Vertical contracting and information spillover in Cournot competition

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Abstract

This paper revisits a standard model of Cournot competition with private (demand) information by introducing a common strategic input supplier. We characterize a unique equilibrium in which such information is endogenously flowed within a vertical chain by signaling and screening behaviors in vertical contracting. Under certain conditions, the equilibrium outcome coincides with the public information case, whereas otherwise, systemic quantity distortions arise for truthful information transmission. Compared to the competitive upstream market benchmark, the equilibrium in our model induces more dispersed equilibrium quantities and may generate higher consumer welfare at the cost of lowered vertical surplus.

JEL classification: D83, D86, L13, L14, L15.
Keywords: information transmission; screening; signaling ; vertical contracting.

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1 Introduction

It is well-known that private information and demand uncertainty play a crucial role in determining market outcomes in oligopoly. While the impact of such incomplete information varies across models and assumptions, it is thus one of the most important questions in IO literature whether and how information exchange or spillover between competing firms affects market outcomes and consumer welfare.

A common assumption in this literature is that the input market is competitive. That is, firms compete in quantities or prices taking the input price as exogenously given. This approach is tractable and appealing in some industries, but there are plenty of examples where competing firms interact with a common input supplier. For instance, the products sold in downstream retailers such as Target and Walmart are largely overlapped, and the market for semiconductors, which are used as input for smartphones and other electronic devices, is highly concentrated. This opens up the possibility that private information of each downstream firm is endogenously flowed into others through contracting with their common upstream supplier, which is not captured in the models assuming competitive input markets.

Motivated by this observation, we revisit a standard model of Cournot duopoly competition with private (demand) information by introducing a common strategic input supplier. In our model, each downstream firm receives a private demand signal and bilaterally contracts with a common upstream supplier. The upstream supplier has no priori private information, but it may learn the downstream firms' signals by observing which input contract is offered. In turn, each downstream firm may want to infer the rival's demand information by offering a contract that screens the information the upstream firm has learned from contracting with the rival.

Our main result is the characterization of a unique equilibrium in which such information spillover is perfect. That is, private information of each downstream firm is endogenously flowed within a vertical chain by signaling and screening behaviors in vertical contracting. Under certain conditions, the equilibrium outcome coincides with the public information case, whereas otherwise, systemic quantity distortions arise for truthful information transmission. Compared to the competitive upstream market benchmark, the equilibrium in our model induces more dispersed equilibrium quantities and may generate higher consumer welfare at the cost of lowered vertical surplus. These observations, therefore, suggest that more concentration in the upstream market could positively affect consumers in the downstream market.

1.1 Literature Review

Our paper is related to the common agency literature. Applications of a common agency game are extensive, including antitrust regulation, tacit collusion, multilateral lobbying, and vertical contracting (Bernheim & Whinston, 1986). Galperti (2015) considers a general mechanism design framework where each principal has private information and provides a delegation principle type of result, which is useful to characterize equilibrium allocations. Lima, Moreira & Verdier (2017), Lima & Moreira (2014), and Martimort & Moreira (2010) consider common agency games with privately informed principals in specific environments such as lobbying and public-good provision problems. We contribute to this literature by showing how downstream firms can use a common manufacturer to exchange relevant information in a vertically-related market. Our paper is also related to the mechanism design literature on an informed principal (Myerson, 1983; Maskin & Tirole, 1990, 1992; Mylovanov & Tröger, 2012). In contrast to these works, we consider multiple privately informed principals that strategically choose mechanisms to signal information to a common agent, whose private information is endogenously generated. In our equilibrium construction, the principals signal their information through the design of a mechanism while screening the agent's information at the same time.

In the context of vertical contracting, Marx & Shaffer (2007) analyze a model in which two competing retailers simultaneously make take-or-leave-it offers to their common upstream manufacturer. They show that a retailer can use upfront payments to exclude its rival in equilibrium. In a similar setup, Miklos-Thal, Rey & Vergé (2011) derive an opposite result by showing that this type of exclusion appears only when the contract cannot be contingent on whether the relationship is exclusive or not. While the main interests of these works are on the effects of competitive externalities between downstream firms on market outcomes, we abstract away from these effects by assuming that each downstream firm is a local monopolist. This allows us to focus on the issue of information transmission through a common upstream supplier, which has not been explored in the existing literature.[1]

Indeed, it is well understood that parties may use contracts as a screening or signaling device in asymmetric information supply chain frameworks (Kong, Rajagopalan & Zhang, 2017). For instance, Chu (1992) studies a model in which a supplier is the one that holds private information about the demand state of a new product. [2] We contribute to this literature by examining the use of contracts as a screening and signaling device simultaneously, which is necessary when supply contracts are secret.

Information sharing between competing firms has been extensively studied. In terms of information-sharing incentives Vives (1984) show that not sharing information is a dominant strategy for each firm in Cournot and then the market outcome is never optimal. Vives (1984) suggest that public policy should encourage information sharing. In a similar setup Clarke (1983) shows that there is never a mutual incentive for all firms to share information unless

[1] Lima (2021) study information transmission in vertical contracting when there is a common retailer. Unlike ours, however, they assume that (i) private information is held by upstream firms; (ii) it is verifiable; and (iii) the upstream firms have all the bargaining power. Also, his focus lies on comparing fixed fee contracts and RPM, while our paper studies the possibility of efficient collusion.

[2] See also Li (2002), Zhang (2002), Li & Zhang (2008), Gal-Or, Geylani & Dukes (2008), Shin & Tunca (2010), Jain, Seshadri & Sohoni (2011), Jain & Sohoni (2015), Anand & Goyal (2009), Shamir (2012), Shamir (2017), and Kong, Rajagopalan & Zhang (2013).

they may cooperate on actions once information has been shared.

In terms of efficiency, Vives (1984), analyzes a setting where firms receive signals about the uncertain demand and show that Bertrand competition is more efficient than Cournot competition, in the sense that the Bertrand equilibrium price (quantity) is lower (higher) than the Cournot equilibrium price. Vives (2002) studies effects of market power and incomplete information on a Cournot market. He shows that when the market is large enough, abstracting from market power provides a much better approximation than abstracting from private information, highlighting the important role of information on the competition outcome.

2 Model

We consider a setting with one upstream firm U and two downstream firms D_1 and D_2. The upstream firm supplies input to D_i, which is transformed to a final product and sold by D_i at zero distribution cost. The upstream firm's marginal production cost is constant at $c \geq 0$.

Market demand is subject to unobservable shock, but each downstream firm D_i receives a private signal $s_i \in \{h, l\}$ about the underlying demand, which we refer to as the firm's type. Given a pair of signals (s_1, s_2) and quantities (q_1, q_2), the inverse demand for D_i's product is

$$P\left(q_i, q_{-i}; s_i, s_{-i}\right) = \theta_{s_1 s_2} - q_i - \beta q_{-i},$$

where $\beta \in (0, 1]$ captures the degree of product differentiation. The "demand shifter" parameter $\theta_{s_1 s_2}$ is assumed to satisfy

$$\theta_{hh} > \theta_{hl} = \theta_{lh} > \theta_{ll} > c,$$

and the signals are drawn from i.i.d. distribution: for each $i = 1, 2$,

$$\rho \equiv \Pr\left(s_i = h\right) \in (0, 1).$$

Each downstream firm D_i simultaneously offers a price schedule $f_i(q)$ to the upstream firm. Given the contracts $f = (f_1, f_2)$, the upstream firm then decides whether to accept each contract or not, and if it accepts the contract f_i, how much it sells its input to D_i. To be precise, let $g_i \in \{1, 0\}$ denote U's acceptance decision on D_i's contract, and $q_i \geq 0$ denote the amount of input supply to D_i (conditional on acceptance). Letting $z_i = (f_i, q_i, g_i)$ be the contracting outcome between U and D_i, the aggregate profit of U is

$$u\left(z_1, z_2\right) = \sum_{i=1,2} g_i \cdot \left(f_i\left(q_i\right) - c q_i\right),$$

and the profit of D_i is

$$\pi(z_i, z_{-i}; s_i, s_{-i}) = g_i \cdot \left(P(q_i, g_{-i} \cdot q_{-i}; s_i, s_{-i}) q_i - f_i\left(q_i\right)\right).$$

Note that the bilateral surplus of a pair U and D_i is

$$v(z_i, z_{-i}; s_i, s_{-i}) = g_i \cdot \left(P\left(q_i, g_{-i} \cdot q_{-i}; s_i, s_{-i}\right) q_i - cq_i\right).$$

We summarize the timing of the game as follows:

1. Each D_i privately observes a demand signal s_i.

2. Each D_i simultaneously offers a price schedule $f_i(\cdot)$ to U.

3. U decides either to accept or reject each offer.

4. If U accepts f_i, it decides the amount of input supply q_i to D_i, and the payments are made accordingly.

Before analyzing this game, we discuss several benchmark cases that are useful to evaluate the effect of information spillover through vertical contracting.

Benchmark 1 (complete information) Suppose that demand signals are publicly observable. Given that the downstream firms have bargaining power and there are no upstream externalities, each downstream firm will fully extract the bilateral surplus by setting

$$f_i(q_i) = \begin{cases} cq_i & \text{if } q_i = q_i^*\left(s_i, s_{-i}\right), \\ -\infty & \text{otherwise.} \end{cases}$$

where $q_i^*\left(s_i, s_{-i}\right)$ is the equilibrium quantity of D_i, which is given by

$$q_i^*\left(s_i, s_{-i}\right) = \arg\max_{q_i} \left(P\left(q_i, q_{-i}^*\left(s_{-i}, s_i\right); s_i, s_{-i}\right) q_i - cq_i\right),$$

for each $i = 1, 2$. Given the linear demand specification, these are given by

$$q_i^*\left(s_i, s_{-i}\right) = q_{s_1 s_2}^* = \frac{\theta_{s_1 s_2} - c}{2 + \beta},$$

for each $i = 1, 2$ and $s_1, s_2 \in \{h, l\}$. This quantity profile corresponds to the unique Nash equilibrium outcome when two firms compete under complete information when they produce goods at the marginal cost of c. For this reason, we will call it the "first-best" quantity profile throughout the paper. $\diamond$

Benchmark 2 (competitive input market without communication) Next, suppose that each downstream firm privately observes a demand signal but can purchase intermediate goods from a competitive input market at the price of c. In this game, the unique Bayesian equilibrium is given by $\left(q_1^*\left(s_1\right), q_2^*\left(s_2\right)\right)$ satisfying, for each $i = 1, 2$ and $s_1, s_2 \in \{h, l\}$,

$$q_i^*\left(s_i\right) \in \arg\max_{q} E\left(P\left(q_i, q_{-i}^*\left(\tilde{s}\right); s_i, \tilde{s}\right) q_i - cq_i\right),$$

where $\tilde{s}$ denotes the random variable of the rival downstream firm's demand signal. In our model specification, the unique equilibrium quantities are given by

$$q_i^*(l) = q_l^* = \frac{E(\theta_{l\tilde{s}}) - c}{2+\beta} - \frac{\beta\rho\left(E(\theta_{h\tilde{s}}) - E(\theta_{l\tilde{s}})\right)}{2(2+\beta)},$$
$$q_i^*(h) = q_h^* = \frac{E(\theta_{h\tilde{s}}) - c}{2+\beta} + \frac{\beta(1-\rho)\left(E(\theta_{h\tilde{s}}) - E(\theta_{l\tilde{s}})\right)}{2(2+\beta)},$$

for each $i = 1, 2$ and $s_1, s_2 \in \{h, l\}$. Note that the downstream firms never achieve the first-best outcome given the lack of information sharing. As we may expect, $q_h^* > q_l^*$. ◇

Benchmark 3 (competitive input market with communication) As a final benchmark, suppose that the firms still purchase intermediate goods from a competitive input market at the price of c, but can exchange public cheap talk messages. Even with such a direct communication device, however, it can be shown that informative cheap talk communication does not occur, and the first-best quantity profile is never achieved in equilibrium. To see this, let us assume that each downstream firm sends message $\hat{h}$ (resp. $\hat{l}$) after receiving signal h (resp. l). *On the path of plays*, information asymmetry disappears, and the firms' quantities are given by those in Benchmark 1. However, since the profit of D_i will increase as the rival downstream firm sells less, each downstream firm with signal h will find it profitable to send $\hat{l}$ and sells q_i^* such that

$$q_i^* = \arg\max_{q_i} \left(P\left(q_i, q_{-i}^*(\hat{s}, l); h, \hat{s}\right) q_i - cq_i\right),$$

where $\hat{s} \in \{\hat{h}, \hat{l}\}$ is the message sent by the rival downstream firm.[3] This type of double-deviation prevents information exchange, and the equilibrium quantities still remain the same as those in Benchmark 2. ◇

3 Vertical contracting and information spillover

Now, we analyze the model introduced in Section 2. This is a common agency game where two competing downstream firms are the principals, who contract with a common upstream input supplier. Each downstream firm is privately informed about a demand signal, which is unobservable to either its rival downstream firm or the upstream firm. While the upstream firm has no a priori private information, it can learn the rival downstream firm's demand signal when the rival downstream firm's contract offer is separating.

Throughout the analysis, we will focus on *fully revealing* equilibria in which both signaling and screening occur: that is, different types of downstream firms make different offers (separating), and different types of the upstream firm choose to supply different amounts of quantity

[3]See also Goltsman and Pavlov (2014), who study cheap talk communication in horizontal Cournot markets where each firm's production cost is private information and drawn from a continuous distribution.

(screening). From the perspective of each downstream firm, the vertical contracting game then becomes a principal-agent problem where both the principal and the agent have private information. As in Martimort & Moreira (2010); Lima & Moreira (2014), we first consider the best response of each downstream firm in this informed principal problem and then find their fixed point, which is a solution to the game we consider.

To be more precise, let us consider the problem of an informed downstream firm that holds the following conjecture about the rival downstream firm's menu offer:

$$\widehat{m} = \left(\widehat{m}^h, \widehat{m}^l\right) = \left(\left(\left(\widehat{q}^h_h, \widehat{f}^h_h\right), \left(\widehat{q}^h_l, \widehat{f}^h_l\right), \phi\right), \left(\left(\widehat{q}^l_h, \widehat{f}^l_h\right), \left(\widehat{q}^l_l, \widehat{f}^l_l\right), \phi\right)\right),$$

where $\left(\widehat{q}^s_{s'}, \widehat{f}^s_{s'}\right)$ is a pair of a quantity and a fixed fee offered by the rival downstream firm of type-s targeted to the upstream firm of s'-type, and ϕ is a null contract of contract rejection.[4] To characterize fully revealing equilibria, we restrict the set of conjectures to those satisfying the following two conditions:

Condition (SI)**.** $q^*_{hh} \geq \widehat{q}^h_s > \widehat{q}^l_s \geq 0$ for each $s \in \{h, l\}$.

Condition (SC)**.** For each $s \in \{h, l\}$, there exists $\widehat{u}^s \geq 0$ such that

$$\widehat{u}^s \equiv \widehat{f}^s_h - c\widehat{q}^s_h = \widehat{f}^s_l - c\widehat{q}^s_l.$$

Note that Condition (SI) imposes the boundaries on quantities under conjectured offers and ensures signaling—that is, $\widehat{m}^h$ and $\widehat{m}^l$ are distinct—in that

$$\max\{\widehat{q}^h_h, \widehat{q}^h_l\} \geq \widehat{q}^h_h > \widehat{q}^l_h \geq \min\{\widehat{q}^l_h, \widehat{q}^l_l\}.$$

Because of separating offers, the upstream supplier learns the rival downstream firm's private information under Condition (SI). The condition is intuitive since it is profitable to sell more quantities in the market when the demand signals are higher.

On the other hand, Condition (SC) ensures screening—that is, the upstream firm is willing to accept an offer and choose a quantity targeted to its own type. In particular, if Condition (SC) is satisfied, we can alternatively write a pair of offers as

$$\widehat{m} = \left(\widehat{m}^h, \widehat{m}^l\right) = \left(\left(\widehat{q}^h_h, \widehat{q}^h_l, \widehat{u}^h\right), \left(\widehat{q}^l_h, \widehat{q}^l_l, \widehat{u}^l\right)\right).$$

Consequently, the set of offers that we consider is given by

$$\mathcal{M} \equiv \left\{(q_h, q_l, u) : (q_h, q_l, u) \text{ satisfies Condition (SI) and } u \geq 0\right\}.$$

Given conjectured menus $\widehat{m}$, the problem of the type-s downstream firm is then defined as follows:

$$\max_{\substack{0 \leq q_h, q_l \leq q^*_{hh} \\ f_h, f_l \in \mathbb{R}}} \rho\left(P\left(q_h, \widehat{q}^h_s; s, h\right) q_h - f_h\right) + (1-\rho)\left(P\left(q_l, \widehat{q}^l_s; s, l\right) q_l - f_l\right)$$

$$s.t. \quad (IC_{-s}), (ICU), \text{ and } (IRU),$$

[4]That is, $\widehat{f}^s\left(\widehat{q}^s_{s'}\right) = \widehat{f}^s_{s'}$ for each $s, s' \in \{h, l\}$ and $\widehat{f}^s(q) = -\infty$ for all other q's.

where

$$\begin{aligned}&\rho\left(P\left(\widehat{q_h}^{-s},\widehat{q}^{h}_{-s};-s,h\right)-c\right)\widehat{q_h}^{-s}+(1-\rho)\left(P\left(\widehat{q_l}^{-s},\widehat{q}^{l}_{-s};-s,l\right)-c\right)\widehat{q_l}^{-s}-\widehat{u}^{-s}\\ \geq&\rho\left(P\left(q_h,\widehat{q}^{h}_{s};-s,h\right)-c\right)q_h+(1-\rho)\left(P\left(q_l,\widehat{q}^{l}_{s};-s,l\right)-c\right)\widehat{q}_l-u \qquad (IC_{-s})\end{aligned}$$

$$f_h-cq_h\geq f_l-cq_l \text{ and } f_l-cq_l\geq f_h-cq_h \qquad (ICU)$$

and

$$f_h-cq_h\geq 0 \text{ and } f_l-cq_l\geq 0 \qquad (IRU)$$

Note that (IC_{-s}) is the incentive constraint for the downstream firm of type $-s\neq s$ not to mimic type s and ensures that private is transmitted from a downstream firm to the upstream firm. When this constraint is binding, it generates distortion for separating types. On the other hand, (ICU) and (IRU) guarantee that the upstream firm is willing to accept a menu and choose a quantity targeted to its own type. In particular, since (ICU) reduces to $f_h-cq_h=f_l-cq_l=u$ for some u, we can alternatively write the informed downstream firm's problem as

$$\begin{aligned}\max_{\substack{0\leq q_h,q_l\leq q^*_{hh}\\ u\in\mathbb{R}}}\ &\rho\left(P\left(q_h,\widehat{q}^{h}_{s};s,h\right)-c\right)q_h+(1-\rho)\left(P\left(q_l,\widehat{q}^{l}_{s};s,l\right)-c\right)q_l-u \qquad \text{(P-}s\text{)}\\ s.t.\quad &(IC_{-s}),\ \text{and } (IRU'),\end{aligned}$$

where

$$u\geq 0 \qquad (IRU')$$

Clearly, the upstream firm enjoys a positive information rent if and only if $u>0$.

Denoting the set of solutions to the h-type problem by $BR^h(\widehat{m})$, we define equilibrium in our game as follows:

Definition 1. An equilibrium is $m=\left(m^h,m^l\right)\in\mathcal{M}^2$ such that $m^s\in BR^s(m)$ for each $s\in\{h,l\}$.

In characterizing equilibrium, we make the following restriction on the parameters to assure interior solutions.

Assumption 1. $\theta_{hh}-\theta_{hl}\geq\theta_{hl}-\theta_{ll}\geq\theta_{ll}-c$ and $\frac{\theta_{ll}-c}{\theta_{hl}-c}\geq\frac{2-\beta}{2}\left(\frac{\theta_{hl}-\theta_{ll}}{\theta_{hh}-\theta_{hl}}\right)$

Intuitively, this restriction requires that the demand size under high signals is large enough so that selling negative quantities in the market is never profitable. Note that the assumption holds for all $0\leq\beta\leq 1$ if, for example, the parameters satisfy $\theta_{hh}-\theta_{hl}=6>\theta_{hl}-\theta_{ll}=2>\theta_{ll}-c=1$.

3.1 Equilibrium analysis

We begin our analysis by setting up the Lagrangian function for the s-type problem (P-s):

$$\begin{aligned}\mathcal{L}^s &= \rho\left(P\left(q_h, \widehat{q}_s^h; s, h\right) - c\right) q_h + (1-\rho)\left(P\left(q_l, \widehat{q}_s^l; s, l\right) - c\right) q_l - u \\ &+ \lambda_{-s}\Big(\rho\left(P\left(\widehat{q_h}^{-s}, \widehat{q}_{-s}^h; -s, h\right) - c\right)\widehat{q_h}^{-s} + (1-\rho)\left(P\left(\widehat{q_l}^{-s}, \widehat{q}_{-s}^l; -s, l\right) - c\right)\widehat{q_l}^{-s} \\ &- \rho\left(P\left(q_h, \widehat{q}_s^h; -s, h\right) - c\right) q_h - (1-\rho)\left(P\left(q_l, \widehat{q}_s^l; -s, l\right) - c\right)\widehat{q}_l + u - \widehat{u}^{-s}\Big) \\ &+ \lambda_u u,\end{aligned}$$

where λ_{-s} and λ_u are the multipliers for (IC_{-s}) and (IRU'), respectively. The boundary condition for the quantities—which we ignore in the above—will be checked later.

Note that the marginal revenue of the downstream firm at quantity q when the rival downstream firm sells q' and the demand signals are s and s' is

$$MR\left(q, q'; s, s'\right) \equiv \frac{\partial P\left(q, q'; s, s'\right)}{\partial q} q + P\left(q, q'; s, s'\right) = \theta_{ss'} - \beta q' - 2q.$$

Using this notation, the Kuhn-Tucker first-order necessary conditions are given as follows:

- First-order condition with respect to q_h:

$$MR(q_h, \widehat{q}_s^h; s, h) - c - \lambda_{-s}\left(MR(q_h, \widehat{q}_s^h; -s, h) - c\right) = 0 \tag{1}$$

- First-order condition with respect to q_l:

$$MR(q_l, \widehat{q}_s^l; s, l) - c - \lambda_{-s}\left(MR(q_l, \widehat{q}_s^l; -s, l) - c\right) = 0 \tag{2}$$

- First-order condition with respect to u:

$$-1 + \lambda_{-s} + \lambda_u = 0 \tag{3}$$

- Complementary slackness condition for (IC_{-s}):

$$\begin{aligned}&\lambda_{-s}\Big(\rho\left(P\left(\widehat{q_h}^{-s}, \widehat{q}_{-s}^h; -s, h\right) - c\right)\widehat{q_h}^{-s} + (1-\rho)\left(P\left(\widehat{q_l}^{-s}, \widehat{q}_{-s}^l; -s, l\right) - c\right)\widehat{q_l}^{-s} \\ &- \rho\left(P\left(q_h, \widehat{q}_s^h; -s, h\right) - c\right) q_h - (1-\rho)\left(P\left(q_l, \widehat{q}_s^l; -s, l\right) - c\right)\widehat{q}_l + u - \widehat{u}^{-s}\Big) \\ &= 0\end{aligned} \tag{4}$$

- Complementary slackness condition for (IRU'):

$$\lambda_u u = 0 \tag{5}$$

- Nonnegative Lagrangian multipliers:

$$\lambda_{-s}, \lambda_u \geq 0 \tag{6}$$

- Constraints:

$$(IC_{-s}),\ (IRU) \tag{7}$$

Note that if $\lambda_{-s} = 0$, each downstream firm equates the upstream marginal cost to its marginal revenue at given demand signals, but otherwise, the quantities sold in the market will be distorted by the signaling effect. On the other hand, the downstream firms' screening efforts generate information rents to the upstream firm only if $\lambda_u = 0$. Given the constant returns to scale production technology, however, it turns out that the downstream firms need not leave information rents to the upstream firm for screening.

Lemma 1. For each $s \in \{h, l\}$, we have $\lambda_u > 0$ at a solution to (P-s). Thus, $u^h = u^l = 0$ in equilibrium.

Proof. See the Appendix. □

Next, consider the incentive compatibility condition (IC_{-s}), which requires that the offer made by the downstream firm of type s is not profitable for the other type $-s$ to mimic. Intuitively, each downstream firm makes more profits when the rival downstream firm sells less, which is the case when the rival downstream firm believes that the market demand is lower. In other words, the downstream firm of l-type has no incentive to mimic the offer made by the h-type downstream firm. The following result formalizes this intuition.

Lemma 2. In equilibrium, $\lambda_l = 0$ and $0 \leq \lambda_h < 1$. Therefore, equilibrium quantities must satisfy

$$q_h^h = q_{hh}^* = \frac{\theta_{hh} - c}{2 + \beta},\ q_l^h = \frac{\theta_{hl} - c - \beta q_h^l}{2}$$

Proof. See the Appendix. □

Given the above observations, we now consider the reduced problems in which the h-type downstream firm solves

$$\max_{0 \leq q_l \leq q_{hh}^*} \left(P\left(q_l, \widehat{q}_h^l; h, l\right) - c\right) q_l \tag{RP-h}$$

and the l-type downstream firm solves

$$\max_{0 \leq q_h, q_l \leq q_{hh}^*} \rho\left(P\left(q_h, \widehat{q}_l^h; l, h\right) - c\right) q_h + (1 - \rho)\left(P\left(q_l, \widehat{q}_l^l; l, l\right) - c\right) q_l \quad s.t. \quad (\widetilde{IC}_h) \tag{RP-l}$$

where $(\widetilde{IC}_h)$ is given by:

$$\begin{aligned}&\rho\left(P\left(q_{hh}^*, q_{hh}^*; h, h\right)-c\right) q_{hh}^*+(1-\rho)\left(P\left(\widehat{q}_l^h, \widehat{q}_h^l; h, l\right)-c\right) \widehat{q}_l^h \\ \geq &\rho\left(P\left(q_h, \widehat{q}_l^h; h, h\right)-c\right) q_h+(1-\rho)\left(P\left(q_l, \widehat{q}_l^l; h, l\right)-c\right) q_l .\end{aligned} \tag{8}$$

We denote the set of solutions by $\widetilde{BR}^s\left(\widehat{q}_l^h,\left(\widehat{q}_h^l, \widehat{q}_l^l\right)\right)$ for each $\left(\widehat{q}_l^h,\left(\widehat{q}_h^l, \widehat{q}_l^l\right)\right) \in\left[0, q_{hh}^*\right]^3$ and $s \in\{h, l\}$. A fixed point of the reduced problem is then defined as $\left(q_l^h\left(q_h^l, q_l^l\right)\right)$ such that

$$\left(q_l^h\left(q_h^l, q_l^l\right)\right) \in \widetilde{BR}^h\left(\left(q_l^h\left(q_h^l, q_l^l\right)\right)\right) \times \widetilde{BR}^l\left(\left(q_l^h\left(q_h^l, q_l^l\right)\right)\right) .$$

The following result is an immediate corollary of the previous lemmas.

Proposition 1. Suppose $m=\left(m^h, m^l\right)=\left(\left(q_h^h, q_l^h, u^h\right),\left(q_h^l, q_l^l, u^s\right)\right)$ is an equilibrium. Then, (i) $u^h=u^s=0$; (ii) $q_h^h=\frac{\theta_{hh}-c}{2+\beta}$; and (iii) $\left(q_l^h,\left(q_h^l, q_l^l\right)\right)$ is a fixed point of the reduced problem.

Thus, our analysis reduces to obtaining a fixed point of the reduced problem. From the first-order conditions with respect to quantities, it is straightforward to check that the equilibrium quantities should be of the following form:

$$\begin{pmatrix} q_h^h\left(\tilde{\lambda}_h\right) \\ q_l^h\left(\tilde{\lambda}_h\right) \\ q_h^l\left(\tilde{\lambda}_h\right) \\ q_l^l\left(\tilde{\lambda}_h\right)\end{pmatrix}=\begin{pmatrix} q_{hh}^* \\ q_{hl}^* \\ q_{hl}^* \\ q_{ll}^*\end{pmatrix}+\begin{pmatrix} 0 \\ \kappa_{hl} \\ -\kappa_{lh} \\ -\kappa_{ll}\end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} \kappa_{hl} & =\frac{\beta \tilde{\lambda}_h\left(\theta_{hh}-\theta_{hl}\right)}{(2-\beta)(2+\beta)\left(1-\lambda_h\right)} \geq 0, \\ \kappa_{lh} & =\frac{2 \tilde{\lambda}_h\left(\theta_{hh}-\theta_{hl}\right)}{(2-\beta)(2+\beta)\left(1-\tilde{\lambda}_h\right)} \geq 0, \\ \kappa_{ll} & =\frac{\tilde{\lambda}_h\left(\theta_{hl}-\theta_{ll}\right)}{(2+\beta)\left(1-\tilde{\lambda}_h\right)} \geq 0,\end{aligned}$$

and $\tilde{\lambda}_h \geq 0$ is the Lagrangian multiplier for $(\widetilde{IC}_h)$. The terms $\kappa_{ss'}$ measure the degree of quantity distortions away from the first-best level generated by information spillover. Notably, the equilibrium quantities achieve the first-best levels when $\tilde{\lambda}_h=0$, which turns out to be the case if the following inequality is satisfied:

$$\begin{aligned}&\rho\left(\theta_{hh}-\theta_{hl}\right)\left(\theta_{hh}-c-(1+\beta)\left(\theta_{hl}-c\right)\right) \\ &+(1-\rho)\left(\theta_{hl}-\theta_{ll}\right)\left(\theta_{hl}-c-(1+\beta)\left(\theta_{ll}-c\right)\right) \geq 0 .\end{aligned} \tag{10}$$

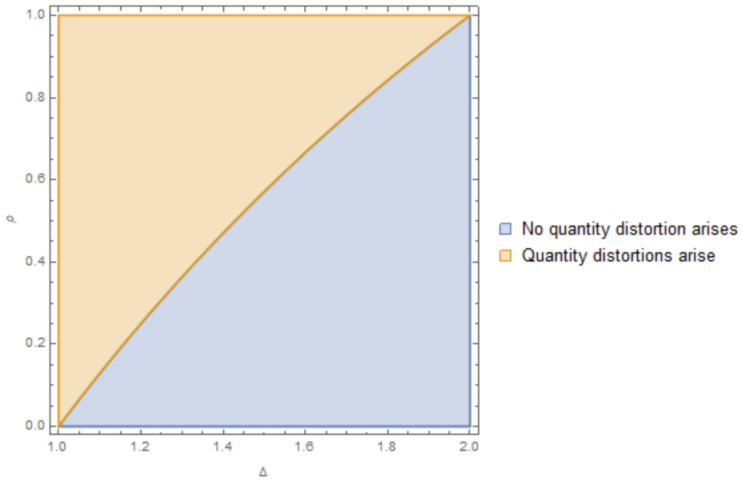

Figure 1: The equilibrium quantity distortions arise in the yellow region when $\theta_{hh} - \theta_{hl} = \frac{3\Delta}{2} > \theta_{hl} - \theta_{ll} = \Delta > \theta_{ll} - c = 1$ and $\beta = 1$.

As shown in the Appendix, this condition is a simple rewriting of the constraint $\left(\widetilde{IC}_h\right)$ when every quantity is replaced by its first-best level $q^*_{ss'}$.

Our main result is a complete characterization of the unique equilibrium. For simplicity, we write as $q^s_{s'} = q^s_{s'}\left(\tilde{\lambda}_h\right)$.

Theorem 1. There exists a unique equilibrium. In the equilibrium, the quantities are given by (9) for some $\widetilde{\lambda}_h \geq 0$ and ordered as $q^h_h > q^h_l \geq q^l_h > q^l_l$. In particular, quantity distortions arise if and only if the inequality (10) does not hold.

Proof. See the Appendix. □

Figure 1 illustrates the case where the quantity distortions arise when the intervals between θ's are parameterized by $\Delta > 0$. It shows that, for a fixed ρ (resp. Δ), a smaller Δ (resp. higher ρ) induces these distortions. The intuition behind this observation can be understood as follows. In the equilibrium, each downstream firm perfectly learns true demand signals, so they want to sell larger quantities when the signals indicate higher demand states. At the same time, however, this generates an incentive for the high-type downstream firm to mimic the low-type downstream firm in an attempt to reduce the rival downstream firm's quantity.

Thus, information spillover is not possible unless the low-type downstream firm commits to sell sufficiently small quantities q^l_h and q^l_l by offering such contracts with distortions. When Δ is high enough, the former effect dominates the latter, and the constraint (IC_h) becomes slack. On the other hand, the low-type downstream firm does not have an incentive to mimic the high type, so the latter effect is absent. Thus, the quantity distortions vanish when ρ is small (note that the similar argument holds for small β).

Proposition 2. The left-hand side of (10) is strictly increasing in $\theta_{hh} - \theta_{hl}$ and strictly decreasing in β. Thus, if the quantity distortion is absent for given parameters, it continues to be

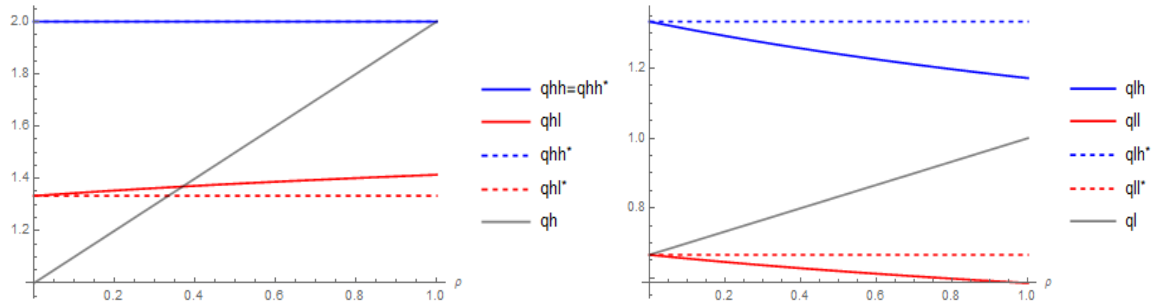

Figure 2: Quantities when $2 = \theta_{hh} - \theta_{hl} = \theta_{hl} - \theta_{ll} = \theta_{ll} - c$, $\beta = 1$, and $c = 1$.

absent for larger $\theta_{hh} - \theta_{hl}$ or smaller β.

3.2 Impact of common supplier and information spillover

When the upstream market is competitive, the equilibrium quantities, q_h^* and q_l^*, are given by the unique Bayesian Nash equilibrium in standard Cournot competition models with private information (Benchmark 2). In contrast, Theorem 1 shows that, with a common input supplier, there exists an equilibrium in which information spillover occurs and firms perfectly learns true demand signals. Now, we analyze how such information spillover affects the firms' profits and consumer welfare.

To simplify the analysis and focus on the case where downstream competition is severe, we will assume $\Delta = \theta_{hh} - \theta_{hl} = \theta_{hl} - \theta_{ll} = \theta_{ll} - c > 0$. and $\beta = 1$. Under this parameterization, it is easy to show that Assumption 1 holds but the inequality (10) does not—meaning that the quantity distortions always arise.

In the following, we will refer to CS and $\widehat{CS}$ as the ex-ante consumer surplus in the unique equilibrium under competitive input markets and common input supplier, respectively. Similarly, we use notations PS and $\widehat{PS}$ for the ex-ante profit of each downstream firm under competitive input markets and common input supplier, respectively. Since the upstream firm makes zero profit in both environments, these are also vertical surplus of each downstream-upstream pair.

Proposition 3. Suppose $\theta_{hh} - \theta_{hl} = \theta_{hl} - \theta_{ll} = \theta_{ll} - c = \Delta > 0$ and $\beta = 1$. Then,

(i) There exist $\bar{\rho}_h \in (0, 1)$ such that if $\rho > \bar{\rho}_h$, then $q_h^h > q_h^* > q_l^h$; and if $\rho \leq \bar{\rho}_h$, then $q_h^h \geq q_l^h \geq q_h^*$.

(ii) There exist $\bar{\rho}_l \in (0, 1)$ such that if $\rho > \bar{\rho}_l$, then $q_l^* > q_h^l > q_l^l$; if $\rho \leq \bar{\rho}_l$, then $q_h^l > q_l^* > q_l^l$.

(iii) The Lagrangian multiplier is given by $\lambda^* = \frac{(3+2\rho)-3\sqrt{1+\rho}}{(3+4\rho)} > 0$ and strictly increasing in ρ.

(iv) For all $\rho \in (0, 1)$, we have $CS < \widehat{CS}$ and $\widehat{PS} < PS$.

Proof. See the appendix. □

Roughly speaking, the first three items (i) and (ii) imply that the quantities are more dispersed with a common input supplier compared to the competitive input market. This is intuitive in that each downstream firm learns the true demand signals and attempt to match its quantity level with the actual market demands.

In particular, as the high demand is more (resp. less) likely, the downward quantity distortions become more (resp. less) severe, and it often leads to quantities set by the high-type downstream firm (resp. low-type downstream firm) smaller than (resp. larger than) the quantity level q_h^* (resp. q_l^*) that is obtained under the competitive input market. The item (iii) indeed shows that the quantity distortion becomes more severe when ρ is high. With a higher ρ, intuitively, each downstream firm believes that the rival downstream firm is more likely high type and sells a larger quantity in the market, which in turn, by strategic substitutability, creates incentives for the downstream firm to offer a contract inducing smaller quantities of its own.

Despite such mixed effects on the equilibrium quantities, the last item (iv) shows that consumer surplus improves when the input is supplied by the common upstream supplier, while the opposite holds for the downstream firms. This is because the common input supplier allows the downstream firms to learn each other's private signal, so they sell larger quantities of products in the market when the actual demand size is large. Although the reversed situation occurs when the market demand is small, the impact of such events on consumer welfare (and the downstream profits) is relatively small because the changes in the equilibrium quantities as well as prices are limited by the demand size. In the equilibrium, therefore, consumers are better off whereas the firms are worse off compared to the case where the upstream market is competitive.

4 Conclusion

In this paper, we revisit standard Cournot competition models with private (demand) information. Unlike the previous literature assuming competitive input market, however, we introduced a common input supplier, which—we believe—is a widely observed and important feature of modern industries. We characterized a unique equilibrium in which private information held by each downstream firms is endogenously flowed within a vertical chain by signaling and screening behaviors in vertical contracting. Compared to the competitive upstream market case, such information spillover induces more dispersed equilibrium quantities and may increase consumer welfare at the cost of lowered downstream profits.

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Appendix: Omitted Proofs

Proof of Lemma 1. Consider the h-type problem. Suppose $\lambda_u = 0$. Then, by (3), we have $\lambda_l = 1$, implying that $MR(q_h, \widehat{q}_h^h; h, h) = MR(q_h, \widehat{q}_h^h; l, h) \iff \theta_{hh} - \theta_{hl} = 0$ from (1). This is a contradiction. The same argument applies to the l-type problem.

Proof of Lemma 2. By Lemma 1, it is immediate that $0 \leq \lambda_{-s} < 1$. Now, suppose that $\lambda_l > 0$. From (1), we have

$$\theta_{hh} - \beta \bar{q}_h^h - 2q_h - c = \lambda_l \left(\theta_{hl} - \beta \bar{q}_h^h - 2q_h - c\right).$$

The left-hand side is nonnegative because equilibrium quantities cannot exceed $q_{hh}^* = \frac{\theta_{hh}-c}{2+\beta}$ To derive a contradiction, suppose that the left-hand side is zero. Then, the right-hand side is strictly negative since $\theta_{hh} > \theta_{hl}$. Suppose now that the left-hand side is strictly positive. Rearranging the terms, one can show that the equation is equivalent to

$$\frac{1}{\lambda_l}(\theta_{hh} - \beta \bar{q}_h^h - 2q_h - c) = \theta_{hl} - \beta \bar{q}_h^h - 2q_h - c.$$

Since $\lambda_l < 1$, this implies

$$\theta_{hh} - \beta \bar{q}_h^h - 2q_h - c < \frac{1}{\lambda_l}(\theta_{hh} - \beta \bar{q}_h^h - 2q_h - c) = \theta_{lh} - \beta \bar{q}_h^h - 2q_h - c,$$

which is a contradiction. Therefore, $\lambda_l = 0$, and the remaining results follow from the first-order conditions (1) and (2) of the high-type problem.

Proof of Theorem 1. We prove the theorem in several steps.

Lemma 3. Suppose $\tilde{\lambda}_h \leq \overline{\lambda}_h \equiv \frac{2-\beta}{2}\left(\frac{\theta_{hl}-\theta_{ll}}{\theta_{hh}-\theta_{hl}}\right) \leq \frac{1}{2}$. Then,

$$0 \leq q_l^l(\tilde{\lambda}_h) \leq q_h^l(\tilde{\lambda}_h) \leq q_l^h(\tilde{\lambda}_h) \leq q_h^h(\tilde{\lambda}_h) = q_{hh}^*.$$

Proof. A simple computation shows that

$$q_h^h(\tilde{\lambda}_h) \geq q_l^h(\tilde{\lambda}_h) \iff \tilde{\lambda}_h \leq \frac{2-\beta}{2},$$

$$q_l^l(\tilde{\lambda}_h) \geq 0 \iff \tilde{\lambda}_h \leq \frac{\theta_{ll}-c}{\theta_{hl}-c},$$

$$q_h^l(\tilde{\lambda}_h) \geq q_l^l(\tilde{\lambda}_h) \iff \tilde{\lambda}_h \leq \frac{2-\beta}{2}\left(\frac{\theta_{hl}-\theta_{ll}}{\theta_{hh}-\theta_{hl}}\right),$$

Under Assumption 1, these inequalities are satisfied.

□

Now, consider $(\widetilde{IC}_h)$ where the quantities in the both sides are replaced by the candidate equilibrium quantities in (9). By tedious computations, one can show that it is equivalent to

$$
\begin{aligned}
&\rho q_{hh}^{*2} + (1-\rho)q_{hl}^{*2} - \rho(\theta_{hh} - (1+\beta)q_{hl}^* - c)q_{hl}^* - (1-\rho)(\theta_{hl} - (1+\beta)q_{ll}^* - c)q_{ll}^* \\
\geq &((1-\rho)(\kappa_{hl} - \beta\kappa_{lh}) + \rho(\kappa_{lh} - \beta\kappa_{hl}))q_{hl}^* - \rho(\theta_{hh} - (1+\beta)q_{hl}^* - c + \kappa_{lh} - \beta\kappa_{hl})\kappa_{lh} \\
-&(1-\rho)(\theta_{hl} - (1+\beta)q_{hl}^* - c + \beta\kappa_{lh} - \kappa_{hl})\kappa_{hl} \\
-&(1-\rho)(\theta_{hl} - 2(1+\beta)q_{ll}^* - c + (1+\beta)\kappa_{ll})\kappa_{ll} \qquad (11)
\end{aligned}
$$

Clearly, the left-hand side is independent of $\tilde{\lambda}_h$ and non-negative if and only if $(\widetilde{IC}_h)$ is satisfied at the first-best quantities $q_{ss'}^*$.

Lemma 4. The right-hand side of (11) is zero at $\tilde{\lambda}_h = 0$, non-positive and strictly decreasing in $\tilde{\lambda}_h \in [0,1)$.

Proof. Letting $\mu = \frac{\tilde{\lambda}_h}{1-\tilde{\lambda}_h}$, we have

$$
\begin{aligned}
\kappa_{hl} &= \frac{\beta\tilde{\lambda}_h(\theta_{hh} - \theta_{hl})}{(2-\beta)(2+\beta)(1-\lambda)} = \mu\left(\frac{\beta}{2-\beta}\right)(q_{hh} - q_{hl}) \\
\kappa_{lh} &= \frac{2\lambda(\theta_{hh} - \theta_{hl})}{(2-\beta)(2+\beta)(1-\tilde{\lambda}_h)} = \mu\left(\frac{2}{2-\beta}\right)(q_{hh} - q_{hl}) \\
\kappa_{ll} &= \frac{\tilde{\lambda}_h(\theta_{hl} - \theta_{ll})}{(2+\beta)(1-\tilde{\lambda}_h)} = \mu(q_{hl} - q_{ll})
\end{aligned}
$$

Thus,

$$
\kappa_{hl} - \beta\kappa_{lh} = \mu(q_{hh} - q_{hl})\left(\frac{\beta}{2-\beta} - \frac{2\beta}{2-\beta}\right) = -\mu\left(\frac{\beta}{2-\beta}\right)(q_{hh} - q_{hl})
$$

and

$$
\kappa_{lh} - \beta\kappa_{hl} = \mu(q_{hh} - q_{hl})\left(\frac{2}{2-\beta} - \frac{\beta^2}{2-\beta}\right) = \mu(q_{hh} - q_{hl})\left(\frac{2-\beta^2}{2-\beta}\right)
$$

Thus, the right-hand side of (11) can be written as

$$
\begin{aligned}
&\mu\left(\rho\left(\frac{2-\beta^2}{2-\beta}\right) - (1-\rho)\left(\frac{\beta}{2-\beta}\right)\right)(q_{hh} - q_{hl})q_{hl} \\
-&\rho\mu\left(\theta_{hh} - (1+\beta)q_{hl} - c + \mu\left(\frac{2-\beta^2}{2-\beta}\right)(q_{hh} - q_{hl})\right)\left(\frac{2}{2-\beta}\right)(q_{hh} - q_{hl}) \\
-&(1-\rho)\mu\left(\theta_{hl} - (1+\beta)q_{hl} - c + \mu\left(\frac{\beta}{2-\beta}\right)(q_{hh} - q_{hl})\right)\left(\frac{\beta}{2-\beta}\right)(q_{hh} - q_{hl}) \\
-&(1-\rho)\mu\left(\theta_{hl} - 2(1+\beta)q_{ll} - c + \mu(1+\beta)(q_{hl} - q_{ll})\right)(q_{hl} - q_{ll}).
\end{aligned}
$$

Note that $\mu = \frac{\tilde{\lambda}_h}{1-\tilde{\lambda}_h}$ is zero at $\tilde{\lambda}_h = 0$, non-negative, and strictly increasing in $\tilde{\lambda}_h \in [0, 1)$. Thus, it suffices to show that

$$\begin{aligned} G \equiv &\left(\rho\left(\frac{2-\beta^2}{2-\beta}\right) - (1-\rho)\left(\frac{\beta}{2-\beta}\right)\right)(q_{hh}-q_{hl})q_{hl} \\ &-\rho\left(\theta_{hh} - (1+\beta)q_{hl} - c + \mu\left(\frac{2-\beta^2}{2-\beta}\right)(q_{hh}-q_{hl})\right)\left(\frac{2}{2-\beta}\right)(q_{hh}-q_{hl}) \\ &-(1-\rho)\left(\theta_{hl} - (1+\beta)q_{hl} - c + \mu\left(\frac{\beta}{2-\beta}\right)(q_{hh}-q_{hl})\right)\left(\frac{\beta}{2-\beta}\right)(q_{hh}-q_{hl}) \\ &-(1-\rho)\left(\theta_{hl} - 2(1+\beta)q_{ll} - c + \mu(1+\beta)(q_{hl}-q_{ll})\right)(q_{hl}-q_{ll}) \end{aligned}$$

is non-positive and strictly decreasing in $\tilde{\lambda}_h \in [0, 1)$. Clearly, G is strictly decreasing in μ, and so is in $\tilde{\lambda}_h$. To show that it is non-positive, observe that G is a linear function of ρ, which implies that it is enough to consider $\rho = 1$ and $\rho = 0$.

First, suppose $\rho = 1$. Then,

$$\begin{aligned} G &= \left(\left(\frac{2-\beta^2}{2-\beta}\right)q_{hl} - \left(\theta_{hh} - (1+\beta)q_{hl} - c + \mu\left(\frac{2-\beta^2}{2-\beta}\right)(q_{hh}-q_{hl})\right)\left(\frac{2}{2-\beta}\right)\right)(q_{hh}-q_{hl}) \\ &= \frac{1}{2-\beta}\left((2-\beta^2)q_{hl} - 2\left(\theta_{hh} - (1+\beta)q_{hl} - c + \mu\left(\frac{2-\beta^2}{2-\beta}\right)(q_{hh}-q_{hl})\right)\right)(q_{hh}-q_{hl}). \end{aligned}$$

This is strictly negative since

$$\begin{aligned} &(2-\beta^2)q_{hl} - 2\left(\theta_{hh} - (1+\beta)q_{hl} - c + \mu\left(\frac{2-\beta^2}{2-\beta}\right)(q_{hh}-q_{hl})\right) \\ \leq &(2-\beta^2)q_{hl} - 2\left(\theta_{hh} - (1+\beta)q_{hl} - c\right) \\ = &\frac{1}{2+\beta}\left(-4\theta_{hh} + \beta^2(c-\theta_{hl}) - 2\beta(\theta_{hh}-\theta_{hl}) + 4\theta_{hl}\right) \\ = &\frac{1}{2+\beta}\left(-4(\theta_{hh}-\theta_{hl}) - \beta\left(2(\theta_{hh}-\theta_{hl}) + \beta(\theta_{hl}-c)\right)\right) \\ < &0. \end{aligned}$$

Next, suppose $\rho = 0$. Then,

$$\begin{aligned} G = &\left(-\frac{\beta}{2-\beta}q_{hl} - \left(\theta_{hl} - (1+\beta)q_{hl} - c + \mu\left(\frac{\beta}{2-\beta}\right)(q_{hh}-q_{hl})\right)\frac{\beta}{2-\beta}\right)(q_{hh}-q_{hl}) \\ &- \left(\theta_{hl} - 2(1+\beta)q_{ll} - c + \mu(1+\beta)(q_{hl}-q_{ll})\right)(q_{hl}-q_{ll}) \\ \leq &\left(-\frac{\beta}{2-\beta}q_{hl} - \left(\theta_{hl} - (1+\beta)q_{hl} - c\right)\frac{\beta}{2-\beta}\right)(q_{hh}-q_{hl}) \\ &- (\theta_{hl} - 2(1+\beta)q_{ll} - c)(q_{hl}-q_{ll}) \\ < &0 \end{aligned}$$

since

$$\theta_{hl} - (1+\beta)q_{hl} - c = (\theta_{hl} - c)\left(1 - \frac{1+\beta}{2+\beta}\right) > 0$$

and

$$\begin{aligned}
\theta_{hl} - 2(1+\beta)q_{ll} - c &= \theta_{hl} - c - \frac{2(1+\beta)(\theta_{ll} - c)}{2+\beta} \\
&= \theta_{hl} - \theta_{ll} + \theta_{ll} - c - \frac{2(1+\beta)(\theta_{ll} - c)}{2+\beta} \\
&\geq 2(\theta_{ll} - c) - \frac{2(1+\beta)(\theta_{ll} - c)}{2+\beta} \\
&= (\theta_{ll} - c)\left(2 - \frac{2+2\beta}{2+\beta}\right) \\
&> 0.
\end{aligned}$$

□

Lemma 5. The left-hand side of (11) is non-negative if and only if

$$\begin{aligned}
&\rho\left(\theta_{hh} - \theta_{hl}\right)\left(\theta_{hh} - c - (1+\beta)\left(\theta_{hl} - c\right)\right) \\
&+ (1-\rho)\left(\theta_{hl} - \theta_{ll}\right)\left(\theta_{hl} - c\left(1+\beta\right)\left(\theta_{ll} - c\right)\right) \geq 0.
\end{aligned}$$

Proof. This directly follows from computation. □

Lemma 6. If (10) does not hold, then there exists unique $\tilde{\lambda}_h \in \left(0, \overline{\lambda}_h\right]$ such that the inequality (11) holds as equality.

Proof. Note that the right-hand side of (11) is zero when $\tilde{\lambda}_h = 0$. Thus, if (10) does not hold, then the left-hand side of (11) is strictly lower than the right-hand side of (11) when $\tilde{\lambda}_h = 0$. By the intermediate value theorem, therefore, it suffices to show that (11) is satisfied at $\tilde{\lambda}_h = \overline{\lambda}_h$ for the existence of $\tilde{\lambda}_h$ inducing equality of (11). Clearly, the uniqueness follows from Lemma 4.

For notational ease, we define the following function

$$F\left(\lambda, \rho\right) \equiv \text{LHS of } (11) - \text{RHS of } (11) \text{ at } \tilde{\lambda}_h = \lambda$$

By tedious computations, one can check that $F\left(\overline{\lambda}_h, \rho\right)$ is linear in ρ with

$$\begin{aligned}
&\operatorname{sgn}\left(\frac{\partial}{\partial \rho} F\left(\overline{\lambda}_h, \rho\right)\right) \\
&= \operatorname{sgn}\Big(2\left(\theta_{hh} - \theta_{hl}\right)\left(\theta_{hh} - \theta_{ll}\right) - \beta^2\left(\theta_{hl} - \theta_{ll}\right)\left(\theta_{hl} - c\right) \\
&\quad + 2\beta\left(\theta_{hh}\left(\theta_{hl} - \theta_{ll}\right) - \left(\theta_{hh} - \theta_{hl}\right)\theta_{ll} + c\left(\theta_{hh} + \theta_{ll} - 2\theta_{hl}\right)\right)\Big).
\end{aligned}$$

Note that

$$\begin{aligned}
&2\left(\theta_{hh}-\theta_{hl}\right)\left(\theta_{hh}-\theta_{ll}\right)-\beta^2\left(\theta_{hl}-\theta_{ll}\right)\left(\theta_{hl}-c\right)\\
&+2\beta\left(\theta_{hh}\left(\theta_{hl}-\theta_{ll}\right)-\left(\theta_{hh}-\theta_{hl}\right)\theta_{ll}+c\left(\theta_{hh}+\theta_{ll}-2\theta_{hl}\right)\right)\\
\geq&2\beta\left(\theta_{hh}-\theta_{hl}\right)\left(\theta_{hh}-\theta_{ll}\right)-\beta\left(\theta_{hl}-\theta_{ll}\right)\left(\theta_{hl}-c\right)\\
&+2\beta\left(\theta_{hh}\left(\theta_{hl}-\theta_{ll}\right)-\left(\theta_{hh}-\theta_{hl}\right)\theta_{ll}+c\left(\theta_{hh}+\theta_{ll}-2\theta_{hl}\right)\right)\\
=&\beta\left(2\left(\theta_{hh}-\theta_{hl}\right)\left(\theta_{hh}-\theta_{ll}-\left(\theta_{ll}-c\right)\right)+\left(\theta_{hl}-\theta_{ll}\right)\left(\theta_{hh}-\theta_{hl}+\theta_{hh}-c\right)\right)\\
\geq&0.
\end{aligned}$$

This shows that $F\left(\overline{\lambda}_h,\rho\right)$ is strictly increasing in ρ. To prove that (11) is satisfied at $\overline{\lambda}_h$, it is thus sufficient to show that $F\left(\overline{\lambda}_h,0\right)\geq 0$. Indeed, this is true since

$$F\left(\overline{\lambda}_h,0\right)=\frac{\left(\theta_{hh}-\theta_{hl}\right)^2\left(\theta_{hl}-\theta_{ll}\right)^2}{\left(2\theta_{hh}-\left(4-\beta\right)\theta_{hl}+\left(2-\beta\right)\theta_{ll}\right)^2}>0.$$

This completes the proof of the lemma. □

The previous results have proven the existence and uniqueness of the Lagrangian multiplier $\tilde{\lambda}_h$ (so, the candidate equilibrium quantities) satisfying the first-order necessary conditions. It is immediate that these quantities satisfy Condition (SI) and Condition (SC). The remaining step is to show that the second-order sufficient conditions are also satisfied. In doing so, we first prove the following general result in optimization problems.

Lemma 7. Consider the optimization problem

$$\max_{x,y\in\mathbb{R}} H(x,y)$$
$$S.T. g(x,y)\leq c$$

and the associated Lagrangian function

$$\mathcal{L}_\lambda(x,y)=H(x,y)-\lambda\left(g(x,y)-c\right),$$

where $H(\cdot)$ and $g(\cdot)$ are continuously differentiable functions. Suppose that (x^*,y^*,λ^*) satisfies $\nabla\mathcal{L}_{\lambda^*}(x^*,y^*)=0$, $g(x^*,y^*)\leq c$, $\lambda^*\left(g(x^*,y^*)-c\right)=0$, $\lambda^*\geq 0$, and that $\mathcal{L}_{\lambda^*}(x,y)$ is strictly concave. Then, (x^*,y^*) solves the optimization problem.

Proof. Since $\mathcal{L}_{\lambda^*}(x,y)$ is strictly concave, $\mathcal{L}_{\lambda^*}(x^*,y^*)>\mathcal{L}_{\lambda^*}(x,y)$ for all $(x,y)\in S$. This directly implies that $H(x^*,y^*)-\lambda^*\left(g(x^*,y^*)-c\right)>H(x,y)-\lambda^*\left(g(x,y)-c\right)$ for all $(x,y)\in\mathbb{R}^2$. If we consider a pair (x,y) that satisfies $g(x,y)\leq c$ we obtain $H(x^*,y^*)>H(x,y)$. Thus, (x^*,y^*) solves the optimization problem. □

Now, we apply this lemma to our problem, which completes the proof of the theorem.

Lemma 8. Suppose (10) does not hold. Consider the reduced l-type problem,

$$\max_{0\leq q_h,q_l\leq q^*_{hh}} \rho\left(P\left(q_h, q^h_l(\lambda^*);h,l\right)-c\right)q_h+(1-\rho)\left(P\left(q_l,q^l_l(\lambda^*);l,l\right)-c\right)q_l$$
$$S.T.(\widetilde{IC}_h)$$

where $(\widetilde{IC}_h)$ is given by

$$\begin{aligned}&\rho\left(P\left(q^*_{hh},q^*_{hh};h,h\right)-c\right)q^*_{hh}+(1-\rho)\left(P\left(q^h_l(\lambda^*),q^l_h(\lambda^*);h,l\right)-c\right)q^h_l(\lambda^*)\\ \geq&\rho\left(P\left(q_h,q^h_l(\lambda^*);h,h\right)-c\right)q_h+(1-\rho)\left(P\left(q_l,q^l_l(\lambda^*);h,l\right)-c\right)q_l,\end{aligned} \tag{12}$$

and λ^* is defined in Lemma 6. Then, $(q_h, q_l) = (q^l_h(\lambda^*), q^l_l(\lambda^*))$ is the solution to this maximization problem.

Proof. By construction, $(q_h, q_l, \lambda) = \left(q^l_h(\lambda^*), q^l_l(\lambda^*), \lambda^*\right)$ satisfies (i) $\mathcal{L}_{q_h} = \mathcal{L}_{q_l} = 0$; (ii) $(\widetilde{IC}_h)$ is binding; and (iii) $\lambda = \lambda^* \in \left(0, \frac{1}{2}\right]$. Also, the Lagrangian function for this problem is strictly concave since

$$D^2\mathcal{L}=\begin{bmatrix}\frac{\partial^2\mathcal{L}}{\partial q_h^2} & \frac{\partial^2\mathcal{L}}{\partial q_h\partial q_l}\\ \frac{\partial^2\mathcal{L}}{\partial q_l\partial q_h} & \frac{\partial^2\mathcal{L}}{\partial q_l^2}\end{bmatrix}=\begin{bmatrix}-2\rho(1-\lambda^*) & 0\\ 0 & -2(1-\rho)(1-\lambda^*)\end{bmatrix} \tag{13}$$

By Lemma 7, we conclude that $(q_h, q_l) = \left(q^l_h(\lambda^*), q^l_l(\lambda^*), \lambda^*\right)$ is the solution to this problem.

□

Proof of Proposition 3. Under the assumptions in the proposition, one can compute to show that

$$q^*_l=\frac{\Delta}{3}+\frac{\rho\Delta}{6}<\Delta\left(\frac{1+\rho}{2}\right)=q^*_h.$$

In addition,

$$q^*_{hh}=\Delta>q^*_{hl}=q^*_{lh}=\frac{2\Delta}{3}>q^*_{ll}=\frac{\Delta}{3},$$

and

$$q^h_h=\Delta,\ q^l_h=\frac{2\Delta}{3}+\kappa_{hl},\ q^h_l=\frac{2\Delta}{3}-\kappa_{lh},\ q^l_l=\frac{\Delta}{3}-\kappa_{ll},$$

where $\kappa_{hl}=\frac{\tilde{\lambda}_h\Delta}{3(1-\lambda_h)}$, $\kappa_{lh}=\frac{2\tilde{\lambda}_h\Delta}{3\left(1-\tilde{\lambda}_h\right)}$, $\kappa_{ll}=\frac{\tilde{\lambda}_h\Delta}{3\left(1-\tilde{\lambda}_h\right)}$. Given these, one can directly compute to show the items (i) and (ii).

Now, to show (iii), we compute the unique $\tilde{\lambda}_h$ that is determined by the binding constraint $(\widetilde{IC}_h)$. By a tedious computation, one can check that it is either of the following two roots:

$$\tilde{\lambda}_h=\lambda^*\equiv\frac{(3+2\rho)-3\sqrt{1+\rho}}{(3+4\rho)}$$

or

$$\tilde{\lambda}_h=\lambda^{**}\equiv\frac{(3+2\rho)+3\sqrt{1+\rho}}{(3+4\rho)}$$

Note that the first root λ^* lies in the interval $\left[0, \frac{1}{2}\right]$. To see this, observe that

$$\lambda^* \leq \frac{1}{2} \iff \frac{1}{2} \leq \sqrt{1+\rho},$$

which is always true, and

$$\lambda^* > 0 \iff (1 + \frac{2}{3}\rho) > \sqrt{1+\rho},$$

which always holds since $\rho > 0$. It follows from Lemma 6 that $\tilde{\lambda}_h = \lambda^*$ since $\overline{\lambda}_h = \frac{1}{2}$, where $\overline{\lambda}_h$ is the upper bound on $\tilde{\lambda}_h$ defined in Lemma 3. Furthermore, since $\rho > 0$, we have

$$\frac{\partial \hat{\lambda}_h}{\partial \rho} = \frac{3}{2}\left(\frac{5 + 4\rho - 4\sqrt{1+\rho}}{\sqrt{1+\rho}(3+4\rho)^2}\right) > 0.$$

To show part (iv), we first compute the (ex-ante) consumer surplus under the complete information benchmark (Benchmark 1), which is denote by CS^*:

$$\begin{aligned}
CS^* &= \frac{1}{2}\left(\rho^2 \left(2q^*_{hh}\right)^2 + 2\rho(1-\rho)\left(q^*_{hl} + q^*_{lh}\right)^2 + (1-\rho)^2\left(2q^*_{ll}\right)^2\right) \\
&= 2\rho^2\Delta^2 + \rho(1-\rho)\frac{16}{9}\Delta^2 + \frac{2}{9}(1-\rho)^2\Delta^2. \\
&= \Delta^2\left(2\rho^2 + \rho(1-\rho)\frac{16}{9} + \frac{2}{9}(1-\rho)^2\right) \\
&= \frac{2\Delta^2}{9}\left(1 + 6\rho + 2\rho^2\right)
\end{aligned}$$

Recall that the equilibrium with a common input supplier entails quantity distortions from the first-best quantities, which yields the following consumer surplus:

$$\widehat{CS} = CS^* + DT,$$

where

$$\begin{aligned}
DT &= \Delta^2\left(\left(\frac{\lambda}{1-\lambda}\right)^2 \frac{2}{9}(1-\rho)(2-\rho) - \left(\frac{\lambda}{1-\lambda}\right)\frac{8}{9}(1-\rho^2)\right) \\
&= -\Delta^2\left(\frac{2(1-\rho)(3+2\rho-3\sqrt{1+\rho})(-6+10\rho^2+18\sqrt{1+\rho}+\rho(7+9\sqrt{1+\rho}))}{9(2\rho+3\sqrt{1+\rho})^2}\right) \\
&\leq 0.
\end{aligned}$$

Note that DT equals to zero only when ρ is 0 or 1, and for any $\rho \in (0,1)$ is strictly negative. Now, we compute the consumer surplus in the competitive input market case (Benchmark 2):

$$\begin{aligned}
CS &= \frac{1}{2}\left(\rho^2\left(2q^*_h\right)^2 + 2\rho(1-\rho)\left(q^*_h + q^*_l\right)^2 + (1-\rho)^2\left(2q^*_l\right)^2\right) \\
&= \rho^2\Delta^2(1+\rho)^2\frac{1}{2} + \rho(1-\rho)\frac{1}{36}\Delta^2(5+4\rho)^2 + \frac{1}{18}(1-\rho)^2\Delta^2(2+\rho)^2. \\
&= \Delta^2\left(\rho^2(1+\rho)^2\frac{1}{2} + \rho(1-\rho)\frac{1}{36}(5+4\rho)^2 + \frac{1}{18}(1-\rho)^2(2+\rho)^2\right) \\
&= \frac{\Delta^2}{36}\left(8 + 17\rho + 27\rho^2 + 16\rho^3 + 4\rho^4\right).
\end{aligned}$$

Therefore, we have

$$CS^* - CS = \frac{\Delta^2\rho}{36}\left(31 - 11\rho - 16\rho^2 - 4\rho^3\right) > 0.$$

Given these, we have

$$\begin{aligned}
&\widehat{CS} - CS \\
=&DT + CS^* - CS \\
=&\Delta^2\left(\frac{2(1-\rho)(-3-2\rho+3\sqrt{1+\rho})(-6+10\rho^2+18\sqrt{1+\rho}+\rho(7+9\sqrt{1+\rho}))}{9(2\rho+3\sqrt{1+\rho})^2}\right) \\
&+\frac{\Delta^2\rho}{36}\left(31 - 11\rho - 16\rho^2 - 4\rho^3\right) \\
=&\Delta^2 K(\rho),
\end{aligned}$$

where

$$\begin{aligned}
K(\rho) =&\frac{1}{36}\rho(31 - 11\rho - 16\rho^2 - 4\rho^3) \\
&+\frac{1}{36}\left(\frac{8(1-\rho)(-3-2\rho+3\sqrt{1+\rho})(-6+10\rho^2+18\sqrt{1+\rho}+\rho(7+9\sqrt{1+\rho}))}{(2\rho+3\sqrt{1+\rho})^2}\right)
\end{aligned}$$

It is direct to see that $K(1) = K(0) = 0$. We will show that, $K(\rho) > 0$ for all $\rho \in (0,1)$. It is sufficient to show that

$$\begin{aligned}
&\rho\left(31 - 11\rho - 16\rho^2 - 4\rho^3\right)\left(2\rho + 3\sqrt{1+\rho}\right)^2 \\
&+8\left(1-\rho\right)\left(-3-2\rho+3\sqrt{1+\rho}\right)\left(-6+10\rho^2+18\sqrt{1+\rho}+\rho(7+9\sqrt{1+\rho})\right) > 0
\end{aligned}$$

Rearranging terms, the previous inequality is equivalent to

$$\begin{aligned}
&\left(576 + 279\rho - 532\rho^2 - 143\rho^3 - 64\rho^4 - 100\rho^5 - 16\rho^6\right) \\
&> \sqrt{1+\rho}\left(576 - 240\rho - 804\rho^2 + 228\rho^3 + 192\rho^4 + 48\rho^5\right)
\end{aligned}$$

Both terms in parenthesis are strictly positive. The first result can be obtained by directly comparing the positive and negative individual terms. For the second one, note that the parenthesis is strictly positive when $\rho = 0$ and equals zero when $\rho = 1$. Also, the term is strictly decreasing, which implies that it must be strictly positive for $\rho \in (0,1)$. Thus, the previous inequality is equivalent to

$$\begin{aligned}
&\left(576 + 279\rho - 532\rho^2 - 143\rho^3 - 64\rho^4 - 100\rho^5 - 16\rho^6\right)^2 \\
&> (1+\rho)\left(576 - 240\rho - 804\rho^2 + 228\rho^3 + 192\rho^4 + 48\rho^5\right)^2
\end{aligned}$$

After calculation, this inequality is reduced to

$$\rho\left(3 + \rho - 4\rho^2\right)^2\left(29.568 + 48.073\rho + 16.674\rho^2 - 3.895\rho^3 - 3.896\rho^4 - 488\rho^5 + 64\rho^6 + 16\rho^7\right) > 0$$

We have that $29.568 > 3.895\rho^3$, $48.073\rho > 3.896\rho^4$ and $16.674\rho^2 > 488\rho^5$ which completes the proof. In the other side

$$\begin{aligned}\widehat{PS} =& \rho^2(\theta_{hh} - 2q_h^h(\tilde{\lambda}_h) - c)q_h^h(\tilde{\lambda}_h) + \rho(1-\rho)(\theta_{hl} - q_l^h(\tilde{\lambda}_h) - q_h^l(\tilde{\lambda}_h) - c)q_l^h(\tilde{\lambda}_h) \\ &+ (1-\rho)\rho(\theta_{lh} - q_h^l(\tilde{\lambda}_h) - q_l^h(\tilde{\lambda}_h) - c)q_h^l(\tilde{\lambda}_h) + (1-\rho)^2(\theta_{ll} - 2q_l^l(\tilde{\lambda}_h) - c)q_l^l(\tilde{\lambda}_h) \\ PS =& \rho^2(\theta_{hh} - 2q_h^* - c)q_h^* + \rho(1-\rho)(\theta_{hl} - q_h^* - q_l^* - c)q_h^* \\ &+ (1-\rho)\rho(\theta_{lh} - q_l^* - q_h^* - c)q_l^* + (1-\rho)^2(\theta_{ll} - 2q_l^* - c)q_l^*\end{aligned}$$

We have that

$$\widehat{PS} - PS = \Delta^2 Q(\rho)$$

with

$$\begin{aligned}Q(\rho) =& -\frac{(1-\rho)\left(16\rho^5 + \rho(153 - 60\sqrt{1+\rho}) - 180(-1+\sqrt{1+\rho})\right)}{36(2\rho + 3\sqrt{1+\rho})^2} \\ &- \frac{(1-\rho)\left(16\rho^3(5 + 6\sqrt{1+\rho}) + 5\rho^2(5 + 12\sqrt{1+\rho}) + \rho^4(68 + 48\sqrt{1+\rho})\right)}{36(2\rho + 3\sqrt{1+\rho})^2}\end{aligned}$$

Again, it is direct to see that $Q(1) = Q(0) = 0$. We will show that $Q(\rho) < 0$ for all $\rho \in (0, 1)$. It is sufficient to show that

$$\left(\rho(153 - 60\sqrt{1+\rho}) - 180(-1+\sqrt{1+\rho}) + 16\rho^3(5 + 6\sqrt{1+\rho})\right) > 0$$

Rearranging terms, the previous inequality is equivalent to

$$\left(153\rho + 180 + 80\rho^3\right) > \sqrt{1+\rho}\left(60\rho + 180 - 96\rho^3\right)$$

Both terms in parenthesis are positive. Thus, the previous is equivalent to

$$\left(153\rho + 180 + 80\rho^3\right)^2 > (1+\rho)\left(60\rho + 180 - 96\rho^3\right)^2$$

After calculation, this inequality is reduced to

$$\rho\left(1.080 - 1.791\rho + 59.760\rho^2 + 70.560\rho^3 + 11.520\rho^4 - 2.816\rho^5 - 9.216\rho^6\right) > 0$$

We have that $70.560\rho^3 > 2.816\rho^5$ and $11.520\rho^4 > 9.216\rho^6$. It is sufficient to show that

$$1.080 - 1.791\rho + 59.760\rho^2 > 0$$

The left-hand side term is a strictly convex function in ρ with a minimum in $\rho_m = \frac{1.791}{119.520}$. Evaluating the left-hand side in $\rho = \rho_m$, we obtain a strictly positive number, which completes the proof.