Decentralized Many-to-One Matching with Bilateral Search

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Abstract

In this paper, I analyze a finite decentralized many-to-one search model, where firms and workers meet randomly. In line with the existing literature, stable matchings of the many-to-one market can be enforced as search equilibria. However, in many-to-one search, firms collect workers in a cumulative manner. For this reason, unlike centralized matching markets, the collective structure of the firms affects the search process fundamentally. Furthermore, although stability in many-to-one markets can be analyzed through their related one-to-one markets, the many-to-one search model is essentially different from its related one-to-one counterpart. One sufficient condition for the equilibria in many-to-one markets to coincide with the equilibria of the related one-to-one market is that firms have additively separable utility over workers. The equilibria also coincide if time is costless for the agents. The paper provides a new matching matrix formulation for many-to-one matchings as well, which is more suitable in environments where matches evolve over time and there are complementarities between workers.

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1 Introduction

One of the most extensively studied topics in the literature on two-sided matching is many-to-one matching. Such matching problems arise when one side of the market can be matched to only one partner at most, while the other side can be matched to multiple partners. For instance, in the labor market, a worker can work for one firm, while the firms can hire multiple workers. The same applies to other examples, such as school choice, college admissions, and resident placement problems, which have been studied even before the development of matching theory.

The literature on centralized matching theory has provided valuable insights into many-to-one matchings. By obtaining preferences from both sides of the market, a centralized matchmaker can impose a plausible allocation in an economy. The National Resident Matching Program (NRMP) is a prime example of a centralized matching problem. The NRMP, a non-profit organization, collects preferences from residents and residency programs and determines the allocation of residents to programs. Similar to the NRMP, the Measurement, Selection, and Placement Center (MSPC) in Turkey conducts centralized exams such as the university entrance exam, collects students' preferences, and assigns them to colleges.

Nevertheless, many many-to-one matchings evolve in a rather decentralized search framework. For instance, students apply to graduate programs over time, and programs decide whether to accept them without an external entity orchestrating the market. Potential employees apply to jobs in a decentralized manner as well. In order to accommodate these kinds of environments, this paper is the first to study the many-to-one matchings in a finite decentralized search model. Over time, firms can employ as many workers as their capacity, whereas one worker can work for one company at most. The two sides randomly meet each other, and employment occurs upon the application by the worker and acceptance by the firm.

In the decentralized finite many-to-one search model, there are two sources of friction: First, time is costly for the agents. Second, the meeting process is random. The game is finite and dynamic, thus the appropriate equilibrium concept is subgame perfect equilibrium. Several information restrictions are introduced to distinguish between settings where agents can only observe the remaining market, past meetings and their conclusions, or anything that has occurred along any history.

Similar to Wu (2015), who analyzes one-to-one matchings in a similar fashion, I show that there is an equilibrium strategy that results in a stable matching. The equilibrium strategy is described based on the stable matching of the market and requires workers to apply to firms that they at least like as much as their stable partners. Firms, on the other hand, accept all the workers that they like more than their least preferred workers under the recommended stable matching. This positive result suggests that centralized corporations such as NRMP and MSPC can suggest strategies to the agents instead of imposing the allocation. The agents following the suggested strategies indeed constitute an equilibrium in the decentralized game, and the stable matching arises in a decentralized manner.

The centralized matching literature shows that many of the results in a marriage market often apply to many-to-one matchings. If firms' preferences over subsets of workers stem actually from their preferences over individual workers, there is a direct projection between many-to-one matchings and their one-to-one replicas, which is obtained by replicating each firm by their capacity. Therefore, whether we will find similarities while analyzing decentralized many-to-one models arises as a natural question. Namely, when we analyze a finite search model, what happens when firms search as collective entities for workers? How is it different from the one-to-one search, and how does the collective structure of the firms affect the search process? In fact, the connection in the centralized matching literature breaks down easily when we consider a decentralized model with frictions. The collective structure of the firm affects the search process on the firm side fundamentally, and therefore enters the workers' decisions as well. One way to overcome this issue would be to assume linear preferences for the firms, that is, firms value each worker individually even if it considers sets of workers.

Furthermore, every search problem is in a sense a stopping problem. The stopping problem requires calculating expected utilities from continuing the search, which was not possible with the many-to-one matrix definition that has been used in the existing literature, even when a cardinal utility structure is implemented. Therefore, I propose an alternative way of describing centralized allocations as a conceptual contribution.

This paper is organized as follows: In the rest of this section, I present a summary of the related literature and a motivating example. In section 2, I introduce the finite decentralized many-to-one bilateral search model. In Section 3, I present the results that connect the search model to the centralized many-to-one matching model. In section 4, I construct the related one-to-one search model and compare the relation between the search models to the centralized framework. I conclude in Section 5.

1.1 Literature Review

In two-sided matchings, the agents belong to one of two disjoint sets and are to be matched to the agents on the other set. The theory is accepted to have started with the famous Gale and Shapley (1962) paper and has revealed a lot about the structure of two-sided matching markets so far. The existence of a stable matching was shown via construction, in which the deferred acceptance algorithm was introduced, and it was proven that the resulting matching of the algorithm is stable. Following this rather early start-off, the theory has deepened slowly but with sound steps through other famous works by Roth in the 1980s (such as Roth (1985)). Finally, the basic results were collected in a book, *Two-sided matching*, by Roth and Sotomayor (1992). These three works constitute the benchmark in most of the papers in the literature, as well as your current read.

There is also a wide literature on the connection between search and matching models, where the stability of outcomes that arise as search equilibria are analyzed. One of the earliest examples of such papers is Eeckhout (1999), where agents have vertical preferences over the other side of the market. The paper shows that as the Poisson meeting process becomes instantaneous, the search model equilibria are equal to the stable matchings. Later, Adachi (2003) considers general preferences with an exogenous distribution of agents, where the search is costly in terms of time. Similarly, as search costs disappear, search model equilibria converge to the stable matchings in the underlying marriage market. Lauermann and Nöldeke (2014) contribute to this literature in the sense that they also allow for an endogenously evolving stock of agents. In their more relaxed model, search model equilibria converge to stable outcomes if there is a unique stable matching in the underlying market.

Search problems are widely studied not only on micro levels to understand the structure, but also on macro levels to answer bigger questions and have a better understanding of the economies. Most famous examples of the search literature include Ken Burdett and Coles (1997), Kenneth Burdett and Coles (1999), Mortensen (1982), and Pissarides (1985) search models.

However, current work is the most closely related to Wu (2015), in the sense that Wu considers a **finite** one-to-one search market. All the papers that analyze the search focus on a steady-state, where there is a continuum of both sides of agents. However, the very nature of the many-to-one problem, the steady-state analysis with a continuum of agents allows for meeting the same agent over and over again with time for the firms. Clearly, replicas of the same agent are not defined in the preferences of any underlying matching market. Thus, additional assumptions on the meeting process would be needed, which might fundamentally affect the randomness of the meeting process. For this reason, a finite market as in Wu (2015) would be a better and more realistic fit for the many-to-one markets. The equilibrium concept is then not a steady-state analysis but a subgame perfect equilibrium of the finite game.

Another closely related work would be Altınok (2019). In his job market paper, Altınok implements the existing many-to-one matching into a centralized model with two periods. His work is closely related to this paper in the sense that he analyzes how this slightly dynamized many-to-one market relates to the static version as well as its one-to-one counterpart. However, this paper is different from Altinok's dynamic but still centralized setup in the sense that it does not put any constraints on the number of periods and the market is fully decentralized.

1.2 Motivating Example

In the following, I will give a basic example of how the collective structure of the firms affects the search process so that taking the projection onto a one-to-one environment does not suffice to reveal all the insights.

Private Investigation Company:

Suppose a private investigation company is looking for detectives to hire. There are three detective candidates in the market: Sherlock (S), Watson (W), and Molly (M). There is no other firm in the market, and all the candidates prefer working for the firm to being unemployed. Therefore, the candidates will accept the firm whenever they meet, it suffices to consider the firm side. The firm has a capacity for two detectives at most, and the table summarizes the utility the firm receives for each subset that has at most two detectives.

	utility
S,W	1000
S,M	16
W,M	15
S	12
W	11
M	10
$\emptyset$	0

In the centralized market, there is a unique stable matching is where the firm employs both Sherlock and Watson. In the decentralized search model, the firm fills its vacancies one by one. That is, each period, the firm meets one of the candidates. Since the firm is their only option, the candidates will accept. Even without describing the model rigorously, it is clear that for low enough waiting costs, there is an equilibrium where the firm accepts only Sherlock and Watson and rejects Molly. This equilibrium ends in the same outcome as the unique stable matching, as long as Sherlock and Watson meet the firm at some point. In a similar manner, I will show how we can enforce stable matchings as search equilibria for low waiting costs.

Furthermore, consider the subgame in which the firm has already hired Sherlock and will hire at most one more detective. At this point, let me compare two search structures: Many-to-one search where the firm searches for detectives as a collective entity vs. the seats of the investigation firm searching individually for workers. In the remaining market without Sherlock, the firm meets Watson with probability p and Molly with probability $1 - p$.

1. Single Seats

   Suppose as in a one-to-one search and matching model, each vacancy (referred to as "seat") of the firm is searching for detectives on its own and has a separate meeting process. Once the firm has hired Sherlock, the remaining vacancy will meet Watson or Molly in the market. Suppose the single seat meets Molly. If the seat accepts Molly, it will receive a one-time utility of 10 and will leave the market.

   Suppose the seat rejects Molly. Since there is no other firm in the market, the remaining market does not change until the seat hires a detective. Therefore, if the seat rejects Molly once, it will keep rejecting her and wait only for Watson. The seat meets Watson with probability p, therefore it expects to meet Watson on day $\frac{1}{p}$. Since the workers accept the seat anyway, the expected utility from waiting for Watson over Molly is $\delta^{\frac{1}{p}}11$. Therefore, the seat accepts Molly if: $10 \geq \delta^{\frac{1}{p}}11$.

2. Collective Firms On the other hand, if the firm searches as a collective entity, the search process as well as the acceptance strategy upon meeting changes significantly. However, the meeting process does not change. Similar to the case above, if the firm (that is holding Sherlock) rejects Molly, it will keep rejecting her and wait only for Watson for its remaining vacancy. The firm's expected *additional* utility from waiting for Watson over Molly is $\delta^{\frac{1}{p}}(1000-12)$. Therefore, the firm accepts Molly if: $16-12 \geq \delta^{\frac{1}{p}}(1000-12)$.

   The left-hand side of the equation reflects the difference between having Sherlock and Molly together and having only Sherlock, whereas the right-hand side reflects the discounted difference between having Sherlock and Watson vs. having Sherlock only.

Taking δ and p as given, the condition for the collective entity to accept Molly is a lot harder to satisfy than the condition with the single seat. Because of super-additivity, Sherlock and Watson together as a pair is so valuable to the firm that having Sherlock already employed makes the incentives to wait for Watson higher when the firm searches collectively. It does not concern a single seat whom its replica has employed, whereas a collective firm cares about the already employed workers while admitting them one by one throughout time. The two conditions become equivalent when the utility is additively separable, i.e. the utility from employing two detectives is equal to the sum of having the detectives individually, which eliminates the effect of the collective structure.

1.3 Literature Review

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Following the conjecture in Roth and Sotomayor (1992) that search frictions would not disturb the stability of the outcome, the connection between the centralized model and its search component has been extensively studied. In these papers, the stability of outcomes that arise as search equilibria is analyzed. One of the earliest examples of such papers is Eeckhout (1999), where agents have vertical preferences over the other side of the market. The paper shows that as the Poisson meeting process becomes instantaneous, the search model equilibria are equal to the stable matchings. Later, Adachi (2003) considers general preferences with an exogenous distribution of agents, where the search is costly in terms of time. Similarly, as search costs disappear, search model equilibria converge to the stable matchings in the underlying marriage market. Lauermann and Nöldeke (2014) contribute to this literature in the sense that they also allow for an endogenously evolving stock of agents. In their more relaxed model, search model equilibria converge to stable outcomes if there is a unique stable matching in the underlying market.

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2 Model

2.1 Environment

There is a finite set of firms and a finite set of workers that are bilaterally and randomly searching for each other. The set of firms is denoted by $F = \{f_1, ..., f_n\}$ and the set of workers is $W = \{w_1, ...w_m\}$, where $(f, w) \in F \times W$ denotes a generic firm-worker pair.

Workers have complete and strict preferences over firms and remaining unemployed. The utility that worker w receives is denoted by $v(i, w)$, where $i \in F \cup \{w\}$. Workers are indifferent between the seats of firms and their coworkers. The many-to-one structure implies that firms can employ many workers, whereas a worker can be matched to one firm at most. The capacity vector $q = (q_{f_1}, ..., q_{f_n})$ specifies the maximum number of workers a firm can employ. Consequently, firms have complete and strict preferences over sets of workers, which is denoted by $u(f, \Omega)$, where $\Omega \subset W$. The utility of not being in a match is normalized for both parties: $u(f, \emptyset) = v(w, w) = 0$. A pair $(f, w) \in F \times W$ is an acceptable pair at u, v if w is *acceptable* for f , and f is acceptable for w, that is, $u(f, \{w\}) > 0$ and $v(f, w) > 0$.

Firms have preferences over sets of workers that actually are induced by an underlying preference relation over individual workers. In other words, if two sets differ in only one worker, the firm prefers the set with the more preferred worker. This condition is referred to as *q-responsive preferences* in the existing literature. Formally:

Definition 1. *The preferences of firms over 2^W are q-responsive if they satisfy the following conditions:*

1. *For all $\Omega \subset W$ such that $|\Omega| > q_f$, we have $u(f, \Omega) < 0$.*
2. *For all $\Omega \subset W$ such that $|\Omega| < q_f$ and $w \notin \Omega$, $u(f, \Omega \cup \{w\}) > u(f, \Omega)$ if and only if $u(f, \{w\}) > u(f, \emptyset) = 0$.*
3. *For all $\Omega \subset W$ such that $|\Omega| < q_f$ and $w, w' \notin \Omega$, $u(f, \Omega \cup \{w\}) > u(f, \Omega \cup \{w'\})$ if and only if $u(f, \{w\}) > u(f, \{w'\})$*

A many-to-one search market is represented by $M = (F, W, q, u, v)$. The assumptions on the market for a search game to start are the following:

1. Both parties have strict preferences.
2. The utility of being single is normalized to zero for both parties.
3. Firms have q-responsive preferences over sets of workers.
4. The market is finite: $|F| < \infty$, $|W| < \infty$, and $q_f < \infty \forall f$.
5. The market is nontrivial in the sense that there are some acceptable pairs.

2.2 The Search Game

The game starts at $t = 1$ with M and continues for an indefinite amount of time. On each day, a random (f, w) meets randomly. I describe the meeting process in detail in the following subsection 2.3. Upon meeting, w first decides whether to apply to f or not. Then f decides whether to hire w. If w does not apply or f rejects w, they separate and return to the market to keep searching. If w applies and f accepts, w leaves the market, f loses one of its vacancies. If f has more vacancies, it stays in the market and leaves if q_f is full after hiring w.

Upon hiring, w receives a one-time payoff of $v(f, w)$. The firm also receives a one-time payoff. However, this depends on the already hired workers. Namely, if f has already hired $\Omega \subset W$ before meeting w, the one-time payoff it gets after hiring w is $u(f, \Omega \cup \{w\}) - u(f, \Omega)$, i.e. it enjoys the additional utility it derives from hiring w. The meeting and the decisions take place on the same day t and if the meeting concludes with hiring, the agents' utilities are discounted by δ^t. The common discount factor refers to the cost of time and the first source of friction in the model.

Leaving the market upon mutual acceptance is permanent, that is, workers can not quit and the firms can not fire workers. As a result, the market shrinks over the course of time. Any submarket or remaining market is then denoted by $M' = (F', W', q', u, v)$, where $F' \subset F$ denotes the remaining firms in the market, $W' \subset W$ the remaining workers, and $q' \subset Q$ the remaining capacities of the firms F'. The capacities of the firms will decrease over time, therefore $q' \leq q$ necessarily.

2.3 The Contact Function

The second source of friction in the decentralized many-to-one search game is the contact function, which describes how the agents meet in a search market.

For any day, the contact function $C(f, w, M')$ defines the probability that the pair (f, w) will meet given that the remaining market at that day is M'. There are several but humble assumptions on the contact function such as:

1. The pair (f, w) meets with positive probability only if f has not filled its capacity and w is not matched to any firm: $C(f, w, M') = 0$ if $f \notin M'$ or $w \notin M'$.
2. A meeting occurs each day: $\sum_{(f,w) \in M'} C(f, w, M') = 1$.
3. Every meeting between the pairs in the remaining market is somewhat possible: $\exists \epsilon > 0$ s.t. $C(f, w, M') \geq \epsilon$ if $(f, w) \in M'$.
4. By definition, the contact function does not depend on the history itself. Whenever the remaining market is the same along different histories, the pair (f, w) has the same meeting probability.

The game ends if either one side of the market is fully matched or there are no mutually acceptable pairs remaining in the market, that is for any given submarket M', $\nexists (f, w)$ such that $u(f, \Omega \cup \{w\}) \geq u(f, \Omega)$ for $f \in M'$ holding Ω and $v(f, w) \geq 0$ for $w \in M'$ [1]. To avoid complications, I also assume that once in a submarket every combination of pairs ends in rejection, the meeting cycle does not start over and the game ends. Workers search until they are matched to a firm or the game ends. Similarly, firms search until they fill their capacity or the game ends. The search game is represented by the tuple $\Gamma = (F, W, q, u, v, C, \delta)$.

[1] Due to q-responsive preferences, $u(f, \Omega \cup \{w\}) \geq u(f, \Omega)$ is equivalent to $u(f, w) \geq 0$

2.4 Information Restriction and Equilibria

Because of the finite and dynamic structure of the many-to-one bilateral search game, the appropriate equilibrium concept for analysis is subgame perfect equilibrium. The set of all histories is denoted by $\mathcal{H}$, where $h \in \mathcal{H}$ and $\hat{\mathcal{H}}$ is the set of all non-terminal histories, after which the game continues with a new pair meeting. Additional to the initial search game Γ, any subgame that follows after history h is $\Gamma(h) = (F', W', q', u, v, C, \delta)$, and $M'(h)$ is the remaining market. A strategy profile is denoted by σ and $\sigma|\Gamma(h)$ is its restriction to the subgame $\Gamma(h)$.

Along with the regular subgame perfect equilibrium analysis, the paper considers some refinements on the information structure. For any history h of the search game, let $g(h) = (f_t, w_t, R_t)_{t=1:T(h)}$. This sequence notes all the pairs that meet along h, and the conclusions of these meetings, i.e. $R_t = \{\text{employment, separation}\}$. Furthermore, let $\mu(h)$ denote the instantaneous matching at history h, i.e. all the meetings that have ended with employment so far, and $M'(h)$ the submarket that remains after h.

1. Full awareness condition: This is the benchmark scenario where agents can condition their strategies on any component of history.
2. Private-dinner condition: The meetings take place in private environments. A strategy profile σ satisfies private-dinner condition if for any $h \neq h'$, $g(h) = g(h')$ implies $\sigma|\Gamma(h) = \sigma|\Gamma(h')$.
3. Slice-of-life condition: The meetings that have ended with employment is indirectly observed via the instantaneous matching. A strategy profile σ satisfies the slice-of-life condition if for any $h \neq h'$, $\mu(h) = \mu(h')$ implies $\sigma|\Gamma(h) = \sigma|\Gamma(h')$.
4. Markov condition: Agents can only observe (or condition on) the remaining market after any history. A strategy profile σ satisfies private-condition if for any $h \neq h'$, $M'(h) = M'(h')$ implies $\sigma|\Gamma(h) = \sigma|\Gamma(h')$

Verbally, the Markov condition refers to when agents condition their strategies on only the remaining market. If the agents can observe the instantaneous matching, this is taking a picture of the current stage of the game, so I call this the slice-of-life condition. The Markov condition would prevent firms from conditioning their behavior on their already hired workers. Thus, the most restrictive information structure that is in line with a many-to-one matching is the slice-of-life condition. The private-dinner condition refers to when they also observe and react to past meetings and their realizations[2]. However, under private-dinner

[2]The contact function C is also Markovian, in the sense that it only depends on the remaining market

condition, the agents only observe the conclusions, not the actions taken in the meetings. Only under full awareness, they can condition their strategies on any component of h. The conditions get stronger as we move from full awareness to Markov. Therefore, the set of equilibrium strategies weakly shrinks in the same direction.

Independent of the information structure, the game dictates the following for optimal behavior upon meeting:

1. A worker w applies to f if the expected utility upon application exceeds the expected utility of continuing the search.
2. Any firm f accepts a w for one of its vacancies if the instant utility gain and the expected utility gain for the remaining vacancies exceeds the expected utility gain for continuing search without accepting w.

A strategy profile σ constitutes a subgame perfect equilibrium of the search game $(F, W, q, u, v, C, \delta)$ if σ is optimal for every agent in every subgame.

3 Connection to Centralized Many-to-One

3.1 Evolving Many-to-One Matching

Naturally, at each possible history h, there is an instantaneous matching in the market, that is denoted by μ_h. The instantaneous matching collects all the meetings that have ended with recruitment so far. In compliance with the centralized matching literature, any instantaneous matching is indeed a many-to-one matching and has the following properties.

1. $|\mu_h(w)| = 1$ and if $\mu_h(w) \neq w$, then $\mu_h(w) \subset F$.
2. $\mu_h(f) \in 2^W$ and $|\mu_h(f)| \leq q_f$.
3. $\mu_h(w) = f$ if and only if $w \in \mu_h(f)$.

Observe that any instantaneous matching μ_h implies a remaining market such that already matched agents and seats of the firms are removed from the initial market. Similarly, any contact function C together with a strategy profile σ induce a probability mass function on outcome matchings. A matching μ *arises* if it appears as an outcome matching with positive probability under σ. If μ appears at a terminal history on some equilibrium path, it is an *equilibrium matching*. If μ obtains almost surely under σ, we say that σ *enforces* μ[3].

The most commonly used equilibrium concept in matching theory is stability, which reflects the sustainability of a matching. A matching μ is stable if it is individually rational and pairwise stable. A matching μ is individually rational if all the matched pairs are acceptable. Pairwise stability refers to the absence of blocking pairs. A matching μ is blocked by the pair (f, w) if they are not matched under μ but prefer each other over their matches under μ. That is, μ is blocked by the pair (f, w) if at least one of the following conditions hold:

1. If $|\mu(f)| \leq q_f$ and $\mu(w) \neq f$,
   $v(f, w) > v(\mu(w), w)$ and $u(f, w) > u(f, w')$ for some $w' \in \mu(f)$.
2. If $|\mu(f)| < q_f$ and $\mu(w) \neq f$,
   $v(f, w) > v(\mu(w), w)$ and $u(f, w) > 0$

Since the preferences of the firms are q-responsive, the set of all stable matchings is non-empty for any preference profile. Furthermore, by the famous Rural Hospital Theorem of Roth (1984), the set of matched agents is the same under every stable matching, and each firm that does not fill its quota has the same set of agents matched under every stable matching.

3.2 Equilibrium Outcomes vs Stable Matchings

In this section, the main question is the relation between the SPE outcomes of the search game and the stable outcomes of the underlying many-to-one market. Several results follow directly from Wu (2015) or need an adaptation due to the many-to-one structure.

First of all, small values of δ reflect high costs of time. If waiting is sufficiently costly, the workers would apply to acceptable firms and firms would accept every acceptable worker. Therefore, the rather interesting question is for larger δ, in fact for $\delta \to 1$. A strategy profile σ is said to be a limit equilibrium of the many-to-one search environment (F, W, q, u, v, C) if there exists some $d < 1$ such that σ is an SPE of the many-to-one search game $(F, W, q, u, v, C, \delta)$ for all $\delta > d$.

Proposition 1. *Suppose μ is a stable matching in the many-to-one market $M = (F, W, q, u, v)$. The following strategy profile S^* is a limit equilibrium in any many-to-one search game $(F, W, q, u, v, C, \delta)$ and it enforces μ:*

[3] The set of matching that arise under σ may contain other elements than μ, which all have zero probability of arising.

1. *Workers apply only to firms such that* $v(f,w) \geq v(\mu(w),w)$ *in any submarket.*
2. *Firms accept workers such that* $u(f,w) \geq \min(u(f,w'))$ *such that* $w' \in \mu(f)$*, reject others.*

In other words, workers apply to firms that they weakly prefer to their allocation under μ *and firms accept workers whom they prefer to their least favorite worker under* μ*.*

Corollary 1. *For any stable* μ *of a many-to-one market, there exists a limit equilibrium that enforces* μ*. Earlier results in the literature ensure the existence of a stable matching in any many-to-one market. As a result, the enforcement of a stable matching as a limit equilibrium is ensured.*

In other words, if agents are patient enough, it is possible to obtain stable matchings as equilibrium outcomes. However, the strategy depends on the stable matching of the underlying market. Because of that, we need agents who are perfectly aware of the market with a single stable matching in the underlying market or a centralized corporation advising them about which stable matching to implement.

However, as Wu (2015) shows, unstable matchings can also be enforced in full-awareness equilibria with reward and punishment schemes. Nevertheless, no unstable matching can be enforced in private-dinner equilibria, yet they can arise with positive probability as outcome matching. Those results are simply applicable to our framework since every one-to-one matching is essentially a many-to-one matching, with the specification that $q_f = 1 \quad \forall f$.

4 Connection to Decentralized One-to-One

4.1 The Related One-to-One Search Game

Following Roth and Sotomayor (1992), one can construct a *related* marriage market for every many-to-one matching by simply replicating the seats of a firm or a college. These replicas share the same preferences as the collective entity, and if necessary, the ties in the workers' preferences are broken arbitrarily. They show that this hypothetical replica market is directly linked to its many-to-one original. More specifically, a many-to-one matching is stable if and only if the corresponding one-to-one matching in the related marriage market is stable. This result simplifies the many-to-one matching analysis quite a lot. In this section, I describe the challenges of carrying out a similar analysis for decentralized environments and identify a sufficient condition that establishes the connection between many-to-one and one-to-one frameworks.

It would not be natural to assume the firms not being aware of the workers that they have employed at some history. To avoid these additional information effects that could yield differences between many-to-one and one-to-one markets, I assume the most restrictive information criterion is the slice-of-life information condition throughout this chapter. That is, the replica seats are also aware of who was matched to whom until then.

In order to be able to analyze how the search behavior changes in a many-to-one market, we need to understand how the model differentiates from a one-to-one market intrinsically. Therefore, a related one-to-one search model has to be defined. For the sake of avoiding repetitions, I will only discuss differences in the related one-to-one market. Whatever remains unspecified is the same as in the original many-to-one market.

The *related* one-to-one search model will be a reflection of our many-to-one search model onto a one-to-one environment. As simple as it gets, a related one-to-one market is obtained where each firm is replicated as many times as its capacity. In other words, each seat of the firms is individually present in the search market. Furthermore, the seats are individually searching for workers and hence become competitors.

When we replicate firms, we obtain q_f identical seats. Namely, the set of the seats in the related one-to-one market is $S = \{s_{11}, ..., s_{1q_1}, ..., s_{n1}, ..., s_{nq_n}\}$, where s_{ij} is the jth seat of firm i. Recall that the preferences of firms in the original many-to-one-market are complete and responsive. This means that we can deduct the firms' preferences over individuals. Each seat has the same preference over individuals as the firm.

On the workers' side, there is no replication. Each worker from $W = \{w_1, ...w_m\}$ is still searching for himself. Workers are indifferent between the seats of the same firm and each worker prefers a seat in firm j over a seat in firm k if and only if he prefers firm j over firm k in the many-to-one market. Formally, $v(s_{j1}, w_i) = ... = v(s_{jq_j}, w_i)$ but $v(s_{jn}, w_i) < v(s_{km}, w_i)$ whenever $v(f_j, w_i) < v(f_k, w_i)$. A related one-to-one market is then the tuple $M_R = (S, W, u, v)$. Similarly, any submarket (remaining market) is $M'_R = (S', W', u, v)$ with $S' \subset S$ and $W' \subset W$. The contact function is adjusted such that for each submarket, the sum over the probabilities of w and s meeting in M'_R ans s is replicated from f of M' is equal to $C(f, w, M')$ in the original market.[4] All other assumptions on the contact function remain the same. Then, the related one-to-one game is denoted by the tuple $\Gamma_R = (S, W, u, v, C, \delta)$.

Another result that Wu proved for one-to-one markets can be retrieved using the related

[4]Any instantaneous matching μ translates onto the one-to-one replica such that $\mu_R(w) = s_{in} \Rightarrow \mu(w) = f_i$.

market construction.

Definition 2. *The pair* (s, w) *is called a top-pair for any related (sub)market if: Among the seats that find* w *acceptable,* s *is the best for* w*, and among the workers that find* s *acceptable,* w *is the best for* s*.*

Following Wu, a related marriage market satisfies the Sequential Preference Condition (SPC) if there is an ordering of the seats and workers and a positive integer k *such that:*

1. *For any* $i \leq k$, (s_i, w_i) *is a top-pair in the (sub)market.*
2. *Discarding the top-pairs result in a trivial market with non-acceptable pairs.*

A many-to-one market satisfies the SPC if its related one-to-one market does.

Proposition 2. *If the initial market satisfies the Sequential Preference Condition, no unstable matching can be enforced in any SPE.*

Proof. In a market where preferences satisfy SPC, there is a unique matching that allocates top pairs to each other. In the original many-to-one search game, the top pairs would always accept each other. Since every pair has a positive probability of meeting, there is an equilibrium path in which top pairs are assigned to each other, and this happens with positive probability. Therefore, enforced matching cannot be unstable when the preferences of the initial market satisfy SPC. □

Even though we need a modification for the Sequential Preference Condition for many-to-one markets, the result and the proof method directly apply from Wu (2015). The unique set of workers will apply to the firm, and the firm will accept those. Thus they cannot credibly threaten each other with rejection, which then results in employment upon first meeting.

4.2 Related Equilibria

Now that the original many-to-one search model and the related one-to-one search model are established, I can present the results. In order to describe the connections between the two environments, first, equilibrium equivalence needs to be defined. Intuitively, the equilibria of the many-to-one search model and its related one-to-one search model are equivalent if the workers adapt the same acceptance strategy for the seats of a firm as the strategy they use for the firm, and the firms' seats use the same acceptance strategy for each worker as the firm itself.

Definition 3. *For any many-to-one market and its related market, the equilibria of the many-to-one search game and its related one-to-one search game are equivalent if under the same information restriction and for every related subgame:*

1. *Workers of the related market accept the seats that belong to the firms they apply to in the original market and reject others.*
2. *Seats of the related market accept the same workers as their mother firm.*

The following lemma additionally shows that we can track equilibrium equivalence from remaining markets, even though we can not observe agents' strategy profiles.

Lemma 1. *For each realization of the contact function C, the slice-of-life equilibria of the many-to-one search model and its related one-to-one search model are equivalent if and only if the remaining markets are related for each history.*

The above lemma establishes the equivalence between the equilibrium strategies and the remaining markets. In the following, I will show a sufficient condition that establishes a direct connection between the many-to-one model and its related one-to-one search model.

Proposition 3. *Suppose the firms in a many-to-one search market have linear utility, i.e.* $u(f, \Omega \cup \{w\}) = u(f, \Omega) + u(f, \{w\}) \quad \forall (f, \Omega, w)$. *If a matching is an equilibrium matching in the related one-to-one search model, then the corresponding many-to-one matching is an equilibrium matching in the many-to-one search model, where firms use the same acceptance strategy as in the seats of the related model for any remaining market, regardless of the subset of workers the firm has already hired.*

Observe that linear utility is a much stronger condition than responsiveness. That is, in order to project a many-to-one environment onto a one-to-one environment, we need a stronger conditions. However, that condition is usually too demanding. For instance, because of diminishing marginal returns, instead of $u(f, w) - u(f, \emptyset) = u(f, \Omega \cup w) - u(f, \Omega)$ it might be more plausible that $u(f, w) - u(f, \emptyset) \geq u(f, \Omega \cup w) - u(f, \Omega)$ for all acceptable workers to firm f and for all subsets such that $w \notin \Omega$. This intuitively means that the utility of a worker to an individual worker to firm f weakly decreases with the number of other workers who are already employed in f.

Proposition 4. *For the limit case $\delta \to \infty$, a strategy profile σ is a slice-of-life subgame perfect equilibrium in the original many-to-one search game if and only if its equivalent strategy profile σ_R is a slice-of-life subgame perfect equilibrium in the related search game.*

Proof. Suppose h is a terminal history for the original game Γ and (f, w) meet, such that $\mu(h)(f) = \Omega$. Then, f accepts w if $u(f, \Omega \cup \{w\} \geq u(f, \Omega)$. By responsive preferences, this is equivalent to $u(f, w) \geq 0$ and therefore $u(s, w) \geq 0$ for any replica seat of f. Therefore, the equilibrium strategy profiles imply the same acceptance condition for the mother firm and its replica seats at any terminal history. By induction, it can be shown that they are equal for non-terminal histories as well, which concludes the proof. □

Corollary 2. *The limit SPE of a many-to-one search game and the limit SPE in its related one-to-one search game are equivalent.*

The proposition presented above highlights that the interdependencies between workers that impact the acceptance of equilibrium become insignificant as the cost of time decreases. In such an environment, corporations make distinct hiring decisions, which necessitates the establishment of a human resources department to manage the recruitment process for vacant positions. However, if the agents demonstrate sufficient patience, the market could function effectively without the need for a human resources department. Ultimately, the market would evolve to reflect the optimal hiring decisions for the given conditions.

4.3 Matchings of the Centralized Market à la Search Game

This subsection takes a quick detour into the existing literature of many-to-one matchings and describes the adaptations that will translate the existing results into a many-to-one search environment. In order to find the set of stable matchings, a linear programming approach is developed. As Vate (1989) and Rothblum (1992) characterize stable matchings in a marriage market as extreme points of a convex polytope, Baïou and Balinski (2000) extend the results to a many-to-one matching market. They show that simply replacing the parameters of a marriage market with their many-to-one counterparts does not extend the results of Rothblum, but needs a slight differentiation. Neme and Oviedo (2020) adapt their approach as well, so am I:

Given a matching μ, an assignment matrix $x^B \in \mathbb{R}^{|F| \times |W|}$ (B for Baïou and/or Balinski) is defined where all its elements are denoted by $x^B(f, w)$ where $x^B(f, w) \in \{0, 1\}$ and $x^B(f, w) = 1$ if and only if $\mu(w) = f$.

Following Baïou and Balinski, let CP denote the convex polytope generated by the following linear inequalities:

$$\sum_{j \in W} x^B_{f,j} \leq q_f \qquad \forall f \in F \tag{1}$$

$$\sum_{i \in F} x^B_{i,w} \leq 1 \qquad \forall w \in W \tag{2}$$

$$x^B_{f,w} \geq 0 \qquad \forall (f,w) \in F \times W \tag{3}$$

$$x^B_{f,w} = 0 \qquad \text{for unacceptable pairs } (f,w) \tag{4}$$

The integer solutions to (1)-(3) are assignment matrices of simple many-to-one matchings. A matching, where some entries $x^B(f,w)$ are non-integers in the interval $(0,1)$ is called a fractional matching. We can interpret the fractional matchings as probabilities that the agents are matched to one another as well as the timeshares the respective agents spend with each other.

An example of a many-to-one assignment matrix with 2 firms $\{f_1, f_2\}$ and 2 workers $\{w_1, w_2\}$ would look like as follows:

	f_1	f_2
w_1	$x^B(f_1, w_1)$	$x^B(f_2, w_1)$
w_2	$x^B(f_1, w_2)$	$x^B(f_2, w_2)$

where all the entries are nonnegative and:

$$\sum_F x^B(f,w) \leq 1 \text{ for both workers}$$

$$\sum_W x^B(f,w) \leq q_f \text{ for both firms}$$

Adding (4) imposes the individual rationality constraint, that the match is at least as good as the outside option. As Baïou and Balinski show, adding another linear inequality to the CP system:

$$\sum_{u(f,j)>u(f,w)} x^B_{f,j} + q_f \sum_{v(i,w)>v(f,w)} x^B_{i,w} + q_f x^B_{f,w} \geq q_f \qquad \forall (f,w) \in A \tag{5}$$

defines the stable convex polytope SCP and the integer solutions to SCP define stable simple matchings, which are individually rational and pairwise stable.

In Kojima and Manea (2010) and Kesten and Ünver (2015), "The School-Choice Birkhoffvon Neumann Theorem states that any fractional matching can be represented as a lottery (not necessarily unique) over simple matchings", which allows us to interpret a fractional matching in a third way. Nevertheless, the intuition about the stable matchings turns out to be incorrect and the non-integer solutions of inequalities (1)-(5) do not immediately give us stability when it comes to fractional matchings.

In fact, as shown by Baïou and Balinski Baïou and Balinski (2000) and elaborated further in Neme and Oviedo Neme and Oviedo (2020), the non-integer solutions to the *SCP* might be blocked in a *fractional way*, by a firm and worker, who want to increase their timeshare together, at the expense of those they like less at a non-integer solution to *SCP*. Formally:

Definition 4. *A matching is blocked by the firm-worker pair* (f, w) *in a fractional way, when* $x^B(f, w) < 1$, $v(f, w) > v(f', w)$ *for some* f' *such that* $x^B(f', w) > 0$ *and* $u(f, w) > u(f, w')$ *for some* w' *such that* $x^B(f, w') > 0$.

Example: An example from Baïou and Balinski (2000) and Neme and Oviedo (2020) which shows an assignment matrix, which is a solution to *SCP* and blocked in a fractional way by is as follows:

	f_1	f_2
w_1	1	0
w_2	0.5	0.5
w_3	0.5	0.5
w_4	0	1

where the preferences (over individuals, derived from the preferences over sets) are such that:

$$f_1 : u(f_1, w_1) > u(f_1, w_2) > u(f_1, w_3) > u(f_1, w_4) \quad \text{and} \quad q_{f_1} = 2$$
$$f_2 : u(f_2, w_4) > u(f_2, w_3) > u(f_2, w_2) > u(f_2, w_1) \quad \text{and} \quad q_{f_2} = 2$$
$$w_1 : v(f_2, w_1) > v(f_1, w_1)$$
$$w_2 : v(f_2, w_2) > v(f_1, w_2)$$
$$w_3 : v(f_2, w_3) > v(f_1, w_3)$$
$$w_4 : v(f_1, w_4) > v(f_2, w_4)$$

In the example above, it can easily be checked that the numbers solve the linear problem *SCP*. However, w_3 likes f_2 better than f_1, but his time is shared equally between the firms.

In addition, f_2 likes w_3 better than w_2 but one seat is shared equally between those workers. In such a case, the pair (f_2, w_3) blocks the assignment above in a fractional way so that they can increase their time spent together in exchange for their other partners in the matching, f_1, and w_2.

As it is not mentioned in either of the papers, the lottery interpretation of the fractional matchings helps us understand the underlying misfunction in this example. Although the lottery over simple matchings which represents a fractional matching is generically not unique, in this example it actually is unique. The fractional matching is a lottery over the following two simple matchings with equal probability 0.5:

	f_1	f_2
w_1	1	0
w_2	1	0
w_3	0	1
w_4	0	1

	f_1	f_2
w_1	1	0
w_2	0	1
w_3	1	0
w_4	0	1

The reason why there is a blocking pair in a fractional way can be observed in the lottery as well. Although the first simple matching of the lottery is stable, the second one is not. Furthermore, the fact that it is not stable pins down the fractional blocking pair: The second simple matching is blocked by the pair (f_2, w_3).

Neme and Oviedo (2020) refer to matchings in which there are no incentives to block (neither as in the usual way nor in the fractional interpretation) as "strong stable matchings" and prove that they can be found adding the additional constraint to SCP:

Definition 5. *Let (F, W, q, u, v) be a many-to-one matching market. A fractional matching is strongly stable if for each acceptable pair (f, w), x satisfies the strong stability condition:*

$$\left[q_f - \sum_{u(f,j)\geq u(f,w)} x^B_{f,j}\right] \cdot \left[1 - \sum_{v(i,w)\geq v(f,w)} x^B_{i,w}\right] = 0 \tag{6}$$

Observe that the assignment matrix of a simple stable matching fulfills (6). This follows from the simple fact that if f and w are matched, the second multiplier is 0. If they are not matched with each other, at least one of them is consuming its own capacity.

For fractional matchings, when (6) does not hold for some (f, w), $q_f > \sum_{u(f,j)\geq u(f,w)} x^B_{f,j}$ and $1 > \sum_{v(i,w)\geq v(f,w)} x^B_{i,w}$, it means that there are f' and w' such that $u(f, w) > u(f, w')$, $v(f, w) > v(f', w)$ and $x(f, w') > 0$, $x(f', w) > 0$ and $x(f, w) < 1$. In such a scenario, the

(f, w) would block the assignment and increase their time shared together.

Insightful Theorem 1 from Neme and Oviedo (2020) concludes: "If x^B is a strongly stable fractional matching, it can be represented as a convex combination of stable matchings. Furthermore, a lottery over simple stable matchings is strongly stable as well." This establishes the lottery interpretation as in Lauermann and Nöldeke (2014).

Both Baïou and Balinski (2000) and Neme and Oviedo (2020) constructed the matching as an assignment matrix with two sides in rows and columns respectively. This approach is quite useful to visualize and point out simple stable matchings. However, the structure in Lauermann and Nöldeke (2014) requires a different approach. The main difference is that the many-to-one papers of Baïou and Balinski and Neme and Oviedo use an ordinal utility approach, whereas Lauermann and Nöldeke employ a cardinal utility in their model, which allows them to calculate the expected utilities for the agents as well. As discussed before, cardinal utility adaptation is vital for a search structure.

If $u(f, \Omega)$ is a linear function of individual values, i.e. additively separable such that $u(f, \Omega) = \sum_{i \in \Omega} u(f, w_i)$, we could still use the assignment matrix with the agents in rows and columns to calculate expected utilities. In order to employ such a structure in a many-to-one framework and be able to calculate expected utilities at the same time, we would need a different assignment matrix $x \in \mathbb{R}^{|F| \times |2^W|}$, satisfying:

$$\sum_{F} \sum_{\substack{\Omega \subset 2^W \\ w \in \Omega}} x(f, \Omega) \leq 1 \qquad \forall w \tag{7}$$

$$\sum_{\Omega \subset 2^W} x(f, \Omega) \leq 1 \qquad \forall f \tag{8}$$

$$x(f, \Omega) = 0 \qquad \forall |\Omega| > q_f \tag{9}$$

$$x(f, \Omega) \geq 0 \qquad \forall (f, \Omega) \in F \times 2^W \tag{10}$$

The above-described assignment matrix has firms in the columns and all possible subsets of the workers in the rows. In a many-to-one matching, a specific worker can not be matched to two firms. Hence, considering a worker would now require considering each subset that this worker appears in.

The expected utilities from a matching x then can be calculated as follows:

$$U(f;x) = \sum_{\Omega} x(f,\Omega)u(f,\Omega)$$
$$V(w;x) = \sum_{F} \sum_{\substack{\Omega \subset 2^W \\ w \in \Omega}} x(f,\Omega)v(f,w)$$

Example: An assignment matrix of a many-to-one matching market with 2 workers $\{w_1, w_2\}$ and 2 firms $\{f_1, f_2\}$ would be as follows according to the latter description:

	f_1	f_2
$\{w_1, w_2\}$	$x(f_1, \{w_1, w_2\})$	$x(f_2, \{w_1, w_2\})$
w_1	$x(f_1, w_1)$	$x(f_2, w_1)$
w_2	$x(f_1, w_2)$	$x(f_2, w_2)$

where all the entries are nonnegative and:

$$x(f_1, \{w_1, w_2\}) + x(f_1, w_1) + x(f_2, \{w_1, w_2\}) + x(f_2, w_1) \leq 1$$
$$x(f_1, \{w_1, w_2\}) + x(f_1, w_2) + x(f_2, \{w_1, w_2\}) + x(f_2, w_2) \leq 1$$
$$x(f, \{w_1, w_2\}) + x(f, w_1) + x(f, w_2) \leq 1 \text{ for both firms}$$
$$x(f, \Omega) = 0 \quad \text{if} \quad |\Omega| > q_f \text{ for all subsets}$$

With expected utilities:

$$U(f;x) = x(f, \{w_1, w_2\})u(f, \{w_1, w_2\}) + x(f, w_1)u(f, w_1) + x(f, w_2)u(f, w_2)$$
$$V(w_1;x) = \sum_{f \in F} \left[x(f, \{w_1, w_2\}) + x(f, w_1)\right] v(f, w_1)$$
$$V(w_2;x) = \sum_{f \in F} \left[x(f, \{w_1, w_2\}) + x(f, w_2)\right] v(f, w_2)$$

Proposition 5. *The assignment matrix described before by Baïou-Balinski and Neme-Oviedo, which has the workers instead of sets of workers in the rows can easily be calculated from the matrix described above, by setting* $x^B(f,w) = \sum_{\substack{\Omega \subset 2^W \\ w \in \Omega}} x(f, \Omega)$.

Similarly, the entries of the assignment matrix x can be calculated if all the entries of x^B are integers, i.e. $x^B(f,w) = \{0,1\} \ \ \forall (f,w)$. *This does not necessarily hold if some entries of* $x^B \in (0,1)$.

When we consider the finite decentralized many-to-one search game, the acceptance decisions of the agents represent a stopping agreement. Therefore, we need to be able to calculate the expected utilities from continuing the search and compare them to the gains from an immediate acceptance decision. As the next subsection shows, this new definition of a matching matrix will enable us to do such comparisons.

4.4 From Equilibria to Assignment Matrices

In pursuance of the analysis of how the subgame perfect equilibria of the many-to-one search model relate to the stable matchings of the centralized many-to-one market, I will describe the assignment matrix that can be obtained from the search equilibrium. In fact, any equilibrium of the search model implies an assignment matrix. Intuitively, the assignment matrix will show the probability of a firm and a subset of workers being matched in equilibrium. In the following revisions, this methodology will be implemented to show whether equilibrium matchings are stable.

In order to calculate the assignment matrix from an equilibrium of the search game, we look at the terminal histories. For any finite terminal history $h \in \mathcal{H} \setminus \hat{\mathcal{H}}$ that takes T periods, let $C_h^t(f, w, M_h^t) \in \{0, 1\}$ denote the meeting function realization for any $t \in \{0, ..., T\}$, where M_h^t is the remaining market implied by μ_h^t, the instantaneous matching in the beginning of period t. [5] The meeting function realization is such that $C_h^t(f, w, M_h^t) = 1$ for firm f and worker w who meet at period t along h and $C_h^t(f, w, M_h^t) = 0$ for all other pairs. The pair (f, w) such that $C_h^t(f, w, M_h^t) = 1$ will be referred to as i_h^t, since they are the agents of the stage game.

Similarly, $a_h^t(i_h^t, \mu_h^t) \in \{A, R\}^2$ denotes the action realization for i_h^t.[6] After the action profile of t realizes, the instantaneous matching is updated to μ_h^{t+1} and remaining market is M_h^{t+1}. Since the market does not change if any of the parties reject, $M_h^{t+1} = M_h^t$ unless $a_h^t(i_h^t, \mu_h^t) = (A, A)$. If both parties accept, the worker leaves the market and the firm leaves the market if the capacity is full.

With this formulation, we can now calculate the probability of a terminal history h occurring on the equilibrium path of the search game. The probability of meeting is simply determined by the choice function, $\mathbb{P}(C_h^t(f, w, M_h^t) = 1) = C(f, w, M_h^t)$.

In order to reach day 1 on the equilibrium path of h, the agents who meet on day 0 should be aligned with h and they should decide accordingly as well. The probability of reaching day 1 under history h, denoted by $\mathbb{P}(\mu_h^1)$ and satisfies $\mathbb{P}(\mu_h^1) = C(i_h^0, M_h^0)\mathbb{P}(a_h^0(i_h^0, \mu_h^0))$. By

[5] Clearly, $M_h^0 = M$ for any history.

[6] Recall that we restrict attention to pure strategies.

induction, the probability of reaching any day $t \leq T$ along history h can be calculated by multiplying the probability of agents meeting and behaving according to the history along the equilibrium path of h:

$$\mathbb{P}(\mu_h^t) = \prod_{k=1}^{t} C(i_h^{k-1}, M_h^{k-1})\mathbb{P}(a_h^{k-1}(i_h^{k-1}, \mu_h^{k-1}))$$

Subsequently, for any given many-to-one search game $(F, W, q, u, v, C, \delta)$ once the equilibrium acceptance strategies of the agents are calculated, we can restrict attention to terminal histories and easily calculate the probability of any f, Ω being matched at the end of the game by simply adding up the probabilities of different terminal histories at the end of which f and Ω are together. By simply taking this probability equal to $x(f, \Omega)$, a matching matrix can be constructed.[7]

5 Conclusion

This paper is the first paper that describes and analyzes a finite decentralized many-to-one bilateral search model. Considering the components of the model, a finite search model is described à la Wu (2015). Comparing the subgame perfect equilibria to the stable matchings of the underlying many-to-one market, the existence of an equilibrium is guaranteed by the finite structure of the game. Furthermore, as the search frictions vanish, stable matchings of the underlying market can be enforced by equilibrium matchings. This implies that we can expect stable outcomes in a decentralized many-to-one search market where agents are sufficiently patient.

Different than the centralized matching models, the many-to-one search model differentiates fundamentally from its projection onto one-to-one environments. The cumulative structure of the firm affects the search process due to the complementarities between workers. The paper presents linear utility or costless time as sufficient conditions that re-establish the connection between the two models. Without these conditions, the firms might need a separate human resources department to look after its benefits.

Moreover, I show that the existing matrix definitions of many-to-one matchings cannot be used in an environment where matchings evolve over time. When calculating expected utilities from multiple matchings, the current structure only allows for linear utilities or

[7]The game ends almost surely, and it can easily be shown that this matrix satisfies the properties of a many-to-one matching.

simple matchings (there is a unique equilibrium matching). To overcome this obstacle, I propose an alternative many-to-one matching definition for the underlying matching market.

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6 Appendix

Proposition 1:

Proof.

- S^* enforces μ.
  S^* enforces μ: μ obtains almost surely on the equilibrium path. For any meeting function realization, another outcome than μ arising from S^* has probability of 0. Suppose f and w are not matched under μ but they end up together under S^* for some meeting function realization. Mutual acceptance requires $u(f, w) \geq \min(u(f, w'))$ such that $w' \in \mu(f)$ and $v(f, w) \geq v(\mu(w), w)$ which contradicts with μ being stable under responsive preferences. This part concludes all pairs in the outcome are consistent with μ.
  Also, note that all existing agents meet with some positive probability and the game only ends when all meeting combinations end in rejection. Since the pairs under μ accept each other, and the probability of a rejecting pair (or a rejection pair cycle) occuring has a probability of 0, μ obtains almost surely on the equilibrium path.
- S^* constitutes a limit equilibrium of any search game. For any firm f, the expected utility of S^* is $u(f, \mu(f))$ and for any worker w the expected utility of S^* is $v(\mu(w), w)$. Now consider a one-step deviation by w such that w accepts f instead of rejecting as under S^*. Rejection under S^* implies $v(f, w) < v(\mu(w), w)$. If f rejects w, the subgame does not change and the expected utility of w does not change. If f accepts, w receives a lower utility. Therefore, it is not benefical for w to accept f. The same logic applies to f and a one step deviation towards rejecting.

□

Lemma 1:

Proof. Let C be any realization of the contact function. I will prove the lemma by proving the if statements from both directions.

1. The equilibria of many-to-one search and the related one-to-one search are equivalent $\Rightarrow$ remaining markets are related for each history.
   Easily proven by induction. Start with the initial market M. Equivalent acceptance

strategies imply:

$$\begin{aligned} f \text{ accepts } w &\iff s \text{ accepts } w \\ w \text{ accepts } f &\iff w \text{ accepts } s \end{aligned}$$

This means, for the same realized related contact function, M' after the first day is the same. Apply this to every step, the first part of the lemma concludes.

2. Remaining markets are related for each history $\Rightarrow$ The equilibria of both search models are equivalent.

   Suppose s is a seat of f. If when s, w and f, w meet after h at the related remaining markets, and the remaining market after this is also the same s, f use the same acceptance strategies.

   If this holds for each remaining market and history, the equilibria are equivalent.

□

Proposition 3:

Proof. When both sides use the same strategies as in a related one-to-one search model, the mutual acceptance probabilities are equal.

It also has to be shown that such acceptance strategies are indeed optimal. First, suppose under the related model the type s does not accept w in the market M'. This means that the continuation value from the search is higher for the single-seat, $u(s, w) < U(s, \mu)$ for μ implying M'. What is to be shown is that with linear utility, the same strategy will be consistent with an equilibrium, i.e. the continuation value from the search will be higher for the same type firm which has employed $\Omega = \mu(f)$ than accepting w and this will hold for every subset Ω. With the same acceptance strategies, the remaining markets are the same for each realization of the contact function. Then:

$$u(f, \mu(f) \cup \{w\}) > U(f, \mu) \Leftrightarrow u(s, w) > U(s, \mu_R)$$

By the rules of the search game, the game ends almost surely. Let $\bar{\tau}$ denote the longest duration of the game. For any market M', let $x_t(f, w)$ be the probability with which f and w are matched t days after M' in an equilibrium matching. Then:

$$U(f, \mu) = \sum_{t=1}^{\bar{\tau}} \sum_{W} x_t(f, w, \mu) u(f, \mu(f) \cup \{w\})$$

Then, for each realization of the contact function, $x_t(f, w, \mu) = x_t(s, w, \mu_R)$. Writing $u(f, \mu(f) \cup \{w\}) = u(f, \mu(f)) + u(f, w)$ satisfies $U(f, \mu) = U(s, \mu_R)$ concludes the proof. □

Proposition 5:

Proof. The first part of the lemma is trivial with the given equality. For the second part, the example below illustrates the calculation of the assignment matrix x from x^B for simple matchings. Furthermore, the second part of the example serves as a proof that x calculated from x^B is not necessarily unique for fractional matchings.

Example: Consider again the example of a many-to-one matching market with 2 workers $\{w_1, w_2\}$ and 2 firms $\{f_1, f_2\}$. In the first scenario, let us take a simple matching with all entries are either 0 or 1. In that case, if $u(f, H)$ and $v(h, w)$ are known, the expected utilities can easily be calculated because the assignment matrix x^B implies a unique x.

x^B	f_1	f_2
w_1	1	0
w_2	1	0

x	f_1	f_2
$\{w_1, w_2\}$	1	0
w_1	0	0
w_2	0	0

However, if x^B is a fractional matching, the corresponding x is not necessarily unique. Under different x representations of x^B, the expected utility of the workers will be equal, whereas the expected utilities of the firms might differ. There is an example of a fractional matching x^B with multiple x representations below, where both x_1 and x_2 imply x^B, which serves as a proof to the Proposition 5 above.

x^B	f_1	f_2
w_1	0.5	0.5
w_2	0.3	0.4

x_1	f_1	f_2
$\{w_1, w_2\}$	0.3	0
w_1	0.2	0.5
w_2	0	0.4

x_2	f_1	f_2
$\{w_1, w_2\}$	0.1	0
w_1	0.4	0.5
w_2	0.2	0.4

□