continuation payoff is

$$V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = \pi_t^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \left\{ \gamma^0(c^{(t)}) + \theta_t^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \left[\left(1-\phi(c^{(t)})\right) \left(d + V^{(t+1|\mathbf{b}^{[t-1]}\cup\{t\})}(\mathbf{c})\right) - a \right] \right\}.$$

To maximize $V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c})$, it must necessarily be that $V^{(t+1|\mathbf{b}^{[t-1]}\cup\{t\})} \geq 0$ (the procurer can always set this latter quantity to zero by simply not assigning the project in round $t+1$). Therefore, given that, by assumption, $\gamma^1(c^{(t)}) \geq \gamma^0(c^{(t)})$, i.e. $\left(1-\phi(c^{(t)})\right) d \geq a$, then it is certainly optimal to audit firm t with probability one. With $\theta_t^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 1$, the continuation payoff becomes

$$V^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = \pi_t^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) \left[\gamma^1(c^{(t)}) + \left(1-\phi(c^{(t)})\right) V^{(t+1|\mathbf{b}^{[t-1]}\cup\{t\})}(\mathbf{c}) \right],$$

and it is clearly optimal to set $\pi_t^{(t|\mathbf{b}^{[t-1]})}(\mathbf{c}) = 1$ (because, by assumption, $\gamma^1(c^{(t)}) \geq 0$). Notice that the way in which this mechanisms allocates in the first $\hat{n}(\mathbf{c})$ rounds (when $\hat{n}(\mathbf{c}) \geq 1$) cannot be improved upon because it pointwise maximizes the procurer's payoff in the first $\hat{n}(\mathbf{c})$ rounds, and, moreover, is indeed incentive compatible.

Now, consider the *random* mechanism with audit M_R: this mechanism differs from M_A only when $\hat{n}(\mathbf{c}) \geq 2$, so let's focus on this case. The fact that $\Gamma^1(c)$ is strictly increasing for all $c \leq \hat{c}$ implies that this mechanism yields a strictly higher payoff to the procurer than the best standard mechanism with audit. The argument is exactly the same as the one used in the proof of Proposition 2, but reversed. Specifically, with $\Gamma^1(c)$ strictly increasing on $(\underline{c}, \hat{c})$, allocating the project, in the first $\hat{n}(\mathbf{c})$ rounds, according to the increasing order of firms' costs yields a strictly lower payoff to the procurer than following any other ordering. Hence, since the random mechamism with audit, in the first $\hat{n}(\mathbf{c})$ rounds, attaches the same probability to any ordering, it is certainly better than the best standard mechanism with audit. In all other circumstances, the two mechanisms are identical and yield the same payoff. Finally, notice that the random mechanism with audit preserves incentive compatibility: all firms with cost $c \leq \hat{c}$ have the same probability of being assigned the project throughout the allocation process. For the other firms, this probability is unaffected. Hence, if the best standard mechanism with audit was incentive compatible for all firms' types, also this random mechanism with audit is.

F Proof of Proposition 4.2

Let $c^{(t)}$ denote the t-th lowest element of the vector of firms' costs $\mathbf{c}$. Consider M_S, i.e. the (not necessarily incentive compatible) standard mechanism with audit that pointwise maximizes the expected payoff of the procurer. It works as follows:

(i) if $\hat{n}(\mathbf{c}) = n$, then, in the first n rounds, the project is assigned to the eligible firm with the lowest cost and this firm is audited (this follows from the fact that, for $c \leq \hat{c}$, $\max\left\{\gamma^1(c), \gamma^0(c), 0\right\} = \gamma^1(c)$, so that it is always optimal to assign the project and audit the firm);

(ii) if $\hat{n}(\mathbf{c}) = n-1$, then, in the first $(n-1)$ rounds, the project is assigned to the eligible firm with the lowest cost and this firm is audited; in round n, the project is assigned to firm n and this firm is not audited (this follows from the fact that it is always optimal to assign the project to a firm with cost lower than $\hat{c}$ and audit it, and it is also optimal to assign the project to the unique firm with cost higher than $\hat{c}$, but not to audit it because, for such firm, $\gamma^1 \leq \gamma^0$);

(iii) if $\hat{n}(\mathbf{c}) \leq n-2$, then, in the first $\hat{n}(\mathbf{c}) + s^*$ rounds, the project is assigned to the eligible firm with the lowest cost and this firm is audited; in round $\hat{n}(\mathbf{c}) + s^* + 1$, the project is assigned to the firm with the $(\hat{n}(\mathbf{c}) + s^* + 1)$-th lowest cost, and this firm is not audited, where $s^* = \arg\max_{s \in \{0,\ldots,n-\hat{n}(\mathbf{c})-1\}} W^s(c^{(\hat{n}(\mathbf{c})+1)}, c^{(\hat{n}(\mathbf{c})+2)}, \ldots, c^{(\hat{n}(\mathbf{c})+1+s)})$.

To understand case *(iii)*, notice that, from round $\hat{n}(\mathbf{c}) + 1$ onwards (i.e. when all the eligible firms have cost higher than $\hat{c}$), the procurer faces a trade-off: auditing, per se, generates more costs than benefits (because, for these firms, $\gamma^1 \leq \gamma^0$), but allows the procurer to possibly go to the next round, where it can get the corresponding continuation payoff. In practice, if round $\hat{n}(\mathbf{c}) + 1$ is reached, the procurer has to decide for how many additional rounds it is worthwhile to go on assigning the project (according to the ordering of firms' costs). In particular, if the procurer goes on for other s rounds, the additional payoff the procurer can get is W^s, so the procurer will choose s to maximize this quantity. Notice, that the mechanism described above is not necessarily incentive compatible for types above $\hat{c}$. To see this, consider case *(iii)* above, and suppose $\hat{n}(\mathbf{c}) = n-2$. In this case, we have

$$W^0(c^{(n-1)}) = \gamma^0(c^{(n-1)}),$$

and

$$W^1(c^{(n-1)}, c^{(n)}) = \gamma^0(c^{(n-1)}) - a + \left(1 - \phi(c^{(n-1)})\right)\left(d + \gamma^0(c^{(n)})\right),$$

and the last round of assignment will be $n-1$ or n depending on whether a is greater or lower than $\left(1 - \phi(c^{(n-1)})\right)\left(d + \gamma^0(c^{(n)})\right)$: hence, for given $c^{(n-1)}$, an increase in $c^{(n)}$ (i.e. an increase in $\gamma^0(c^{(n)})$ because γ^0 is strictly increasing for $c > \hat{c}$), may lead to passing from a situation in which the last round of assignment is $n-1$ (firm n has a null probability of being assigned the project) to a situation in which the last round of assignment is n (firm n has a strictly positive probability of being assigned the project).

Consider now M_R, which differs from M_A only in case *(iii)*, i.e. when $\hat{n}(\mathbf{c}) \leq n-2$. In this case, M_R prescribes that, in the first $\hat{n}(\mathbf{c})$ rounds, the allocation is as in M_A, but in any round $\hat{n}(\mathbf{c})+1 \leq t < n$, the project is assigned randomly, with equal probability, to any of the eligible firms, and, letting $i(t)$ be the firm that is drawn in round t and $c_{i(t)}$ its cost,

- if $\arg\max_{s=0,\dots,n-t}\left\{\overline{W}^s(c_{i(t)})\right\} \neq 0$, firm $i(t)$ is audited (and, if the outcome of the audit is negative, we move to the round $t+1$);
- if $\arg\max_{s=0,\dots,n-t}\left\{\overline{W}^s(c_{i(t)})\right\} = 0$, firm $i(t)$ is not audited (no other round takes place).

Finally, if round n is reached, the project is assigned to the only remaining firm which is not audited.

Notice, first, that this mechanism is incentive compatible also for types above $\hat{c}$. In fact, consider firm i with cost $c_i > \hat{c}$. Whenever all other firms' costs are below $\hat{c}$, clearly an increase in c_i does not affect the probability that firm i is assigned the project. Suppose, instead, that $\hat{n}(\mathbf{c}) \leq n-2$: then, in any round $t > \hat{n}(\mathbf{c})$ in which firm i is still eligible, all eligible firms (including firm i) are assigned the project with the same probability, and, moreover, if firm i is not drawn in round t, the probability of moving to the next round (where firm i may be possibly assigned the project) does not depend on firm i's cost (but only on the cost of the firm that was drawn in round t). Hence, we conclude that the expected probability of being assigned the project is constant for all $c > \hat{c}$.

Finally, we have to show that M_R yields a higher expected payoff to the procurer than M_A. Now, let $\mathbf{c}$ be the vector of firms' costs, where, again, $c^{(t)}$ denotes the t-th lowest element of this vector. Since the two mechanisms are equal in cases *(i)* and *(ii)* above, we concentrate on case *(iii)*, i.e. $\mathbf{c}$ is such that $\hat{n}(\mathbf{c}) \leq n-2$; and, since M_R and M_A allocate in the same way in the first $\hat{n}(\mathbf{c})$ rounds, we focus on the procurer's continuation payoff evaluated at round $\hat{n}(\mathbf{c})+1$. Now, under M_S, such continuation payoff is

$$V^{(\hat{n}(\mathbf{c})+1|[c^{(1)},\dots,c^{(\hat{n}(\mathbf{c}))}])}(\mathbf{c}) = \max_{s\in\{0,\dots,n-\hat{n}(\mathbf{c})-1\}} W^s(c^{(\hat{n}(\mathbf{c})+1)}, c^{(\hat{n}(\mathbf{c})+2)}, \dots, c^{(\hat{n}(\mathbf{c})+1+s)}).$$

Consider now M_R. In M_R, from round $\hat{n}(\mathbf{c})+1$ onwards, the allocation is made randomly. We can think as if the procurer makes a series of $n-\hat{n}(\mathbf{c})$ draws without replacement from the set $\{c^{(\hat{n}(\mathbf{c})+1)}, c^{(\hat{n}(\mathbf{c})+2)}, \dots, c^{(n)}\}$, i.e. the set containing the $n-\hat{n}(\mathbf{c})$ highest element of $\mathbf{c}$. Let $[c_1, c_2, \dots, c_{n-\hat{n}(\mathbf{c})}]$ the sequence of such draws, i.e. c_t is the t-th draw from the $\{c^{(\hat{n}(\mathbf{c})+1)}, c^{(\hat{n}(\mathbf{c})+2)}, \dots, c^{(n)}\}$. In practice, the procurer assigns the project to the firm with cost c_1 in round $\hat{n}(\mathbf{c})+1$, to firm with cost c_2 in round $\hat{n}(\mathbf{c})+2$ (if this round is reached), and so on. Hence, if $[c_1, c_2, \dots, c_{n-\hat{n}(\mathbf{c})}]$ is the realized sequence of (potential) random assignments from round $\hat{n}(\mathbf{c})+1$ onwards, the continuation payoff of the procurer is

$$V^{(\hat{n}+1|[c^{(1)},\dots,c^{(\hat{n})}])}(\mathbf{c}) = \begin{cases} W^0(c_1) & \text{if} \quad \arg\max_{s=0,\dots,n-\hat{n}-1}\left\{\overline{W}^s(c_1)\right\} = 0 \\ W^1(c_1,c_2) & \text{if} \quad \arg\max_{s=0,\dots,n-\hat{n}-1}\left\{\overline{W}^s(c_1)\right\} \neq 0 \\ & \text{and} \quad \arg\max_{s=0,\dots,n-\hat{n}-2}\left\{\overline{W}^s(c_2)\right\} = 0 \\ W^2(c_1,c_2,c_3) & \text{if} \quad \arg\max_{s=0,\dots,n-\hat{n}-1}\left\{\overline{W}^s(c_1)\right\} \neq 0 \\ & \text{and} \quad \arg\max_{s=0,\dots,n-\hat{n}-2}\left\{\overline{W}^s(c_2)\right\} \neq 0 \\ & \text{and} \quad \arg\max_{s=0,\dots,n-\hat{n}-3}\left\{\overline{W}^s(c_3)\right\} = 0 \\ \vdots & \\ W^{n-\hat{n}-1}(c_1,\dots,c_{n-\hat{n}}) & \text{if} \quad \arg\max_{s=0,\dots,n-\hat{n}-r}\left\{\overline{W}^s(c_r)\right\} \neq 0 \quad r < n-\hat{n} \end{cases}.$$

The condition stated in the proposition ensures that, for any given $\hat{n}$, the expected value of the latter continuation payoff is higher than the expected value of the former. □

References

Birulin, O. (2020). Optimality of simple procurement auctions. *International Journal of Industrial Organization 70*, 102610.

Birulin, O. and S. Izmalkov (2022). On advance payments in tenders with budget constrained contractors. *The RAND Journal of Economics 53*(4), 733–762.

Board, S. (2007). Bidding into the red: A model of post-auction bankruptcy. *The Journal of Finance 62*(6), 2695–2723.

Burguet, R., J.-J. Ganuza, and E. Hauk (2012). Limited liability and mechanism design in procurement. *Games and Economic Behavior 76*(1), 15–25.

Calveras, A., J.-J. Ganuza, and E. Hauk (2004). Wild bids. gambling for resurrection in procurement contracts. *Journal of Regulatory Economics 26*, 41–68.

Chang, W.-S., B. Chen, and T. C. Salmon (2015). An investigation of the average bid mechanism for procurement auctions. *Management Science 61*(6), 1237–1254.

Chillemi, O. and C. Mezzetti (2014). Optimal procurement mechanisms: bidding on price and damages for breach. *Economic Theory 55*(2), 335–355.

Conley, T. G. and F. Decarolis (2016). Detecting bidders groups in collusive auctions. *American Economic Journal: Microeconomics 8*(2), 1–38.

Decarolis, F. (2014). Awarding price, contract performance, and bids screening: Evidence from procurement auctions. *American Economic Journal: Applied Economics 6*(1), 108–132.

Decarolis, F. (2018). Comparing public procurement auctions. *International Economic Review 59*(2), 391–419.

Galavotti, S., L. Moretti, and P. Valbonesi (2018). Sophisticated bidders in beauty-contest auctions. *American Economic Journal: Microeconomics 10*(4), 1–26.

Krishna, V. (2009). *Auction theory.* Academic press.

Manelli, A. M. and D. R. Vincent (1995). Optimal procurement mechanisms. *Econometrica: Journal of the Econometric Society*, 591–620.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of operations research 6*(1), 58–73.

Spulber, D. F. (1990). Auctions and contract enforcement. *The Journal of Law, Economics, and Organization 6*(2), 325–344.

Waehrer, K. (1995). A model of auction contracts with liquidated damages. *Journal of Economic Theory 67*(2), 531–555.

Wan, Z. and D. R. Beil (2009). Rfq auctions with supplier qualification screening. *Operations Research 57*(4), 934–949.

Zheng, C. Z. (2001). High bids and broke winners. *Journal of Economic theory 100*(1), 129–171.