Interim Information and Seller's Revenue in Standard Auctions*

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Abstract

We study the interim revenue — i.e., the expected seller's revenue coWe study the interim revenue — i.e., the expected seller's revenue conditional on the valuation of one of the bidders — in standard auctions, a class of sealed-bid auctions that are ex-ante equivalent by the Revenue Equivalence Theorem. The first-price auction yields a higher (lower) interim revenue than the second-price auction if the valuation is lower (higher) than a threshold. The first-price auction also has the highest interim revenue among all standard auctions if the valuation is the lowest possible one. By contrast, when the valuation is sufficiently high, the first-price auction has the lowest interim revenue and the last-pay auction — an atypical mechanism where only the lowest bidder pays — yields an unbounded interim revenue.

1 Introduction

Consider an auction with risk-neutral bidders who have independent private valuations, and assume that the seller privately learns the valuation of one of the bidders, which we refer to as the *special* bidder. For example, this information about a bidder's valuation may emerge exogenously if a potential buyer in the auction has a score derived from his past purchases. Alternatively, the seller may obtain information endogenously, for example in multiple auctions where a bidder is the winner of a past auction who has revealed his valuation through his bid.[1] In other environments, the seller may simply know whether the valuation of a bidder is higher or lower than a threshold, for example because she observed whether or not the bidder previously agreed to pay a fixed price for an object similar to the one on sale.

We analyze the expected seller's revenue conditional on the information about the valuation of the special bidder — the *interim* seller's revenue. We focus on single-object sealed-bid auctions that award the

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[1] Carannante *et al.* (2023) analyzes the optimal choice of reserve prices in repeated auctions for online display advertising, where bidders bid to acquire multiple impressions to the same user. In this dynamic model, the special bidder in an auction is the winner of a past auction who has a constant valuation for acquiring another impression, while all other bidders draw new time-specific valuations.

object to the bidder with the highest valuation, and where the bidder with the lowest possible valuation obtains 0 surplus.

In the environment that we analyze, the celebrated Revenue Equivalence Theorem (Vickrey, 1961; Myerson, 1981; Riley and Samuelson, 1981) characterizes the *ex-ante* expected seller's revenue. In particular, the theorem shows that any efficient equilibrium of any auction where the lowest bidder's type obtains 0 surplus yields the same expected revenue for the seller, and also results in the same expected payment by every type of bidder. This second result (which actually implies the first one) is sometimes called the bidders' Payoff Equivalence Theorem (see, e.g., Milgrom, 2004). Hence, the theorem implies that the choice of the auction format is irrelevant both for the seller and — ex interim — for all types of bidders, because in all auctions they win with the same probability and expect to pay the same.

Does this indifference also hold conditional on the valuation of one of the bidders? Or does knowing the valuation of the special bidder induce the seller to prefer a specific auction format, depending on this valuation? To address these questions, we first focus on the two most common auction formats: the first-price auction (FPA) — where the winner pays his bid — and the second-price auction (SPA) — where the winner pays the second-highest bid. We show that the there is a unique threshold such that the interim seller's revenue is higher in the FPA than in the SPA if the bidder's valuation is lower than the threshold, while it is higher in the SPA than in the FPA if the bidder's valuation is higher than the threshold (Theorem 3).

To see the intuition behind this result, consider an auction with only two bidders and suppose that the seller learns that one of them — the special bidder — has a very low valuation, say close to zero. This is bad news regardless of the auction format, but especially so in the SPA. The reason is that the special bidder's bid is likely to lose and to determine the auction price in the SPA. In the FPA, by contrast, the auction price is determined by the bid of the other bidder, whose expected valuation is no lower than the unconditional expectation of a bidder's valuation. Hence, the seller expects to earn a very low revenue in the SPA, close to zero and bounded above by the special bidder's valuation, but not in the FPA. This intuition extends to more bidders and any (relatively) low valuation of the special bidder: such a low valuation has a stronger negative effect on the expected seller's revenue in the SPA than in the FPA, since it reduces the expected second-highest value more than the expected highest one.

Suppose now that the seller learns that the special bidder has a very high value, say close to the maximum possible one. This is good news regardless of the auction formats, but especially so in the SPA. The reason is the following. If the special bidder wins the auction, then the seller's expected revenue is the same in both auction formats (by the Payoff Equivalence Theorem). If the special bidder loses the auction, however, the seller's revenue is equal to the special bidder's valuation in the SPA, while it is only equal to the expected bid of the other bidder in the FPA, who also has a valuation close to maximum but shades his bid below his valuation. This intuition extends to more bidders and any (sufficiently) high valuation of the special bidder: when a bidder with a very high valuation loses, the seller's revenue in the SPA is at least equal to his valuation, while in the FPA it is equal to the bid by the highest bidder, who has a valuation close to the special bidder's one but shades his bid below this valuation.

An analogous intuition suggests that the same ranking of the FPA and SPA also holds when both auctions include a (common) reserve price. In fact, the interim seller's revenue is higher in the FPA than in the SPA if and only if the special bidder's valuation is lower than a threshold, which is higher than the reserve price (Proposition 4).

We also generalize this analysis by considering *standard auctions*, a broad class of sealed-bid auctions that assign the object to the highest bidder and are characterized by the set of bidders who make a transfer to the seller and by a function that determines the transfer made by each of these bidders. This class includes, in addition to the FPA and SPA, other winner-pay auctions where only the highest bidder pays one of the submitted bids, and pay-as-bid auctions (like the the all-pay auction) where one or more bidders pay their own bids. When the special bidder's valuation is the lowest possible one, or close to it, the FPA is interim revenue dominant among all standard auctions (Proposition 9).

We prove this results in two steps. First, we show that the FPA maximizes the interim seller's revenue among all pay-as-bid auctions (Proposition 8). This first result implies, for example, that close to zero the FPA yields a higher interim revenue than the all-pay auction. Second, we show that, when the special bidder's valuation is the lowest possible one, the interim seller's revenue is also higher in the FPA than in any alternative auction where payers do not necessarily pay their bids. This second result generalizes the comparison close to zero between the FPA and SPA, since the highest bidder pays both auctions, but he pays his bid only in the FPA.

By contrast, when the special bidder's valuation is close to the highest possible one, the two previous results are reversed and the FPA yields the lowest interim seller's revenue among all standard auctions (Proposition 9). In fact, the special bidder's payment represents a lower bound for the interim revenue (Lemma 1), and the FPA achieves this bound when the special bidder has the highest possible value. The reason is that the special bidder always wins in this case and, hence, his payment represents the total seller's revenue (since no other bidder pays). Moreover, the Payoff Equivalence Theorem implies that this lower bound is also achieved by the SPA and by any other auction where only the winner pays (Proposition 8).

Other formats with a different set of payers, however, yield a higher interim seller's revenue when the special bidder has the highest possible value. For example, the interim revenue is strictly higher in the all-pay auction, where all bidders pay their bids, than in any winner-pay auction. The reason is that, in the all-pay auction, the seller obtains payments by all other bidders, in addition to the special bidder. Finally, the interim revenue is even higher when *only* losing bidders pay: when the special bidder's valuation is close to the highest possible one, the interim revenue tends to infinity in the Last-Pay Auction (LPA) — an atypical mechanism that we introduce, where only the *lowest* bidder pays his bid (Proposition 8). Conversely, the LPA yields the lowest interim seller's revenue when the special bidder's valuation is the lowest possible one.

Since the FPA maximizes the interim revenue when the special bidder's value is 0, and minimizes it when the special bidder's value is the highest possible one (and since the interim revenue is increasing in the special bidder's value), the FPA yields the smallest range of possible interim revenues for the seller among standard auctions, and it maximizes the minimum interim revenue (Corollary 5.4). This provides a new interpretation for why the FPA may be considered the less risky standard auction format for the seller.[2]

Our theoretical analysis has a direct application to environments where the seller is privately informed

[2]The common interpretation is that a risk-averse seller ex-ante prefers the FPA because, although the expected seller's revenue is the same, the realized seller's revenue in other standard auctions is a mean-preserving spread of the one in the FPA (Waehrer *et al.*, 1998).

about a particular bidder who participates in the auction, and other competitors are unaware of that. In fact, our result suggests that the seller may obtain a higher revenue by choosing a particular auction format conditional on this information, if other bidders do not draw any inference from the seller's choice and always bid as in an auction with symmetric competitors. This may be a natural assumption in environments with inexperienced or unsophisticated bidders.

The rest of the paper is organized as follows. Section 2 describes the environment and Section 3 considers a simple example. Section 4 compares the interim seller's revenue in the FPA and the SPA. In Sections 5, we introduce the broad class of standard auctions. Sections 5.2 and 5.3 consider special cases of standard auctions, the winner-pay and the pay-as-bid auctions, and Section 5.4 analyzes the auctions that maximizes the interim revenue, when the special bidder's value is close to the extrema. Section 6 concludes. Extensions of our analysis, omitted proofs and derivations are in the Appendix.

2 Environment

There is a single-object auction with n risk-neutral bidders. Each bidder i is privately informed of his valuation v_i. Valuations are independently drawn from the same CDF $F(\cdot)$ with full support $[0,1]$ and virtual value ψ. Throughout the paper, unless otherwise specified, the expectation $\mathbb{E}$ integrates over F. Given a set of bidders' values $\boldsymbol{v} \in [0,1]^n$, we denote by $v_{(k)}(\boldsymbol{v})$ its k^{th}-order statistic, suppressing the argument when convenient.

We consider sealed-bid auction mechanisms that have an equilibrium with the following characteristics: (i) the object is awarded to the bidder with the highest valuation and (ii) a bidder with a valuation equal to 0 obtains 0 surplus. Throughout the paper, an auction mechanism is always associated with such efficient equilibrium, assuming it exists.[3] The equilibrium bidding function in the FPA with reserve price R is

$$b^F(v_i, R) \equiv \frac{R \cdot F(R) + \int_0^{v_i} x \mathrm{d}F^{n-1}(x)}{F^{n-1}(v_i)}.$$

If there is no reserve price, we simply write $b^F(v_i)$. We assume that there is a unique $\hat{v}$ such that $b^F(\hat{v}, R) = \psi(\hat{v})$ for every R.[4] The assumption is satisfied by the uniform distribution, the Beta distribution with different parameters, the Logit-Normal distribution, as well as many other distributions.

Let $\Pi^a : [0,1]^n \to \mathbb{R}$ be the seller's revenue in the equilibrium of auction mechanism a as a function of the bidders' valuations. The Revenue Equivalence Theorem (Vickrey, 1961; Myerson, 1981; Riley and Samuelson, 1981) states that

Theorem. *In any auction a where the lowest bidder's type obtains 0 surplus:*

- *Every bidder's type v_i makes expected payment $t(v_i) = \int_0^{v_i} x \, dF^{n-1}(x)$ and therefore*
- *The seller earns expected revenue $\mathbb{E}_{\boldsymbol{v}}[\Pi^a] = \bar{\Pi} = n\mathbb{E}[t(v_i)]$.*

[3] In particular, statements of the form 'in auction format a' are shorthand for 'in any efficient equilibrium of auction format a.'

[4] This assumption is a sufficient (but not necessary) condition for the uniqueness of $\tilde{v}$, as defined in Theorem 3 and Proposition 4. No other result requires this assumption. Alternative sufficient conditions for the uniqueness of $\tilde{v}$ are: (i) $\psi'(v_i) > b^{F\prime}(v_i, R)$ at any v such that $b^F(v_i, R) = \psi(v_i)$; (ii) bid shading in the FPA is increasing in the bidder's value, but less so as n increases.

We refer to the first statement as the Payoff Equivalence Theorem (PET) and to the second (weaker, ex-ante) statement as the Revenue Equivalence Theorem (RET).

We analyze the properties of the *interim revenue function* (IRF)

$$\mathit{\Pi}^a(v) \equiv \mathbb{E}_{\boldsymbol{v}}\left[\Pi^a | v_1 = v\right],$$

where $\mathit{\Pi}^a : [0,1] \to \mathbb{R}$ is the expected seller's revenue in auction a conditional on the valuation of one of the n bidders, say bidder 1. We denote $\mathbb{P}_{\boldsymbol{v}|v}$ as the probability measure conditional on the event $v_1 = v$ and $\mathbb{E}_{\boldsymbol{v}|v}$ the associated expectation operator, so that an equivalent rewriting is $\mathit{\Pi}^a(v) = \mathbb{E}_{\boldsymbol{v}|v}[\Pi^a]$.

In the Appendix, we show that the IRF can be written as the sum of the special bidder's transfer $t(v)$, which is independent of the auction format, and the expected transfers of the other $(n-1)$ generic bidders — i.e.,

$$\mathit{\Pi}^a(v) = t(v) + (n-1)\,\mathbb{E}\left[t_v^a(x)\right], \tag{2.1}$$

where $t_v^a(x)$ is the *interim transfer* of a bidder with valuation x in auction a, conditional on the information that one competitor has valuation v. This highlights that the auction format does not affect the interim revenue through the special bidder's transfer, because of the PET, but only through the transfers of the other symmetric bidders.

Lemma 1. *The IRF has the following properties:*

- $\mathit{\Pi}^a(v)$ *is increasing.*
- $\mathit{\Pi}^a(v) \geq t(v)$*, for all* v*.*
- $\mathbb{E}_v\left[\mathit{\Pi}^a(v)\right] = \bar{\Pi}$.

The fist property in Lemma 1 is a consequence of the auction efficiency. The second property follows from (2.1) and provides a lower bound for the IRF of any auction: the expected seller's revenue is at least equal to the transfer of the special bidder. The last property follows form the LIE:

$$\mathbb{E}_v\left[\mathit{\Pi}^a(v)\right] = \mathbb{E}_v\left[\mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^a\right]\right] = \mathbb{E}_{\boldsymbol{v}}\left[\Pi^a\right] = \bar{\Pi}.$$

This property simply reflects the fact that the expectation with respect to v of the IRF is the ex-ante expected seller's revenue, which is independent of the auction format by the RET.

3 Example

Suppose there are two bidders, each with a private valuation which is independently and uniformly distributed on $[0,1]$. We compare the interim seller's revenue in the First-Price sealed-bid Auction (FPA) and the Second-Price sealed-bid Auction (SPA), when the seller privately learns that one of the bidders has a valuation equal to v. Recall that in the SPA it is a dominant strategy for every bidder to bid his value, while in the FPA the equilibrium bidding function is $b^F(x) = \frac{x}{2}$.

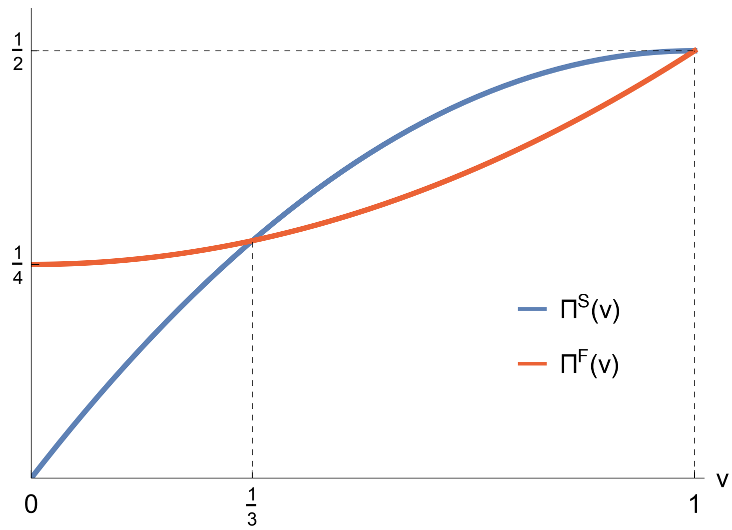

Figure 3.1: Interim revenue functions in the FPA and SPA with $n = 2$ and $v_i \sim \mathcal{U}[0,1]$.

Let x denote the valuation of the other bidder, which is unknown to the seller. In the SPA, the interim seller's revenue is

$$\begin{aligned} \Pi^S(v) &= \mathbb{P}(x < v) \cdot \mathbb{E}[x|x < v] + \mathbb{P}(x > v) \cdot v \\ &= v \cdot \frac{v}{2} + (1-v) \cdot v = v\left(1 - \frac{v}{2}\right), \end{aligned} \tag{3.1}$$

because the seller's revenue is v when the other bidder's value is higher than v, and it is the expectation of the other bidder's value when v is the highest value. Similarly, in the FPA the interim seller's revenue is

$$\begin{aligned} \Pi^F(v) &= \mathbb{P}(x < v) \cdot b^F(v) + \mathbb{P}(x > v) \cdot \mathbb{E}\left[b^F(x)|x > v\right] \\ &= v \cdot \frac{v}{2} + (1-v) \cdot \frac{1+v}{4} = \frac{1+v^2}{4}, \end{aligned} \tag{3.2}$$

because the seller's revenue is the winner's bid, which is $b^F(v)$ when v is the highest value and the expectation of other bidder's bid when he has a value higher than v.

Figure 3.1 shows the interim revenue functions in the two auction formats. Notice that, if the seller knows that a bidder has valuation 0, his interim revenue in the SPA is 0, since this bidder always loses and the auction price is equal to his value. By contrast, in the FPA the auction price is given by the other bidder's bid; hence it is strictly positive and equal to $\Pi^F(0) = \frac{1}{4}$. If instead the seller knows that a bidder has valuation 1, this bidder always wins in both auction formats and the seller's revenue is equal to the bidder's payment which, by the PET, is the same in the FPA and SPA ($\Pi^F(1) = \Pi^S(1) = \frac{1}{2}$).

Finally, comparing equations (3.2) and (3.1): (i) $\Pi^F(v) > \Pi^S(v)$ if $v < \frac{1}{3}$, (ii) $\Pi^F(v) < \Pi^S(v)$ if $\frac{1}{3} < v < 1$, and (iii) $\Pi^F(v) = \Pi^S(v)$ if $v \in \left\{\frac{1}{3}, 1\right\}$. Therefore, the interim seller's revenue is higher in the FPA than in the SPA if and only if v is sufficiently low. Moreover, the difference between the interim revenue in the FPA and SPA — i.e., $\Pi^F(v) - \Pi^S(v)$ — is maximized at $v = 0$ and minimized at $v = \frac{2}{3}$. It is also straightforward to check that the two auction formats are ex-ante revenue equivalent (as implied by the RET): $\mathbb{E}_v\left[\mathit{\Pi}^S(v)\right] = \mathbb{E}_v\left[\mathit{\Pi}^F(v)\right] = \frac{1}{3}$. Graphically, this implies that the areas below each of the interim revenue functions in Figure 3.1 are equal.

In the next section, we show that the qualitative properties of the IRFs in the FPA and SPA hold more generally, with an arbitrary number of bidders and distribution of values.

4 FPA vs. SPA

The following decomposition of the IRF will be useful in our analysis:[5]

Lemma 2. *Let a be an auction where only the winner pays. Then,*

$$\mathit{\Pi}^a(v) = t(v) + \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} \neq v\right)\mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^a|v_{(1)} \neq v\right].$$

Lemma 2 highlights that, when the special bidder is the winner, all auction formats in which only the winner pays are interim revenue equivalent. In this case, the seller's revenue is the expected payment of the special bidder, and hence it is independent of the auction format. Moreover, since a special bidder with the highest possible valuation wins for sure, these auctions are interim revenue equivalent when $v = 1$.

By Lemma 2, the difference between the interim revenues in the FPA and SPA only depends on the event that the special bidder loses:

$$\begin{aligned}\Delta(v) &\equiv \mathit{\Pi}^F(v) - \mathit{\Pi}^S(v) \\ &= \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} \neq v\right)\left(\mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^F|v_{(1)} \neq v\right] - \mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^S|v_{(1)} \neq v\right]\right).\end{aligned} \tag{4.1}$$

The following theorem shows that this difference is positive — i.e., the FPA yields a higher interim revenue than the SPA — if and only if the value of the special bidder is lower than a threshold (see Figure 4.1 for an example).

Theorem 3. *There is a unique $\tilde{v}$ such that*

$$\begin{cases}\Delta(v) > 0 & \text{if } v \in [0, \tilde{v}) \\ \Delta(v) < 0 & \text{if } v \in (\tilde{v}, 1) \\ \Delta(v) = 0 & \text{if } v \in \{\tilde{v}; 1\}\end{cases}$$

Moreover, $\Delta(v)$ has a global maximum at $v = 0$ and global minimum at $\hat{v}$ such that $b^F(\hat{v}) = \psi(\hat{v})$.

Proof. By (4.1), the Interim revenue difference between the FPA and SPA is

$$\begin{aligned}\Delta(v) &= \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} \neq v\right) \cdot \left(\mathbb{E}_{\boldsymbol{v}|v}\left[b^F\left(v_{(1)}\right)|v_{(1)} \neq v\right] - \mathbb{E}_{\boldsymbol{v}|v}\left[v_{(2)}|v_{(1)} \neq v\right]\right) \\ &= \int_v^1 b^F(x)\,\mathrm{d}F^{n-1}(x) - \int_v^1 \int_0^x \max\{v, y\} \frac{\mathrm{d}F^{n-2}(y)}{F^{n-2}(x)} \mathrm{d}F^{n-1}(x).\end{aligned} \tag{4.2}$$

[5] This decomposition relies on the characteristics of the FPA and SPA and separates the events that the special bidder wins (and hence is the only payer in each of the two formats) and loses. This is conceptually different from the decomposition (2.1), which holds in all auction formats (including those that we analyze in Section 5, where the winner does not pay or is not the only payer).

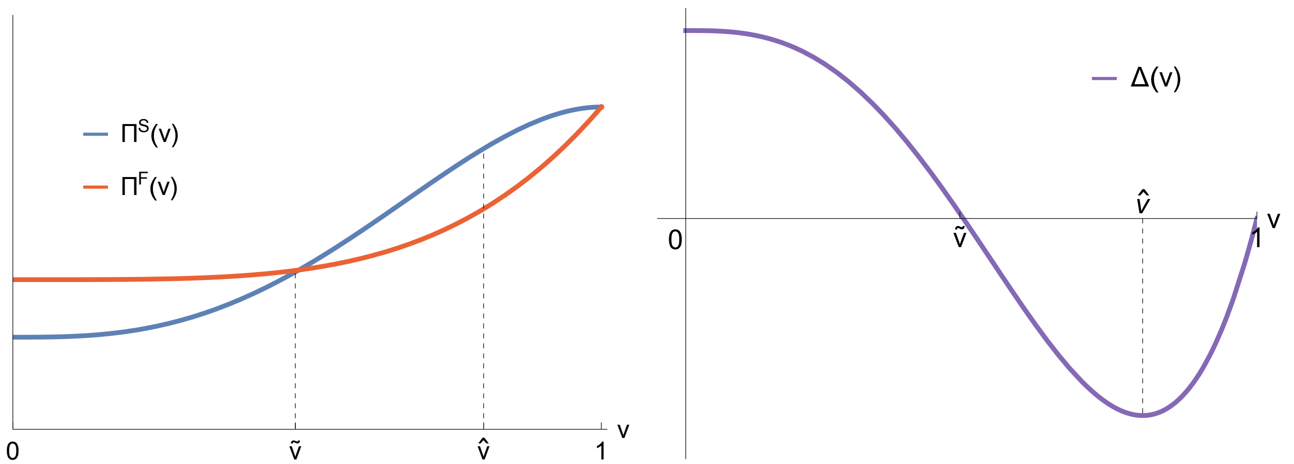

Figure 4.1: Interim revenue functions in the FPA and SPA (left) and interim revenue difference $\Delta(v)$ (right) with $n = 4$ and $v_i \sim \mathcal{U}[0,1]$.

The first term in the previous equation is the expected highest bid in the FPA when v loses, while the second term is the expected second-highest bid in the SPA — which is the expected second-highest value — when v loses.

In the Appendix, we show the second term is equal to $\int_v^1 \psi(x)\,\mathrm{d}F^{n-1}(x)$. Therefore,

$$\Delta(v) = \int_v^1 \left(b^F(x) - \psi(x)\right) \mathrm{d}F^{n-1}(x), \tag{4.3}$$

and

$$\Delta'(v) = -\left(b^F(v) - \psi(v)\right) \mathrm{d}F(v)^{n-1}.$$

Hence, critical points of $\Delta(v)$ are either 0 or such that $b^F(v) = \psi(v)$. In the Appendix, we show that $\Delta(v)$ has a maximum at $v = 0$ and $\Delta(0) > 0$ (see Lemma 13). Moreover, it is straightforward that $\Delta(1) = 0$, $\Delta'(1) \propto 1 - b^F(1) > 0$ and, by the RET, $\mathbb{E}_v[\Delta(v)] = 0$.

It follows that $\Delta(v)$ must cross the horizontal axis (from above) before 1 and must have minimum at $\hat{v}$ such that $b^F(\hat{v}) = \psi(\hat{v})$. Under our assumption that there is a unique v such that $b^F(v) = \psi(v)$, there are exactly two critical points of $\Delta(v)$: (i) the maximum at 0 is unique and (ii) the minimum at $\hat{v}$ is unique. Hence, $\Delta(v)$ cannot cross the horizontal axis more than once and so there is a unique $\tilde{v} \neq 1$ such that: (i) $\Delta(\tilde{v}) = 0$, (ii) $\Pi^F(v) > \Pi^S(v)$ if $v < \tilde{v}$, and (iii) $\Pi^F(v) < \Pi^S(v)$ if $\tilde{v} < v < 1$. □

To shed some light on equation (4.3), we notice that, perhaps surprisingly, the second term in equation (4.2) is equal to the ex-ante expected revenue in *any* auction with *reserve price* v *and* $n-1$ *bidders*. The reason is the following. First, the probability that in an auction with n bidders a bidder with value v loses is exactly equal to the probability that, in an auction with $n-1$ bidders and reserve price v, there is at least one bidder with value higher than the reserve price. Hence, $\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} \neq v\right)$ represents the probability that the seller's revenue is different from 0 in any such auction.

Second, $\mathbb{E}_{\boldsymbol{v}|v}\left[v_{(2)} | v_{(1)} \neq v\right]$ — i.e., the expected second-highest value when one bidder among n has value v and this is not the highest value — is exactly equivalent to the expected seller's revenue in a SPA with $n-1$ bidders and reserve price v, when the highest bidder has a value higher than the reserve price.

In fact, the seller's revenue in this auction is the reserve price v when only one bidder has a value higher than v and is the second-highest value otherwise. Intuitively, when a bidder with value v loses in a SPA with n bidder she acts exactly as a reserve price in a SPA with $n-1$ bidders, since his bid represents a lower bound to the price that the remaining bidders have pay.

Finally, the SPA with $n-1$ bidders and reserve v is revenue equivalent to any auction with $n-1$ bidders and reserve v, and yields an expected revenue equal to the expected virtual value of the highest bidder in the auction, conditional on this bidder having a value higher than the reserve price.

Theorem 3 says that the IRFs of the FPA and SPA cross only once, so that the seller obtains a higher interim revenue in the FPA if $v < \tilde{v}$, while she obtain a higher interim revenue in the SPA if $v > \tilde{v}$. Single crossing is relevant because it allows the seller to determine her preferred auction format not only when she knows the exact valuation of the special bidders, but also with different and arguably more realistic information structures. For example, in many application the seller may only know that the special bidder's value is either above or below an *arbitrary* threshold, possibly different from $\tilde{v}$. Because of single crossing, the seller prefers the FPA (resp. SPA) when she knows that the special bidder is below (resp. above) *any* threshold, since this is the auction format that yields the higher *expected* interim revenue conditional on this information.

4.1 Reserve Price

Our main result on the comparison between the FPA and SPA extends to a setting with a common reserve price R. Let $\mathit{\Pi}^a(v, R)$ be the IRF in auction a with reserve price R, when the special bidder has value v, and let the interim revenue difference between the FPA and SPA be

$$\Delta(v, R) \equiv \mathit{\Pi}^F(v, R) - \mathit{\Pi}^S(v, R).$$

Proposition 4. *There is a unique $\tilde{v} > R$ such that*

$$\begin{cases} \Delta(v, R) > 0 & \text{if } v \in [0, \tilde{v}) \\ \Delta(v, R) < 0 & \text{if } v \in (\tilde{v}, 1) \\ \Delta(v, R) = 0 & \text{if } v \in \{\tilde{v}; 1\} \end{cases}$$

Moreover, $\Delta(v, R)$ is maximized at all $v \leq R$ and has a global minimum at $\hat{v}$ such that $b^F(\hat{v}, R) = \psi(\hat{v})$.

Figure 4.2 compares the IRFs of the FPA and SPA. Notice that, when the special bidder's value is lower R, the interim revenue is constant in both auction formats, but it is higher in the FPA than in the SPA. This is obvious when there are only two bidders: if the seller knows that one bidder has a value lower than R, then her revenue is positive if and only if the other bidder's value is higher than R. In this case, however, the seller's revenue is equal to the reserve price in the SPA, while it is equal to the expectation of the other bidder's bid in the FPA (which is higher than R because the bidder has a value higher than the reserve price). Similarly, with multiple bidders, when the highest bidder's value is higher than R (and the special bidder's value is lower than R): in the SPA the seller's revenue is the maximum between R and the second-highest among $n-1$ bidders, while in the FPA it is equal to the highest bidder's bid, which is the expectation of the maximum between R and the second-highest *among n bidders*.

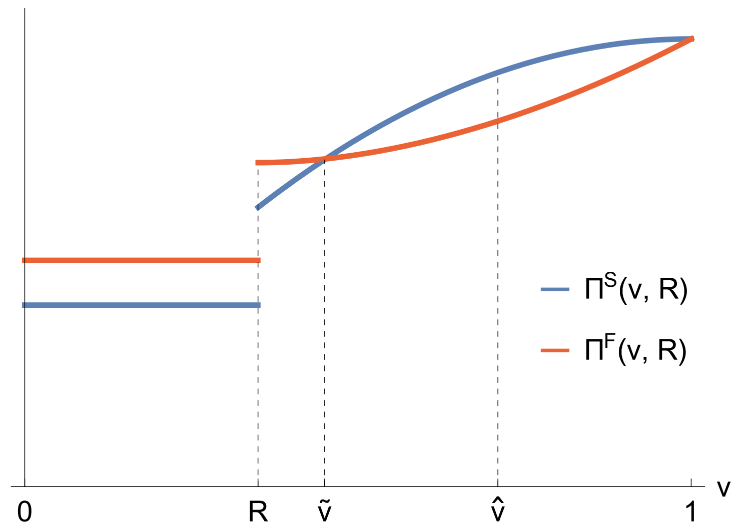

Figure 4.2: Interim revenue functions in the FPA and SPA with $n = 2$, $R = 0.35$ and $v_i \sim \mathcal{U}[0, 1]$.

Moreover, in both auction formats the interim revenue is discontinuous when the special bidder's value is exactly equal to the reserve price. The reason is that, if the seller knows that at least one bidder meets the reserve price, then her revenue is always strictly positive. In both auction formats, the discontinuous increase in the interim revenue when $v = R$ is equal to the probability that all other bidders have a value lower than the reserve price, times the reserve price.[6] Finally, the intuition for the single crossing of the IRFs and for the equivalence of the two auction formats at $v = 1$ is analogous to the case without a reserve price.

Since Proposition 4 holds for any reserve price, our comparison between the FPA and SPA also holds when the seller sets the optimal (ex-ante) reserve prices, which are the same in both auctions. Therefore, while the (ex-ante) expected revenues in the optimal FPA and SPA are equal, the optimal FPA yields a higher (lower) interim revenue than the optimal SPA if v is low (high), exactly as with FPA and SPA without reserve prices.

5 Interim Revenue in Standard Auctions

In this section, we analyze the IRFs in different auction formats (without a reserve price) to investigate which properties of the IRFs of the FPA and SPA generalize to other formats, and to identify the format which is preferred by the seller, depending on the special bidder's valuation.

In Section 5.1, we introduce *standard auctions*, a broad set of sealed-bid auctions that assign the object to the highest bidder. A standard auction is characterized by the set of bidders who make a transfer to the seller and, given such set, by a function that determines the transfer made by each of these bidders. We then consider special cases of standard auctions: the winner-pay auctions in Section 5.2 and the pay-as-bid auctions in Section 5.3, where we also introduce the last-pay auction, an atypical auction format that plays a special role in our analysis. Finally, Section 5.4 characterizes the auction formats that maximize the interim seller's revenue when the special bidder's value is close to the extrema.

[6] In both auction formats, when all other bidders have a value lower than R, the seller's revenue is 0 when the special bidder's value is also lower than R, and is R when the special bidder's value is equal to R.

5.1 Standard Auctions

In a standard auction, each bidder submits a sealed bid, which is a non-negative real number. Bids (and their associated bidders) are ranked and the object is assigned to the bidder who submits the highest bid. Where standard auctions differ is in how transfers are determined — i.e., who pays what. Since we focus on the efficient equilibrium in which a bidder with the lowest possible value obtains zero surplus, the auction winner is the bidder with the highest value and the RET applies. Moreover, the order statistic of the bids coincide with the order statistic of bidders' valuations.

Let $[n] \coloneqq \{1, \dots n\}$. An element $i \in [n]$ is interpreted as the i^{th}-order statistic of submitted bids.

Definition 5. The standard auction a is characterized by a non-empty set $\mathcal{P}_a \subseteq [n]$ and by a function $T_a : \mathcal{P}_a \to [n]$ such that $T_a(j) \geq j$ for all $j \in \mathcal{P}_a$.

The set $\mathcal{P}_a$ specifies the order statistics — or, equivalently, the bidders — that pay. The function T_a associates to each payer the order statistic — or, equivalently, the bid — that determines his transfer, with the constraint that a bidder cannot pay a bid higher than his own.[7] Although we only consider auctions where a bidder's payment is equal to one of the bids, it is straightforward to extend our results to payments that are combinations of multiple bids. $T_a(j) = k$ indicates that the bidder with the j^{th}-highest valuation, who hence submits the j^{th}-highest bid, pays a price equal to the k^{th}-highest bid.[8]

Standard auctions encompass the following familiar types of auctions.

Definition 6. A Winner-Pay Auction (WPA) is standard auction with $\mathcal{P}_a = \{1\}$. A Pay-as-Bid-Auction (PBA) is a standard auction with $T_a(i) = i$, $\forall i \in \mathcal{P}_a$.

WPAs are auctions where only the bidder who obtains the object pays. The FPA and SPA are WPAs with $T_a(1) = 1$ and $T_a(1) = 2$, respectively. In general, WPAs are characterized by the order of the bid that determines the transfer: we refer to the WPA with $T_a(1) = k$ as the k^{th}-price auction (kPA).

PBAs are auctions where bidders pay their own bids. Notice that the FPA is also a PBA while any other kPA is not; indeed to each set of payers is associated a unique PBA. The All-Pay Auction (APA) is the PBA with $\mathcal{P} = [n]$. A rather atypical standard auction that will be important for our analysis is the Last-Pay Auction (LPA), which is the PBA with $\mathcal{P} = \{n\}$ — i.e., the auction where *only the lowest bidder* pays (and pays his bid).

5.2 Winner-Pay Auctions

Each WPA is a kPA, for some k — i.e., an auction where the highest bidder wins and pays the k^{th}-highest bid. We denote by Π^k the IRF of the kPA.

Proposition 7. *$\Pi^k(0)$ is decreasing in k. $\Pi^k(1) = t(1)$ for every k, and there is a threshold $\hat{v}$ such that $\Pi^k(v)$ is increasing in k for $v \in (\hat{v}, 1)$.*

[7] More concisely, we can represent a static auction as a partial function from $[n]$ into itself. A partial function specifies the subset of $[n]$ over which it is defined (the domain $\mathcal{P}_a$) and the function over this domain ($T_a : \mathcal{P}_a \to [n]$, with the inequality constraint). If $T_a(j) < j$ for some $j \in \mathcal{P}_a$, then a bidder with a valuation close the lowest possible one would obtain negative expected utility (in a monotone equilibrium).

[8] For example, in the standard auction with $\mathcal{P}_a = \{2, 4\}$, $T_a(2) = 2$, and $T_a(4) = 6$, the second-highest bidder pays his own bid, while the fourth-highest bidder pays the sixth-highest bid. No other bidder pays.

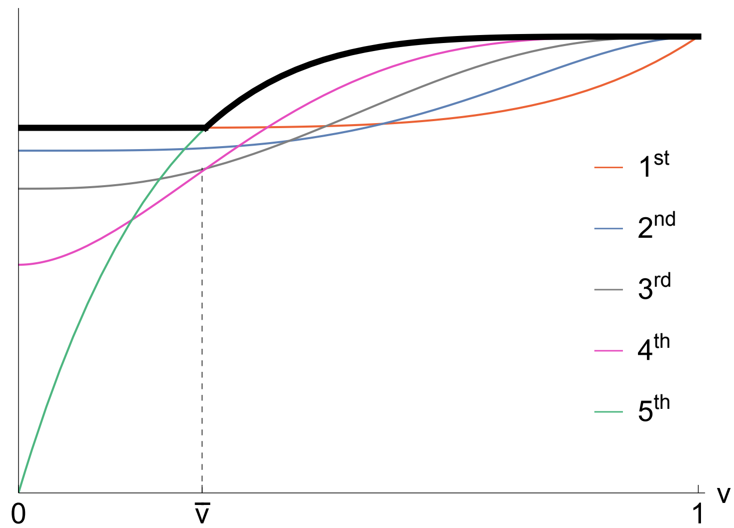

Figure 5.1: Interim revenue functions in kPA with $k = 1, ..., 5$, $n = 5$ and $v_i \sim \mathcal{U}[0,1]$.

Proposition 7 generalizes Theorem 3 for extreme values of v, establishing that if the special bidder has very low (resp. high) valuation, the seller prefers the kPA with lower (resp. higher) k. Therefore, among all WPAs: (i) the FPA — i.e., the kPA with $k = 1$ — yields the highest interim revenue when $v = 0$; (ii) the nPA — i.e., the auction where the winner pays the lowest bid — yields the highest interim revenue for $v \approx 1$. When $v = 1$, all WPAs yield the same interim revenue, which is consistent with Lemma 2. These results are displayed in Figure 5.1.

A key step in the proof of Proposition 7 is that, for all $j \geq k$,

$$\mathbb{E}_{\boldsymbol{v}}\left[\Pi^j | \Pi^k\right] = \Pi^k. \tag{5.1}$$

Consider first $k = 1$. Since the realized revenue in the FPA only depends on the valuation of the highest bidder, conditioning on Π^1 is equivalent to conditioning on $v_{(1)}$, which is the value of the only bidder who pays in any kPA. Therefore, $\mathbb{E}_{\boldsymbol{v}}\left[\Pi^j | \Pi^1\right] = \Pi^1$ for all j is a direct consequence of the PET. A similar argument holds for general k. First, in a monotone equilibrium conditioning on Π^k is equivalent to conditioning on the realization of the k^{th}-order statistic. Moreover, using PET we show that the bid of a generic type x in any kPA is equal to the expected revenues of all jPAs with $j > k$, conditional on the k^{th}-order statistic being x.[9]

Notice that in (5.1) the expectation is taken under the unconditional distribution of $\boldsymbol{v}$. By contrast, integrating under the interim measure $\boldsymbol{v}|0$ reduces the expectation of Π^j given Π^k. This is intuitive as the distribution of the j^{th}-order statistic is negatively affected by the presence of a bidder with value 0, and this is true both unconditionally and also conditional on the k^{th}-order statistic, with $k < j$. This implies the result at $v = 0$ in Proposition 7.[10]

[9] Incidentally, (5.1) implies that a risk-averse seller (ex-ante) prefers the WPA with a lower k because Π^j is a MPS of Π^k if $j > k$. This is a generalization of Waehrer *et al.* (1998), who show that the FPA is the preferred auction format for a risk-averse seller.

[10] This is a direct extension of the argument for the FPA and SPA in Section 4, where the loss of a bidder affects the expected revenue in the SPA, even conditional on the valuation of the winner — i.e., on Π^F.

Using (5.1), it also follows that

$$\mathit{\Pi}^k(v) - \mathit{\Pi}^j(v) = \sum_{i=k+1}^{n} \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)\left(\mathbb{E}_{\boldsymbol{v}|\boldsymbol{v}}\left[\Pi^k | v_{(i)} = v\right] - \mathbb{E}_{\boldsymbol{v}|v}\left[\Pi^j | v_{(i)} = v\right]\right). \tag{5.2}$$

Hence, the interim revenue difference between any two WPAs is driven by the events in which the special bidder is an order statistic above k, where k is the lowest order in the comparison. If $k = 1$ — i.e., comparing the FPA with all other WPAs — we obtain that "only losers matter" (Lemma 2).

Since $\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = 1\right) = 0$ for all $i \geq 2$, (5.2) also implies that $\mathit{\Pi}^k(1) = \mathit{\Pi}^j(1)$ for all k, j. If $v = 1$, the special bidder is the only payer in all WPA, and the PET implies that all WPAs achieve the lower bound of the IRF (Lemma 1). Local to $v = 1$, since bidders bid more aggressively in the jPA than in the kPA when $j > k$, when the special bidder is an order statistic higher than k (which is unlikely since this implies that he does not have the highest value), the jPA yields a higher interim revenue.[11]

Finally, with uniformly distributed valuations, the IRFs of two kPAs cross only once, and there is a threshold $\overline{v}$ such that the FPA is interim dominant (among all WPAs) if $v < \overline{v}$, while the nPA is interim dominant if $v > \overline{v}$, see Figure 5.1. There are no easy conditions on primitives, however, that ensure that these results hold for any distribution.

5.3 Pay-as-Bid Auctions

PBAs play a special role in the analysis of the interim dominant auction, and also reveal that some of the properties of the IRFs of WPAs do not generalize to formats where losers may pay.

Define by PB-$\mathcal{P}$ the PBA with set of payers $\mathcal{P}$. In the PB-$\mathcal{P}$, the transfer of a bidder with value x is equal to his bid times the ex-ante probability of belonging to $\mathcal{P}$, denoted by $\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})$.[12] Hence, by the PET, the equilibrium bidding function is

$$b^{\text{PB-}\mathcal{P}}(x) = \frac{t(x)}{\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})}. \tag{5.3}$$

Moreover, the interim transfer of a generic bidder x is[13]

$$t_v^{\text{PB-}\mathcal{P}}(x) = b^{\text{PB-}\mathcal{P}}(x) \cdot \mathbb{P}_{\boldsymbol{v}|x,v}(x \in \mathcal{P}) = t(x)\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})}, \tag{5.4}$$

where the ratio $\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})}$ measures how the probability that bidder x pays changes with the information that a competitor has value v.

Substituting in (2.1) yields the IRF

$$\mathit{\Pi}^{\text{PB-}\mathcal{P}}(v) = t(v) + (n-1)\,\mathbb{E}\left[t(x)\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x \in \mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P})}\right]. \tag{5.5}$$

[11] Figure 5.1 shows that, with uniformly distributed values, the higher is k, the flatter is the interim revenue of the kPA around 1. Formally, in the Appendix we show that, at $v = 1$, all derivatives of $\mathit{\Pi}^k$ up to order $k - 1$ are equal to 0.

[12] Formally, $\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P}) = \sum_{j \in \mathcal{P}} \mathbb{P}_{\boldsymbol{v}|x}\left(v_{(j)} = x\right)$, where $\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(j)} = x\right)$ is the probability that x is the j^{th}-order statistic in a sample of n i.i.d. draws from F.

[13] We denote $\mathbb{P}_{\boldsymbol{v}|v,x}$ as the probability measure conditional on the event that one bidder has valuation v and another bidder has valuation x.

All-Pay Auction Consider the APA, which is the PB-$[n]$ where every bidder pays. In the APA, since $\mathbb{P}_{\boldsymbol{v}|x}(x \in \mathcal{P}) = \mathbb{P}_{\boldsymbol{v}|x,v}(x \in \mathcal{P}) = 1$, a bidder's interim transfer is independent of v. Substituting in (5.3) and (5.5) yields $b^{APA}(x) = t(x)$ and

$$\Pi^{APA}(v) = t(v) + (n-1)\,\mathbb{E}[t(x)]. \tag{5.6}$$

Therefore, when $v = 1$ the APA dominates all WPAs since $\Pi^k(1) = t(1)$; when $v = 0$ the APA dominates the nPA since $\Pi^n(0) = 0$ (see Section 5.2).

This implies that the IRFs of the APA and the nPA cross at least twice: the APA yields a higher interim revenue both at $v = 0$ and at $v = 1$, and the IRFs of the two auctions integrate to the same value. Moreover, a sufficient condition for multiple crossing between the IRFs of the APA and a WPA is that the APA yields a higher interim revenue at $v = 0$. This condition is not met by the FPA,[14] which indeed crosses the APA only once (see the Appendix). By contrast, at $v = 0$ the APA yields a higher interim revenue than all kPA with $k \geq 2$ if and only if the expected transfer of a bidder is increasing in the number of competitors.[15]

The mechanics behind the interim dominance of the APA over WPAs at $v = 1$ is somehow counterintuitive. The APA does better because, in addition to the transfer of the special bidder — who wins for sure and pays $t(1)$ both in the APA and in all WPAs — the seller collects bids from all other bidders. Hence, in the APA the seller exploits her information advantage over other bidders and receives positive payments from bidders who (she knows) lose for sure. This intuition aligns with (2.1): ex-interim, the special bidder's transfer is pinned down by the PET and the auction format only affects the transfers by other bidders, which are 0 in a WPA when $v = 1$.

Last-Pay Auction When v is high, is the interim revenue even higher in auctions where *only* losing bidders pay? We show that the answer is positive by considering the LPA, which is the PB-$\{n\}$. By (5.3), the equilibrium bidding function in the LPA is

$$b^L(x) = \frac{t(x)}{[1 - F(x)]^{n-1}},$$

which is unbounded. To see why bids are unbounded, suppose otherwise and let M be an upper bound to the equilibrium bids. Then bidding above M is a profitable deviation, as it allows to win at price 0.

Clearly, unbounded bids do not imply an unbounded ex-ante revenue (because of RET): although in the LPA high-value bidders bid arbitrarily high because they want to win and do not expect to pay their bids, low-value bidders bid arbitrarily low because they expect to lose and pay their bids. Moreover, unbounded bids do not even imply unbounded interim revenue as $v \to 1$, because the winner does not pay his bid. Nevertheless, the next section shows that the interim revenue is indeed unbounded in the LPA (for examples of this, see Figure 5.2 and the Appendix).[16]

[14] That the FPA dominates the APA at $v = 0$ is as a special case of Proposition 8, but it also follows from a direct comparison. If the special bidder has value 0, in the FPA the interim transfer of a generic bidder is $F^{n-2}(x) \cdot b^F(x) = \frac{t(x)}{F(x)}$, which is higher than $t(x)$, his interim expected transfer in the APA.

[15] This condition ensures that the APA yields a higher interim revenue than the SPA which, by Proposition 7, implies that the APA also dominates all other kPA with $k > 2$. The condition is satisfied if n is large; for example, with uniformly distributed bidders, it requires $n > 3$.

[16] More generally, unbounded bids and interim revenue arise in all standard auctions where the winner does not pay:

Interim Ranking of PBAs By (5.5), given v the interim optimal PBA has the set of payers $\mathcal{P}$ that maximizes $\mathbb{E}\left[t\left(x\right)\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})}\right]$. The next Proposition builds on the observation that, for extreme values of v, there exists a $\mathcal{P}$ that maximizes the probability ratio $\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})}$ *for every* x.

Proposition 8. *For any* $\mathcal{P}\subseteq[n]$*,* $\Pi^{FPA}\left(0\right)>\Pi^{PB\text{-}\mathcal{P}}\left(0\right)>\Pi^{LPA}\left(0\right)$ *and* $\Pi^{LPA}\left(v\right)>\Pi^{PB\text{-}\mathcal{P}}\left(v\right)>\Pi^{FPA}\left(v\right)$ *for* $v\approx 1$*. Moreover,* $\lim_{v\to 1}\Pi^{LPA}\left(v\right)-\Pi^{PB\text{-}\mathcal{P}}\left(v\right)=\infty$*.*

Therefore, among all PBAs, the LPA yields the highest (resp. lowest) interim revenue when $v\approx 1$ (resp. $v=0$) and the FPA yields the highest (lowest) interim revenue when $v=0$ (resp. $v\approx 1$). The intuition is as follows. When $v=0$ the special bidder is the n^{th}-order statistic. This increases the likelihood that a generic bidder is *any other* order statistics, but the highest increase occurs for being the maximum. Hence, the seller wants to receive payments only from the first-order statistic, since this maximizes the probability ratio of being a payer. The argument is reversed for $v\approx 1$: in this case the special bidder is the first-order statistic and the highest increase in probability occurs for a generic bidder being the minimum, so the seller wants only the n^{th}-order statistic to pay.

Proposition 8 also provides a quantitative ranking of auction formats for $v\approx 1$, showing that the difference between the interim revenue of the LPA and any other PBA diverges. This also implies that the LPA has unbounded interim revenue. A straightforward extension the proof of this result shows that a stronger version holds, namely that for $v\approx 1$ the limit behavior of the interim revenue in PBAs is determined by the lowest payer: $\lim_{v\to 1}\Pi^{\text{PB-}\mathcal{P}}\left(v\right)-\Pi^{\text{PB-}\mathcal{P}'}\left(v\right)=\infty$ if and only if $\min\mathcal{P}>\min\mathcal{P}'$. Therefore, increasing the minimum of $\mathcal{P}$ (e.g., removing the winner or the highest payer from the set of payers) induces an infinite increase in the interim revenue for $v\approx 1$.

5.4 Interim Dominant Auctions

In this section we complete our analysis of interim-dominant auctions by establishing that the FPA yields higher (resp., lower) interim revenue than any other standard auction when $v=0$ (resp, $v\approx 1$). This generalizes the comparisons between the FPA and the SPA (Theorem 3), WPAs (Proposition 7), and PBAs (Proposition 8).

Proposition 9. *The FPA interim dominates all other standard auctions at* $v=0$ *and is interim dominated by all other standard auctions for* $v\approx 1$*.*

The proof of this result uses (2.1) and establishes that, when the special bidder has value 0, the interim transfer of a generic bidder, $t_0^a\left(x\right)$, is higher in the FPA than in any other auction format a, regardless of his value x. In particular, in the Appendix we show that the difference $t_0^{FPA}\left(x\right)-t_0^a\left(x\right)$ is proportional to

$$\sum_{j\in\mathcal{P}_a}\mathbb{E}_w\left[b^a\left(w\right)|v_{(j)}=x,\,v_{(T_a(j))}=w\right]-\frac{n-j}{n-1}\mathbb{E}_w\left[b^a\left(w\right)|v_{(j)}=x,\,v_{(T_a(j))}=w,\,v_{(n)}=0\right], \tag{5.7}$$

where b^a is the equilibrium bidding function in auction format a. The conditional expectations in (5.7) represent the expected payment of a bidder with value x, when he is the j^{th}-order statistic, this order static is a payer, and his payment is the bid of (the associated lower) bidder $T_a(j)$.

$\lim_{v\to 1}\Pi^a\left(v\right)=\infty$ if and only if $1\notin\mathcal{P}_a$.

If auction a is not a PBA, then the difference (5.7) is positive because the first expectation is higher than the second one: both expectations represent the expected bid of the same order statistic, but the second one uses $n-1$ draws (reflecting the fact that one bidder has value 0) rather than n. This is the effect that drives the interim dominance of the FPA vs. the SPA (which is not a PBA) when $v=0$, as in the SPA the transfer of the winner is lower because of the information that a competitor has value 0 (see Section 4).[17]

If auction a is a PBA, instead, then $T_a(j)=j$ and the two conditional expectations in (5.7) are both equal to $b^a(x)$. In this case, the difference (5.7) is positive because $\frac{n-j}{n-1}<1$ for all $j>1$. This effect reflects the change in the probability ratio of a generic bidder being a payer that occurs with the information that $v=0$, which drives the interim dominance of the FPA among PBAs (see Section 5.3).

Propositions 9, however, does not yield a revenue ranking for all auction formats. For $v\approx 1$, for example, they only imply that both the SPA and the LPA interim dominate the FPA. While the revenue difference between the SPA and the FPA vanishes for $v\to 1$, however, the revenue difference between the LPA and the FPA diverges, implying that the LPA dominates the SPA. This suggests that changing the set of payers has a stronger effect than only changing the bidders who determine the transfer. Indeed, raising the *minimum* order statistic in the set of payers yields an unbounded revenue increase in PBAs for $v\to 1$.[18]

Figure 5.2 gives a graphical representation of our results for standard auctions with 3 bidders and uniformly distributed valuations (see the Appendix for analytical derivations). The bottom-left panel displays the behavior of standard auctions at $v\approx 0$, where the FPA (red line) is best and the LPA (yellow line) is worst. Notice that each PBA (continuous line) lies above the corresponding non-PBA with the same set of payers (dashed line of the same color). The bottom-right panel displays the behavior of standard auctions at $v\approx 1$, where the LPA is best and the FPA is worst. In this case, each non-PBA (dashed line) lies above the corresponding PBA (continuous line of the same color). Finally, for $v\to 1$ all standard auctions with $1\notin\mathcal{P}$ diverge, but the LPA does it faster. Moreover, the difference between the LPA and the PB-$\{2\}$ (purple solid line) is infinite (Proposition 8) while the difference between the PB-$\{2\}$ and the non-PBA with $\mathcal{P}=\{2\}$ (purple dashed line) remains finite.

Finally, Proposition 9 and monotonicity of the IRF imply that

Corollary. *For any standard auction a, $Im\left(\Pi^{FPA}\right)\subset Im\left(\Pi^{a}\right)\subseteq Im\left(\Pi^{LPA}\right)$.*

This results suggests that, although a generic standard auction is not a mean-preserving spread of the FPA (in contrast to what happens with WPAs — see (5.1)), the FPA can still be considered the less risky standard auction format for the seller. First, the FPA has the smallest range of possible interim revenues, and hence the lowest variability among standard auctions. Second, the FPA maximizes the lowest possible interim seller's revenue, so it provides the best insurance for the event that $v=0$.

[17] In fact we conjecture that, when $v=0$, among all auctions with the same set of payers, the PBA yields the highest interim revenue. For example, the difference in the transfers conditional on the realization of the order statistic of a payer drives the interim dominance of the PBA vs. non-PBAs when $v=0$ for single-payer auctions (i.e., when $|\mathcal{P}_a|=1$).

[18] Quantifying the gain from changing the bidders who determine the transfer, given an arbitrary set of payers, is challenging because there is no closed form for the equilibrium bid. In the Appendix we show that, when $\mathcal{P}=\{n-1\}$, the limit revenue difference between the (unique) non-PBA and the PBA is *strictly positive but finite* (see Figure 5.2).

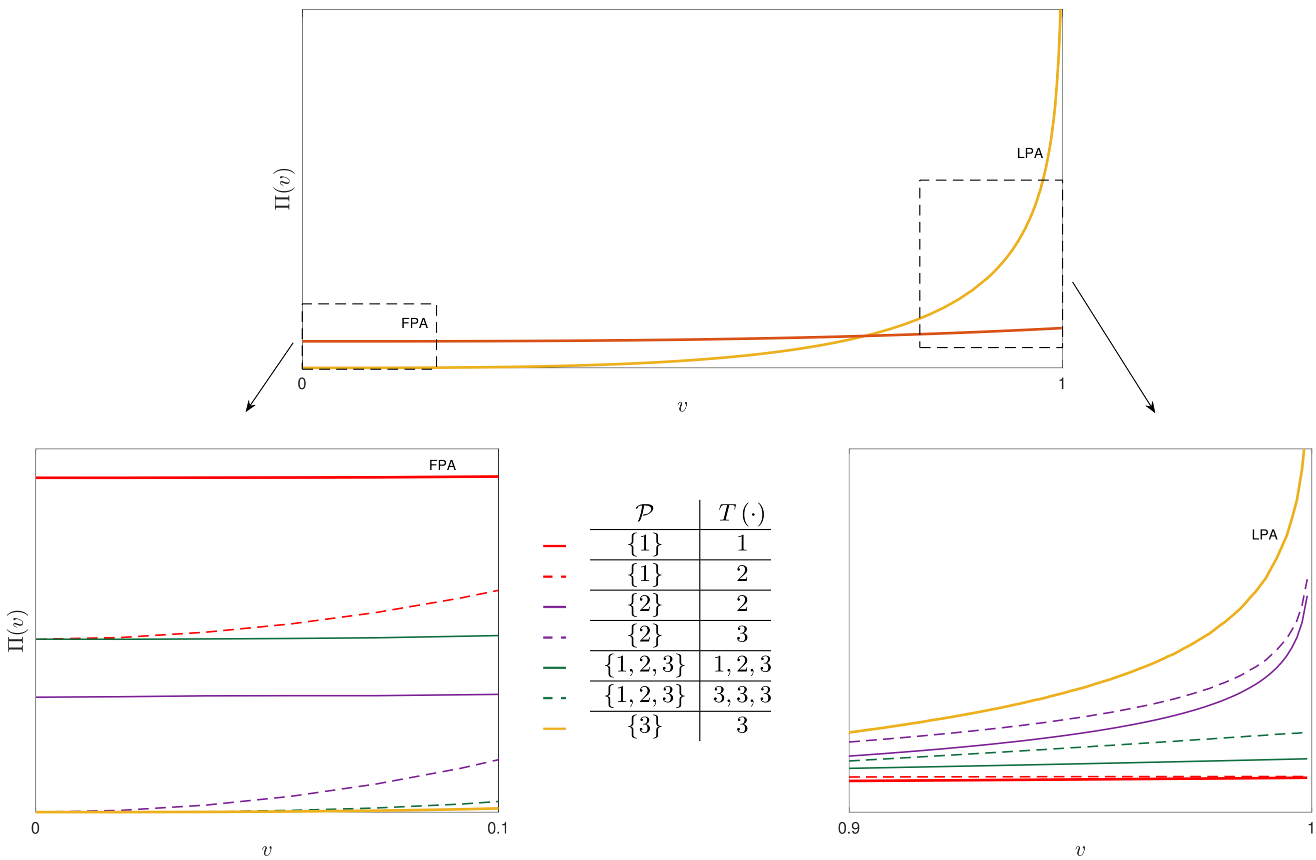

Figure 5.2: Interim revenue functions in standard auctions with $n = 3$ and $v_i \sim \mathcal{U}[0, 1]$. Column $T(\cdot)$ reports the image of the corresponding element of $\mathcal{P}$. The bottom panels zoom around $v \approx 0$ and $v \approx 1$.

6 Conclusions

We have analyzed interim revenue of the seller in a wide class of ex-ante revenue equivalent auction formats, when the seller learns the valuation v of one of the bidders. For example, the seller may obtain this information through an exogenous rating based on the bidder's purchasing history (Bonatti and Cisternas, 2020), or when a bidder participates repeatedly in sequential auctions for similar objects.

From a theoretical perspective, our analysis highlights how the valuation of a single bidder affects the seller's revenue in various auction formats. In fact, the marginal contribution to the seller's revenue of a bidder with a specific value v rather than a generic bidder can be interpreted as the difference between the interim revenue at v and the ex-ante revenue. Hence, this represents the seller's willingness to pay for replacing, in a given auction format, a generic bidder with a specific one whose valuation is v.

Our results also identify the type of information that is most useful for the seller. Because of single crossing, the seller can determine if her expected revenue is higher in either the FPA or SPA simply by knowing whether the bidder's valuation is lower than *any* threshold. For example, in sequential auctions a seller learns that a bidder has a low valuation if he observes that he lost a previous auction, even if she does not observe any of the bids in the previous auction. Or the seller may only observe whether the bidder has previously completed a purchase at a fixed price, even without observing the actual price.

Our analysis contributes to the literature exploring why sellers prefer specific auction formats.[19] It is natural to interpret our results as suggesting that the seller may choose a particular auction format to obtain a higher revenue, given her information about the special bidder. Of course, this application requires to assume that bidders always bid as in an auction with symmetric competitors, regardless of the format. This requires that (i) the seller's information is private — i.e., other bidders do not know the value of the special bidder — and (ii) bidders do not draw any inference on their competitors from the seller's choice of auction — e.g., because they are unsophisticated or unaware that the seller is informed about one of the bidders.[20]

[19]For example, other studies focus on risk preferences of the seller (Waehrer *et al.*, 1998) or bidders (Maskin and Riley, 1984), and the seller's ability to manipulate bids (Akbarpour and Li, 2020).

[20]Multiple equilibria exist in a model where bidders are aware that the seller chooses either an FPA or an SPA, contingent on the valuation v of the special bidder. For example, in the Appendix we show that, with uniformly distributed valuations, there is an equilibrium where the seller chooses the SPA for any $v > 0$ and the FPA if $v = 0$. In the SPA, all bidders bid their values. In the FPA, non-special bidders bid as in an auction with $n - 1$ symmetric bidders, while the special bidder bids the best response to his competitors' strategy.

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7 Appendix

7.1 Extensions

Savvy Bidders

Consider a setting where bidders are aware that the seller observes the valuation v of the special bidder before choosing the auction format. The identity (but not the valuation) of this special bidder, as well as the fact that the seller selects the auction format based on his valuation, is common knowledge.

Define the function $\mathcal{E} : [0,1] \to \mathcal{A}$, where $\mathcal{A}$ represents the set of standard auctions available to the seller. $\mathcal{E}$ partitions the set of possible special bidder's valuation, depending on the associated auction format. By choosing the function $\mathcal{E}(v)$, the seller chooses an auction format for each special bidder's valuation. Let $\mathcal{E}^{-1}(a)$ denote the subset of special bidder's values to which the function $\mathcal{E}$ associates the auction format a, i.e. $\mathcal{E}^{-1}(a) = \{w \in [0,1] : \mathcal{E}(w) = a\}$. Then, $\mathcal{E}^{-1}(\mathcal{E}(v))$ denotes the atom of the partition to which v belongs (i.e., all bidders' valuations that induce the same auction format as v).

For a standard auction $a \in \mathcal{A}$ and a set $V \subset [0,1]$, construct the function $\Pi_V^a : [0,1]^n \to \mathbb{R}$ as follows.

1. *Equilibrium Bids.* Compute an equilibrium of auction a with asymmetric bidders in $V \times [0,1]^{n-1}$. The equilibrium specifies the bidding functions $b_V^{a,S} : V \to \mathbb{R}$ and $b_V^{a,N} : [0,1] \to \mathbb{R}$, which are mutual best responses for the special and normal bidders, respectively.

2. *Equilibrium Bids Extension.* Extend the equilibrium to the full space $[0,1]^n$, by computing, for each $v \notin V$, the best response to $n-1$ bidders playing $b_V^{a,N}$. This means that types of the special bidders for which the seller should not choose format a, play a best response to the equilibrium in auction a. Define this function $\tilde{b}_V^{a,S} : [0,1] \to \mathbb{R}$, noting that $\tilde{b}_V^{a,S}$ extends $b_V^{a,S}$ on $[0,1] \setminus V$.

3. *Interim Revenue Function.* Let $\Pi_V^a : [0,1]^n \to \mathbb{R}$ denote the revenue in such an auction as a function of the entire profile of bidders' valuations. Define the interim counterpart $\mathit{\Pi}_V^a(v) := \mathbb{E}[\Pi_V^a | v_1 = v]$, which is also defined for $v \notin V$.

In words, Π_V^a represents the seller's revenue in auction format a assuming that: *i*) Auction format a is chosen only if the special bidder's valuation is in V; *ii*) Non-special bidders always play according to the equilibrium strategy for an asymmetric auction where one competitor is in the set V; *iii*) The special bidder responds optimally to the equilibrium strategy, even when his valuation is not in V.[21]

Definition 10. The function $\mathcal{E}(v)$ is a *savvy-bidder equilibrium* if the following conditions are satisfied.
1. $\Pi_{\Omega_a}^a$ is well-definite $\forall a \in \mathcal{A}$, i.e. there exist bid functions as defined in Step 1 and Step 2
2. For all $v \in [0,1]$ and $a \in \mathcal{A}$, $\mathit{\Pi}_{\mathcal{E}^{-1}(\mathcal{E}(v))}^{\mathcal{E}(v)}(v) \geq \mathit{\Pi}_{\mathcal{E}^{-1}(a)}^a(v)$.

In the SPA, knowing that a competitor belongs to the set $V \subset [0,1]$ does not affect bids. Therefore, for all sets V, $\mathit{\Pi}_V^{SPA}(v)$ exists and is equal to $\mathit{\Pi}^{SPA}(v)$, the interim revenue studied in this paper. For any other auction format a, computing $\Pi_{\mathcal{E}^{-1}(a)}^a$ requires the full characterization of the bid functions

[21]The rationale behind this selection criterion, as formalized in Step 2, is that only the special bidder can detect any deviation by the seller, as he is the only one aware of his own valuation.

with asymmetric bidders (Step 1), a problem that presents notorious difficulties.[22] However, savvy-bidder equilibria that partition $[0, 1]$ into $\{0\}$ and $(0, 1]$ are immune to this problem, as the equilibrium played when there is common knowledge that a bidder has value 0 coincides with the equilibrium of the auction with $n-1$ bidders. In the FPA, with uniform bidders, we can compute the best response under deviation (Step 2) and analytically characterize the interim revenue function (Step 3).

Proposition 11. *Suppose F is the uniform CDF and $\mathcal{A} = \{FPA, SPA\}$. Then, for each n*

$$\mathcal{E}(v) = \begin{cases} FPA & v = 0 \\ SPA & v > 0 \end{cases}$$

constitutes a savvy-bidder equilibrium in which

$$b_0^{FPA,N}(x) = \frac{n-2}{n-1}x, \quad \tilde{b}_0^{FPA,S}(x) = \max\left\{\frac{n-1}{n}x, \frac{n-2}{n-1}\right\}, \quad b_{(0,1]}^{SPA,N}(x) = \tilde{b}_{(0,1]}^{SPA,S}(x) = x$$

Proposition 11 characterizes an equilibrium in which the seller chooses the FPA if and only if the special bidder's valuation is zero, and chooses the SPA otherwise. The intuition behind this strategy forming a savvy-bidder equilibrium is as follows. In this scenario, the equilibrium bids of non-special bidders in the FPA coincide with the equilibrium bids in a standard FPA with $n-1$ bidders. This is because, by participating in an FPA, bidders learn that the special bidder's valuation is 0, which is equivalent to competing with one fewer bidder (Step 1). The deviation strategy of the special bidder in the event the FPA is chosen when he has value $v > 0$ is the best response to these bids (Step 2). With uniform distribution, this best response, $\tilde{b}_0^{FPA,S}(x)$, can be derived in closed form.[23] Given the strategies on and off equilibrium, we show that $\Pi_0^F(v)$ equals $\Pi^S(v)$ at $v = 0$, while it is lower than it for all $v > 0$. At $v = 0$, the interim revenue in the FPA coincides with the interim revenue in the SPA, as non-special bidders bid as in an auction with one fewer competitor in both auction formats. Local to $v = 0$, gains are of order n in the FPA, lower than gains of order $n-1$ in the SPA, and the FPA never catches up.

Intuitively, when bidders are unaware that the seller's choice of auction format reflects her information about the value of the special bidder (as in our main analysis), the FPA maximizes the seller's revenue for low values of v. However, when bidders are aware that the seller is using information about a competitor, observing that they play in a FPA indicates that they are competing against a bidder with a low value. Consequently, they adjust their bids accordingly, and the equilibrium tends to favor the auction format where interim information does not influence bidding behavior.

Comparing Proposition 11 with Theorem 3 we see that in the presence of savvy bidders, the seller cannot exploit the information about a bidder, as the others will respond and adjust their bids leading

[22] For the FPA, Maskin and Riley (2000) compute the equilibrium in asymmetric auction with two asymmetric bidders, supposing that one of the valuation is drawn from a set of the form $[0, v_u]$. Kaplan and Zamir (2012) extend the result considering an asymmetric auction with two bidders having uniform distributions defined on sets of the form $[\underline{v}_i, \overline{v}_i]$. Olszewski et al. (2023) provide conditions for the existence of equilibria in first-price auctions with asymmetric bidders; in particular they provide sufficient conditions for the existence of equilibria when one bidder is drawn from a set V which can be of arbitrary form, but do not provide characterization of the equilibrium.

[23] Notice that the responsive part of the special bidder's best response is the equilibrium bidding function in the FPA with n symmetric bidders. This is also a bidders' best response when $n-1$ competitors adopt *any* linear bidding strategy (a property that relies on the uniform assumption). Of course, the special bidder never bids more than $b^F(1, n-1) = \frac{n-2}{n-1}$, the highest bid by non-special bidders.

to an unraveling process that ultimately favors the choice of an auction format where bidding behavior is unaffected by information about competitors.

Notice that within the class of standard auctions only the SPA is immune to manipulations given information about a competitor. A natural question is, then, whether *only* this format can be played in a savvy-bidder equilibrium, i.e. whether the equilibrium of Proposition 11 is unique. Guided by Theorem 3, we look in the class of cutoff strategies, that is where the seller selects a cutoff $\bar{v} \in [0,1]$ and chooses the FPA if and only if $v \leq \bar{v}$ and the SPA otherwise. Notice that the equilibrium in Proposition 11 belongs to this family, but is degenerate as it sets $\bar{v} = 0$. However, proving uniqueness (even in this family) has proven elusive as a direct unraveling argument is not possible and equilibria in the FPA with asymmetric bidders (Step 1 in the derivation of $\mathit{\Pi}^F_{[0,\bar{v}]}$ for $\bar{v} > 0$) do not have a closed-form representation, except in special cases (uniform with two bidders, see Kaplan and Zamir (2012)).

General Information and Implications of Single Crossing

We have analyzed the auction format preferred by the seller when she knows that one bidder has valuation v. A natural extension of our analysis is to assume instead that the seller has some, though not perfect, information about v. Let $\tilde{F}$ be a distribution over $[0,1]$ that represents such information. The ex-ante case corresponds to $\tilde{F} = F$, while the analysis of this paper corresponds to $\tilde{F} = \delta_v$ for some $v \in [0,1]$.[24] Computing the interim revenue in this setting is a straightforward extension of our analysis since

$$\mathit{\Pi}^a\left(\tilde{F}\right) := \mathbb{E}\left[\Pi^a | v_1 \sim \tilde{F}\right] = \mathbb{E}_{\tilde{F}}\left[\mathit{\Pi}^a\left(v\right)\right]$$

Some qualitative implications of Theorem 3 do not extend to this stochastic setting. In particular, although Theorem 3 implies that

$$\mathit{\Pi}^S\left(v\right) - \mathit{\Pi}^F\left(v\right) > 0 \quad \Rightarrow \quad \mathit{\Pi}^S\left(v'\right) - \mathit{\Pi}^F\left(v'\right) > 0$$

whenever $v' > v$, improving (in FOSD sense) the distribution of the special bidder does not necessarily make the SPA more desirable. In fact, it is possible that $\tilde{F}_2$ first-order stochastically dominates $\tilde{F}_1$ but[25]

$$\mathbb{E}_{\tilde{F}_1}\left[\mathit{\Pi}^S\left(v\right) - \mathit{\Pi}^F\left(v\right)\right] > 0 \text{ and } \mathbb{E}_{\tilde{F}_2}\left[\mathit{\Pi}^S\left(v\right) - \mathit{\Pi}^F\left(v\right)\right] < 0.$$

There are, however, signal structures where single crossing is sufficient to determine unambiguously the seller's preference. For example, suppose that the seller observes only whether v is above or below a random threshold, but is ignorant of the precise threshold and even of the distribution from which it is drawn. This type of information emerges naturally in settings where a seller is only able to observe whether one of his bidders has previously completed a purchase or not from a competitor, without observing the

[24] We still work at an interim stage and not model explicitly where $\tilde{F}$ comes from, though the natural interpretation is to think of $\tilde{F}$ as the posterior resulting from observing some signal about the special bidder's valuation, the ex-ante (resp., interim model of the main paper) model being an uninformative (resp., perfectly informative) signal.

[25] This observation follows immediately from the fact that the interim revenue difference is non-monotonic. Thus, shifting mass above $\hat{v}$ shrinks the advantage of the SPA. As an example of $\tilde{F}_1, \tilde{F}_2$ that yield the desired contradition, take $\tilde{F}_1 = \begin{cases} 0 & p \\ \hat{v} & 1-p \end{cases}$ where p is chosen such that $\mathbb{E}_{\tilde{F}_1}\left[\mathit{\Pi}^S\left(v\right) - \mathit{\Pi}^F\left(v\right)\right] = \epsilon$ for $\epsilon > 0$ small. Then, $\tilde{F}_2 = \begin{cases} 0 & p \\ 1 & 1-p \end{cases}$ dominates $\tilde{F}_1$ but is such that $\mathbb{E}_{\tilde{F}_2}\left[\mathit{\Pi}^S\left(v\right) - \mathit{\Pi}^F\left(v\right)\right] < 0$.

price or the competitor's pricing strategy. If the interim revenue functions of two formats intersect only once, this information is sufficient to determine which format the seller prefers.[26]

Proposition 12. *Suppose the seller observes* $s = \mathbb{I}\{v < P\}$ *for some random variable* P *with support in* $[0, 1]$*, independent of* v *but of unknown distribution. If* $s = 1$*, seller's interim revenue is higher in the FPA. If* $s = 0$ *seller's interim revenue is higher in the SPA.*

Proposition 12 implies that the seller can understand whether the interim revenue is higher in the FPA or SPA by knowing only whether the special bidder has a value higher or lower than a threshold, even without knowing the exact value of either the bidder or the threshold.

7.2 Proofs

Proof of Lemma 1

We first prove expression (2.1) by writing the revenue in auction a as the sum of all bidders' transfers, where the revenue and such transfers are random variables that depend on the realization of all bidders' values. Formally, letting $t_i^a(\boldsymbol{v})$ denote bidder i's transfers to the seller in auction a, given bidders' values $\boldsymbol{v}$,

$$\Pi^a(\boldsymbol{v}) = \sum_{i=1}^{n} t_i^a(\boldsymbol{v}).$$

Notice that

$$t(v) = \mathbb{E}_{\boldsymbol{v}}\left[t_i^a(\boldsymbol{v})\,|v_i = v\right],$$

because the expectation depends neither on i, by symmetry, nor on a, by the PET. By linearity of expectations, for *any measure,*

$$\mathbb{E}\left[\Pi^a(\boldsymbol{v})\right] = \sum_{i=1}^{n} \mathbb{E}\left[t_i^a(\boldsymbol{v})\right].$$

Using the unconditional measure yields the RET because

$$\mathbb{E}_{\boldsymbol{v}}\left[\Pi^a(\boldsymbol{v})\right] = n\mathbb{E}_v\left[\mathbb{E}_{\boldsymbol{v}}\left[t_i^a(\boldsymbol{v})\,|v_i = v\right]\right] = n\mathbb{E}_v\left[t(v)\right],$$

where the first equality follows by LIE and symmetry. Using the interim measure $\boldsymbol{v}|v$ instead (recall that, by convention, the special bidder is bidder 1),

$$\mathit{\Pi}^a(v) = \mathbb{E}_{\boldsymbol{v}}\left[\Pi^a(\boldsymbol{v})\,|v_1 = v\right] = \mathbb{E}_{\boldsymbol{v}}\left[t_1^a(\boldsymbol{v})\,|v_1 = v\right] + \sum_{i=2}^{n} \mathbb{E}_{\boldsymbol{v}}\left[t_i^a(\boldsymbol{v})\,|v_1 = v\right]$$

$$= t(v) + \sum_{i=2}^{n} \mathbb{E}_x\left[\mathbb{E}_{\boldsymbol{v}}\left[t_i^a(\boldsymbol{v})\,|v_1 = v, v_i = x\right]\right] = t(v) + (n-1)\,\mathbb{E}_x\left[t_v^a(x)\right],$$

where the last equality follows because $\mathbb{E}_{\boldsymbol{v}}\left[t_i^a(\boldsymbol{v})\,|v_1 = v, v_i = x\right]$ is the interim transfer.

[26] Although we state the result for FPA vs. SPA, it is immediate to extend it to any pair of auction formats (e.g., the FPA and APA) whose interim revenue functions cross only once.

Proof of Lemma 2

The statement is a special case of (5.2) with $k = 1$ and $j = 2$.

Proof of Theorem 3

We complete the proof in the main text by proving (4.3) and Lemma 13. Notice that

$$\int_v^1 \int_0^x \max\{v, y\} \frac{\mathrm{d}F^{n-2}(y)}{F^{n-2}(x)} \mathrm{d}F^{n-1}(x) = \int_v^1 \left[v \frac{F^{n-2}(v)}{F^{n-2}(x)} + \int_v^x y \frac{\mathrm{d}F^{n-2}(y)}{F^{n-2}(x)} \right] \mathrm{d}F^{n-1}(x)$$
$$= \int_v^1 \left[x - \int_v^x \frac{F^{n-2}(y)}{F^{n-2}(x)} \mathrm{d}y \right] \mathrm{d}F^{n-1}(x),$$

where the second equality uses integration by parts. Now let

$$g(v) \coloneqq \int_v^1 \left[x - \int_v^x \frac{F^{n-2}(y)}{F^{n-2}(x)} \mathrm{d}y \right] \mathrm{d}F^{n-1}(x).$$

Notice that $g(1) = 0$ and

$$g'(v) = -v\mathrm{d}F^{n-1}(v) + \int_v^1 \frac{F^{n-2}(v)}{F^{n-2}(x)} \mathrm{d}F^{n-1}(x)$$
$$= -\mathrm{d}F^{n-1}(v) \left[v - \frac{1 - F(v)}{f(v)} \right] = -\mathrm{d}F^{n-1}(v)\,\psi(v).$$

Finally, by the fundamental theorem of calculus:

$$g(v) = g(1) - \int_v^1 g'(w)\,\mathrm{d}w$$
$$= \int_v^1 \psi(w)\,\mathrm{d}F^{n-1}(w).$$

Lemma 13. *$\Delta(v)$ has a maximum at $v = 0$ and $\Delta(0) > 0$.*

Proof. We denote by $b^F(x, R, n)$ be the equilibrium bidding function in the FPA with n bidders and reserve price R. Integration by parts yields

$$\int_v^1 b^F(x, v, n)\,\mathrm{d}F^n(x) = \int_v^1 \psi(x)\,\mathrm{d}F^n(x),$$

for every n. Therefore, using (4.3),

$$\Delta(v) = \int_v^1 \left[b^F(x, 0, n) - b^F(x, v, n-1) \right] \mathrm{d}F^{n-1}(x).$$

Since $b^F(x, 0, n) - b^F(x, 0, n-1) > 0$, this implies that $\Delta(0) > 0$.

To show that $v = 0$ is a local maximum, write $\Delta'(v) = -\Delta_1(v) \cdot \Delta_2(v)$ where $\Delta_1(v) = b^F(v) - \psi(v)$

and $\Delta_2(v) = \mathrm{d}F(v)^{n-1}$. By the Leibniz rule, for $k \geq 1$,

$$\Delta^{(k-1)}(v) = \sum_{j=0}^{k-1} \binom{j}{k} \Delta_1^{(k-j)}(v)\, \Delta_2^{(j)}(v).$$

Notice that, for all $j < n-1$, $\Delta_2^{(j)}(0) \propto F(0) = 0$. Then, $\Delta^{(k)}(0) = 0$ for all $k < n-1$ while

$$\Delta^{(n-1)}(0) = \binom{n-2}{n-1} \Delta_1(0)\, \Delta_2^{(n-2)}(0) \propto -\Delta_1(0) = \psi(0) < 0,$$

proving that 0 is a local maximum. Since the other critical point $b^F(v) = \psi(v)$ must be a minimum, 0 is also the global maximum. □

Proof of Proposition 4

Suppose that $v < R$. Then,

$$\mathit{\Pi}^F(v) = \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} > v\right) \cdot \mathbb{E}_{\boldsymbol{v}|v}\left[b^F\left(v_{(1)}, R\right) | v_{(1)} > R\right]$$

and

$$\mathit{\Pi}^S(v) = \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} > R\right) \cdot \mathbb{E}_{\boldsymbol{v}|v}\left[\max\left\{v_{(2)}; R\right\} | v_{(1)} > R\right].$$

Therefore, when $v < R$, the IRFs of the FPA and SPA do not depend on v and the IRF of the FPA is strictly higher than the SPA. At $v = R$ the IRFs of both formats have an upward jump equal to $R \cdot \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} < R\right)$.

For $v > R$, Lemma 2 yields

$$\Delta(v, R) = \int_v^1 b^F(x, R) - \psi(x)\, \mathrm{d}F^{n-1}(x),$$

and the arguments in the proof of Theorem (3) without reserve directly extend, giving a unique minimum at $b^F(x, R) = \psi(x)$ and hence a unique $\widetilde{v} \neq 1$ such that $\Delta(\widetilde{v}, R) = 0$.

Proof of Proposition 7

Let

$$\Delta^k(v) \equiv \mathit{\Pi}^{k+1}(v) - \mathit{\Pi}^k(v).$$

To prove the statement, we show that, for all k, $\Delta^k(v) > 0$ at $v = 0$, $\Delta^k(v) < 0$ for $v \approx 1$ and $\Delta^k(1) = 0$. We start by establishing the following result.

Lemma 14. *For all k, j with $k \geq j$, $\mathbb{E}\left[\Pi^k | \Pi^j\right] = \Pi^j$. Moreover,*

$$\Delta^k(v) = \sum_{i=k+1}^{n} \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right) \left[\mathbb{E}\left(\Pi^{k+1} | v_{(i)} = v\right) - \mathbb{E}\left(\Pi^k | v_{(i)} = v\right)\right]. \tag{7.1}$$

Proof. Since $\Pi^k = b^k\left(x_{(k)}\right)$ and $b^k(\cdot)$ is a strictly monotonic function, conditioning on Π^k is equivalent to conditioning on the realization of the k^{th}-order statistic — i.e.

$$\mathbb{E}\left[\Pi^{k+1}|\Pi^k\right] = \mathbb{E}\left[\Pi^{k+1}|x_{(k)} = \left(b^k\right)^{-1}\left(\Pi^k\right)\right].$$

We first prove that $\mathbb{E}\left[\Pi^k|\Pi^{k-1}\right] = \Pi^{k-1}$ by showing that

$$\mathbb{E}[\underbrace{b^{k+1}\left(x_{(k+1)}\right)}_{\Pi^{k+1}}|x_{(k)}] = x = \underbrace{b^k\left(x\right)}_{\Pi^k},$$

using the following result in probability theory (see, e.g. Casella and Berger, Ex. 5.27.a).

Fact 15. *The pdf of the* $(k+j)^{th}$*-order statistic, conditioned on the realization of the* k^{th}*-order statistic,* x*, is the pdf of the* j^{th}*-order statistic from a sample of* $(n-k)$ *draws from the distribution truncated at* x*.*

Let $G^y_{(j,n)}(x)$ denote the CDF of the j^{th}-order statistic among n draws below the threshold y. By the PET,

$$\int_0^v b^k(x)\,\mathrm{d}G^v_{(k-1,n-1)}(x) = \int_0^v b^{k+1}(y)\,\mathrm{d}G^v_{(k,n-1)}(y)\,, \tag{7.2}$$

where the RHS (LHS) is the expected payment conditional on winning of a generic bidder with valuation v in the $(k+1)^{\text{th}}(k^{\text{th}})$ -price auction. Applying the LIE,

$$\int_0^v b^{k+1}(y)\,\mathrm{d}G^v_{(k,n-1)}(y) = \int_0^v \mathbb{E}\left[b^{k+1}(y)\,|x\right]\mathrm{d}G^v_{(k-1,n-1)}(x)\,. \tag{7.3}$$

Combining equations (7.2) and (7.3) and differentiating with respect to v yields

$$b^k(v) = \mathbb{E}\left[b^{k+1}(y)\,|v\right],$$

and the result follows from Fact 15. The first statement of Lemma 14 follows from repeatedly applying the LIE:

$$\mathbb{E}[\Pi^k|\Pi^j] = \mathbb{E}[\ldots\overbrace{\mathbb{E}[\underbrace{\mathbb{E}[\Pi^k|\Pi^{k-1}]}_{\Pi^{k-1}}|\Pi^{k-2}]}^{\Pi^{k-2}}\ldots|\Pi^j] = \Pi^j.$$

We now prove (7.1). Using the Law of Total Probability,

$$\Delta^k(v) = \sum_{i=1}^n \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)\left[\mathbb{E}\left(\Pi^{k+1}|v_{(i)} = v\right) - \mathbb{E}\left(\Pi^k|v_{(i)} = v\right)\right].$$

However, all terms in the summation up to $i = k$ are equal to 0. The reason is that, by strict monotonicity of the bidding function, $\mathbb{E}\left[\Pi^k|v_{(i)} = v\right] = \mathbb{E}\left[\Pi^k|\Pi^i = b^i(v)\right]$ and, as we have shown,

$$\mathbb{E}\left[\Pi^{k+1}|v_{(i)} = v\right] = \Pi^i = \mathbb{E}\left[\Pi^k|v_{(i)} = v\right] \qquad \forall i \le k.$$

□

Consider first $v = 0$.

$$\begin{aligned}\Delta^k(0) &= \mathbb{E}_{\boldsymbol{v}|0}\left[\Pi^k - \Pi^{k+1}\right] \\ &= \mathbb{E}_{\boldsymbol{v}|0}\left[\mathbb{E}_{\boldsymbol{v}|0}\left[\Pi^k - \Pi^{k+1}|\Pi^k\right]\right] \\ &> \mathbb{E}_{\boldsymbol{v}|0}\left[\Pi^k - \mathbb{E}_{\boldsymbol{v}}\left[\Pi^{k+1}|\Pi^k\right]\right] \\ &= \mathbb{E}_{\boldsymbol{v}|0}\left[\Pi^k - \Pi^k\right] = 0.\end{aligned}$$

The first equality is a definition, the second equality is the LIE, the inequality uses $\boldsymbol{v} > \boldsymbol{v}|0$, and the last line uses Lemma 14.

Consider now $v \approx 1$. We establish that, for all k, there is a neighborhood of 1 where $\Delta^k(v) < 0$. By a Taylor expansion of order N around 1,

$$\Delta^k(v) = \sum_{j=0}^{N} \Delta^k(1)^{(j)} \frac{(v-1)^j}{j!} + o\left((v-1)^N\right),$$

where $\Delta^k(v)^{(j)}$ denotes the j^{th}-order derivative of $\Delta^k(v)$. Since $\Delta^k(1) = 0$, the sign of the the function Δ^k in a neighborhood of 1 is determined by the sign of its first non-zero derivative at 1. By Leibniz's rule,

$$\Delta^k(v)^{(j)} = \sum_{i=k+1}^{n} \sum_{s=0}^{j} \binom{j}{s} \mathbb{P}\left(v_{(i)} = v\right)^{(s)} \Delta_i^k(v)^{(j-s)}, \tag{7.4}$$

where $\Delta_i^k(v) \equiv \mathbb{E}\left(\Pi^k|v_{(i)} = v\right) - \mathbb{E}\left(\Pi^{k+1}|v_{(i)} = v\right)$. Since $\Delta_i^k(v)$ has finite derivatives, the corresponding term in (7.4) is equal to 0 if $\mathbb{P}\left(v_{(i)} = v\right)^{(s)} = 0$.

Lemma 16. *For all i,*

$$\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)^{(s)}\big|_{v=1} = 0 \qquad \forall s < i-1,$$

and

$$(-1)^{i-1}\, \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)^{(i-1)}\big|_{v=1} > 0.$$

Proof. For all k and i, $\mathbb{E}\left[\Pi^k|v_{(i)} = v\right]$ is n times continuously differentiable with finite derivatives. Using

$$\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right) = (1 - F(v))^{i-1} F(v)^{n-i} \binom{n-1}{i-1}$$

and Leibniz's rule, we have

$$\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)^{(j)} = \binom{n-1}{i-1} \sum_{s=0}^{j} \binom{j}{s} \left[(1-F(v))^{i-1}\right]^{(s)} \left[F(v)^{n-i}\right]^{(j-s)}.$$

Therefore, for all $s < i-1$, $\left[(1-F(v))^{i-1}\right]^{(s)}\big|_{v=1} = 0$ because $1 - F(1) = 0$, and $\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)} = v\right)^{(s)}\big|_{v=1} = 0$ because it is the sum of addenda that are all zero. By contrast,

$$\left[(1-F(v))^{i-1}\right]^{(i-1)}\big|_{v=1} = (i-1)!\,(-1)^{i-1} f(1)^{i-1},$$

and hence $\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(i)}=v\right)^{(i-1)}\big|_{v=1}\propto(-1)^{i-1}$. □

Lemma 16 implies that, for all $j<k$, $\Delta^{k}(1)^{(j)}=0$ since all addenda in (7.4) are equal to 0. Moreover, for $j=k$ the only non-zero addendum in (7.4) at $v=1$ is

$$\mathbb{P}\left(v_{(k+1)}=v\right)^{(k)}\Delta_{k+1}^{k}(v)\bigg|_{v=1}.$$

Notice that $\Delta_{k+1}^{k}(v)=\mathbb{E}\left[b^{k}(v')\,|v'>v\right]-b^{k+1}(v)$ and converges to $b^{k}(1)-b^{k+1}(1)<0$ as $v\to1$. Therefore, by Fact 16,

$$\Delta^{k}(1)^{(k)}=\mathbb{P}\left(v_{(k+1)}=1\right)^{(k)}\Delta_{k+1}^{k}(1)\propto(-1)^{k-1}.$$

Summing up, we have shown that

$$\Delta^{k}(1)^{(j)}(-1)^{j}=\begin{cases}0 & \text{if } j<k\\ <0 & \text{if } j=k\end{cases}\tag{7.5}$$

which, using (7.4), proves the statement that $\Delta^{k}(v)<0$ for $v\approx1$.

Proofs of the Statements in Section 5.3

The IRFs of the FPA and APA cross only once. To see this notice that, by (5.4), the IRF of the PB-$\mathcal{P}$ is

$$\begin{aligned}\mathit{\Pi}^{\text{PB-}\mathcal{P}}(v)&=t(v)+(n-1)\,\mathbb{E}\left[b^{\text{PB-}\mathcal{P}}(x)\,\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})\right]\\&=t(v)+(n-1)\,\mathbb{E}\left[t(x)\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})}\right].\end{aligned}\tag{7.6}$$

Ex-interim, the seller obtains from the special bidder his transfer. From each other bidder, she obtains the transfer weighted by the ratio of the likelihood of belonging to the set of payers, conditional on v, relative to its ex-ante counterpart.

In the APA, $\mathcal{P}=[n]$ and $\frac{\mathbb{P}_{\boldsymbol{v}|v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}}(x\in\mathcal{P})}=1$ for all v and x. In the FPA, $\mathcal{P}=\{1\}$ and

$$\frac{\mathbb{P}_{\boldsymbol{v}|v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}}(x\in\mathcal{P})}=\begin{cases}\frac{F^{n-2}(x)}{F^{n-1}(x)}=\frac{1}{F(x)} & x>v\\ 0 & x<v\end{cases}$$

Therefore,

$$\mathit{\Pi}^{FPA}(v)-\mathit{\Pi}^{APA}(v)=(n-1)\left[\mathbb{E}\left[\frac{t(x)}{F(x)}\mathbb{I}[x>v]\right]-\mathbb{E}[t(x)]\right].$$

Notice that $\mathit{\Pi}^{FPA}(0)-\mathit{\Pi}^{APA}(0)\propto\mathbb{E}\left[\frac{t(x)}{F(x)}\right]-\mathbb{E}[t(x)]>0$ and $\mathit{\Pi}^{FPA}(1)-\mathit{\Pi}^{APA}(1)\propto-\mathbb{E}[t(x)]<0$. Moreover, the difference is monotonically decreasing in v, proving single crossing.

Compare now the APA with other WPAs. By (7.6), $\mathit{\Pi}^{APA}(0)=(n-1)\,\mathbb{E}[t(x,n)]$. By contrast, since at $v=0$ the SPA is equivalent to any efficient auction with $n-1$ bidders, $\mathit{\Pi}^{APA}(0)=(n-1)\,\mathbb{E}[t(x,n-1)]$. Therefore,

$$\mathit{\Pi}^{APA}(0)-\mathit{\Pi}^{SPA}(0)\propto\mathbb{E}[t(x,n)-t(x,n-1)].$$

Proof of Proposition 8

By (7.6),

$$\mathit{\Pi}^{\text{PB-}\mathcal{P}}(v) > \mathit{\Pi}^{\text{PB-}\mathcal{P}'}(v) \qquad \Leftrightarrow \qquad \mathbb{E}\left[t(x)\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})}\right] > \mathbb{E}\left[t(x)\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P}')}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P}')}\right]. \tag{7.7}$$

The following Lemma proves the statements by showing that, for every x, the ratio $\frac{\mathbb{P}_{\boldsymbol{v}|x,v}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})}$ is highest (lowest) in the FPA (LPA) than in any other PBA when $v=0$, while it is highest (lowest) in the LPA (FPA) when $v=1$. Since $t(x)$ is positive, this implies the result.

Lemma 17. *For all x and $\mathcal{P}$,*

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,0}\left(x=v_{(1)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(1)}\right)} > \frac{\mathbb{P}_{\boldsymbol{v}|x,0}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})} > \frac{\mathbb{P}_{\boldsymbol{v}|x,0}\left(x=v_{(n)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(n)}\right)} \tag{7.8}$$

and

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(x=v_{(n)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(n)}\right)} > \frac{\mathbb{P}_{\boldsymbol{v}|x,1}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})} > \frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(x=v_{(1)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(1)}\right)}. \tag{7.9}$$

Proof. Let $p_j^n(x)$ be the probability that x is the j^{th}-highest among n bidders — i.e., $p_j^n(x) = \mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(j)}\right)$ — so that $\mathbb{P}_{\boldsymbol{v}|x,0}\left(x=v_{(j)}\right) = p_j^{n-1}(x)$ and $\mathbb{P}_{\boldsymbol{v}|x,1}\left(x=v_{(j)}\right) = p_{j-1}^{n-1}(x)$. Then,

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,0}\left(x=v_{(1)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(1)}\right)} = \frac{p_1^{n-1}(x)}{p_1^n(x)}, \qquad \frac{\mathbb{P}_{\boldsymbol{v}|x,0}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})} = \frac{\sum_{j\in\mathcal{P}} p_j^{n-1}(x)}{\sum_{j\in\mathcal{P}} p_j^n(x)}.$$

Notice that

$$\frac{p_j^{n-1}(x)}{p_j^n(x)} = \frac{(1-F(x))^{j-1}F(x)^{n-j-1}\binom{n-2}{j-1}}{(1-F(x))^{j-1}F(x)^{n-j}\binom{n-1}{j-1}} = \frac{n-j}{F(x)(n-1)}. \tag{7.10}$$

This ratio is decreasing in j, which proves (7.8).[27]

Similarly,

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(x=v_{(n)}\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(x=v_{(n)}\right)} = \frac{p_{n-1}^{n-1}(x)}{p_n^n(x)}, \qquad \frac{\mathbb{P}_{\boldsymbol{v}|x,1}(x\in\mathcal{P})}{\mathbb{P}_{\boldsymbol{v}|x}(x\in\mathcal{P})} = \frac{\sum_{j\in\mathcal{P}} p_{j-1}^{n-1}(x)}{\sum_{j\in\mathcal{P}} p_j^n(x)}$$

and

$$\frac{p_{j-1}^{n-1}(x)}{p_j^n(x)} = \frac{(1-F(x))^{j-2}F(x)^{n-j}\binom{n-2}{j-2}}{(1-F(x))^{j-1}F(x)^{n-j}\binom{n-1}{j-1}} = \frac{j-1}{(1-F(x))(n-1)}. \tag{7.11}$$

This ratio is increasing in j, which proves (7.9) (see footnote 27). □

Finally, we show that $\lim_{v\to 1}\mathit{\Pi}^{\text{LPA}}(v) - \mathit{\Pi}^{\text{PB-}\mathcal{P}}(v) = \infty$. First, from the proof of Lemma (17), $\mathit{\Pi}^{\text{PB-max}\mathcal{P}}(1) > \mathit{\Pi}^{\text{PB-}\mathcal{P}}(1)$. Consider single-payer auctions, and compare the PB-$\{j\}$ and PB-$\{k\}$. Using

[27] We use the following algebraic fact (whose proof follows from a simple induction argument on the cardinality of M). Let $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^N$ be sequences of positive numbers such that $\frac{a_i}{b_i}$ is (strictly) decreasing in i. Then for, any $M\subseteq\{1,\ldots,N\}$, $\frac{a_1}{b_1} \geq \frac{\sum_{i\in M}a_i}{\sum_{i\in M}b_i} \geq \frac{a_N}{b_N}$, with strict inequality if $M\neq\{1\}$. Moreover, if $\min\{i : i\in M\}\geq j$, then $\frac{a_j}{b_j} \geq \frac{\sum_{i\in M}a_i}{\sum_{i\in M}b_i}$, with strict inequality if $M\neq\{j\}$.

(7.7) and (7.11),

$$\begin{aligned}\mathit{\Pi}^{\text{PB-}\{j\}}(1)-\mathit{\Pi}^{\text{PB-}\{k\}}(1)&=\mathbb{E}\left[t(x)\left(\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(j)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(j)}=x\right)}-\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(k)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(k)}=x\right)}\right)\right]\\&=\mathbb{E}\left[t(x)\left(\frac{j-1}{(1-F(x))(n-1)}-\frac{k-1}{(1-F(x))(n-1)}\right)\right]\\&=\frac{j-k}{n-1}\mathbb{E}\left[\frac{t(x)}{1-F(x)}\right].\end{aligned}\tag{7.12}$$

Notice that, for generic $l\in(0,v)$,

$$\begin{aligned}\mathbb{E}\left[\frac{t(x)}{1-F(x)}\right]&=\lim_{v\to1}\int_0^v\frac{t(x)}{1-F(x)}f(x)\,\mathrm{d}x\\&>t(l)\lim_{v\to1}\int_l^v\frac{f(x)}{1-F(x)}\mathrm{d}x\\&=-t(l)\lim_{v\to1}\int_l^v\mathrm{d}\log(1-F(x))\\&\propto-t(l)\lim_{v\to1}\log(1-F(v))=\infty,\end{aligned}\tag{7.13}$$

where the inequality holds because the function $\frac{t(x)}{1-F(x)}f(x)$ is bounded in the closed interval $[0,v]$ and $0<t(l)\le t(x)\le t(1)<\infty$. Therefore, if $n\notin\mathcal{P}$ then $\mathit{\Pi}^{\text{PB-}\{n\}}(1)-\mathit{\Pi}^{\text{PB-max}\mathcal{P}}(1)$ diverges and $\mathit{\Pi}^{\text{PB-max}\mathcal{P}}(1)>\mathit{\Pi}^{\text{PB-}\mathcal{P}}(1)$.

For the case $n\in\mathcal{P}$, let $j=\max\mathcal{P}\setminus\{n\}$. Then, from the proof of Lemma (17), $\mathit{\Pi}^{\text{PB-}\{n,j\}}(1)>\mathit{\Pi}^{\text{PB-}\mathcal{P}}(1)$. Moreover,

$$\mathit{\Pi}^{\text{PB-}\{n\}}(1)-\mathit{\Pi}^{\text{PB-}\{n,j\}}(1)$$

$$\begin{aligned}&=\mathbb{E}\left[t(x)\left(\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(n)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(n)}=x\right)}-\frac{\mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(j)}=x\right)+\mathbb{P}_{\boldsymbol{v}|x,1}\left(v_{(n)}=x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(j)}=x\right)+\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(n)}=x\right)}\right)\right]\\&=\mathbb{E}\left[t(x)\left(\frac{p_{n-1}^{n-1}(x)}{p_n^n(x)}-\frac{p_{j-1}^{n-1}(x)+p_{n-1}^{n-1}(x)}{p_j^n(x)+p_n^n(x)}\right)\right]\\&=\mathbb{E}\left[t(x)\left(\frac{F(x)^{n-j}\left[\binom{n-1}{j-1}-\binom{n-2}{j-2}\right]}{(1-F(x))\left(F(x)^{n-j}\binom{n-1}{j-1}+(1-F(x))^{n-j}\right)}\right)\right]\propto\mathbb{E}\left[\frac{t(x)}{1-F(x)}\right],\end{aligned}\tag{7.14}$$

where the asymptotic comparison holds because $F(x)^{n-j}\binom{n-1}{j-1}+(1-F(x))^{n-j}$ is bounded away from 0. As the last term is infinite by (7.13), $\mathit{\Pi}^{\text{PB-}\{n\}}(1)-\mathit{\Pi}^{\text{PB-}\{n,j\}}(1)$ diverges and $\mathit{\Pi}^{\text{PB-}\{n,j\}}(1)>\mathit{\Pi}^{\text{PB-}\mathcal{P}}(1)$.

Proof of Proposition 9

Let the increasing function $b^a : [0,1] \to \mathbb{R}$ denote the equilibrium bidding function in auction a. By PET and using Fact 15 (and the corresponding notation), for all v,

$$t(v) = \sum_{j \in \mathcal{P}_a} p_n^j(v) \, \mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n - j \text{ below } v\right] \tag{7.15}$$
$$= \sum_{j \in \mathcal{P}_a} p_n^j(v) \int_0^v b^a(w) \, \mathrm{d}G^v_{(T_a(j)-j, n-j)}(w)$$

because, by definition, in auction a a bidder with value v pays if and only if he is an order statistic $j \in \mathcal{P}_a$ (so that there are $n-j$ bidders with values lower than v) and, in this case, pays the bid submitted by the (lower) bidder $T_a(j)$. To maintain this compact notation, we use following conventions (so that unfeasible order statistics have zero probability and pick the maximum/minimum of the support):

1. $\mathbb{E}\left[b^a(w) \,|w \text{ is } 0^{th} \text{ of } n \text{ below } v\right] = b^a(v)$
2. $\mathbb{E}\left[b^a(w) \,|w \text{ is } n^{th} \text{ of } n-1 \text{ below } v\right] = b^a(0) = 0$
3. $p_n^j(\cdot) = 0$ if $j > n$ or $j < 1$.

Moreover, notice that we can write the interim transfer $t_v^a(x)$ as

$$t_v^a(x) = \begin{cases} \sum_{j \in \mathcal{P}_a} p_{n-1}^{j-1}(x) \, \mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n-j \text{ below } x\right] & \text{if } x < v \\ \sum_{j \in \mathcal{P}_a} p_{n-1}^{j}(x) \, \mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n-j-1 \text{ below } x \text{ and one is } v\right] & \text{if } x > v \end{cases}$$

Therefore, at $v = 0$, $t_0^a(x) = \sum_{j \in \mathcal{P}_a} p_{n-1}^j(x) \, \mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n-j-1 \text{ below } x\right]$: when a bidder with value x is the j^{th}-order statistic and pays, his expected payment is the bid submitted by the lower bidder $T_a(j)$, taking into account that there are $n-j-1$ bidders with unknown values lower than x and one bidder with value 0. Moreover, using (5.4) and (7.10), $t_0^{FPA}(x) = \frac{t(x)}{F(x)}$.

Recall that $\Pi^a(v) = t(v) + (n-1)\int_0^1 t_v^a(x) \, \mathrm{d}F(x)$. The result at $v = 0$ follows because, for every standard auction a and $x \in (0,1]$,[28]

$$t_0^{FPA}(x) > t_0^a(x) \quad \Leftrightarrow \quad \frac{t(x)}{F(x)} > \sum_{j \in \mathcal{P}_a} p_{n-1}^j(x) \, \mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n-j-1 \text{ below } x\right]$$

$$\Leftrightarrow \quad t(x) > \sum_{j \in \mathcal{P}_a} p_{n-1}^j(x) \, F(x) \, \mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n-j-1 \text{ below } x\right]$$

$$\Leftrightarrow \quad \sum_{j \in \mathcal{P}_a} p_n^j(x) \, \mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n-j \text{ below } x\right] >$$

$$> \sum_{j \in \mathcal{P}_a} \frac{n-j}{n-1} p_n^j(x) \, \mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n-j-1 \text{ below } x\right],$$

[28] A type $x = 0$ transfers nothing in all standard auctions, so we can disregard this case and assume $F(x) > 0$.

where the last step uses (7.15) and the fact that $p_{n-1}^{j}(x) F(x) = \frac{n-j}{n-1} p_{n}^{j}(x)$ (see (7.10)). The inequality holds because $\frac{n-j}{n-1} < 1$ for all $j > 1$ and, for all x and j,

$$\mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n-j \text{ below } x\right] \geq \mathbb{E}\left[b^a(w) \,|w \text{ is } T_a(j) - j \text{ of } n-j-1 \text{ below } x\right]$$

because b^a is increasing and the k^{th}-order statistic of n independent draws FOSD the k^{th}-order statistic of $n' < n$ independent draws from the same distribution. Moreover, the inequality is strict except for degenerate cases induced by our conventions — e.g., if $T_a(j) = j$ so that both expectations are equal to $b^a(x)$.

Computations of the Interim Revenue Functions in Figure 5.2

Let $n = 3$ and $v \sim \mathcal{U}[0,1]$, so that $t(v) = \frac{2}{3}v^3$. The bid and interim revenue functions in the FPA, SPA and APA are standard and omitted.

Consider first the PB- $\{2\}$ and PB- $\{3\}$. By (5.3), the bidding functions are

$$b^{\text{PB-}\{2\}} = \frac{t(v)}{\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(2)} = v\right)} = \frac{v^2}{3(1-v)},$$

and

$$b^{\text{PB-}\{3\}} = \frac{t(v)}{\mathbb{P}_{\boldsymbol{v}|v}\left(v_{(3)} = v\right)} = \frac{2}{3}\frac{v^3}{(1-v)^2}.$$

We compute the interim revenue using (5.5). The likelihood rations are

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,v}\left(v_{(2)} = x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(2)} = x\right)} = \begin{cases} \frac{1}{2x} & \text{if } x > v \\ \frac{1}{2(1-x)} & \text{if } x < v \end{cases}$$

$$\frac{\mathbb{P}_{\boldsymbol{v}|x,v}\left(v_{(3)} = x\right)}{\mathbb{P}_{\boldsymbol{v}|x}\left(v_{(3)} = x\right)} = \begin{cases} 0 & \text{if } x > v \\ \frac{1}{1-x} & \text{if } x < v \end{cases}$$

and the interim revenues are

$$\mathit{\Pi}^{\text{PB-}\{2\}}(v) = t(v) + \frac{4}{3}\left[\int_0^v \frac{x^3}{2(1-x)}\mathrm{d}x + \int_v^1 \frac{x^2}{2}\mathrm{d}x\right]$$

$$\mathit{\Pi}^{\text{PB-}\{3\}}(v) = t(v) + \frac{4}{3}\left[\int_0^v \frac{x^3}{1-x}\mathrm{d}x\right].$$

It is immediate to check that

$$\mathit{\Pi}^{FPA}(0) > \mathit{\Pi}^{\text{PB-}\{3\}}(0) > \mathit{\Pi}^{\text{PB-}\{3\}}(0) = 0$$

and

$$\lim_{v\to 1} \mathit{\Pi}^{\text{PB-}\{2\}}(v) = \lim_{v\to 1} \mathit{\Pi}^{\text{PB-}\{3\}}(v) = \lim_{v\to 1} \mathit{\Pi}^{\text{PB-}\{3\}}(v) - \mathit{\Pi}^{\text{PB-}\{2\}}(v) = \infty.$$

Consider now the standard auction with $\mathcal{P} = \{2\}$ and $T(2) = 3$, which we denote compactly as the

2, 3 auction. By the PET,

$$b^{\text{PB-}\{2\}}(v)\,v = \int_0^v b^{2,3}(w)\,\mathrm{d}w,$$

and hence the bidding function is

$$b^{2,3}(v) = \frac{v^2(3-2v)}{3(1-v)^2}.$$

The interim revenue function is

$$\begin{aligned}
\Pi^{2,3}(v) &= \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(2)} = v\right)\mathbb{E}\left[b^{2,3}(w)\,|w<v\right] + \\
&\qquad + \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(3)} = v\right)\mathbb{E}\left[b^{2,3}(w)\,|w \text{ is second below } v\right] + \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(1)} = v\right) b^{2,3}(v) \\
&= \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(2)} = v\right)\frac{1}{v}\int_0^v b^{2,3}(w)\,\mathrm{d}w + \int_0^v 2(v-w)\,b^{2,3}(w)\,\mathrm{d}w + (1-v)^2\,b^{2,3}(v) \\
&= t(v) + \int_0^v 2(v-w)\,b^{2,3}(w)\,\mathrm{d}w.
\end{aligned}$$

It is immediate to see that $\Pi^{2,3}(0) = 0 < \Pi^{\text{PB-}\{2\}}(0)$. The interim revenue difference with the associated PBA is

$$\begin{aligned}
\Pi^{2,3}(v) - \Pi^{\text{PB-}\{2\}}(v) &= \mathbb{P}_{\boldsymbol{v}|v}\left(v_{(3)} = v\right)\left(b^{2,3}(v) - \int_v^1 b^{\text{PB-}\{2\}}(w)\,2(1-w)\,\mathrm{d}w\right) \\
&= \frac{v^2(3-2v)}{3} - \int_v^1 \frac{2}{3}w^2\mathrm{d}w,
\end{aligned}$$

from which it follows that $\lim_{v\to 1}\Pi^{2,3}(v) - \Pi^{\text{PB-}\{2\}}(v) = \frac{1}{3} > 0$.

Finally, consider the standard auction with $\mathcal{P} = \{1,2,3\}$ and $T(1) = T(2) = T(3) = 3$, which we denote compactly as the All-Pay-Last (APL). The associated PBA is the APA so, by the PET,

$$\begin{aligned}
b^{APA}(v) = \frac{2}{3}v^3 &= \mathbb{E}\left[b^{APL}(w)\,|w \text{ is last of 3 draws when one is } v\right] \\
&= b^{APL}(v)(1-v)^2 + \int_0^v b^{APL}(w)\,2(1-w)\,\mathrm{d}w.
\end{aligned}$$

Differentiating and simplifying, we obtain the differential equation $\frac{\mathrm{d}}{\mathrm{d}v}b^{APL}(v) = \frac{2v^2}{(1-v)^2}$, which implies

$$b^{APL}(v) = \frac{2v(2-v)}{1-v} + 4\log(1-v),$$

a divergent bid. This is natural: although the winning bidder pays, he does not pay his own bid, and bounded bids unravel in any equilibrium. However, in this case the interim revenue remains finite. To see this, write[29]

$$\Pi^{APL}(v) = 3\mathbb{E}\left[b^{APL}(w)\,|w \text{ is last of 3 draws when one is } v\right] = 3t(v).$$

Recall that the IRF of the APA, the associated PBA, is $\Pi^{APA}(v) = t(v) + 2\mathbb{E}[t(x)]$. Hence, $\Pi^{APL}(v) -$

[29] Notice how none of the arguments for the IRF rely on the uniform distribution (or the number of bidders); the APL is the auction format in which $t^a(v) = t(v)$, that is when given the information that the special bidder is v

$\mathit{\Pi}^{APA}(v) = 2(t(v) - \mathbb{E}[t(x)])$ from which we immediately get interim dominance (domination) of the PBA at 0 (at 1, strict) and single crossing of the IRFs.

Online Appendix

Proof of Proposition 11

To show that $\mathcal{E}$ is a savvy-bidder equilibrium we need

$$\mathit{\Pi}_0^F(0) \geq \mathit{\Pi}_{(0,1]}^S(0) \tag{7.16}$$

and

$$\mathit{\Pi}_0^F(v) \leq \mathit{\Pi}_{(0,1]}^S(v), \forall v > 0. \tag{7.17}$$

The argument made in the text establishes $b_{(0,1]}^{SPA,N}(x) = \tilde{b}_{(0,1]}^{SPA,S}(x) = x$ and that

$$\mathit{\Pi}_{(0,1]}^S(v) = \mathit{\Pi}^S(v) = \frac{n-2}{n} + v^{n-1} - \frac{n-1}{n}v^n. \tag{7.18}$$

We construct $\mathit{\Pi}_0^F$ following the three steps in its definition.

Step 1. The FPA is chosen only when $v = 0$, so non-special bidders play $b_0^{FPA,N}(x)$, the equilibrium in the $n-1$ bidders auction, while $b_0^{FPA,S}$ is defined only on $\{0\}$.

Step 2. If the seller chose the FPA even when $v > 0$ the special bidder chooses

$$\begin{aligned}\tilde{b}_0^{FPA,S}(v) &= \arg\max_b (v-b)\,\mathbb{P}\left(b > \max b_0^{FPA,N}(x)\right)\\ &= \arg\max_b (v-b)\max\left\{\left(\frac{n-1}{n-2}b\right)^{n-1}, 1\right\}\end{aligned}$$

if $\left(\frac{n-1}{n-2}b\right)^{n-1} < 1$ then the problem is equivalent to maximizing $(v-b)\,b^{n-1}$, yielding the equilibrium bid with n bidders; bidding above the value $\frac{n-2}{n-1}$ that ensures to win is dominated so the equilibrium bid is capped at this level and we obtain $\tilde{b}_0^{FPA,S}(v) = \max\left\{\frac{n-1}{n}v, \frac{n-2}{n-1}\right\}$.

Step 3. By deviating to the FPA when $v > 0$, the seller obtains $\max\left\{\tilde{b}_0^{FPA,S}(v), \frac{n-2}{n-1}y\right\}$, where y denotes the maximum of the valuation of the non-special bidders.[30] That is

$$\Pi_0^F = \begin{cases}\tilde{b}_0^{FPA,S}(v) & \text{if } \frac{n-2}{n-1}y < \tilde{b}_0^{FPA,S}(v)\\ \frac{n-2}{n-1}y & \text{else}\end{cases}$$

[30] Notice that the equilibrium is inefficient whenever $v < y < \frac{(n-1)^2}{n(n-2)}v$; by bidding more aggressively than his competitors, the special bidder wins too often.

which yields interim revenue

$$\Pi_0^F(v) = \begin{cases} \frac{n-1}{n} v \left(\frac{(n-1)^2}{n(n-2)} v\right)^{n-1} + \int_{\frac{(n-1)^2}{n(n-2)}v}^1 \frac{n-2}{n-1} y \mathrm{d}y^{n-1} & v < \frac{n(n-2)}{(n-1)^2} \\ \frac{n-2}{n-1} & \text{else} \end{cases}$$
$$= \min\left\{\frac{n-2}{n} + \frac{(n-1)^{2n-1}}{(n-2)^{n-1} n^{n+1}} v^n, \frac{n-2}{n-1}\right\}.$$

Comparing with (7.18) we obtain that $\Pi_{(0,1]}^S(0) = \Pi_0^F(0) = 0$, which establishes (7.16). Moreover, if $v \leq \frac{n(n-2)}{(n-1)^2}$ then

$$\Pi_{(0,1]}^S(v) - \Pi_0^F(v) \propto 1 - \left(\frac{n-1}{n} + \frac{(n-1)^{2n-1}}{(n-2)^{n-1} n^{n+1}}\right) v$$

which is positive for all $v \in \left[0, \frac{n(n-2)}{(n-1)^2}\right]$ since it is decreasing in v and positive at $v = \frac{n(n-2)}{(n-1)^2}$ for all $n \geq 2$. If $v > \frac{n(n-2)}{(n-1)^2}$, then $\Pi_{(0,1]}^S(v) > \Pi_{(0,1]}^S\left(\frac{n(n-2)}{(n-1)^2}\right) > \Pi_0^F\left(\frac{n(n-2)}{(n-1)^2}\right) = \Pi_0^F(v)$ as Π_0^F is flat in that region. Therefore, $\Pi_{(0,1]}^S(v) > \Pi_0^F(v)$ for all $v > 0$, which establishes (7.17) and completes the proof.

Proof of Proposition 12

Using the LIE,

$$\mathbb{E}_{s=1}\left[\Pi^F(v) - \Pi^S(v)\right] = \mathbb{E}_P\left[\mathbb{E}\left[\Pi^F(v) - \Pi^S(v) | v < P\right]\right]$$

we show that this expectation is positive no matter the distribution of P, which is unknown to the seller, by showing that for each realization p of P,

$$\mathbb{E}\left[\Pi^F(v) - \Pi^S(v) | v < p\right] > 0.$$

using independence, we get

$$\mathbb{E}\left[\Pi^F(v) - \Pi^S(v) | v < p\right] = \frac{1}{F(p)} \int_0^p \Pi^F(v) - \Pi^S(v) \mathrm{d}F(v)$$
$$= -\frac{1}{F(p)} \int_p^1 \Pi^F(v) - \Pi^S(v) \mathrm{d}F(v)$$

where the last line uses $\int_0^1 \Pi^F(v) - \Pi^S(v) \mathrm{d}F(v) = 0$. Let $\tilde{v}$ be the threshold from Theorem 3 such that $\text{sign}\left(\Pi^F(v) - \Pi^S(v)\right) = \text{sign}(\tilde{v} - v)$. If $p < \tilde{v}$, then $\Pi^F(v) - \Pi^S(v) > 0$ for all $v < p$, implying that the integral in the first line is positive. If $p > \tilde{v}$ then $\Pi^F(v) - \Pi^S(v) \leq 0$ for all $v > p$ (and strictly for $v < 1$),[31] implying that the integral in the second line is positive. The argument for $s = 0$ is the same.

[31]If $P \sim \delta_1$, then $v < P$ is uninformative and auctions are equivalent by the RET.