Bertrand Competition with Search Goods*

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Abstract

We study a search market in which two firms compete in prices to entice consumers to check and buy their products. Consumers optimally inspect first the product of the firm quoting the lower price, and then, if so they wish, they check the product of the rival firm. There does not exist an equilibrium in pure strategies. We derive some general properties of the symmetric equilibrium in mixed strategies and show that, when the distribution of match values is polynomial, the equilibrium price distribution can be characterised as the solution to an ordinary differential equation. To illustrate our result, we compute the solution for the case of uniformly-distributed match values. Interestingly, as the cost of checking products goes up, firms' profits decrease, consumer surplus increases, while social welfare is non-monotonic, first increasing and then decreasing.

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1 Introduction

In many consumer search situations on the Internet (but not only), after a search query, consumers first obtain a list of distinct offers ordered by price and then search through them (from lowest to highest price) until they find, if any, one that suits their preferences. Retail platforms such as *ebay* or *bonanza* are good examples in which consumers encounter themselves in this situation.[1]

Despite the prevalence of this search problem, except in special cases (see Armstrong and Zhou (2011) and Ding and Zhang (2018)), the search literature has so far not provided a characterisation of the pricing equilibrium in such a context. The reason for this is that such a characterisation is not without difficulty. The complexity arises from two properties of the problem that interact with one another to make it relatively intricate. The first is that an equilibrium in pure strategies fails to exist; the second is that products are genuinely differentiated.

Specifically, we consider a search setting in which two symmetric firms, each offering a specific product (in a given product category), compete in prices. Consumers observe firms' prices before inspecting the products. Because firms' products are ex-ante identical, price is the only attribute that guides consumers' search. Hence, all consumers optimally choose to inspect the product of the firm charging the lower price. After having checked its product, consumers learn their taste for that product and decide whether to buy it, leave the market, or to also inspect the more expensive product of the rival firm. In the latter case, consumers choose the best of the two deals or none at all.

A pure-strategy equilibrium fails to exist because, as is standard in Bertrand environments, a firm that undercuts the price charged by the rest of the firms will see its demand jump up. Here, this occurs because the firm that charges the lower price is automatically ranked first and, since consumers check product suitability orderly from lowest to highest price, the demand of this firm is much higher than if it charged the same price as the rival firm. This pressure to lower prices unravels if the initial price level is sufficiently low. In this case, because firms' products are differentiated, an individual firm has an incentive to deviate by raising its price, thereby catering to the (fewer) consumers whose preferences

[1]Enter for instance the search query "Gotham Steel Kitchen Set" in *ebay.com*.

match sufficiently better with the product of the deviating firm. In contrast to homogeneous-product market models such as Burdett and Judd (1983) and Stahl (1989), here product differentiation and mixed strategies interact in such a way that the demand of a firm depends on the actual price charged by the rival firm and not just on whether its price is higher or lower (see also Armstrong (2017)).

The characterisation of the mixed-strategy equilibrium in this common setting has, to the best of our knowledge, not yet been provided. Our paper tries to fill this gap. We provide some general properties of the symmetric equilibrium in mixed strategies and show that, when the distribution of match values is polynomial, the equilibrium price distribution can be characterised as the solution to an ordinary differential equation. To illustrate our result, we compute the solution for the case of uniformly-distributed match values and study how the market performs as consumers' inspection costs increase.

We find that firms' prices and profits strictly decrease in search costs. The intuition is that firms understand that when the cost of inspecting products goes up, consumers are less likely to transit from firm to firm to check products. This increases the value of attracting the consumers in the first place, which the firms try to do by quoting lower prices. When consumers' first search is free, the increased competition induced by higher search costs always more than offsets the negative impact of a higher search friction, such that consumer surplus strictly increases. Search frictions thus work in the favour of consumers, not to their detriment as it is the case in standard models of search. In the limiting case in which the likelihood of continued search reaches zero, products become *de facto* homogeneous and firms end up competing à la Bertrand, thus obtaining zero profits.

Putting together firms' profits and consumer surplus, we find that social welfare is non-monotonic in the search cost, first strictly increasing and then strictly decreasing. This is because, when the inspection cost is initially low, consumers benefit more from an increase in the cost of checking products than the firms suffer. When the search cost is high to start with, the opposite occurs. The message for policymakers is that, because of the weakening of competition, a reduction in search costs is not necessarily welfare improving.

Related literature

In terms of its model, our paper is related to the seminal paper of Wolinsky (1986), further analysed by Anderson and Renault (1999). What is different in our paper is that we model Bertrand competition in its own tradition, which means that consumers observe the prices the firms charge and thus they serve to guide consumers' search. While in Wolinsky (1986) and Anderson and Renault (1999) there exists a unique pure-strategy equilibrium (under standard regularity conditions), as explained above, when search is directed by prices such an equilibrium fails to exist.

The question how search markets operate when firms' prices have a bearing on the order of search has recently intrigued several authors who have incorporated this feature in various types of models. Armstrong and Zhou (2011) introduce search frictions in a duopolistic market with Hotelling-type preferences. The special Hotelling structure of consumer tastes allows the authors to compute the equilibrium price distribution in closed form. Like in our paper, a higher search cost induces firms to charge lower prices and profits thus decrease. Ding and Zhang (2018) modify the well-known model of Stahl (1989) to allow for product differentiation. To obtain a tractable model of mixed pricing, they assume that firms carry products that may or may not fit consumers' tastes. They find that prices may, but need not, decrease as search costs go up. The reason for this is that a higher search cost decreases the maximum price firms can quote to non-shoppers in order to induce them to check their products.

Haan et al. (2018) and Choi et al. (2018) take an alternative route and escape the analysis of the mixed-strategy equilibrium by differentiating firms' products in terms of non-search characteristics. Specifically, they show that when products are sufficiently differentiated and this differentiation is observed by consumers prior to search, then the demand of a firm is no longer discontinuous in own price, which restores the existence of a pure-strategy equilibrium. A similar insight obtains if consumers have search costs that are heterogeneous and firm-specific (Armstrong (2017)).

The remainder of this article is structured as follows. We introduce our model in Section 2. In Section 3, we provide some general characterisation results for the ensuing mixed-strategy equilibria. In Section 4, we solve for the mixed-strategy equilibrium when match values are uniformly distributed. We also provide the comparative-statics effects of higher search costs. Section 5 concludes. Some of the proofs are relegated to Appendix A. In Appendix B, we apply our general methodology to a more complex case than the one with uniformly-distributed match values.

2 Model

There are two firms $i = 1,2$ competing in prices to sell their products to a mass 1 of consumers, indexed by $m \in [0,1]$. The products are differentiated but the differentiation can only be appreciated after inspecting the products. Each consumer is interested in buying at most one of the firms' products. Production costs are normalised to zero.

Initially, consumers observe firms' prices. After this, they decide whether they wish to inspect the products, and in which order. Only after inspecting a product, the consumer learns her value for the product. Let $\tilde{\varepsilon}_{im}$ denote consumer m's value of the match with the product of firm i. As is standard in the literature, we assume that match values are independent and identically distributed (i.i.d.) across firms and consumers. Let F be the distribution of match values, with support $[\underline{\varepsilon}, \overline{\varepsilon}]$, where $0 = \underline{\varepsilon} < \overline{\varepsilon} \leq \infty$, $F(\underline{\varepsilon}) = 0$, $F(\overline{\varepsilon}) = 1$.[2] We assume that F is smooth and strictly increasing in the interior of its support, and that $1-F$ is strictly log-concave. Thus, the finite match value density $f(\varepsilon)$ exists everywhere in the support and is strictly positive in its interior. For later use, let $\mathbb{E}_{\tilde{\varepsilon}} := \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \varepsilon f(\varepsilon) d\varepsilon$ be the expected match value with the product of any firm.

For simplicity, we assume in the baseline model that consumers' first inspection is for free, while checking the product of a second firm involves an inspection cost s, with $0 < s < \mathbb{E}_{\tilde{\varepsilon}}$.[3] During the inspection process we assume that recall is costless.

[2] We can easily allow for match-value distributions that put positive probability mass on zero (i.e., marginal cost) or below, without having to adapt our methodology. For technical reasons, it is however convenient to rule out match-value distributions that are bounded away from zero, as otherwise, depending on the degree of competition, case distinctions would have to be made (e.g., for certain parameter values, firms may always price below $\underline{\varepsilon} > 0$ in equilibrium, while for others, they may not).

[3] The assumption that the search cost is sufficiently low is necessary to make the problem interesting. If this assumption is violated, no consumer would transit from firm to firm to inspect products. At the end of Section 3, we also consider the case in which consumers' first search is costly.

If a consumer purchases at a firm i after having checked $n \in \{1,2\}$ products, her payoff is given by

$$u_m = \tilde{\varepsilon}_{im} - p_i - (n-1)s.$$

If a consumer chooses to not buy any of the products after having inspected $n \in \{1,2\}$ products, her payoff is given by $u_m = -(n-1)s$. Consumers maximise their expected utility.

We look for symmetric equilibria in which (i) consumers search optimally, given the prices the firms quote, and (ii) each firm prices optimally, given its rival's pricing strategy and consumers' optimal search behaviour.

3 Analysis

Given the i.i.d. nature of the match values, it is obvious that all consumers find it optimal to inspect the products in ascending order of price; if the products sell at equal prices, let us assume that consumers choose randomly which one to inspect first (though this will not occur in equilibrium). As explained in the Introduction, a symmetric pure-strategy equilibrium does not exist here: any putative symmetric equilibrium price $p > 0$ would be destabilised by the incentive of an individual firm to undercut it in order to have its product inspected first by all consumers (rather than by half of them); a putative symmetric equilibrium price $p = 0$ would also be destabilised but in this case by the incentive of an individual firm to raise the price and cater to the consumers who find its product sufficiently better than the rival's one. We therefore look for a symmetric equilibrium in mixed strategies in which each firm charges prices according to the cumulative distribution function $G(p) = Pr(\tilde{p} \leq p)$, with support $[\underline{p}, \overline{p}]$, $\overline{p} > \underline{p}$. By the same reasoning, note that G must be atomless.

We now compute the demands for the firms' products for an arbitrary realization of the prices they charge. Because consumers optimally search orderly, it is convenient to label firms' prices as p_L and p_H, p_L indicating the price charged by the lower-priced firm and p_H indicating the price charged by the higher-priced firm. All consumers find it optimal to first inspect the product of the firm charging p_L. Take a consumer m who has inspected the product of the firm charging p_L and has realized a match-value ε_{Lm}. For this consumer, either $\varepsilon_{Lm} < p_L$ or $\varepsilon_{Lm} \geq p_L$. Let us now discuss in turn how the consumer will proceed further in these two cases.

First, consider the case in which $\varepsilon_{Lm} < p_L$. The consumer will never buy at firm L. Her problem is then whether to search firm H or drop out of the market. Inspecting the product of the firm charging p_H is optimal if the gains from search are (weakly) positive, i.e., if

$$GS := \int_{p_H}^{\overline{\varepsilon}} (\varepsilon - p_H) f(\varepsilon) d\varepsilon - s \geq 0.$$

Let $\hat{p}$ be implicitly defined by the solution to

$$\int_{\hat{p}}^{\overline{\varepsilon}} (\varepsilon - \hat{p}) f(\varepsilon) d\varepsilon = s, \tag{1}$$

which is unique and well-defined ($\hat{p} \in [0, \overline{\varepsilon}]$) so long as $0 \leq s \leq \mathbb{E}_{\tilde{\varepsilon}}$. Moreover, implicitly differentiating equation (1) with respect to s gives

$$\frac{d\hat{p}}{ds} = -\frac{1}{1 - F(\hat{p})} < 0, \tag{2}$$

which implies that $\hat{p}$ is strictly decreasing in s and reaches zero for $s = \mathbb{E}_{\tilde{\varepsilon}}$.

Note that $\hat{p}$ can be interpreted as a consumer's reservation price for searching the second firm, given that her realized match value at the first firm fell short of this firm's price (so that the consumer's current best option is taking her outside option). In other words, having no positive surplus at hand after having checked the product of the firm charging the lower price, a consumer will only inspect the product of the rival firm if it charges a price $p_H \leq \hat{p}$. This immediately implies the following:

Lemma 1. *The upper bound $\overline{p}$ of the equilibrium price distribution G cannot lie above $\hat{p}$, that is, $\overline{p} \leq \hat{p}$.*

Proof. By contradiction, suppose that $\overline{p} > \hat{p}$ and take a firm i charging a price $p_i = \overline{p}$. Because G is atomless, this firm would only see its product inspected by the consumers after they inspect the product of the rival firm. However, because $p_i > \hat{p}$, not even consumers obtaining a match value $\varepsilon_{Lm} \leq p_L$ at the rival firm would find it optimal to inspect the product of firm i, in which case firm i would make zero profits. Deviating to a price $p_i \in (0, \hat{p})$ would instead generate positive demand and profit. $\square$

Consider now the second case, that is, that in which $\varepsilon_{Lm} \geq p_L$. In this case, the consumer has to decide whether to buy directly the product sold by the firm charging the lower price, or to inspect the product of the other firm and compare the deals. Let $GS(\varepsilon_{Lm}, p_L)$ denote the gains from searching the product of the firm charging the higher price given $\varepsilon_{Lm} \geq p_L$. Because the consumer will only find a better deal at that firm if $\tilde{\varepsilon}_{Hm} - p_H \geq \varepsilon_{Lm} - p_L$, i.e. if $\tilde{\varepsilon}_{Hm} \geq \varepsilon_{Lm} + p_H - p_L$, these gains are:

$$GS(\varepsilon_{Lm}, p_L) = \int_{\varepsilon_{Lm}+p_H-p_L}^{\overline{\varepsilon}} [(\varepsilon - p_H) - (\varepsilon_{Lm} - p_L)] f(\varepsilon) d\varepsilon - s.$$

Note that GS decreases in p_H and ε_{Lm}. Therefore, $GS(p_L, p_L) = \int_{p_H}^{\overline{\varepsilon}} (\varepsilon - p_H) f(\varepsilon) d\varepsilon - s \geq 0$ for any price p_H because $p_H \leq \hat{p}$ and $\hat{p}$ satisfies $\int_{\hat{p}}^{\overline{\varepsilon}} (\varepsilon - \hat{p}) f(\varepsilon) d\varepsilon - s = 0$ (see equation (1) and Lemma 1). Moreover, $GS(\hat{p} - (p_H - p_L), p_L) = \int_{\hat{p}}^{\overline{\varepsilon}} (\varepsilon - \hat{p}) f(\varepsilon) d\varepsilon - s = 0$. Hence, combining these observations, a consumer with match value $\varepsilon_{Lm} \geq p_L$ at hand will find it worthwhile to check the product of firm H if and only if $\varepsilon_{Lm} \leq \overline{\varepsilon}_L(p_H, p_L) := \hat{p} - (p_H - p_L)$. Clearly, for arbitrary price realizations satisfying $p_L \leq p_H$, this threshold lies weakly above p_L (and strictly so for $p_H < \hat{p}$), as $\overline{\varepsilon}_L(p_H, p_L) \geq p_L$ for $p_H \leq \hat{p}$.

Altogether, the firm charging the lower price has a demand $D_L(p_L, p_H)$ made up of all consumers with a match value $\varepsilon_{Lm} \geq \overline{\varepsilon}_L(p_H, p_L)$, who immediately buy its product, and those consumers with a match value $\varepsilon_{Lm} \in [p_L, \overline{\varepsilon}_L(p_H, p_L))$ who also check the product of the rival firm and choose the one of this firm if $\varepsilon_{Hm} < \varepsilon_{Lm} + p_H - p_L$.

Similarly, the demand for the product of the firm charging the higher price, $D_H(p_H, p_L)$, stems from the consumers with a match value at firm ε_{Lm} below $\overline{\varepsilon}_L(p_H, p_L)$: those with a match value $\varepsilon_{Lm} \leq p_L$ buy at the higher-priced firm if $\varepsilon_{Hm} \geq p_H$; those with a match value $\varepsilon_{Lm} > p_L$ buy at firm H if $\varepsilon_{Hm} \geq \varepsilon_{Lm} + p_H - p_L$. We may hence state the following lemma.

Lemma 2. *Given any two prices $0 \le p_L < p_H \le \hat{p}$, the demands for the products of the two firms are given by:*

$$D_L(p_L, p_H) = 1 - F(\hat{p} + p_L - p_H) + \int_{p_L}^{\hat{p}+p_L-p_H} F(\varepsilon - p_L + p_H) f(\varepsilon) d\varepsilon \qquad (3)$$

$$D_H(p_H, p_L) = F(p_L)(1 - F(p_H)) + \int_{p_L}^{\hat{p}+p_L-p_H} [1 - F(\varepsilon - p_L + p_H)] f(\varepsilon) d\varepsilon. \qquad (4)$$

Figure 1 gives a graphical representation of firms' demands for a given pair of prices p_H and p_L (with $\underline{\varepsilon} < p_L < p_H < \hat{p}$). As a consistency check, note that $D_L(p_L, p_H) + D_H(p_H, p_L) = 1 - F(p_L)F(p_H)$. It can also be verified that for $s = 0$ such that $\hat{p} = \overline{\varepsilon}$, the demand system becomes exactly identical to that under perfect information (see also the discussion in the paragraph "Equilibrium without Search Costs" below).

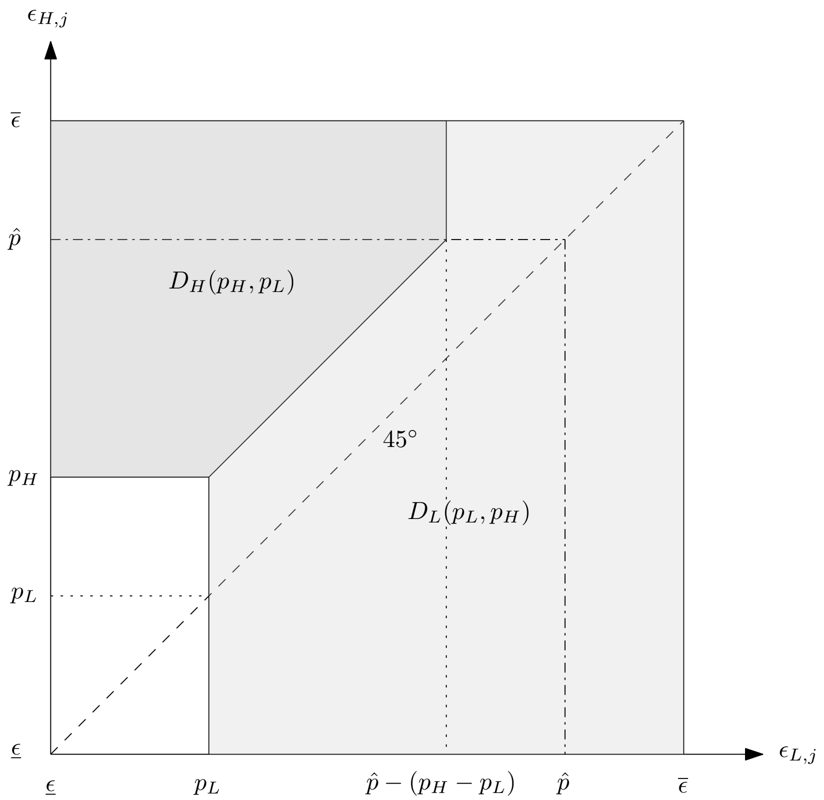

Figure 1: Illustration of firms' demands for a given pair of prices satisfying $p_H > p_L$. Firm L's demand is given by the match-value probability mass spread out over the light-gray area, whereas firm H's demand is given by the match-value probability mass spread out over the dark-gray area. Consumers with a match-value combination in the white area search both firms, but eventually take their outside option.

We now report some general properties of demand for fixed prices.

Lemma 3. *Firms' demands are decreasing in own price and increasing in the rival firm's price. For given prices* $0 \leq p_L < p_H < \hat{p}$*, a marginal increase in the search cost s increases firm L's demand at the expense of firm H's, while overall demand remains constant.*

Proof. The first two statements follow immediately from differentiation. For the statement on search costs, notice that

$$\begin{aligned}\frac{\partial D_L(p_L, p_H)}{\partial s} &= -\frac{d\hat{p}}{ds}(1 - F(\hat{p}))f(\hat{p} + p_L - p_H) \\ &= f(\hat{p} + p_L - p_H) \geq 0,\end{aligned}$$

where the second equality follows from equation (2). Likewise, we have that

$$\begin{aligned}\frac{\partial D_H(p_H, p_L)}{\partial s} &= \frac{d\hat{p}}{ds}(1 - F(\hat{p}))f(\hat{p} + p_L - p_H) \\ &= -f(\hat{p} + p_L - p_H) \leq 0.\end{aligned}$$

The sum of these derivatives is clearly zero. □

Since the products are substitutes, the behaviour of firms' demands with respect to prices is standard. More importantly, the lemma shows that firms' wedge in demand increases as consumers' search cost goes up. The intuition is that with a higher search cost, consumers are less willing to check the product of the firm charging the higher price after having first inspected the product of the firm charging the lower price, and will therefore more readily accept low match values without checking the second product. *Ceteris paribus* this makes it more valuable to quote the lower price in the market.

Equilibrium without Search Costs. The above results allow us to characterise the (symmetric) equilibrium. Before doing so for the interesting case with strictly positive search costs (which, as argued above, necessarily leads to mixed-strategy pricing in equilibrium) we briefly discuss the simpler benchmark case in which $s = 0$. Given our assumptions, the game then collapses to a standard random-utility model à la Perloff and Salop (1985), in which consumers always know both match values. Our setup is however slightly different because consumers are assumed to have an outside option.

Since consumers know both match values and have an outside-option of value zero, they will only ever buy from firm i if $\varepsilon_i \geq p_i$ and $\varepsilon_i - p_i \geq \varepsilon_j - p_j$. Hence, firm i's demand and profit can be written as

$$D_{i,0}(p_i, p_j) := \int_{p_i}^{\bar{\varepsilon}} f(\varepsilon)F(\varepsilon - p_i + p_j)d\varepsilon, \tag{5}$$

$$\pi_{i,0}(p_i, p_j) := p_i \int_{p_i}^{\bar{\varepsilon}} f(\varepsilon)F(\varepsilon - p_i + p_j)d\varepsilon, \tag{6}$$

where the 0-subscript stands for the model variant with zero search costs.[4] Following Perloff and Salop (1985), we look for a symmetric equilibrium in pure strategies. Firm i's first order condition is given by

$$\frac{\partial \pi_i^0(p_i, p_j)}{\partial p_i} = \int_{p_i}^{\bar{\varepsilon}} f(\varepsilon)F(\varepsilon - p_i + p_j)d\varepsilon - p_i \left[f(p_i)F(p_j) + \int_{p_i}^{\bar{\varepsilon}} f(\varepsilon)f(\varepsilon - p_i + p_j)d\varepsilon \right] \stackrel{!}{=} 0.$$

In a symmetric pure-strategy equilibrium, it has to hold that $p_i = p_j = p$, such that in particular $\frac{\partial \pi_{i,0}(p_i,p_j)}{\partial p_i}\Big|_{p_i=p_j=p} \stackrel{!}{=} 0$ is necessary for its existence. Noting that $\int_p^{\bar{\varepsilon}} f(\varepsilon)F(\varepsilon)d\varepsilon = \frac{1-F(p)^2}{2}$, we hence need that

$$h(p) := \frac{1 - F(p)^2}{2} - pf(p)F(p) - p\int_p^{\bar{\varepsilon}} f(\varepsilon)^2 d\varepsilon \stackrel{!}{=} 0. \tag{7}$$

Our assumption that $1 - F(\varepsilon)$ is strictly log-concave now guarantees that equation (7) has a unique solution $p^* \in (0, \bar{\varepsilon})$.[5] This allows us to state the following lemma, which concludes our discussion of the case without search costs.

Lemma 4. *Suppose that $s = 0$ and that $1 - F(\varepsilon)$ is strictly log-concave. Then, if a symmetric pure-strategy equilibrium exists, it is characterised by the unique solution $p^* \in (0, \bar{\varepsilon})$ to equation (7).*[6]

[4] For $p_i < p_j$, $F(\varepsilon - p_i + p_j)$ in the integrand of $D_{i,0}(p_i, p_j)$ equals 1 for $\varepsilon \geq \bar{\varepsilon} + p_i - p_j$. Hence, for $p_i < p_j$, firm i's demand can alternatively by written as

$$D_{i,0}(p_i, p_j | p_i < p_j) = \int_{p_i}^{\bar{\varepsilon}+p_i-p_j} f(\varepsilon)F(\varepsilon - p_i + p_j)d\varepsilon + [1 - F(\bar{\varepsilon} + p_i - p_j)].$$

[5] The proof of this is rather lengthy and therefore omitted. It is available from the authors upon request.

[6] A sufficient condition for existence of this equilibrium is that $D_{i,0}(p_i, p_j)$ as defined in equation (5) is log-concave in p_i. If this is not the case *and* there is a profitable unilateral deviation from the equilibrium candidate, no symmetric pure-strategy equilibrium exists for $s = 0$.

Equilibrium with Search Costs. We now turn to the characterisation of the symmetric mixed-strategy equilibria for $s > 0$. These are described by an atomless distribution function G over the range of possible prices $[0, \hat{p}]$, see Lemma 1, where $\underline{p}$ and $\overline{p}$ denote the infimum and supremum, respectively, of the support of G. The expected profit of a firm setting a price p is equal to:

$$\pi(p) := p\left(\int_{\underline{p}}^{p} D_H(p,\tilde{p})dG(\tilde{p}) + \int_{p}^{\overline{p}} D_L(p,\tilde{p})dG(\tilde{p})\right), \tag{8}$$

where D_H and D_L are defined in Lemma 2. The distribution function G characterises a mixed-strategy equilibrium if there exists a scalar $\pi^* \geq 0$ such that $\pi(p) = \pi^*$ in the support of G and $\pi(p) \leq \pi^*$ outside the support. By the next lemma, one can restrict attention to prices less than the unique[7] solution to $1 - F(p) - pf(p) = 0$ that we denote by p^M.

Lemma 5. *To characterise the set of symmetric equilibria, there is no loss in restricting attention to prices less than* p^M.

Proof. See Appendix A. □

Lemma 5 is intuitive: Not even a monopolist would ever find it optimal to price above the monopoly price p^M. From firms' perspective, this is even more so true under duopoly, as decreasing one's price starting from some $p > p^M$ now has the added benefit of potentially beating the rival's price, for a discontinuous upward jump in demand.

In what follows, we therefore restrict attention to prices less than p^M. The next lemma provides conditions under which the equilibrium distribution has convex support and is characterised by an integral equation.

Lemma 6. *Suppose that* $pD_L(p,\tilde{p})$ *and* $pD_H(p,\tilde{p})$ *are strictly concave*[8] *in* $p \in [0, p^M]$. *Then, an atomless distribution function G is a symmetric equilibrium in mixed strategies if and only if G has a convex support* $[\underline{p}, \overline{p}]$, $G'(\overline{p}) = 0$ *and there exists a* $\pi^* \in [0, \infty)$ *such that:*

$$\frac{\pi^*}{p} = \int_{\underline{p}}^{p} D_H(p,\tilde{p})dG(\tilde{p}) + \int_{p}^{\overline{p}} D_L(p,\tilde{p})dG(\tilde{p}), \forall p \in [\underline{p}, \overline{p}]. \tag{9}$$

[7] As a consequence of our assumption that $1 - F$ is strictly log-concave.

[8] Unfortunately, we could not find an obvious condition on F that guarantees the concavity of $pD_L(p,\tilde{p})$ and $pD_H(p,\tilde{p})$ up to p^M.

Proof. See Appendix A. □

The requirement that $pD_L(p,\tilde{p})$ and $pD_H(p,\tilde{p})$ are strictly concave for $p \in [0, p^M]$ is only necessary to guarantee the convexity of the support of G. Any symmetric equilibrium G with convex support solves the integral equation in (9) and satisfies $G'(\overline{p}) = 0$. In what follows, we shall study the solutions to the latter mathematical problem. Unfortunately, the integral equation does not have an explicit solution in general. However, one can compute an approximate solution by noting that by application of the Stone-Weierstrass theorem (see e.g. Rudin (1976), Theorem 7.26) any continuous match-value distribution function $F(\varepsilon)$ with compact support can be uniformly approximated arbitrarily closely by a polynomial function. We may then apply the following lemma.[9]

Lemma 7. *Suppose $F(\varepsilon)$ is polynomial of order $K \geq 1$, with $F(\varepsilon) = \sum_{k=0}^{K} a_k \varepsilon^k$, $a_K \neq 0$. Then $D_L(p,\tilde{p})$ is polynomial of order K in p and of order $2K$ in $\tilde{p}$, while $D_H(p,\tilde{p})$ is polynomial of order $2K$ in p and of order K in $\tilde{p}$.*

Proof. See Appendix A. □

Since, with polynomially-distributed match values, also the demand functions D_L and D_H are polynomial, one can transform the above integral equation (9) into a linear homogeneous ordinary differential equation taking derivatives with respect to p successively until the integrals vanish. To pave the way for this result, we shall use the simplification arising from the subsequent lemma.

Lemma 8. *For any $\tau \geq 0$, $\Lambda_\tau := D_L^{(\tau,0)}(p,p) - D_H^{(\tau,0)}(p,p)$ is constant in p. For any τ odd, $\Lambda_\tau = 0$.*

Proof. See Appendix A. □

We are now ready to convert the integral equation (9) to a linear homogeneous differential equation of order $2K+1$ with a total of $2K+3$ boundary conditions, defined at the yet unknown equilibrium support bounds $\overline{p}$ and $\underline{p}$.

[9]For the proof of the subsequent Lemma 7 and all of what follows, to have a more compact notation, we adopt, for $a \in \{0,1,\ldots\}$ and $b \in \{0,1,\ldots\}$, that $D_i^{(a,b)}(p,\tilde{p}) := \frac{\partial^{a+b} D_i(p',\tilde{p}')}{\partial(p')^a \partial(\tilde{p}')^b}\Big|_{p'=p,\ \tilde{p}'=\tilde{p}}$, with $D_i^{(0,0)}(p,\tilde{p}) = D_i(p,\tilde{p})$, $G^{(a)}(p) := \frac{\partial^a G(p')}{\partial(p')^a}\Big|_{p'=p}$, with $G^{(0)}(p) = G(p)$, and $F^{(a)}(\varepsilon) := \frac{\partial^a F(\varepsilon')}{\partial(\varepsilon')^a}\Big|_{\varepsilon'=\varepsilon}$, with $F^{(0)}(\varepsilon) = F(\varepsilon)$.

Lemma 9. *Suppose $F(\varepsilon)$ is polynomial of order $K \geq 1$, with $F(\varepsilon) = \sum_{k=0}^{K} a_k \varepsilon^k$, $a_K \neq 0$. Suppose moreover that the induced demand functions $D_L(p,\tilde{p})$ and $D_H(p,\tilde{p})$ are strictly concave in p up to the corresponding p^M. Then, a smooth function $G : [\underline{p},\overline{p}]$ that satisfies $G(\underline{p}) = 0$, $G(\overline{p}) = 1$ and $G^{(1)}(\overline{p}) = 0$ solves (9) if and only if:*

$$-(-1)^K (a_K K!)^2 \left[(2K+1)G(p) + pG^{(1)}(p)\right] - \sum_{\tau=0}^{2K-2} \Lambda_\tau \left[(2K+1)G^{(2K-\tau)}(p) + pG^{(2K+1-\tau)}(p)\right] = 0 \quad \forall p \in [\underline{p},\overline{p}], \tag{10}$$

$$\sum_{l=0}^{K} (-1)^l G^{(-l)}(\overline{p}) \left[(k+1)D_H^{(k,l)}(\overline{p},\overline{p}) + \overline{p} D_H^{(k+1,l)}(\overline{p},\overline{p})\right] - \sum_{\tau=0}^{k-1} \Lambda_\tau \left[(k+1)G^{(k-\tau)}(\overline{p}) + \overline{p} G^{(k+1-\tau)}(\overline{p})\right] = 0 \quad \forall k \in \{0,...,2K-1\} \tag{11}$$

$$a_K (K+1)! + \sum_{\tau=0}^{K-1} \Lambda_\tau \left[(K+1)G^{(K-\tau)}(\underline{p}) + \underline{p} G^{(K+1-\tau)}(\underline{p})\right] + \underline{p} \Lambda_K G^{(1)}(\underline{p}) = 0 \qquad (k = K) \tag{12}$$

$$\sum_{\tau=0}^{k-1} \Lambda_\tau \left[(k+1)G^{(k-\tau)}(\underline{p}) + \underline{p} G^{(k+1-\tau)}(\underline{p})\right] + \underline{p} \Lambda_k G^{(1)}(\underline{p}) = 0 \qquad \forall k \in \{K+1,...,2K-1\}. \tag{13}$$

Conditional on $\overline{p}$, firms' expected profit is given by

$$\pi^*(\overline{p}) = \frac{\overline{p}^{2K+1}}{(2K)!} \left[-(-1)^K (a_K K!)^2 - \sum_{\tau=0}^{2K-2} \Lambda_\tau G^{(2K-\tau)}(\overline{p})\right]. \tag{14}$$

Proof. See Appendix A. $\square$

Lemma 9 establishes that in case of polynomially-distributed match values of order K, finding the equilibrium CDF is equivalent to finding the solution to an *overspecified* linear homogeneous ODE of order $2K+1$, see equation (10), with a total of $2K+3$ boundary conditions. To see where the number of boundary conditions comes from, note that equation (11) specifies $2K$ boundary conditions at $\overline{p}$ with K unknowns $G^{(-1)}(\overline{p}),...,G^{(-K)}(\overline{p})$ entering linearly – which can be reduced to just K boundary conditions at $\overline{p}$ without any unknowns[10] – equations (12)+(13) provide in total K boundary conditions at $\underline{p}$, while the remaining 3 boundary conditions are given by $G(\underline{p})=0$, $G(\overline{p})=1$ and $G^{(1)}(\overline{p})=0$.

Our approach to find the equilibrium is now as follows. First, we may fix any $2K+1$ of the $2K+3$ boundary conditions and use them to compute the solution to the resulting *boundary value problem* (in our case, a linear homogeneous ODE of order $2K+1$ with $2K+1$ *mixed* boundary conditions) as a function of the admissible $0<\underline{p}<\overline{p}\leq p^M$.[11] Second, we need to find those values of $\overline{p}$ and $\underline{p}$ such that also the two still unused boundary conditions are satisfied. The solutions to these final consistency conditions is what pins down the equilibrium.

[10]Letting

$$a_{kl} := (-1)^l \left[(k+1)D_H^{(k,l)}(\overline{p},\overline{p}) + \overline{p}D_H^{(k+1,l)}(\overline{p},\overline{p})\right],$$

we can write e.g. the first K equations (for $k=0,\dots,K-1$) in (11) as

$$A\begin{pmatrix} G^{(-1)}(\overline{p}) \\ \vdots \\ G^{(-K)}(\overline{p}) \end{pmatrix} = b,$$

where

$$A := \begin{pmatrix} a_{01} & \cdots & a_{0K} \\ \vdots & & \vdots \\ a_{K-1,1} & \cdots & a_{K-1,K} \end{pmatrix}$$

is the $K\times K$ matrix with entry $a_{k-1,l}$ in row k and column l, and b is the $K\times 1$ vector with element b_k, $k=0,\dots K-1$, given by

$$b_k := \sum_{\tau=0}^{k-1}\Lambda_\tau\left[(k+1)G^{(k-\tau)}(\overline{p}) + \overline{p}G^{(k+1-\tau)}(\overline{p})\right] - a_{k0}.$$

If A is invertible, we can then find

$$\begin{pmatrix} G^{(-1)}(\overline{p}) \\ \vdots \\ G^{(-K)}(\overline{p}) \end{pmatrix} = A^{-1}b.$$

These concrete expressions for $G^{(-1)}(\overline{p}),\dots,G^{(-K)}(\overline{p})$ can in turn be plugged back into the K still unused equations in (11) for $k=K,\dots,2K-1$. We are then left with K mixed boundary conditions at $\overline{p}$ which relate sums of the various derivatives of G evaluated at $\overline{p}$ (i.e., $G^{(2)}(\overline{p}),\dots,G^{(2K)}(\overline{p})$) to functions of $\overline{p}$ only.

[11]Fortunately, even in cases where this cannot be done analytically, standard numerical methods such as the so-called "shooting method" exist which can do this efficiently.

In practice, it will often be convenient to exploit all boundary conditions *apart from* $G(\underline{p}) = 0$ and $G(\overline{p}) = 1$, as these values do not depend on the match-value distribution and no derivatives need to be considered when checking the final consistency conditions. Doing so, the solution to the boundary value problem can be computed for any given values of $(\underline{p}, \overline{p})$ by employing the boundary condition $G^{(1)}(\overline{p}) = 0$ and the $2K$ additional boundary conditions stemming from equations (11) to (13). We denote this solution by $\hat{\mathcal{G}}(\cdot; \underline{p}, \overline{p})$. To find the equilibrium it then only remains to solve the corresponding consistency conditions

$$\hat{\mathcal{G}}(\overline{p}; \underline{p}, \overline{p}) = 1 \tag{15}$$

$$\hat{\mathcal{G}}(\underline{p}; \underline{p}, \overline{p}) = 0, \tag{16}$$

with unknowns $(\underline{p}, \overline{p})$ such that $\underline{p} \in (0, p^M)$, $\overline{p} \in (\underline{p}, p^M]$. Then one can compute π^* using e.g. equation (14).

We will illustrate this approach in the next section. Before doing so, we finish our general analysis by briefly discussing the case where also consumers' first search is costly.

Costly First Search. Consider a variant of the baseline model in which consumers also have to incur a search cost $s \in (0, \mathbb{E}_{\tilde{\varepsilon}})$ when conducting their first search. As before, they can conduct a second search after observing the realization of the first search for the same cost s.

Then clearly, consumers would only conduct a first search if they expect to make a non-negative surplus from their whole search process. A lower bound for the expected surplus when conducting the first search to a firm fixing a price p is:

$$\int_p^{\bar{\varepsilon}} (\varepsilon - p) dF(\varepsilon) - s,$$

i.e., the expected utility of sticking to the first offer and not searching a second time. But the fact that firms do not fix prices greater than $\hat{p}$, see Lemma 1, means that this expected surplus is always non-negative. Thus, any equilibrium of the model in which the first search is for free[12] is an equilibrium of the model in which the first search is as costly as the second one. Therefore, we may apply the same equilibrium analysis as the one outlined above

[12]Or, more generally, in which the first search is less costly than the second search.

for costless first search. The only difference is that when computing total social welfare and consumer surplus, also the frictions stemming from consumers' first search need to be considered.

4 Uniformly Distributed Match Values

In this section, we illustrate how Lemma 9 can be used to transform the integral equation in (9) into an ordinary differential equation that can be solved. To do so, we assume in the rest of the section that $F(\varepsilon) = \varepsilon$ for $\varepsilon \in [0,1]$ and $s \in (0,1/2)$, which we call the *uniform case* for brevity.[13] We shall also use the numerical results of the uniform case to study the comparative-statics predictions of a change in the consumers' search cost.

4.1 Computing the Equilibrium

Note first that in the uniform case, $p^M = 1/2$, and thus we can restrict our attention to prices in $[0,1/2]$ by Lemma 5. Besides, one can show after some trivial algebraic manipulations on (1), (3) and (4) that:

$$\begin{aligned} \hat{p} &= 1-\sqrt{2s}, && (17)\\ D_L(p,\tilde{p}) &= \frac{1}{2}+s-p+\tilde{p}-\frac{\tilde{p}^2}{2}, && (18)\\ D_H(p,\tilde{p}) &= \frac{(1-p)^2}{2}-s+(1-p)\tilde{p}. && (19) \end{aligned}$$

Thus, $D_L^{(1,0)}(p,\tilde{p}) = -1$, $D_L^{(j,0)}(p,\tilde{p}) = 0$ for any $j \geq 2$, $D_H^{(1,0)}(p,\tilde{p}) = -(1-p)-\tilde{p}$, $D_H^{(2,0)}(p,\tilde{p}) = 1$, $D_H^{(j,0)}(p,\tilde{p}) = 0$ for any $j \geq 3$, $\Lambda_0 = 2s$, $\Lambda_2 = -1$, $\Lambda_i = 0$ for either $i = 1$ or $i \geq 3$. Besides, one can establish via differentiation that $pD_L(p,\tilde{p})$ and $pD_H(p,\tilde{p})$ are strictly concave in p for $p \in [0,p^M]$, and as a consequence, the application of Lemmas 6 and 9 to characterise the equilibrium is straightforward.

[13] In Appendix B, we showcase the applicability of our general approach also for the more complicated case of a linear-decreasing match-value density.

Lemma 10. *In the uniform case, an atomless distribution function $G : [\underline{p}, \overline{p}] \to [0,1]$ is a symmetric equilibrium in mixed strategies if and only if G solves:*

$$\begin{aligned}
3G(p) + pG^{(1)}(p) - 2s\left[3G^{(2)}(p) + pG^{(3)}(p)\right] &= 0 \qquad \forall p \in [\underline{p}, \overline{p}], && (20)\\
G(\overline{p}) &= 1, \\
G^{(1)}(\overline{p}) &= 0, \\
G^{(2)}(\overline{p}) &= -\frac{1 - 3\overline{p} + 3\overline{p}^2 + 2s}{2s\overline{p}(1 - 2\overline{p})}, && (21)\\
G(\underline{p}) &= 0, \\
1 + 2sG^{(1)}(\underline{p}) + s\underline{p}G^{(2)}(\underline{p}) &= 0. && (22)
\end{aligned}$$

Moreover, the equilibrium profit as function of the upper bound is given by

$$\pi^*(\overline{p}) = \frac{\overline{p}^2\left[(1-\overline{p})^2 + 2s\right]}{2(1-2\overline{p})}, \tag{23}$$

while the expected price as function of the upper bound is given by

$$\mathbb{E}\tilde{p}(\overline{p}) = \frac{-3\overline{p}^2 + 4\overline{p} + 2s - 1}{2(1-2\overline{p})}. \tag{24}$$

Proof. Equations (20) and (22) follow immediately from applying equations (10) and (12) in Lemma 9 for $K = 1$ and $a_K = 1$.[14] For equation (21), note that applying equation (11) in Lemma 9 for $k = 0, 1$ gives the system of equations

$$\begin{aligned}
&G(\overline{p})\left[D_H^{(0,0)}(\overline{p},\overline{p}) + \overline{p}D_H^{(1,0)}(\overline{p},\overline{p})\right] - G^{(-1)}(\overline{p})\left[D_H^{(0,1)}(\overline{p},\overline{p}) + \overline{p}D_H^{(1,1)}(\overline{p},\overline{p})\right] = 0 \\
&G(\overline{p})\left[2D_H^{(1,0)}(\overline{p},\overline{p}) + \overline{p}D_H^{(2,0)}(\overline{p},\overline{p})\right] - G^{(-1)}(\overline{p})\left[2D_H^{(1,1)}(\overline{p},\overline{p}) + \overline{p}D_H^{(2,1)}(\overline{p},\overline{p})\right] \\
&- \Lambda_0\left[2G^{(1)}(\overline{p}) + \overline{p}G^{(2)}(\overline{p})\right] = 0.
\end{aligned}$$

[14] $a_K = 1$ since $F(\varepsilon) = a_0 + a_1\varepsilon$ with $a_0 = 0$, $a_1 = a_K = 1$.

Using that $G(\overline{p})=1$, $G^{(1)}(\overline{p})=0$, as well as, for the uniform case, $D_H^{(0,0)}(\overline{p},\overline{p})=\frac{(1-\overline{p})^2}{2}-s+(1-\overline{p})\overline{p}$, $D_H^{(1,0)}(\overline{p},\overline{p})=-1$, $D_H^{(0,1)}(\overline{p},\overline{p})=1-\overline{p}$, $D_H^{(1,1)}(\overline{p},\overline{p})=-1$, $D_H^{(2,0)}(\overline{p},\overline{p})=1$, $D_H^{(2,1)}(\overline{p},\overline{p})=0$ and $\Lambda_0=2s$, the above system simplifies to

$$\begin{aligned}
&\frac{(1-\overline{p})^2}{2}-s-\overline{p}^2-G^{(-1)}(\overline{p})\,(1-2\overline{p})=0\\
&-2+\overline{p}+2G^{(-1)}(\overline{p})-2s\overline{p}G^{(2)}(\overline{p})=0.
\end{aligned}$$

Eliminating $G^{(-1)}(\overline{p})$ and solving for $G^{(2)}(\overline{p})$ then easily yields the following solution, conditional on $\overline{p}$:

$$\begin{aligned}
G^{(2)}(\overline{p})&=-\frac{1-3\overline{p}+3\overline{p}^2+2s}{2s\overline{p}(1-2\overline{p})}\\
G^{(-1)}(\overline{p})&=\frac{1-2\overline{p}-\overline{p}^2-2s}{2(1-2\overline{p})},
\end{aligned}$$

validating equation (21).

Firms' equilibrium profit conditional on $\overline{p}$, as reported in equation (23), can now be found by applying equation (14) in Lemma 9 for $K=1$ and $a_K=1$. This gives

$$\pi^*(\overline{p})=\frac{\overline{p}^3}{2}\left[1-2sG^{(2)}(\overline{p})\right]=\frac{\overline{p}^2}{2}\left[1+\frac{1-3\overline{p}+3\overline{p}^2+2s}{1-2\overline{p}}\right]=\frac{\overline{p}^2\left[(1-\overline{p})^2+2s\right]}{2(1-2\overline{p})},$$

as reported.

Finally, the expected price conditional on $\overline{p}$, as stated in equation (24), comes from noting that

$$\mathbb{E}\tilde{p}(\overline{p})=\int_{\underline{p}}^{\overline{p}}\tilde{p}dG(\tilde{p})=\overline{p}-\int_{\underline{p}}^{\overline{p}}G(\tilde{p})d\tilde{p}=\overline{p}-G^{(-1)}(\overline{p}).$$

Hence

$$\mathbb{E}\tilde{p}(\overline{p})=\overline{p}-\frac{1-2\overline{p}-\overline{p}^2-2s}{2(1-2\overline{p})}=\frac{-3\overline{p}^2+4\overline{p}+2s-1}{2(1-2\overline{p})}.$$

□

In the uniform case, we henceforth obtain a third order linear homogeneous ODE with a total of five boundary conditions. As outlined in the main text above, a general way to find the equilibrium when $F(\varepsilon)$ is of order K would then be to exploit all $2K+1$ boundary conditions apart from $G(\overline{p})=1$ and $G(\underline{p})=0$, analytically or numerically solve the corresponding boundary value problem for any admissible $(\overline{p},\underline{p})$ to obtain $\hat{\mathcal{G}}(\cdot;\underline{p},\overline{p})$, and then pin down the equilibrium variables $\overline{p}$ and $\underline{p}$ by solving the final consistency conditions $\hat{\mathcal{G}}(\overline{p};\underline{p},\overline{p})=1$, $\hat{\mathcal{G}}(\underline{p};\underline{p},\overline{p})=0$.

However, the uniform case has the unique property that there are actually enough boundary conditions to fully specify the ODE as *initial value problem* with just conditions at the (yet unknown) upper bound of the price distribution.[15] The advantage is that a relatively manageable and *unique* analytic solution to this initial value problem exists, which in turn can be used to semi-analytically find the equilibrium objects $\overline{p}$ and $\underline{p}$.

Lemma 11. *For any $\overline{p}\in\left[0,\frac{1}{2}\right)$, the unique continuous solution $\check{G}$ to (20) with boundary conditions $\check{G}(\overline{p})=1$, $\check{G}^{(1)}(\overline{p})=0$ and $\check{G}^{(2)}(\overline{p})=-\frac{1-3\overline{p}+3\overline{p}^2+2s}{2s\overline{p}(1-2\overline{p})}$ is*

$$\check{G}(p)=A[(B_1(p)+B_2(p))C+D(p)+E_1(p)+E_2(p)], \tag{25}$$

where

$$\begin{aligned}
A &= \frac{1}{8s(1-2\bar{p})}\\
B_1(p) &= e^{\frac{p}{\sqrt{2s}}}\int_{\frac{p}{\sqrt{2s}}}^{\frac{\bar{p}}{\sqrt{2s}}}\left(\frac{e^{-x}}{x}\right)dx\\
B_2(p) &= e^{-\frac{p}{\sqrt{2s}}}\int_{-\frac{\bar{p}}{\sqrt{2s}}}^{-\frac{p}{\sqrt{2s}}}\left(\frac{e^{-x}}{x}\right)dx\\
C &= \bar{p}^2\left((1-\bar{p})^2+2s\right)/\sqrt{2s}\\
D(p) &= -2\bar{p}^2\left((1-\bar{p})^2+2s\right)/p\\
E_1(p) &= e^{\frac{p-\bar{p}}{\sqrt{2s}}}\left[\bar{p}^3-\left(\sqrt{2s}+2\right)\bar{p}^2-\left(\sqrt{2s}-1\right)\left(3\sqrt{2s}+1\right)\bar{p}-\sqrt{2s}\left(1-\sqrt{2s}\right)^2\right]\\
E_2(p) &= e^{-\frac{p-\bar{p}}{\sqrt{2s}}}\left[\bar{p}^3+\left(\sqrt{2s}-2\right)\bar{p}^2-\left(\sqrt{2s}+1\right)\left(3\sqrt{2s}-1\right)\bar{p}+\sqrt{2s}\left(1+\sqrt{2s}\right)^2\right]
\end{aligned}$$

[15] Already for $K=2$ (i.e., a quadratic match-value CDF), our methodology leads to a fifth order linear homogeneous ODE with a total of four boundary conditions at $\overline{p}$ and three boundary conditions at $\underline{p}$ (see Appendix B). Since the number of boundary conditions at the upper and lower bound is in both cases strictly less than five, the ODE cannot be described by an initial value problem. The same argument applies for any $K\geq 2$.

Proof. Uniqueness is guaranteed e.g. by Theorem 7.1, page 22, in Coddington and Levinson (1955). The explicit solution we provide can be computed by any standard software that solves ordinary differential equations.[16] □

Next, we show that the constraints $\check{G}(\underline{p}) = 0$ and $1 + 2s\check{G}^{(1)}(\underline{p}) + s\underline{p}\check{G}^{(2)}(\underline{p}) = 0$ that the solution in (25) must satisfy, see Lemma 10, determine uniquely $\underline{p}$ and $\overline{p}$. First, equation (25) evaluated at $p = \underline{p}$ can be used to transform $\check{G}(\underline{p}) = 0$ into:

$$A\left[\left(B_1(\underline{p}) + B_2(\underline{p})\right)C + D(\underline{p}) + E_1(\underline{p}) + E_2(\underline{p})\right] = 0. \tag{26}$$

Second, differentiating (25) with respect to p and evaluating it at $p = \underline{p}$ gives:

$$\check{G}^{(1)}(\underline{p}) = A\left[\left(\frac{B_1(\underline{p}) - B_2(\underline{p})}{\sqrt{2s}}\right)C - \frac{D(\underline{p})}{\underline{p}} + \frac{E_1(\underline{p}) - E_2(\underline{p})}{\sqrt{2s}}\right]. \tag{27}$$

Moreover, differentiating (25) twice with respect to p and evaluating it at $p = \underline{p}$ gives:

$$\begin{aligned}
\check{G}^{(2)}(\underline{p}) &= A\left[\left(\frac{B_1(\underline{p}) + B_2(\underline{p})}{2s} - \frac{2}{\underline{p}\sqrt{2s}}\right)C + \frac{2D(\underline{p})}{\underline{p}^2} + \frac{E_1(\underline{p}) + E_2(\underline{p})}{2s}\right] \\
&= A\left[-\frac{D(\underline{p})}{2s} - \frac{2C}{\underline{p}\sqrt{2s}} + \frac{2D(\underline{p})}{\underline{p}^2}\right] \\
&= \frac{2AD(\underline{p})}{\underline{p}^2},
\end{aligned} \tag{28}$$

where the second equality follows from equation (26) and the third equality from $-\frac{D(\underline{p})}{2s} - \frac{2C}{\underline{p}\sqrt{2s}} = 0$, compare with Lemma 11. These values for $\check{G}^{(1)}(\underline{p})$ and $\check{G}^{(2)}(\underline{p})$ can be used to transform (22) into:

$$A\left[\left(B_1(\underline{p}) - B_2(\underline{p})\right)C + E_1(\underline{p}) - E_2(\underline{p})\right] + \frac{1}{\sqrt{2s}} = 0. \tag{29}$$

Equations (26) and (29) form a system of two non-linear equations with two unknowns $\overline{p}$ and $\underline{p}$. Any solution to this system in which $0 \leq \underline{p} < \overline{p} < \frac{1}{2}$ defines the equilibrium values of $\overline{p}$ and $\underline{p}$, see Lemma 10. Unfortunately, the solutions to this system of equations cannot

[16] We have used Mathematica version 11.0.

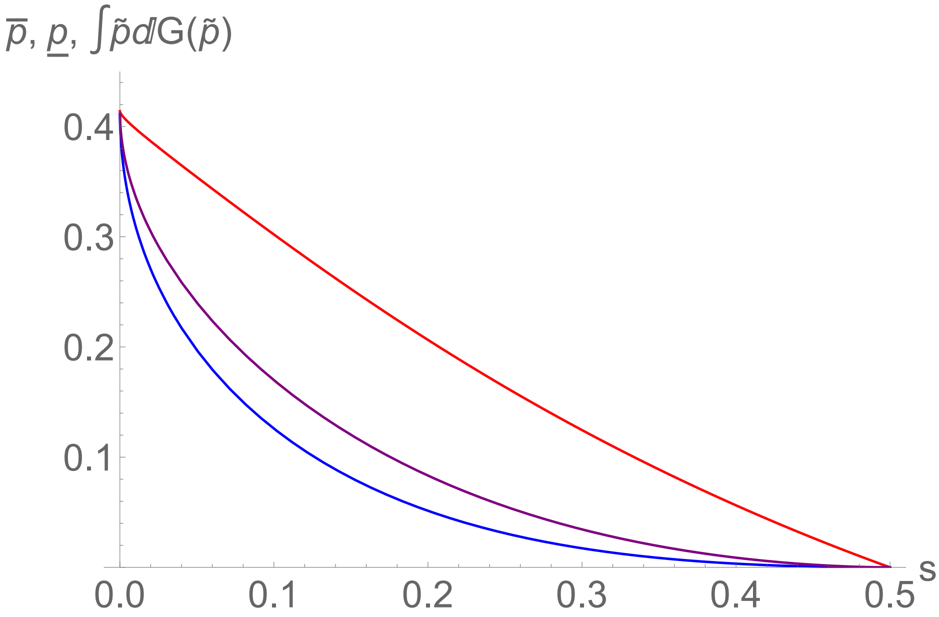

Figure 2: Equilibrium upper support bound (red), lower support bound (blue), and average price (purple) as a function of s.

be obtained analytically, so we proceed by solving it numerically. Our numerical analysis below suggests that it has a unique solution for any $s \in (0, 1/2)$.

Numerical Results 1. *Figure 2 plots the upper support bound $\overline{p}$, lower support bound $\underline{p}$ and the average price charged by firms, (24), in equilibrium as a function of s.*[17] *The three variables are strictly decreasing in s, starting from $\overline{p} = \underline{p} = \sqrt{2} - 1$ for the case $s \to 0$, down to $\overline{p} = \underline{p} = 0$ for the case $s \to 1/2$.*

These results confirm the general intuition that an increase in search costs should tend to intensify price competition. Quite remarkably, the two extreme cases $s \to 0$ and $s \to 1/2$ converge to the unique symmetric pure-strategy equilibrium that exists when $s = 0$ and $s = 1/2$, respectively.[18]

[17] We draw these plots using Mathematica 11.0. The code is available from the authors upon request.

[18] For $s = 0$, equation (7) implies that in the uniform case, the unique solution in $(0, 1)$ to $p^2 + 2p - 1 \stackrel{!}{=} 0$ defines the symmetric candidate equilibrium price. This is clearly given by $p^* = \sqrt{2} - 1$. Moreover, using the profit function in (6), it is straightforward to check that there are also no profitable non-marginal deviations from $p^* = \sqrt{2} - 1$. Hence, this indeed constitutes the unique symmetric pure-strategy equilibrium.

4.2 Numerical Welfare Analysis

In this section, we study how industry profits, consumer surplus and social welfare vary with the consumers' search cost s. Since there are two firms only, the expected industry profits are simply equal to $2\pi^*$, where π^* as a function of $\overline{p}$ is given by (23). Since firms' costs are equal to zero, the social surplus is equal to the buyer's value, minus the search cost if the buyer conducts a second search. Because this second search is conducted only if the buyer's value from the product of the firm charging the lower price ε_L is less than $\hat{p}-(p_H-p_L)$, the expected social surplus conditional on a vector of prices (p_L, p_H), where $p_L < p_H$, is equal to

$$W(p_H, p_L, s) := \int_L \varepsilon_L \, d\lambda(\varepsilon_L, \varepsilon_H) + \int_H \varepsilon_H \, d\lambda(\varepsilon_L, \varepsilon_H) - s(\hat{p} - (p_H - p_L)),$$

where λ denotes the Lebesgue measure and the regions L and H are depicted in light grey (L) and dark grey (H) in Figure 1.[19] One can show after some tedious algebra that the above expression is equal to:

$$W(p_H, p_L, s) = \frac{2p_H^3 - 3p_H^2(2p_L+1) + 6p_H p_L - 3p_L^2 + 4s\sqrt{2s} - 6s + 4}{6}. \tag{30}$$

The unconditional expected social surplus is then equal to:

$$W := 2\int_{\underline{p}}^{\overline{p}} \left[\int_{\underline{p}}^{p_H} W(p_H, p_L, s) dG(p_L) \right] dG(p_H). \tag{31}$$

[19] A general explicit expression for $W(p_H, p_L, s)$ is given by

$$\begin{aligned} W(p_H, p_L, s) = & -F(\hat{p} - p_H + p_L)s + F(p_L) \int_{p_H}^{\overline{\varepsilon}} \varepsilon_H f(\varepsilon_H) d\varepsilon_H \\ & + \int_{p_L}^{\hat{p}-p_H+p_L} \varepsilon_L f(\varepsilon_L) F(\varepsilon_L + p_H - p_L) d\varepsilon_L \\ & + \int_{p_L}^{\hat{p}-p_H+p_L} \left[\int_{\varepsilon_L+p_H-p_L}^{\overline{\varepsilon}} \varepsilon_H f(\varepsilon_H) d\varepsilon_H \right] f(\varepsilon_L) d\varepsilon_L \\ & + \int_{\hat{p}-(p_H-p_L)}^{\overline{\varepsilon}} \varepsilon_L f(\varepsilon_L) d\varepsilon_L. \end{aligned}$$

The expected consumer surplus can be computed as the expected social surplus minus the expected industry profits:

$$CS := 2\int_{\underline{p}}^{\overline{p}} \left[\int_{\underline{p}}^{p_H} W(p_H, p_L, s) dG(p_L) \right] dG(p_H) - 2\frac{\overline{p}^2 \left[(1-\overline{p})^2 + 2s \right]}{2(1-2\overline{p})}. \qquad (32)$$

An increase in the search cost s directly reduces both total surplus and consumer surplus. This effect is reinforced by the induced effect on the search intensity and therefore on the efficiency of the allocation. However, another indirect effect is that the increase in search costs fosters competition and thus reduces prices (Numerical Results 1). Consequently, an increase in search costs induces countervailing effects on the expected consumer surplus and the expected social surplus and it is *a priori* unclear which one dominates.

In the case of industry profits, an increase in search costs shifts demand from the more expensive to the cheaper firm for *given* prices and thereby directly reduces industry profit. Moreover, the lower prices due to enhanced competition (see Numerical Results 1) also indirectly decrease profits.

Numerical Results 2. *Figure 3 plots expected social welfare, consumer surplus and industry profit as a function of s. Welfare is first increasing and then decreasing in search costs. Consumer surplus is strictly increasing and industry profit is strictly decreasing in search costs.*[20]

As anticipated above, industry profits decrease with search costs. In the case of consumer surplus, the numerical results suggest that the competition effect dominates, whereas in the case of social surplus, the results are less clear. For low values of the search cost the competition effect dominates, but for larger values the dominant forces are the increase in search costs and allocative efficiency. The surprising implication is that reducing search frictions may not always be a welfare-enhancing policy, even if this can be done at virtually no cost, and the reason is that competition is softened.

[20] Observe moreover that for $s \to 0$, $W(0) \approx 0.6193 = W(\sqrt{2}-1, \sqrt{2}-1, 0)$ (compare with equation (30)), where $p^* = \sqrt{2}-1$ is the symmetric pure-strategy equilibrium for $s = 0$ (see Footnote 18 above). Also $CS(0)$ and $\Pi(0)$ are consistent with this equilibrium price level.

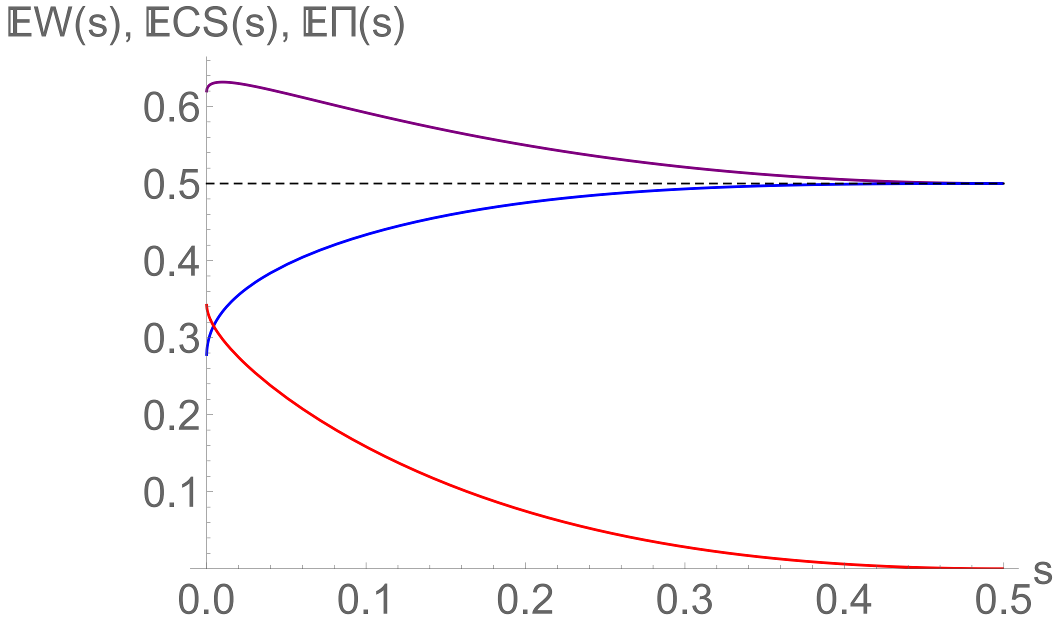

Figure 3: Expected social welfare (purple), consumer surplus (blue), and industry profit (red) as a function of s.

Note finally that this counterintuitive welfare result prevails in the model variant with costly first search.[21] For this variant, as all consumers still conduct a first search (compare with the discussion at the end of Section 3), we just have to deduct s from total social welfare and consumer surplus to find their new values. As it turns out, the only qualitative difference to the case with costless first search is the behaviour of consumer surplus as search costs vary. With costly first search, it is not a strictly increasing function any longer. This is because, as the search cost tends to the expected match value, total social welfare and consumer surplus now tend to zero, such that the consumer surplus cannot increase indefinitely.

Numerical Results 3. *Figure 4 plots the expected total social welfare, the consumer surplus and the industry profit in the model with costly first search, as a function of s. The first two are initially increasing and then decreasing, achieving their maximum value at a strictly positive search cost. The third one is strictly decreasing.*

[21] It also prevails under a linear-decreasing match-value density rather than a uniform density, see Appendix B for details.

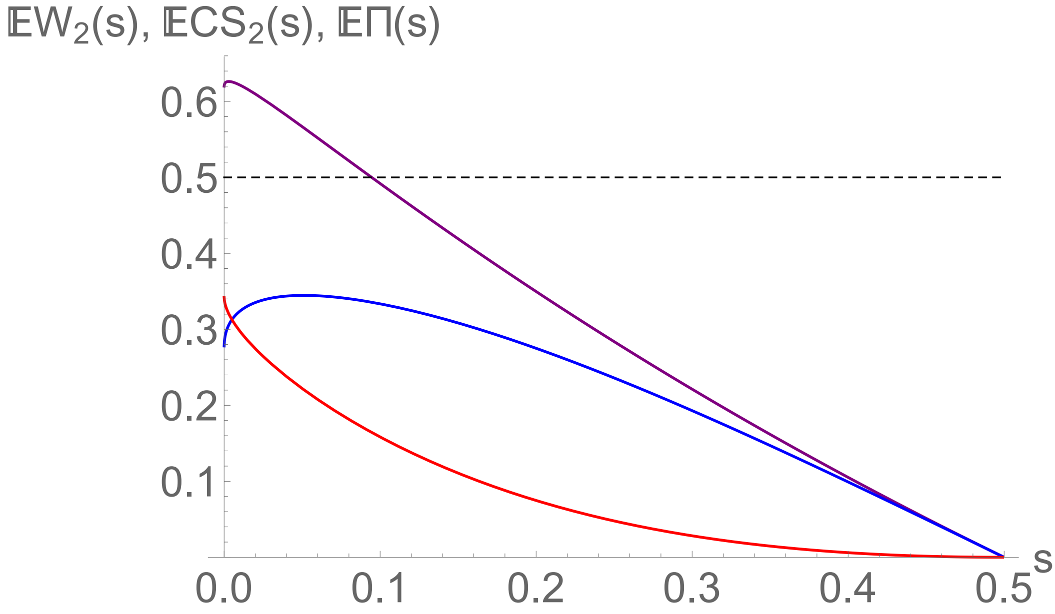

Figure 4: Expected total social welfare (purple), consumer surplus (blue) and industry profit (red) with costly first search as a function of s.

5 Conclusion

On the Internet consumers can easily check prices and therefore firms can use them to influence the way in which consumers search. The question how the ability of firms to direct consumer search impacts market competitiveness has proven to be a difficult one. This paper has added to the small literature on this issue by analysing it within the context of the workhorse model of consumer search for differentiated products of Wolinsky (1986).

We have studied a duopolistic search market in which firms compete in prices to entice consumers to inspect their products. Consumers optimally inspect first the product of the firm quoting the lower price, and then, if so they wish, they check the product of the rival firm. Upon inspection, they learn the value they place on a product. We have shown that there does not exist an equilibrium in pure strategies. We have also seen how the fact that products are differentiated makes the characterisation of the mixed-strategy equilibrium non-standard. We have derived some general properties of the symmetric equilibrium in mixed strategies and show that, when the distribution of match values is polynomial, the equilibrium price distribution can be characterised as the solution to an ordinary differential equation. To illustrate our result, we have computed the equilibrium price distribution for the often-used case in which match values are uniformly distributed. We have finally paid attention to the way the equilibrium changes as the cost of checking products increases. This has revealed that firms' profits decrease, consumer surplus increases, while social welfare is non-monotonic, first increasing and then decreasing in search costs.

An extensive literature, using an array of models with sequential and non-sequential search for homogeneous and differentiated products, has demonstrated that search costs weaken competition. A constant in that literature has been that consumers cannot check prices before they search. This paper has shown that the influence of search frictions on the functioning of the market is totally different when firms can use prices to influence the way in which consumers search. The message for policymakers is that, because cutting search costs softens competition, a reduction in the costs of search need not be welfare improving.

References

Simon P. Anderson and Régis Renault. Pricing, Product Diversity, and Search Costs: A Bertrand-Chamberlin-Diamond Model. *RAND Journal of Economics*, 30(4):719–735, Winter 1999.

Mark Armstrong. Ordered Consumer Search. *Journal of European Economic Association*, 15(5):989–1024, 2017.

Mark Armstrong and Jidong Zhou. Paying for prominence. *The Economic Journal*, 121: 368–395, 2011.

Kenneth Burdett and Kenneth L. Judd. Equilibrium Price Dispersion. *Econometrica*, 51(4): 955–69, July 1983.

Michael Choi, Anovia Y. Dai, and Kyungmin Kim. Consumer Search and Price Competition. *Econometrica*, 86(4):1257–1281, 2018.

Earl A Coddington and Norman Levinson. *Theory of ordinary differential equations*. Tata McGraw-Hill Education, 1955.

Yucheng Ding and Tianle Zhang. Price-directed consumer search. *International Journal of Industrial Organization*, 58:106–135, 2018.

Marco Haan, José L. Moraga-González, and Vaiva Petrikaite. A Model of Directed Consumer Search. *International Journal of Industrial Organization*, 61:223–255, 2018.

Jeffrey Perloff and Steven C. Salop. Equilibrium with product differentiation. *Review of Economic Studies*, 52(1):107–120, 1985.

Walter Rudin. *Principles of mathematical analysis*. McGraw-hill New York, 1976.

Dale O. Stahl. Oligopolistic Pricing with Sequential Consumer Search. *American Economic Review*, 79(4):700–712, September 1989.

Asher Wolinsky. True Monopolistic Competition as a Result of Imperfect Information. *The Quarterly Journal of Economics*, 101(3):493–511, August 1986.

Appendix A Technical Proofs

Proof of Lemma 5. To prove the lemma, we shall show that the derivative of the profit function:

$$\pi'(p) = \int_{\underline{p}}^{p} \frac{\partial(pD_H(p,\tilde{p}))}{\partial p} dG(\tilde{p}) + \int_{p}^{\overline{p}} \frac{\partial(pD_L(p,\tilde{p}))}{\partial p} dG(\tilde{p}) - pG'(p)(D_L(p,p) - D_H(p,p))$$

is strictly negative for $p \in (p^M, \hat{p}]$. Since[22] $D_L(p,p) - D_H(p,p) = (1-F(\hat{p}))^2 > 0$, it is sufficient to show that the terms $\frac{\partial(pD_S(p,\tilde{p}))}{\partial p}$ are strictly negative for $p \in (p^M, \hat{p}]$. To prove so, we deduce an alternative expression for D_L and D_H from the discussion preceding (3) and (4). In particular, the low price sells if and only if:

$$\varepsilon_{Lm} \geq \min(p_L + \hat{p} - p_H, p_L + \max(\varepsilon_{Hm} - p_H, 0)) = p_L + \max(\min(\hat{p}, \varepsilon_{Hm}) - p_H, 0),$$

and the high price sells if and only if $\varepsilon_{Lm} \leq \hat{p} - (p_H - p_L)$ and $\varepsilon_{Hm} \geq p_H + \max(\varepsilon_{Lm} - p_L, 0)$. Thus,

$$D_L(p_L, p_H) = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} (1 - F(p_L + \psi_L(\varepsilon)) f(\varepsilon) d\varepsilon, \tag{33}$$

$$D_H(p_H, p_L) = \int_{\underline{\varepsilon}}^{\hat{p}-(p_H-p_L)} (1 - F(p_H + \psi_H(\varepsilon))) f(\varepsilon) d\varepsilon, \tag{34}$$

where $\psi_L(\varepsilon) := \max(\min(\hat{p}, \varepsilon) - p_H, 0)$ and $\psi_H(\varepsilon) := \max(\varepsilon - p_L, 0)$.

Thus:

$$\frac{\partial(p_L D_L(p_L, p_H))}{\partial p_L} = -\int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left(p_L - \frac{1 - F(p_L + \psi_L(\varepsilon)}{f(p_L + \psi_L(\varepsilon))} \right) f(p_L + \psi_L(\varepsilon)) f(\varepsilon) d\varepsilon,$$

[22] The difference between (3) and (4) evaluated at $p_L = p_H = p$ gives:

$$\begin{aligned} D_L(p,p) - D_H(p,p) &= 1 - F(\hat{p}) - F(p)(1-F(p)) - \int_p^{\hat{p}} (1 - 2F(\varepsilon)) f(\varepsilon) d\varepsilon \\ &= 1 - F(\hat{p}) - F(p)(1-F(p)) - (F(\hat{p}) - F(\hat{p})^2) + (F(p) - F(p)^2) \\ &= (1 - F(\hat{p}))^2. \end{aligned}$$

and,

$$\frac{\partial(p_H D_H(p_H,p_L))}{\partial p_H} = -\int_{\underline{\varepsilon}}^{\hat{p}-(p_H-p_L)} \left(p_H - \frac{1-F(p_H+\psi_H(\varepsilon))}{f(p_H+\psi_H(\varepsilon))}\right) f(p_H+\psi_H(\varepsilon))f(\varepsilon)d\varepsilon$$
$$- p_H(1-F(\hat{p}))f(\hat{p}-(p_H-p_L)).$$

Both expressions are strictly negative for $p_S > p^M$ $(S = L, H)$ as desired, because,

$$p_S - \frac{1-F(p_S+\psi_S(\varepsilon)}{f(p_S+\psi_S(\varepsilon))} \geq p_S - \frac{1-F(p_S)}{f(p_S)} > 0,$$

where both inequalities are a consequence of our assumption of strict log-concavity of $1-F$ and because $\psi_S(\varepsilon) \geq 0$, first inequality, and $p_S > p^M$, second inequality. □

Proof of Lemma 6. First, the "if" part. Eq. (9) implies that $\pi(p) = \pi^*$ for any $p \in [\underline{p}, \overline{p}]$, and thus the firm is indifferent between any $p \in [\underline{p}, \overline{p}]$. To show that there are no incentives to deviate to a price less than $\underline{p}$, it is sufficient to show that for $p < \underline{p}$:

$$\pi'(p) = \int_{\underline{p}}^{\overline{p}} \frac{\partial(p \cdot D_L(p,\tilde{p}))}{\partial p} dG(\tilde{p}) \geq 0.$$

Note that the concavity of $p \cdot D_L(p,\tilde{p})$ with respect to p means that it is sufficient to prove this inequality for $p = \underline{p}$. To prove so, multiply both sides of Eq. (9) by p, differentiate them with respect to $p \in [\underline{p}, \overline{p}]$ and evaluate at $p = \underline{p}$ to get:

$$\int_{\underline{p}}^{\overline{p}} \left. \frac{\partial(p \cdot D_L(p,\tilde{p}))}{\partial p} \right|_{p=\underline{p}} dG(\tilde{p}) = \underline{p}G'(\underline{p})\left(D_L(\underline{p},\underline{p}) - D_H(\underline{p},\underline{p})\right) \geq 0,$$

where the sign follows from the facts that $\underline{p} \geq 0$, $G'(p) \geq 0$ and $D_L(p,p) - D_H(p,p) = (1-F(\hat{p}))^2 > 0$, see Footnote 22.

To show that there are no incentives to deviate to a price higher than $\overline{p}$, it is sufficient to show, by Lemma 5, that for $p \in [\overline{p}, p^M]$:

$$\pi'(p) = \int_{\underline{p}}^{\overline{p}} \frac{\partial(p \cdot D_H(p,\tilde{p}))}{\partial p} dG(\tilde{p}) \leq 0.$$

The concavity of $p \cdot D_H(p, \tilde{p})$ with respect to p means that it is sufficient to prove this inequality for $p = \overline{p}$. Again, multiply by p both sides of Eq. (9), differentiate them with respect to p and evaluate at $p = \overline{p}$ to get:

$$\int_{\underline{p}}^{\overline{p}} \frac{\partial (p \cdot D_H(p, \tilde{p}))}{\partial p} \bigg|_{p=\overline{p}} dG(\tilde{p}) = \overline{p} G'(\overline{p}) \left(D_L(\overline{p}, \overline{p}) - D_H(\overline{p}, \overline{p}) \right). \tag{35}$$

This expression is equal to zero, as desired, when $G'(\overline{p}) = 0$.

To prove the "only if" part, we note first that the equation in (9) is the usual indifference condition that must necessarily hold for prices in the support of G. Thus, (35) must also hold. Consequently, $G'(\overline{p}) = 0$ follows from the fact that the equilibrium condition that payoffs must be lower for prices higher than $\overline{p}$ implies that,

$$\int_{\underline{p}}^{\overline{p}} \frac{\partial (p \cdot D_H(p, \tilde{p}))}{\partial p} \bigg|_{p=\overline{p}} dG(\tilde{p}) \leq 0,$$

and $(D_L(\overline{p}, \overline{p}) - D_H(\overline{p}, \overline{p})) = (1 - F(\hat{p}))^2 > 0$, see Footnote 22.

Finally, to show that G cannot have gaps in the support, we argue by contradiction. Suppose that there exists a gap $(p', p'') \subset (\underline{p}, \overline{p})$ in which G does not put probability mass. In this case, equilibrium requires that $\pi(p) \leq \pi(p') = \pi(p'')$ for any $p \in (p', p'')$. Besides, one can deduce from (14) and our assumptions that $pD_L(p, \tilde{p})$ and $pD_H(p, \tilde{p})$ are strictly concave in p that $\pi(p)$ is strictly concave in $p \in [p', p'']$. This is a contradiction as there is no strictly concave function that satisfies $\pi(p) \leq \pi(p') = \pi(p'')$ for any $p \in (p', p'')$. □

Proof of Lemma 7. It is straightforward to see that for $F(\varepsilon)$ polynomial, the demand functions $D_L(p, \tilde{p})$ and $D_H(p, \tilde{p})$ as specified in equations (3) and (4) are bivariate polynomials. A multivariate polynomial is of order Z in some argument x if the Z'th ($Z \geq 1$) partial derivative of that function with respect to x is constant in x and different from zero.

We start by proving the order of $D_L(p, \tilde{p})$ in p and $\tilde{p}$. Note by letting $p_L = p$ and $p_H = \tilde{p}$ in equation (3) (see Lemma 2) that

$$\begin{aligned} D_L(p, \tilde{p}) &= 1 - F(\hat{p} + p - \tilde{p}) + \int_{p}^{\hat{p}+p-\tilde{p}} F(\varepsilon - p + \tilde{p}) f(\varepsilon) d\varepsilon \\ &= 1 - F(\hat{p} + p - \tilde{p}) + \int_{\tilde{p}}^{\hat{p}} F(\varepsilon) F^{(1)}(\varepsilon + p - \tilde{p}) d\varepsilon, \end{aligned} \tag{36}$$

where the second equality follows from changing the integration variable. One may then directly observe that

$$D_L^{(K,0)}(p,\tilde{p}) = -F^{(K)}(\hat{p}+p-\tilde{p}) + \int_{\tilde{p}}^{\hat{p}} F(\varepsilon)F^{(K+1)}(\varepsilon+p-\tilde{p})d\varepsilon = -a_K K! \neq 0,$$

where the second equality follows from the assumption that $F(\varepsilon) = \sum_{k=0}^{K} a_k \varepsilon^k$, $a_K \neq 0$, such that $F^{(K+1)}(\cdot) = 0$ and $F^{(K)}(\cdot) = a_K K!$. Hence, $D_L(p,\tilde{p})$ is polynomial of order K in p.

Invoking again equation (36), one may further note that

$$\begin{aligned}
D_L^{(0,2K)}(p,\tilde{p}) &= -F^{(2K)}(\hat{p}+p-\tilde{p}) + \frac{\partial^{2K}}{\partial(\tilde{p})^{2K}}\left[\int_{\tilde{p}}^{\hat{p}} F(\varepsilon)F^{(1)}(\varepsilon+p-\tilde{p})d\varepsilon\right] \\
&= 0 + \sum_{k=1}^{2K}(-1)^k F^{(2K-k)}(\tilde{p})F^{(k)}(p) + \int_{\tilde{p}}^{\hat{p}} F(\varepsilon)F^{(2K+1)}(\varepsilon+p-\tilde{p})d\varepsilon \\
&= \sum_{k=1}^{2K}(-1)^k F^{(2K-k)}(\tilde{p})F^{(k)}(p) \\
&= (-1)^K F^{(K)}(\tilde{p})F^{(K)}(p) = (-1)^K (a_K K!)^2 \neq 0,
\end{aligned}$$

where the first equality is obvious, the second equality follows from $F^{(2K)}(\cdot) = 0$ (since $2K > K$ and F is polynomial of order K) and direct calculation, the third equality follows from $F^{(2K+1)}(\cdot) = 0$ (since $2K+1 > K$ and F is polynomial of order K), the fourth equality follows since all sum terms apart for $k = K$ are zero (since either $2K - k > K$ or $k > K$ unless $k = K$, and F is polynomial of order K), and the last equality follows from $F^{(K)}(\cdot) = a_K K!$. Hence, $D_L(p,\tilde{p})$ is polynomial of order $2K$ in $\tilde{p}$.

We now turn to proving the order of $D_H(p,\tilde{p})$ in p and $\tilde{p}$. As mentioned after Lemma 2, it holds that $D_L(p_L,p_H) + D_H(p_H,p_L) = 1 - F(p_L)F(p_H)$. Letting $p_H = p$ and $p_L = \tilde{p}$ and rearranging, we obtain

$$D_H(p,\tilde{p}) = 1 - F(p)F(\tilde{p}) - D_L(\tilde{p},p).$$

Hence, it follows that

$$\begin{aligned}
D_H^{(2K,0)}(p,\tilde{p}) &= 1 - F^{(2K)}(p)F(\tilde{p}) - D_L^{(0,2K)}(\tilde{p},p) \\
&= -(-1)^K (a_K K!)^2 \neq 0,
\end{aligned}$$

where the second equality follows because $2K > K$ and F is polynomial of order K, and the above result that $D_L^{(0,2K)}(p,\tilde{p}) = (-1)^K(a_K K!)^2$ independent of p and $\tilde{p}$, such that also $D_L^{(0,2K)}(\tilde{p},p) = (-1)^K(a_K K!)^2$. Hence, $D_H(p,\tilde{p})$ is polynomial of order $2K$ in p.

Likewise, observe that

$$\begin{aligned} D_H^{(0,K)}(p,\tilde{p}) &= 1 - F(p)F^{(K)}(\tilde{p}) - D_L^{(K,0)}(\tilde{p},p) \\ &= -F(p)a_K K! + a_K K! = a_K K!(1-F(p)) \neq 0, \end{aligned}$$

where the second equality follows from $F^{(K)}(\cdot) = a_K K!$ and the above result that $D_L^{(K,0)}(p,\tilde{p}) = -a_K K!$ independent of p and $\tilde{p}$, such that also $D_L^{(K,0)}(\tilde{p},p) = -a_K K!$. Hence, $D_H(p,\tilde{p})$ is polynomial of order K in $\tilde{p}$. □

Proof of Lemma 8. The case $\tau = 0$ has already been shown in the proof of Lemma 5, Footnote 22. For $\tau \geq 1$, we use that (3) implies that

$$D_L^{(1,0)}(p,\tilde{p}) = -F^{(1)}(\hat{p}+p-\tilde{p})(1-F(\hat{p})) - F^{(1)}(p)F(\tilde{p}) - \int_p^{\hat{p}+p-\tilde{p}} F^{(1)}(\varepsilon-p+\tilde{p})F^{(1)}(\varepsilon)d\varepsilon.$$

Note next that slightly simplifying equation (4), applying integration by parts and changing the integration variable gives an alternative expression for D_H,

$$D_H(p,\tilde{p}) = F(\hat{p}-p+\tilde{p})(1-F(\hat{p})) + \int_p^{\hat{p}} F(\varepsilon-p+\tilde{p})F^{(1)}(\varepsilon)d\varepsilon,$$

such that

$$D_H^{(1,0)}(p,\tilde{p}) = -F^{(1)}(\hat{p}-p+\tilde{p})(1-F(\hat{p})) - F^{(1)}(p)F(\tilde{p}) - \int_p^{\hat{p}} F^{(1)}(\varepsilon-p+\tilde{p})F^{(1)}(\varepsilon)d\varepsilon.$$

Subtracting then yields

$$\begin{aligned} D_L^{(1,0)}(p,\tilde{p}) - D_H^{(1,0)}(p,\tilde{p}) = -(1-F(\hat{p}))\left[F^{(1)}(\hat{p}+p-\tilde{p}) - F^{(1)}(\hat{p}-p+\tilde{p})\right] \\ + \int_{\hat{p}+p-\tilde{p}}^{\hat{p}} F^{(1)}(\varepsilon-p+\tilde{p})F^{(1)}(\varepsilon)d\varepsilon. \end{aligned}$$

Via direct calculation, we therefore obtain for any $\tau \geq 1$ that

$$\begin{aligned}
&D_L^{(\tau,0)}(p,\tilde{p}) - D_H^{(\tau,0)}(p,\tilde{p}) \\
&= \frac{\partial^{\tau-1}\left[D_L^{(1,0)}(p,\tilde{p}) - D_H^{(1,0)}(p,\tilde{p})\right]}{\partial p^{\tau-1}} \\
&= -(1-F(\hat{p}))\left[F^{(\tau)}(\hat{p}+p-\tilde{p}) + (-1)^{\tau}F^{(\tau)}(\hat{p}-p+\tilde{p})\right] \\
&\qquad + \sum_{t=1}^{\tau-1}(-1)^t F^{(t)}(\hat{p})F^{(\tau-t)}(\hat{p}+p-\tilde{p}) - (-1)^{\tau}\int_{\hat{p}+p-\tilde{p}}^{\hat{p}} F^{(\tau)}(\varepsilon-p+\tilde{p})F^{(1)}(\varepsilon)d\varepsilon.
\end{aligned}$$

The last expression evaluated at $\tilde{p} = p$ is equal to:

$$-(1-F(\hat{p}))\left[F^{(\tau)}(\hat{p}) + (-1)^{\tau}F^{(\tau)}(\hat{p})\right] + \sum_{t=1}^{\tau-1}(-1)^t F^{(t)}(\hat{p})F^{(\tau-t)}(\hat{p}),$$

which is constant in p as desired. Moreover, it can easily be seen that for τ odd, the expression is equal to zero. □

Proof of Lemma 9. Recall first from equation (9) that the identity

$$\frac{\pi^*}{p} = \int_{\underline{p}}^{p} D_H(p,\tilde{p})dG(\tilde{p}) + \int_{p}^{\overline{p}} D_L(p,\tilde{p})dG(\tilde{p}) \tag{37}$$

must hold for all prices p in the equilibrium support $[\underline{p},\overline{p}]$. Differentiating (37) k times with respect to p while using from Lemma 8 that $\Lambda_\tau = D_L^{(\tau,0)}(p,p) - D_H^{(\tau,0)}(p,p)$ is constant in p, it is straightforward to see (and prove via induction) that[23]

$$(-1)^k k!\frac{\pi^*}{p^{k+1}} = \int_{\underline{p}}^{p} D_H^{(k,0)}(p,\tilde{p})dG(\tilde{p}) + \int_{p}^{\overline{p}} D_L^{(k,0)}(p,\tilde{p})dG(\tilde{p}) - \sum_{\tau=0}^{k-1} G^{(k-\tau)}(p)\Lambda_\tau \qquad \forall k = 0,1,2,\dots\,. \tag{38}$$

[23] For $k = 0$, we employ the convention that $\sum_{\tau=0}^{-1}(\cdot) = 0$.

Multiplying both sides of (38) by p^{k+1}, differentiating once more with respect to k, and finally dividing both sides by p^k again, we obtain

$$\begin{aligned}0 =&(k+1)\left[\int_{\underline{p}}^{p} D_H^{(k,0)}(p,\tilde{p})dG(\tilde{p}) + \int_{p}^{\overline{p}} D_L^{(k,0)}(p,\tilde{p})dG(\tilde{p}) - \sum_{\tau=0}^{k-1} G^{(k-\tau)}(p)\Lambda_\tau\right] \\ &+ p\left[-\Lambda_k G^{(1)}(p) + \int_{\underline{p}}^{p} D_H^{(k+1,0)}(p,\tilde{p})dG(\tilde{p}) + \int_{p}^{\overline{p}} D_L^{(k+1,0)}(p,\tilde{p})dG(\tilde{p}) - \sum_{\tau=0}^{k-1} G^{(k+1-\tau)}(p)\Lambda_\tau\right] \\ &\forall k = 0,1,2,\ldots, \forall p \in [\underline{p},\overline{p}].\end{aligned} \tag{39}$$

From Lemma 7, we know that if $F(\varepsilon)$ is polynomial of degree $K \geq 1$, $D_L(p,\tilde{p})$ is polynomial of degree K in p, while $D_H(p,\tilde{p})$ is polynomial of degree $2K$ in p. We moreover know from the proof of Lemma 7 that then

$$D_H^{(2K,0)}(p,\tilde{p}) = -(-1)^K(a_K K!)^2.$$

Hence, setting $k = 2K$ in equation (39), all integrals vanish (with $\int_{\underline{p}}^{\overline{p}} D_H^{(2K,0)}(p,\tilde{p})dG(\tilde{p}) = -(-1)^K(a_K K!)^2$), and we are left with

$$0 = -(2K+1)\left((-1)^K(a_K K!)^2 G(p) + \sum_{\tau=0}^{2K-1} G^{(2K-\tau)}(p)\Lambda_\tau\right) - p\left[\Lambda_{2K} G^{(1)}(p) + \sum_{\tau=0}^{2K-1} G^{(2K+1-\tau)}(p)\Lambda_\tau\right]$$

for all $p \in [\underline{p},\overline{p}]$. Noting that $\Lambda_{2K-1} = 0$ since $2K-1$ is odd, such that the last sum terms drop out, as well as using that $\lambda_{2K} = D_L^{(2K,0)}(p,p) - D_H^{(2K,0)}(p,p) = -D_H^{(2K,0)}(p,p) = (-1)^K(a_K K!)^2$, a slight rearrangement finally yields equation (10). Clearly, to achieve profit indifference in the equilibrium pricing support, this equation must be satisfied for all $p \in [\underline{p},\overline{p}]$.

We now turn to showing that the boundary conditions given in equation (11) must hold. For this, note first that evaluating (39) at $\overline{p}$, using $G^{(1)}(\overline{p}) = 0$ and rearranging gives

$$\begin{aligned}0 = \int_{\underline{p}}^{\overline{p}} \left[(k+1)D_H^{(k,0)}(\overline{p},\tilde{p}) + \overline{p}D_H^{(k+1,0)}(\overline{p},\tilde{p})\right] dG(\tilde{p}) - \sum_{\tau=0}^{k-1} \Lambda_\tau \left[(k+1)G^{(k-\tau)}(\overline{p}) + \overline{p}G^{(k+1-\tau)}(\overline{p})\right] \\ \forall k = 0,1,2,\ldots .\end{aligned} \tag{40}$$

The right summation term is already identical to the one in equation (11). It thus remains to show that the integral term can be transformed to the expression in (11). To do so, note that applying integration by parts once yields

$$\int_{\underline{p}}^{\overline{p}} \left[(k+1)D_H^{(k,0)}(\overline{p},\tilde{p}) + \overline{p}D_H^{(k+1,0)}(\overline{p},\tilde{p})\right] dG(\tilde{p}) =$$
$$\left[(k+1)D_H^{(k,0)}(\overline{p},\overline{p}) + \overline{p}D_H^{(k+1,0)}(\overline{p},\overline{p})\right] G(\overline{p}) - \int_{\underline{p}}^{\overline{p}} \left[(k+1)D_H^{(k,1)}(\overline{p},\tilde{p}) + \overline{p}D_H^{(k+1,1)}(\overline{p},\tilde{p})\right] G(\tilde{p})d\tilde{p}$$
$$\forall k = 0,1,2,\dots \,. \tag{41}$$

Applying integration by parts a second time then yields

$$\int_{\underline{p}}^{\overline{p}} \left[(k+1)D_H^{(k,0)}(\overline{p},\tilde{p}) + \overline{p}D_H^{(k+1,0)}(\overline{p},\tilde{p})\right] dG(\tilde{p}) =$$
$$\left[(k+1)D_H^{(k,0)}(\overline{p},\overline{p}) + \overline{p}D_H^{(k+1,0)}(\overline{p},\overline{p})\right] G(\overline{p}) - \left[(k+1)D_H^{(k,1)}(\overline{p},\overline{p}) + \overline{p}D_H^{(k+1,1)}(\overline{p},\overline{p})\right] G^{(-1)}(\overline{p})$$
$$- \int_{\underline{p}}^{\overline{p}} \left[(k+1)D_H^{(k,2)}(\overline{p},\tilde{p}) + \overline{p}D_H^{(k+1,2)}(\overline{p},\tilde{p})\right] G^{(-1)}(\tilde{p})d\tilde{p}$$
$$\forall k = 0,1,2,\dots \,, \tag{42}$$

where $G^{(-1)}(\tilde{p}) := \int_{\underline{p}}^{\tilde{p}} G(x)dx$ (such that in particular $G^{(-1)}(\underline{p}) = 0$). The same pattern repeats, such that applying integration by parts exactly $K+1$ times gives

$$\begin{aligned}
&\int_{\underline{p}}^{\overline{p}} \left[(k+1)D_H^{(k,0)}(\overline{p},\tilde{p}) + \overline{p}D_H^{(k+1,0)}(\overline{p},\tilde{p})\right] dG(\tilde{p}) \\
&= \sum_{l=0}^{K} (-1)^l \left[(k+1)D_H^{(k,l)}(\overline{p},\overline{p}) + \overline{p}D_H^{(k+1,l)}(\overline{p},\overline{p})\right] G^{(-l)}(\overline{p}) \\
&\quad - \int_{\underline{p}}^{\overline{p}} \left[(k+1)D_H^{(k,K+1)}(\overline{p},\tilde{p}) + \overline{p}D_H^{(k+1,K+1)}(\overline{p},\tilde{p})\right] G^{(-(K-1))}(\tilde{p})d\tilde{p} \\
&= \sum_{l=0}^{K} (-1)^l \left[(k+1)D_H^{(k,l)}(\overline{p},\overline{p}) + \overline{p}D_H^{(k+1,l)}(\overline{p},\overline{p})\right] G^{(-l)}(\overline{p})
\end{aligned}$$
$$\forall k = 0,1,2,\dots \,, \tag{43}$$

where $G^{(-Z)}(\tilde{p}), Z \geq 0$, is defined recursively by $G^{(-Z)}(\tilde{p}) := \int_{\underline{p}}^{\tilde{p}} G^{(-(Z-1))}(x)dx$ for all $Z \geq 1$, with $G^{(0)}(\tilde{p}) = G(\tilde{p})$. Note that the last equality follows because $D_H(p,\tilde{p})$ is polynomial of order K in $\tilde{p}$, see Lemma 7, such that the integral term drops out.

Substituting the integral term in (40) by the summation term in (44), the equivalence of (40) to (11) is finally evident.

As next step, we show that the boundary conditions given in equations (12) and (13) must hold. To see this, note that evaluating (39) at $\underline{p}$ gives

$$\begin{aligned}0 =&(k+1)\left[\int_{\underline{p}}^{\overline{p}} D_L^{(k,0)}(\underline{p},\tilde{p})dG(\tilde{p}) - \sum_{\tau=0}^{k-1} G^{(k-\tau)}(\underline{p})\Lambda_\tau\right] \\ &+ \underline{p}\left[-\Lambda_k G^{(1)}(\underline{p}) + \int_{\underline{p}}^{\overline{p}} D_L^{(k+1,0)}(\underline{p},\tilde{p})dG(\tilde{p}) - \sum_{\tau=0}^{k-1} G^{(k+1-\tau)}(\underline{p})\Lambda_\tau\right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall k = 0,1,2,\ldots, \forall p \in [\underline{p},\overline{p}].\end{aligned} \tag{44}$$

Now, recall again from Lemma 7 that for $F(\varepsilon)$ polynomial of degree $K \geq 1$, $D_L(p,\tilde{p})$ is polynomial of degree K in p. Moreover, it is known from the proof of Lemma 7 that then

$$D_L^{(K,0)}(p,\tilde{p}) = -a_K K!.$$

Hence, evaluating equation (44) for $k = K$, all integral terms drop out (with $\int_{\underline{p}}^{\overline{p}} D_L^{(K,0)}(\underline{p},\tilde{p})dG(\tilde{p}) = -a_K K!$), and we obtain

$$0 =(K+1)\left[-a_K K! - \sum_{\tau=0}^{K-1} G^{(K-\tau)}(\underline{p})\Lambda_\tau\right] + \underline{p}\left[-\Lambda_K G^{(1)}(\underline{p}) - \sum_{\tau=0}^{K-1} G^{(K+1-\tau)}(\underline{p})\Lambda_\tau\right] \quad \forall p \in [\underline{p},\overline{p}]. \tag{45}$$

Rearranging, equation (12) easily follows.

Similarly, evaluating equation (44) for $k > K$, all integral terms drop out completely, and we obtain

$$0 = (k+1)\left[-\sum_{\tau=0}^{k-1} G^{(k-\tau)}(\underline{p})\Lambda_\tau\right] + \underline{p}\left[-\Lambda_k G^{(1)}(\underline{p}) - \sum_{\tau=0}^{k-1} G^{(k+1-\tau)}(\underline{p})\Lambda_\tau\right]$$
$$\forall k = K+1, K+2, \ldots, \forall p \in [\underline{p}, \overline{p}]. \tag{46}$$

Again, a slight rearrangement then easily yields equation (13).

We finally prove the statement on firms' expected profit conditional on $\overline{p}$, equation (14). Setting $k = 2K$ in equation (38) and evaluating it at $\overline{p}$, noting that $\int_{\underline{p}}^{\overline{p}} D_H^{(2K,0)}(p,\tilde{p})dG(\tilde{p}) = -(-1)^K(a_K K!)^2$, we obtain

$$(2K)!\frac{\pi^*}{\overline{p}^{2K+1}} = -(-1)^K(a_K K!)^2 - \sum_{\tau=0}^{2K-1} G^{(2K-\tau)}(\overline{p})\Lambda_\tau.$$

Since $\Lambda_{2K-1} = 0$ since $2K-1$ is odd, this directly implies the result. This completes the proof. $\square$

Appendix B Linear-Decreasing Match Value Density

In this appendix we showcase how our general methodology for polynomially-distributed match values can be applied to a situation which is more complex than the uniform case. Adding some realism, we will thereby focus on a setting where high match values are less likely than low match values. The simplest possible such case is where the match-value density is linearly decreasing. Specifically, we will assume in what follows that

$$F(\varepsilon) = \frac{4\varepsilon}{3} - \frac{4\varepsilon^2}{9} \qquad \text{for } \varepsilon \in [0, 3/2], \tag{47}$$

which is the unique match-value CDF that (i) corresponds to a linear-decreasing match-value density reaching from $\underline{\varepsilon} = 0$ to some $\overline{\varepsilon} = a > 0$, with $f(0) > 0$ and $f(\overline{\varepsilon}) = 0$, and (ii) has a mean $\mathbb{E}_{\tilde{\varepsilon}} = 1/2$ as in the uniform case.

Solving $\max_p p(1-F(p))$, it can now easily be seen that $p^M = 1/2$, which also coincides with the uniform case. Hence, for the equilibrium analysis, attention can be restricted to prices in $[0,1/2]$. Using equations (1), (3) and (4), it can moreover be shown that:

$$\begin{aligned}
\hat{p} &= \frac{3}{2}\left(1-(2s)^{\frac{1}{3}}\right), && (48)\\
D_L(p,\tilde{p}) &= \left[\frac{1+(2s)^{\frac{4}{3}}}{2}+\frac{8\tilde{p}(1+s)}{9}-\frac{4\tilde{p}^2}{9}+\frac{8\tilde{p}^4}{243}\right] \\
&\quad +p\left[-\frac{8(1+s)}{9}-\frac{8\tilde{p}}{9}+\frac{16\tilde{p}^2}{27}-\frac{32\tilde{p}^3}{243}\right]+p^2\left[\frac{4}{9}\right], && (49)\\
D_H(p,\tilde{p}) &= \left[\frac{1-(2s)^{\frac{4}{3}}}{2}+\frac{8\tilde{p}(1+s)}{9}-\frac{4\tilde{p}^2}{9}\right]+p\left[-\frac{8(1+s)}{9}-\frac{8\tilde{p}}{9}+\frac{16\tilde{p}^2}{27}\right] \\
&\quad +p^2\left[\frac{4}{9}-\frac{16\tilde{p}^2}{81}\right]+p^3\left[\frac{32\tilde{p}}{243}\right]+p^4\left[-\frac{8}{243}\right]. && (50)
\end{aligned}$$

From these equations it follows that

$$\begin{aligned}
D_L^{(1,0)}(p,\tilde{p}) &= \left[-\frac{8(1+s)}{9}-\frac{8\tilde{p}}{9}+\frac{16\tilde{p}^2}{27}-\frac{32\tilde{p}^3}{243}\right]+p\left[\frac{8}{9}\right], && (51)\\
D_L^{(2,0)}(p,\tilde{p}) &= \frac{8}{9}, && (52)\\
D_L^{(j,0)}(p,\tilde{p}) &= 0 \qquad \text{for any } j\geq 3, && (53)\\
D_H^{(1,0)}(p,\tilde{p}) &= \left[-\frac{8(1+s)}{9}-\frac{8\tilde{p}}{9}+\frac{16\tilde{p}^2}{27}\right]+p\left[\frac{8}{9}-\frac{32\tilde{p}^2}{81}\right] \\
&\quad +p^2\left[\frac{32\tilde{p}}{81}\right]+p^3\left[-\frac{32}{243}\right], && (54)\\
D_H^{(2,0)}(p,\tilde{p}) &= \left[\frac{8}{9}-\frac{32\tilde{p}^2}{81}\right]+p\left[\frac{64\tilde{p}}{81}\right]+p^2\left[-\frac{32}{81}\right], && (55)\\
D_H^{(3,0)}(p,\tilde{p}) &= \left[\frac{64\tilde{p}}{81}\right]+p\left[-\frac{64}{81}\right], && (56)\\
D_H^{(4,0)}(p,\tilde{p}) &= -\frac{64}{81}, && (57)\\
D_H^{(j,0)}(p,\tilde{p}) &= 0 \qquad \text{for any } j\geq 5. && (58)
\end{aligned}$$

Moreover, it is straightforward to establish using equations (49) and (50) that

$$\Lambda_0 = (2s)^{\frac{4}{3}}, \tag{59}$$

while equations (51) to (58) reveal that $\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$,

$$\Lambda_4 = \frac{64}{81}, \tag{60}$$

and $\Lambda_j = 0$ for all $j \geq 5$.

It can finally be proved via differentiation and some manipulations that $pD_L(p, \tilde{p})$ and $pD_H(p, \tilde{p})$ are strictly concave in p for $p \in [0, p^M]$,[24] such that we may apply Lemmas 6 and 9 to further characterise the equilibrium.

Since $F(\varepsilon)$ is polynomial of order $K = 2$, with highest coefficient $a_2 = -\frac{4}{9}$, equation (10) in Lemma 9, in conjunction with the above results on Λ_k, immediately reveals that the linear homogeneous ODE

$$-\frac{64}{81}\left[5G(p) + pG^{(1)}(p)\right] - (2s)^{\frac{4}{3}}\left[5G^{(4)}(p) + pG^{(5)}(p)\right] \stackrel{!}{=} 0 \tag{61}$$

pins down firms' equilibrium CDF $G(p)$ in the (yet to be determined) equilibrium pricing support $[\underline{p}, \overline{p}]$. We will now proceed to find the $K = 2$ missing boundary conditions at $\overline{p}$ and the $K = 2$ missing boundary conditions at $\underline{p}$ (next to the already known boundary conditions $G(\overline{p}) = 1$, $G^{(1)}(\overline{p}) = 0$ and $G(\underline{p}) = 0$). In total, this will give us an overspecified linear homogeneous ODE of order $2K + 1 = 5$ with 7 boundary conditions. Fixing all boundary conditions except for $G(\overline{p}) = 1$ and $G(\underline{p}) = 0$, treating the numerical solution to the (now appropriately specified) ODE (of degree 5 with 5 boundary conditions) as function of $\overline{p}$ and $\underline{p}$, and then numerically finding the combination of $\overline{p}$ and $\underline{p}$ such that the final consistency requirements $G(\overline{p}) = 1$ and $G(\underline{p}) = 0$ are satisfied, will then give us the solution.

Boundary conditions at the upper bound. We start with the boundary conditions at $\overline{p}$. Applying equation (11) for $k = 2K - 1 = 3$ first reveals that

$$\begin{aligned} & 4D_H^{(3,0)}(\overline{p}, \overline{p}) + \overline{p}D_H^{(4,0)}(\overline{p}, \overline{p}) \\ - \quad & G^{(-1)}(\overline{p})\left[4D_H^{(3,1)}(\overline{p}, \overline{p}) + \overline{p}D_H^{(4,1)}(\overline{p}, \overline{p})\right] \\ + \quad & G^{(-2)}(\overline{p})\left[4D_H^{(3,2)}(\overline{p}, \overline{p}) + \overline{p}D_H^{(4,2)}(\overline{p}, \overline{p})\right] - \Lambda_0\left[4G^{(3)}(\overline{p}) + \overline{p}G^{(4)}(\overline{p})\right] \stackrel{!}{=} 0. \end{aligned}$$

[24] Details are available from the authors upon request.

Noting that $D_H^{(3,1)}(p,\tilde{p}) = \frac{64}{81}$ for any $(p,\tilde{p})$ (compare with equation (56)) and $D_H^{(4,0)}(p,\tilde{p}) = -\frac{64}{81}$ for any $(p,\tilde{p})$ (compare with equation (57)), it is obvious that $D_H^{(3,2)}(\overline{p},\overline{p}) = D_H^{(4,1)}(\overline{p},\overline{p}) = D_H^{(4,2)}(\overline{p},\overline{p}) = 0$. In particular, this implies that in the above condition, the term which includes $G^{(-2)}(\overline{p})$ drops out. Solving for $G^{(-1)}(\overline{p})$ then immediately gives

$$
\begin{aligned}
G^{(-1)}(\overline{p}) &= \frac{4D_H^{(3,0)}(\overline{p},\overline{p}) + \overline{p}D_H^{(4,0)}(\overline{p},\overline{p}) - \Lambda_0\left[4G^{(3)}(\overline{p}) + \overline{p}G^{(4)}(\overline{p})\right]}{4D_H^{(3,1)}(\overline{p},\overline{p})} \\
&= -\frac{\overline{p}}{4} - \frac{81}{256}(2s)^{\frac{4}{3}}\left[4G^{(3)}(\overline{p}) + \overline{p}G^{(4)}(\overline{p})\right]. \qquad (62)
\end{aligned}
$$

Next, applying equation (11) for $k = 2K-2 = 2$ gives us the condition

$$
\begin{aligned}
& 3D_H^{(2,0)}(\overline{p},\overline{p}) + \overline{p}D_H^{(3,0)}(\overline{p},\overline{p}) \\
- \;& G^{(-1)}(\overline{p})\left[3D_H^{(2,1)}(\overline{p},\overline{p}) + \overline{p}D_H^{(3,1)}(\overline{p},\overline{p})\right] \\
+ \;& G^{(-2)}(\overline{p})\left[3D_H^{(2,2)}(\overline{p},\overline{p}) + \overline{p}D_H^{(3,2)}(\overline{p},\overline{p})\right] - \Lambda_0\left[3G^{(2)}(\overline{p}) + \overline{p}G^{(3)}(\overline{p})\right] \stackrel{!}{=} 0.
\end{aligned}
$$

Noting that $D_H^{(2,1)}(\overline{p},\overline{p}) = 0$ and $D_H^{(2,2)}(\overline{p},\overline{p}) = -\frac{64}{81}$ (compare with equation (55)), as well as that $D_H^{(3,1)}(\overline{p},\overline{p}) = \frac{64}{81}$ and $D_H^{(3,2)}(\overline{p},\overline{p}) = 0$ (see above), such that in particular $3D_H^{(2,1)}(\overline{p},\overline{p}) + \overline{p}D_H^{(3,1)}(\overline{p},\overline{p}) = \frac{64}{81}\overline{p}$, solving the above boundary condition for $G^{(-2)}(\overline{p})$ gives

$$
\begin{aligned}
G^{(-2)}(\overline{p}) &= \frac{-3D_H^{(2,0)}(\overline{p},\overline{p}) - \overline{p}D_H^{(3,0)}(\overline{p},\overline{p}) + \frac{64}{81}\overline{p}G^{(-1)}(\overline{p}) + \Lambda_0\left[3G^{(2)}(\overline{p}) + \overline{p}G^{(3)}(\overline{p})\right]}{3D_H^{(2,2)}(\overline{p},\overline{p})} \\
&= \frac{-\frac{8}{3} + \frac{64}{81}\overline{p}G^{(-1)}(\overline{p}) + (2s)^{\frac{4}{3}}\left[3G^{(2)}(\overline{p}) + \overline{p}G^{(3)}(\overline{p})\right]}{-\frac{64}{27}} \\
&= \frac{9}{8} - \frac{1}{3}\overline{p}G^{(-1)}(\overline{p}) - \frac{27}{64}(2s)^{\frac{4}{3}}\left[3G^{(2)}(\overline{p}) + \overline{p}G^{(3)}(\overline{p})\right] \\
&= \frac{9}{8} + \frac{\overline{p}^2}{12} + (2s)^{\frac{4}{3}}\left[-\frac{81}{64}G^{(2)}(\overline{p}) + \frac{27}{256}\overline{p}^2G^{(4)}(\overline{p})\right], \qquad (63)
\end{aligned}
$$

where the last equality follows from substituting $G^{(-1)}(\overline{p})$ from equation (62) and simplifying.

Having isolated the moments $G^{(-1)}(\overline{p})$ and $G^{(-2)}(\overline{p})$ (linking them to just $\overline{p}$ and the higher-order derivatives $G^{(2)}(\overline{p})$, $G^{(3)}(\overline{p})$, $G^{(4)}(\overline{p})$ at the upper bound), we can proceed to pin down two (mixed) boundary conditions at the upper bound that do not depend on any unknown moments. For the first of these, equation (11) applied for $k=0$ gives us the condition

$$\begin{aligned} D_H(\overline{p},\overline{p})+\overline{p}D_H^{(1,0)}(\overline{p},\overline{p}) \quad &- \quad G^{(-1)}(\overline{p})\left[D_H^{(0,1)}(\overline{p},\overline{p})+\overline{p}D_H^{(1,1)}(\overline{p},\overline{p})\right] \\ &+ \quad G^{(-2)}(\overline{p})\left[D_H^{(0,2)}(\overline{p},\overline{p})+\overline{p}D_H^{(1,2)}(\overline{p},\overline{p})\right] \stackrel{!}{=} 0. \end{aligned}$$

Inserting

$$D_H(\overline{p},\overline{p}) = \frac{1-(2s)^{\frac{4}{3}}}{2} - \frac{8\overline{p}^2}{9} + \frac{16\overline{p}^3}{27} - \frac{8\overline{p}^4}{81}$$

(compare with equation (50)),

$$D_H^{(1,0)}(\overline{p},\overline{p}) = -\frac{8}{243}\left(27(1+s)-18\overline{p}^2+4\overline{p}^3\right)$$

(compare with equation (54)),

$$D_H^{(0,1)}(\overline{p},\overline{p}) = -\frac{8}{243}\left(-27(1+s)+54\overline{p}-36\overline{p}^2+8\overline{p}^3\right)$$

(taking the partial derivative of equation (50) with respect to $\tilde{p}$ and evaluating at $p=\tilde{p}=\overline{p}$),

$$D_H^{(1,1)}(\overline{p},\overline{p}) = -\frac{8}{9} + \frac{32\overline{p}}{27} - \frac{32\overline{p}^2}{81}$$

(taking the partial derivative of equation (54) with respect to $\tilde{p}$ and evaluating at $p=\tilde{p}=\overline{p}$),

$$D_H^{(0,2)}(\overline{p},\overline{p}) = -\frac{8}{9} + \frac{32\overline{p}}{27} - \frac{32\overline{p}^2}{81}$$

(taking the second partial derivative of equation (50) with respect to $\tilde{p}$ and evaluating at $p=\tilde{p}=\overline{p}$), and

$$D_H^{(1,2)}(\overline{p},\overline{p}) = \frac{32}{27} - \frac{64\overline{p}}{81}$$

(taking the second partial derivative of equation (54) with respect to $\tilde{p}$ and evaluating at $p = \tilde{p} = \overline{p}$), simplification reveals that the above boundary condition can be expressed as

$$\begin{aligned}
&\left[\frac{1-(2s)^{\frac{4}{3}}}{2} - \frac{8\overline{p}(1+s)}{9} - \frac{8\overline{p}^2}{9} + \frac{32\overline{p}^3}{27} - \frac{56\overline{p}^4}{243}\right] \\
&- G^{(-1)}(\overline{p})\left[\frac{8(1+s)}{9} - \frac{8\overline{p}}{3} + \frac{64\overline{p}^2}{27} - \frac{160\overline{p}^3}{243}\right] \\
&+ G^{(-2)}(\overline{p})\left[-\frac{8}{9} + \frac{64\overline{p}}{27} - \frac{32\overline{p}^2}{27}\right] \stackrel{!}{=} 0,
\end{aligned} \tag{64}$$

with $G^{(-1)}(\overline{p})$ and $G^{(-2)}(\overline{p})$ as provided in equations (62) and (63).

For the second mixed boundary condition at $\overline{p}$, equation (11) applied for $k = 1$ implies that

$$\begin{aligned}
& 2D_H^{(1,0)}(\overline{p},\overline{p}) + \overline{p}D_H^{(2,0)}(\overline{p},\overline{p}) \\
- \; & G^{(-1)}(\overline{p})\left[2D_H^{(1,1)}(\overline{p},\overline{p}) + \overline{p}D_H^{(2,1)}(\overline{p},\overline{p})\right] \\
+ \; & G^{(-2)}(\overline{p})\left[2D_H^{(1,2)}(\overline{p},\overline{p}) + \overline{p}D_H^{(2,2)}(\overline{p},\overline{p})\right] - \Lambda_0\overline{p}G^{(2)}(\overline{p}) \stackrel{!}{=} 0,
\end{aligned}$$

where we have already used that $G^{(1)}(\overline{p}) = 0$. Using the above results for $D_H^{(1,0)}(\overline{p},\overline{p})$, $D_H^{(1,1)}(\overline{p},\overline{p})$ and $D_H^{(1,2)}(\overline{p},\overline{p})$, noting moreover that

$$D_H^{(2,0)}(\overline{p},\overline{p}) = \frac{8}{9}$$

(compare with equation (55)),

$$D_H^{(2,1)}(\overline{p},\overline{p}) = 0$$

(taking the partial derivative of equation (55) with respect to $\tilde{p}$ and evaluating at $p = \tilde{p} = \overline{p}$),

$$D_H^{(2,2)}(\overline{p},\overline{p}) = -\frac{64}{81}$$

(taking the second partial derivative of equation (55) with respect to $\tilde{p}$ and evaluating at $p = \tilde{p} = \overline{p}$), another simplification reveals that the above boundary condition can be expressed as

$$\begin{aligned}
&\left[-\frac{16(1+s)}{9}+\frac{8\overline{p}}{9}+\frac{32\overline{p}^2}{27}-\frac{64\overline{p}^3}{243}\right] \\
&-G^{(2)}(\overline{p})(2s)^{\frac{4}{3}}\overline{p} \\
&-G^{(-1)}(\overline{p})\left[-\frac{16}{9}+\frac{64\overline{p}}{27}-\frac{64\overline{p}^2}{81}\right] \\
&+G^{(-2)}(\overline{p})\left[\frac{64}{27}-\frac{64\overline{p}}{27}\right] \overset{!}{=} 0,
\end{aligned} \tag{65}$$

with $G^{(-1)}(\overline{p})$ and $G^{(-2)}(\overline{p})$ as provided in equations (62) and (63). Equations (64) and (65) thus constitute the two boundary conditions at $\overline{p}$ that we were looking for.

Boundary conditions at the lower bound. We now determine the boundary conditions at $\underline{p}$. This is fortunately much easier to do than finding those at $\overline{p}$ (see above). Applying first equation (12), we obtain directly

$$-\frac{8}{3}+(2s)^{\frac{4}{3}}\left[3G^{(2)}(\underline{p})+\underline{p}G^{(3)}(\underline{p})\right] \overset{!}{=} 0. \tag{66}$$

Applying next equation (13), we get, after division through the positive constant $(2s)^{\frac{4}{3}}$,

$$4G^{(3)}(\underline{p})+\underline{p}G^{(4)}(\underline{p}) \overset{!}{=} 0. \tag{67}$$

Equations (66) and (67) thus constitute the two boundary conditions at $\underline{p}$.

Equilibrium. Collecting the above results, we can finally characterise the pricing equilibrium under a linear-decreasing match-value density, analogous to Lemma 10 in the uniform case.

Lemma 12. *Suppose that consumers' match values follow the CDF*

$$F(\varepsilon) = \frac{4\varepsilon}{3}-\frac{4\varepsilon^2}{9} \qquad \textit{for } \varepsilon \in [0, 3/2].$$

Then, an atomless distribution function $G : [\underline{p}, \overline{p}] \to [0,1]$ *is a symmetric equilibrium in mixed strategies if and only if G solves:*

$$\frac{64}{81}\left[5G(p)+pG^{(1)}(p)\right]+(2s)^{\frac{4}{3}}\left[5G^{(4)}(p)+pG^{(5)}(p)\right]=0 \qquad \forall p\in[\underline{p},\overline{p}], \tag{68}$$

$$G(\overline{p})=1, \quad G(\underline{p})=0, \quad G^{(1)}(\overline{p})=0,$$

$$\begin{aligned}&\left[\frac{1-(2s)^{\frac{4}{3}}}{2}-\frac{8\overline{p}(1+s)}{9}-\frac{8\overline{p}^2}{9}+\frac{32\overline{p}^3}{27}-\frac{56\overline{p}^4}{243}\right]\\&-\left\{-\frac{\overline{p}}{4}-\frac{81}{256}(2s)^{\frac{4}{3}}\left[4G^{(3)}(\overline{p})+\overline{p}G^{(4)}(\overline{p})\right]\right\}\left[\frac{8(1+s)}{9}-\frac{8\overline{p}}{3}+\frac{64\overline{p}^2}{27}-\frac{160\overline{p}^3}{243}\right]\\&+\left\{\frac{9}{8}+\frac{\overline{p}^2}{12}+(2s)^{\frac{4}{3}}\left[-\frac{81}{64}G^{(2)}(\overline{p})+\frac{27}{256}\overline{p}^2G^{(4)}(\overline{p})\right]\right\}\left[-\frac{8}{9}+\frac{64\overline{p}}{27}-\frac{32\overline{p}^2}{27}\right]=0,\end{aligned} \tag{69}$$

$$\begin{aligned}&\left[-\frac{16(1+s)}{9}+\frac{8\overline{p}}{9}+\frac{32\overline{p}^2}{27}-\frac{64\overline{p}^3}{243}\right]-G^{(2)}(\overline{p})(2s)^{\frac{4}{3}}\overline{p}\\&-\left\{-\frac{\overline{p}}{4}-\frac{81}{256}(2s)^{\frac{4}{3}}\left[4G^{(3)}(\overline{p})+\overline{p}G^{(4)}(\overline{p})\right]\right\}\left[-\frac{16}{9}+\frac{64\overline{p}}{27}-\frac{64\overline{p}^2}{81}\right]\\&+\left\{\frac{9}{8}+\frac{\overline{p}^2}{12}+(2s)^{\frac{4}{3}}\left[-\frac{81}{64}G^{(2)}(\overline{p})+\frac{27}{256}\overline{p}^2G^{(4)}(\overline{p})\right]\right\}\left[\frac{64}{27}-\frac{64\overline{p}}{27}\right]=0,\end{aligned} \tag{70}$$

$$-\frac{8}{3}+(2s)^{\frac{4}{3}}\left[3G^{(2)}(\underline{p})+\underline{p}G^{(3)}(\underline{p})\right]=0, \tag{71}$$

$$4G^{(3)}(\underline{p})+\underline{p}G^{(4)}(\underline{p})=0. \tag{72}$$

Unfortunately, an analytic solution $G(p)$ to the above problem does not seem to exist. We will therefore provide a numerical solution. As outlined earlier in this appendix, we do so by setting up a procedure that numerically solves the above ODE for arbitrary combinations of $(\overline{p}, \underline{p})$ using all boundary conditions except for $G(\overline{p}) = 1$ and $G(\underline{p}) = 0$, treating the solution as function of $\overline{p}$ and $\underline{p}$, and then numerically finding the combination of $\overline{p}$ and $\underline{p}$ such that the solution indeed satisfies the additional requirements $G(\overline{p}) = 1$ and $G(\underline{p}) = 0$.[25]

[25] A straightforward Mathematica code (written in version 11.0) is available from the authors upon request.

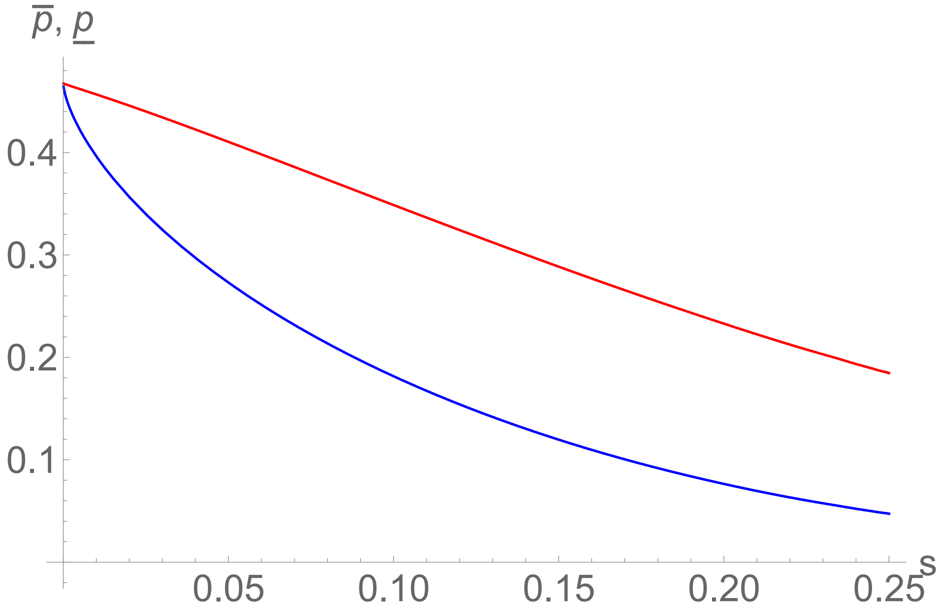

Figure 5: Equilibrium upper support bound (red) and lower support bound (blue) under a linear-decreasing match value density (with match-value CDF $F(\varepsilon) = (4/3)\varepsilon - (4/9)\varepsilon^2$, $\varepsilon \in [0, 3/2]$) as a function of s.

Following this procedure, we can, in principle, compute the solution for any sensible search cost s in $(0, 1/2)$. For example, when $s = 0.1$, we find that for $\overline{p} \approx 0.3488$ and $\underline{p} \approx 0.1816$, the numerical solution to the above fifth-order ODE with boundary conditions $G^{(1)}(\overline{p}) = 0$ and (69) to (72) also satisfies $G(\overline{p}) = 1$ and $G(\underline{p}) = 0$, as required. However, we unfortunately run into convergence problems for relatively large values of s (such that the equilibrium prices are relatively low). Because of this, we only consider the interval $s \in (0, 1/4]$ in what follows.

Numerical Results 4. *Figure 5 plots the upper support bound $\overline{p}$ and lower support bound $\underline{p}$ in equilibrium as a function of s, for $s \in (0, 1/4]$. Both pricing bounds strictly decrease in s, starting from $\overline{p} = \underline{p} \approx 0.4678$ for the case $s \to 0$, down to $\overline{p} \approx 0.1846$ and $\underline{p} \approx 0.0473$ for the case $s = 1/4$.*

As with uniformly-distributed match values, these results confirm the intuition that an increase in search costs should tend to intensify price competition. Moreover, the extreme case $s \to 0$ again corresponds to the unique symmetric pure-strategy equilibrium that exists when $s = 0$.[26]

Welfare. We conclude this appendix with a discussion of welfare. Using the general expression for welfare as reported in Footnote 19 in the main text, it can first be shown that for given prices $0 < p_L < p_H \leq \hat{p}$, total social welfare under the considered setup match-value distribution is given by

$$\begin{aligned} W(p_H, p_L, s) := {} & \frac{7}{10} - s + \frac{3}{10}(2s)^{\frac{5}{3}} - \frac{4(1+s)p_H^2}{9} + \frac{8p_H^3}{27} - \frac{32p_H^5}{1215} \\ & + p_L\left(\frac{8(1+s)p_H}{9} - \frac{8p_H^2}{9} + \frac{32p_H^4}{243}\right) \\ & + p_L^2\left(-\frac{4(1+s)}{9} - \frac{8p_H}{9} + \frac{8p_H^2}{9} - \frac{64p_H^3}{243}\right) + \frac{8p_L^3}{27}. \end{aligned} \tag{73}$$

Moreover, for given prices $0 < p_L < p_H \leq \hat{p}$, the industry profit is simply given by

$$\Pi(p_H, p_L, s) := p_L D_L(p_L, p_H) + p_H D_H(p_H, p_L),$$

with $D_L(p_L, p_H)$ and $D_H(p_H, p_L)$ taken from equations (49) and (50). From these two observations, it also immediately follows that for given prices, the consumer surplus in the market can be written as

$$CS(p_H, p_L, s) := W(p_H, p_L, s) - \Pi(p_H, p_L, s).$$

[26] For $s = 0$, equation (7) implies that for $F(\varepsilon) = (4/3)\varepsilon - (4/9)\varepsilon^2$, the unique solution in $(0, 3/2)$ to

$$-\frac{56p^4}{243} + \frac{32p^3}{27} - \frac{8p^2}{9} - \frac{8p}{9} + \frac{1}{2} \overset{!}{=} 0$$

defines the symmetric candidate equilibrium price. One may check numerically that this is indeed given by $p^* \approx 0.4678$. Moreover, using the profit function in (6), it can be verified that there are also no profitable non-marginal deviations from $p^* = 0.4678$. Hence, this constitutes the unique symmetric pure-strategy equilibrium.

Since our above numerical method allows us to compute the equilibrium CDF $G(p)$ and corresponding density $G'(p)$ for arbitrary values of s, the unconditional expected aggregate welfare, industry profit and consumer surplus in the market are respectively given by

$$W := 2\int_{\underline{p}}^{\overline{p}} \left[\int_{\underline{p}}^{p_H} W(p_H, p_L, s) dG(p_L) \right] dG(p_H), \tag{74}$$

$$\Pi := 2\int_{\underline{p}}^{\overline{p}} \left[\int_{\underline{p}}^{p_H} \Pi(p_H, p_L, s) dG(p_L) \right] dG(p_H), \tag{75}$$

and

$$CS := 2\int_{\underline{p}}^{\overline{p}} \left[\int_{\underline{p}}^{p_H} CS(p_H, p_L, s) dG(p_L) \right] dG(p_H). \tag{76}$$

Applying numerical integration in conjunction with our above methodology, we finally obtain the following.

Numerical Results 5. *Figure 6 plots expected social welfare, consumer surplus, and industry profit as a function of s, for* $s \in (0, 1/4]$. *Welfare is first increasing and then decreasing in search costs. Consumer surplus is strictly increasing and industry profit is strictly decreasing in search costs.*[27]

Figure 6 shows in particular that the non-monotonicity of welfare with respect to search costs prevails under a linear-decreasing match-value density: welfare is again clearly maximised for a strictly positive level of search costs. In addition, the industry profit (consumer surplus) across the considered range is again strictly decreasing (increasing) in s.

[27] Like in the uniform case, it may moreover be observed that for $s \to 0$, now $W(0) \approx 0.6177 \approx W(0.4678, 0.4678, 0)$ (compare with equation (73)), where $p^* = 0.4678$ is the symmetric pure-strategy equilibrium for $s = 0$ (see Footnote 26 above). Also $CS(0)$ and $\Pi(0)$ are consistent with this equilibrium price level.

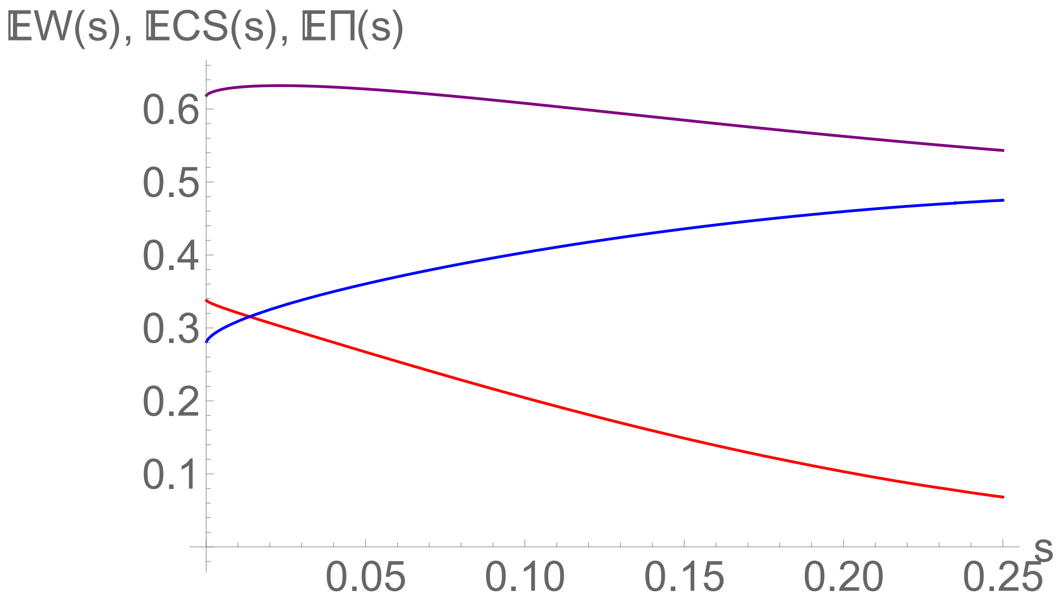

Figure 6: Expected social welfare (purple), consumer surplus (blue), and industry profit (red) under a linear-decreasing match value density (with match-value CDF $F(\varepsilon) = (4/3)\varepsilon - (4/9)\varepsilon^2$, $\varepsilon \in [0, 3/2]$) as a function of s.