Buy price auctions with a resale opportunity *

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Abstract

This paper considers buy price auctions with a resale opportunity where there are two bidders. First, we examine the case in which bidders' valuations are perfectly revealed before resale. When the winner of an auction makes a take-it-or-leave-it offer to the loser, there exists an equilibrium in which bidders who have a lower valuation than a buy price submit the buy price. On the other hand, such an equilibrium does not exist when the loser of an auction makes a take-it-or-leave-it offer to the winner. When bidders' valuations are not perfectly revealed before resale, bidders who have a lower valuation than a buy price submit the buy price at an equilibrium even in the case in which the loser of an auction makes a take-it-or-leave-it offer to the winner.

JEL classification: D44

Key words: Auction, buy price, resale

1 Introduction

A buy price is often observed among Internet auctions. It is one of the options that sellers can choose to use additionally. If a bidder submits a buy price in an auction where a seller sets a buy price, then he immediately wins the auction without waiting for a predetermined end time. On the other hand, the winner who submits a buy price must pay it to a seller even in English auctions or second-price sealed-bid auctions, where the second highest bid may be less than the buy price. For example, a buy price in Yahoo! Japan auctions and a Buy-It-Now price in eBay auctions are popular.[1]

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[1] There is a difference between a buy price and a Buy-It-Now price. On the one hand, bidders can submit a buy price through an auction. On the other hand, bidders need to decide whether or not to submit a

If a seller sets a buy price, she does not obtain an amount that is higher than a buy price.[2] Budish and Takeyama (2001) focus on this point, and then consider English auctions with a buy price by using a two-bidder two-valuation framework. They show that a seller can improve her expected revenue by setting a buy price if bidders are risk-averse. Hidvégi, Wang and Whinston (2006) and Reynolds and Wooders (2009) use a more general framework, and then obtain a similar result. Mathews and Katzman (2006) also give attention to risk attitudes. They consider second-price sealed-bid auctions with a Buy-It-Now price, and then show that a seller who faces risk-neutral bidders can improve her expected revenue if she is risk-averse.

In buy price auctions, all bidders whose valuations are not less than a buy price can win an auction by submitting the buy price. Therefore, a winner of an auction may not have the highest valuation among bidders. Also, bidders whose valuations are less than a buy price never submit the buy price.

However, resale opportunities may change these situations. As transactions among bidders are allowed, the bidder who has the highest valuation but loses an auction may obtain an auctioned item as a result. If there is a resale opportunity, bidders whose valuations are less than a buy price may expect that an auctioned item can be sold at a higher price than the buy price.

We consider buy price auctions with a resale opportunity where there are two bidders. We describe these auctions as a two-stage game. At the first stage, an initial seller conducts second-price sealed-bid auctions with a buy price. At the second stage, a transaction between bidders is to take place.

In most part of the analysis, we assume that bidders' valuations are perfectly revealed before resale. This assumption guarantees that a transaction among bidders is carried out advantageously for a proposer of a transaction. As a first step, however, we consider buy price auctions with a resale opportunity under this strong assumption. We use subgame perfect equilibrium as a solution concept.

First, we consider a resale mechanism that the winner of an auction makes a take-it-or-leave-it offer to the loser. In this case, a bidder who wins against a relatively higher valuation bidder has an advantage at the second stage. When bidders do not have a resale opportunity, bidders whose valuations are greater than a buy price but not greater than a certain threshold do not submit a buy price at any symmetric equilibrium.[3] When the winner of an auction can sell an item, on the contrary, such bidders submit a buy price at a symmetric equilibrium.

Next, we show that there exists an equilibrium in which bidders whose valuation are less than a buy price submit the buy price. In other words, bidders whose valuations are less than

Buy-It-Now price before an auction starts. Once an auction starts, no one can bid a Buy-It-Now price any more.

[2] If a seller sets a Buy-It-Now price, she may obtain an amount that is higher than a Buy-It-Now price.

[3] See Inami (2021) for details.

a buy price tempt to speculate at the first stage. Without resale opportunities, of course, such bidders' behavior never consists of an equilibrium. Moreover, an initial seller can obtain a higher expected revenue than that of the case in which resale is not allowed.

In the next section, we reconsider the case in which the winner of an auction makes a take-it-or-leave-it offer to the loser. We weaken the assumption that bidders' valuations are perfectly revealed before resale. Strictly speaking, we assume that bidders' valuations are not perfectly revealed before resale. We then show that there exists an equilibrium at which all bidders with any valuation submit a buy price, and that an initial seller can obtain a higher expected revenue than that of the case in which resale is not allowed.

We discuss other resale mechanisms. We consider that the loser of an auction makes a take-it-or-leave-it offer to the winner. In this case, a bidder who loses against a relatively lower valuation bidder has an advantage at the second stage. When the winner of an auction makes a take-it-or-leave-it offer to the loser, there exists no equilibrium at which bidders whose valuations are greater than a buy price but not greater than a certain threshold do not submit the buy price. When the loser of an auction has the right to fetch a price, on the other hand, we can observe such bidders' behavior at an equilibrium. We also show that there exists no equilibrium in which bidders who have a lower valuation than a buy price submit the buy price, and that an initial seller cannot obtain a higher expected revenue than that of the case in which resale is not allowed. When bidders' valuations are not perfectly revealed before resale, however, we then show that there exists an equilibrium at which all bidders with any valuation submit a buy price.

In asymmetric first-price auctions without a resale opportunity, equilibrium outcomes are inefficient. Gupta and Lebrun (1999) show that this inefficiency is resolved by allowing resale among bidders. In the analysis, they assume that after an auction and before resale, bidders' valuations are perfectly revealed. Gupta and Lebrun (1999) point out that this assumption is essential to derive the result.[4]

Hafalir and Krishna (2008) examine both first-price auctions with a resale opportunity and second-price auctions with a resale opportunity. They analyze a similar auction environment to that of Gupta and Lebrun (1999). A striking difference between Gupta and Lebrun (1999) and Hafalir and Krishna (2008) is an information structure where the winner of an auction sells an item. Hafalir and Krishna (2008) consider the case in which the winner of an auction knows the loser's valuation very little. They show that an expected revenue for an initial seller from first-price auctions with a resale opportunity is greater than that of second-price auctions with a resale opportunity.

Resale opportunities create a motive for speculation. Garratt and Tröger (2006) explicitly introduce a speculator who values an auctioned item zero, and then analyze standard auctions with a resale opportunity. They suppose that the winner of auction can make a take-it-or-leave-it offer to the loser. Garratt and Tröger (2006) show that a speculator cannot obtain

[4]Krishna (2002) shows precisely that the inefficiency cannot be solved without this assumption.

a positive expected payoff in first-price auctions with a resale opportunity or Dutch auctions with a resale opportunity, but that a speculator can obtain a positive expected payoff in second-price auctions with a resale opportunity or English auctions with a resale opportunity.

There is other reason why bidders want a resale opportunity. In Haile (2003), bidders face uncertainties about their valuations of an item. This property alike common value environment provides a motive for resale to bidders. Haile (2003) shows that revenue equivalence between first-price auctions and second-price auctions at the first stage still holds, but that revenue equivalence between English auctions and second-price auctions at the first stage does not hold when an English auction is held at the second stage.

The rest of this paper is organized as follows. Section 2 describes the model. We analyze buy price auctions with a resale opportunity in Section 3. In the analysis, the winner of an auction makes a take-it-or-leave-it offer to the loser. Section 4 considers the resale mechanism under a weak assumption. Section 5 analyzes the case the loser of an auction makes a take-it-or-leave-it offer to the winner, and then concludes.

2 The model

We describe buy price auctions with a resale opportunity as a two-stage game. At the first stage, a seller conducts a second-price sealed-bid auction with a buy price. At the second stage, bidders do a transaction among them.

First, we explain about the first stage. A seller sells one item at an auction. She sets a buy price $B \in [0, +\infty)$, and then conducts a second-price sealed-bid auction with a buy price.[5] Two bidders attend the auction. And bidders' types are independently drawn from an uniform distribution. Bidders evaluate the item depending on their types only.

Here we provide details of the rule of buy price auctions. If no one bids a buy price, then the highest bidder wins the auction and pays the second highest bid to the seller. If only one bidder bids a buy price, then he certainly wins the auction but must pay it to the seller. If both bidders bid the same amount (it may be a buy price), the winner of the auction pays the second highest bid to the seller. In this case, we adopt a tie-breaking rule that the winner is determined with equal probability.

Next, we give an explanation about the second stage. The winner of the auction makes a take-it-or-leave-it offer to the loser (that is, the other bidder). If the loser accepts the offer, then he pays the offer price to the winner and obtains the item. Otherwise, the winner consumes the item by himself.

Let $N = \{1, 2\}$ be the set of bidders. For each bidder $i \in N$, the set of types is $T_i = [0, \omega]$. For each bidder $i \in N$, the set of actions is $A_i = [0, \bar{b}] \cup \{B\}$, where the bid $\bar{b}$ is the highest

[5] In most part of the analysis, we examine buy price auctions with a resale opportunity given a buy price B.

possible bid except a buy price B. We then assume that the bid $\bar{b}$ is less than the buy price B. In Internet auctions, bidders are not allowed to submit a bid above a buy price.

At the second stage, the winner of an auction makes an offer $x \in [0, +\infty)$, and the loser takes an action $y \in \{\text{accept}, \text{reject}\}$.

A payoff structure is formalized as follows: By taking an action $a_i \in A_i$, t_i-type of bidder i wins an auction, and then obtains:

$$\begin{array}{cl} x - B & \text{if } a_i = B \text{ and an offer } x \text{ is accepted,} \\ t_i - B & \text{if } a_i = B \text{ and an offer } x \text{ is rejected,} \\ x - a_j & \text{if } a_i \neq B \text{ and an offer } x \text{ is accepted,} \\ t_i - a_j & \text{if } a_i \neq B \text{ and an offer } x \text{ is rejected.} \end{array}$$

Similarly, t_i-type of bidder i loses an auction by taking an action $a_i \in A_i$, and then obtains:

$$\begin{array}{cl} t_i - x & \text{if he accepts an offer } x, \\ 0 & \text{if he rejects an offer } x. \end{array}$$

Here let $\mathcal{H}_i$ be an information partition of bidder i and let $H_i \in \mathcal{H}_i$ be an information set of bidder i.

Definition 1. A strategy of bidder i is, for all $H_i \in \mathcal{H}_i$,

$$\sigma_i : H_i \mapsto a_i(H_i) \in A_i(H_i),$$

where $A_i(H_i)$ is the set of available actions at an information set H_i.

We adopt perfect Bayesian equilibrium as a solution concept.

Definition 2. A strategy profile $\sigma = (\sigma_i(\cdot), \sigma_j(\cdot))$ is a subgame-perfect equilibrium if it induces Nash equilibria in all subgames.

In the analysis, we restrict attention to pure strategy equilibria.

Finally, we make the following assumption that plays an important role in the analysis.

Assumption 1. At the end of an auction and before resale, bidders' valuations are perfectly revealed.

Because of this assumption, the winner of an auction can always sell an item for a maximum price that the loser is able to pay. In this sense, it is a strong assumption. As the first step, however, we analyze buy price auctions with a resale opportunity under this assumption.

3 Resale with perfect information

We consider buy price auctions with a resale opportunity. Once an initial seller introduces a buy price, bidders who have a higher valuation than the buy price need to choose between submitting the buy price and submitting other bids. When resale is allowed, moreover, bidders who have a lower valuation than a buy price may submit the buy price. We focus on equilibrium bidding behavior.[6] We then examine how resale opportunities affect an initial seller's expected revenue. In the analysis, we restrict attention to symmetric equilibria.

Let $\beta_i(\cdot) : T_i \to \mathbb{R}$ be a bid function of each bidder $i \in N$. As Inami (2021) shows, in buy price auctions *without* a resale opportunity, a bidder whose valuation is not less than a buy price does not always submit the buy price at a symmetric equilibrium. We then consider following bidding behavior, and examine whether or not it can be observed at an equilibrium.

Proposition 1. *Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price $B \in (0, +\infty)$. Then, there exists no equilibrium in which bidders submit a bid as follows:*

$$\beta_i(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \bar{b}, \\ \bar{b} & \text{if } B \leq t_i \leq t^*, \\ B & \text{if } t^* < t_i \leq \omega. \end{cases}$$

Note that t^* is a threshold in the interval (B, ω).

Proof. See Appendix.

When one bidder wins against the other bidder who has a relatively lower valuation, the former bidder does not want to sell an item at the second stage. When one bidder wins against the other bidder who has a relatively higher valuation, on the contrary, the former bidder wants to sell an item at the second stage.

Next, we consider bidding behavior such that all bidders whose valuations are not less than a buy price submit the buy price. When resale is not allowed, this bidding behavior cannot be observed at a symmetric equilibrium.

Proposition 2. *Suppose that an initial seller conducts second-price sealed-bid auctions with the buy price $B = (\sqrt{2} - 1)\omega$. Then, there exists an equilibrium in which bidders submit a bid as follows:*

$$\beta_i^{\sharp}(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \bar{b}, \\ B & \text{if } B \leq t_i \leq \omega. \end{cases}$$

Proof. See Appendix.

When one bidder wins against the other bidder who has a relatively higher valuation, the former bidder still wants to sell an item at the second stage. As a buy price set by an initial seller goes higher, however, such bidder needs to deviate from submitting the buy price to increase an expected payoff.

Now, we examine whether buy prices enhance bidders' motive of speculation. We consider bidding behavior such that all bidders with any valuation submit a buy price.

[6] In each proposition, equilibrium behavior at the second stage is common.

Proposition 3. *Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price $B \in (0, \frac{\omega}{2}]$. Then, there exists an equilibrium in which bidders submit a bid as follows:*

$$\beta_i^{\flat}(t_i) = B \text{ if } 0 \leq t_i \leq \omega.$$

Proof. See Appendix.

Clearly, a bidder whose valuation is less than a buy price does not submit the buy price at an equilibrium when resale is not allowed. Under the assumption that bidders' valuations are perfectly revealed before resale, one bidder certainly obtains a positive payoff by winning against the other bidder who has a relatively higher valuation.

Inami (2021) shows that the strategy profile corresponding to bidding behavior of Proposition 1 is an equilibrium in second-price sealed-bid auctions with a buy price $B \in (0, \frac{\omega}{2})$. This result and Proposition 3 imply the following theorem.

Theorem 1. *Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price $B \in (0, \frac{\omega}{2}]$. Then, she obtains a higher expected revenue from buy price auctions with a resale opportunity than from buy price auctions without a resale opportunity.*

By Proposition 3, there exists an equilibrium when an initial seller conducts second-price sealed-bid auctions with a buy price $B \in (0, \frac{\omega}{2}]$. On the other hand, it is unclear whether there exists an equilibrium when an initial seller conducts second-price sealed-bid auctions with a buy price $B \in (\frac{\omega}{2}, \omega]$.

Proposition 4. *Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price $B \in (0, +\infty)$. Then, there exists no equilibrium in which bidders submit a bid as follows:*

$$\beta_i^{\natural}(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \bar{b}, \\ \bar{b} & \text{if } B \leq t_i \leq \omega. \end{cases}$$

Proof. See Appendix.

When resale is not allowed, the strategy profile corresponding to above-mentioned bidding behavior is an equilibrium in second-price sealed-bid auctions with a buy price $B \in [\frac{\omega}{2}, \omega]$.[7]

4 Resale with imperfect information

We have considered buy price auctions with a resale opportunity. With the resale mechanism that the winner of an auction makes a take-it-or-leave-it offer to the loser, we have shown that there exists an equilibrium in which all bidders with any valuation submit a buy price. We

[7]Inami (2021) also shows that there is no equilibrium such that a bidder submits a buy price in second-price sealed-bid auctions with a buy price $B \in [\frac{\omega}{2}, \omega]$.

have also shown that an initial seller can obtain a higher expected revenue than that of the case in which resale is not allowed. When the loser of an auction makes a take-it-or-leave-it offer to the winner, on the other hand, an initial seller cannot increase her expected revenue even when resale is allowed.

To derive above the results, we have relied on the important assumption that bidders' valuations are perfectly revealed after an auction and before resale. Here we weaken the assumption. In this section, we assume that bidders' valuations are not perfectly revealed. We adopt perfect Bayesian equilibrium as a solution concept. After the first stage, bidders know the second highest bid because of the auction rule. Therefore, we make a weak assumption. That is, bidders expect that bidders' valuation are not less than the second highest bid on the off-equilibrium path of an equilibrium.

We have a corresponding proposition to Proposition 3.

Proposition 5. *Consider the case in which bidders' valuations are not perfectly revealed. Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price* $B \in (0, \frac{\omega}{2}]$*. Then, there exists an equilibrium in which bidders submit a bid as follows:*

$$\beta_i^{\flat}(t_i) = B \text{ if } 0 \leq t_i \leq \omega.$$

Proof. See Appendix.

As well as the case in which bidders' valuations are perfectly revealed, we then have the following theorem.

Theorem 2. *Consider the case in which bidders' valuations are not perfectly revealed. Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price* $B \in (0, \frac{\omega}{2}]$*. Then, she obtains a higher expected revenue from buy price auctions with a resale opportunity than from buy price auctions without a resale opportunity.*

5 Other resale mechanism

In this section, we discuss resale mechanisms further. When the winner of an auction makes a take-it-or-leave-it offer, a bidder who wins against a relatively higher valuation bidder has an advantage. Here we consider other resale mechanism that the loser of an auction by turns makes a take-it-or-leave-it offer to the winner. At the second stage, on the contrary, a bidder who loses against a relatively lower valuation bidder has an advantage.

When the loser of an auction makes a take-it-or-leave-it offer to the winner, the payoff structure is modified as follows. By taking an action a_i, bidder i whose valuation is t_i loses an auction, and then obtains:

$$\begin{array}{cl} t_i - x & \text{if an offer } x \text{ is accepted,} \\ 0 & \text{if an offer } x \text{ is rejected.} \end{array}$$

Similarly, bidder $j \neq i$ whose valuation is t_j wins an auction by taking an action a_j, and then obtains:

$$\begin{array}{ll} x - B & \text{if } a_j = B \text{ and he accepts an offer } x, \\ t_j - B & \text{if } a_j = B \text{ and he rejects an offer } x, \\ x - a_i & \text{if } a_j \neq B \text{ and he accepts an offer } x, \\ t_j - a_i & \text{if } a_j \neq B \text{ and he rejects an offer } x. \end{array}$$

As well as the case in which the winner of an auction makes a take-it-or-leave-it offer, we examine whether or not following bidding behavior can be observed at an equilibrium.

Proposition 6. *Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price* $B \in \left(\frac{(\sqrt{3}-1)\omega}{2}, (\sqrt{2}-1)\omega\right)$*. Then, there exists an equilibrium in which bidders submit a bid as follows:*

$$\beta_i(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \bar{b}, \\ \bar{b} & \text{if } B \leq t_i \leq t^*, \\ B & \text{if } t^* < t_i \leq \omega. \end{cases}$$

Note that $t^* = \omega - \sqrt{-B^2 - 2B\omega + \omega^2}$*.*

Proof. See Appendix.

On the contrary to the case in which the winner of an auction makes a take-it-or-leave-it offer, a bidder whose valuation is not less than a buy price does not always submit the buy price at an equilibrium. When resale is not allowed, above-mentioned bidding behavior can be observed at an equilibrium in second-price sealed-bid auctions with a buy price $B \in \left(0, \frac{\omega}{2}\right)$.

Next, we consider bidding behavior such that all bidders whose valuations are not less than a buy price submit the buy price. When resale is not allowed, this bidding behavior cannot be observed at an equilibrium.

Proposition 7. *Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price* $B \in (0, +\infty)$*. Then, there exists no equilibrium in which bidders submit a bid as follows:*

$$\beta_i^{\sharp}(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \bar{b}, \\ B & \text{if } B \leq t_i \leq \omega. \end{cases}$$

Proof. See Appendix.

When one bidder loses against the other bidder who has a relatively lower valuation, the former bidder is able to buy an item for a minimum price that the loser is able to pay at the second stage. Therefore, a bidder whose valuation is not less than a buy price has a motive to lose an auction.

Now, we examine whether buy prices also enhance bidders' motive of speculation when the loser of an auction makes a take-it-or-leave-it offer. We consider bidding behavior such that all bidders with any valuation submit a buy price.

Proposition 8. *Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price $B \in (0, +\infty)$. Then, there exists no equilibrium in which bidders submit a bid as follows:*

$$\beta_i^{\flat}(t_i) = B \text{ if } 0 \leq t_i \leq \omega.$$

Proof. See Appendix.

When the loser of an auction makes a take-it-or-leave-it offer, a bidder whose valuation is less than a buy price does not have a motive to win against a bidder who has a relatively higher valuation. This is because even if one bidder wins against the other bidder who has a relatively higher valuation, the former bidder is forced to sell an item for a minimum price that he is able to sell at the second stage.

When the loser of an auction makes a take-it-or-leave-it offer, a bidder whose valuation is not less than a buy price is likely to want to lose an auction.

Proposition 9. *Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price $B \in (0, +\infty)$. Then, there exists an equilibrium in which bidders submit a bid as follows:*

$$\beta_i^{\natural}(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \bar{b}, \\ \bar{b} & \text{if } B \leq t_i \leq \omega. \end{cases}$$

Proof. See Appendix.

When resale is not allowed, above-mentioned bidding behavior can be observed at an equilibrium in second-price sealed-bid auctions with a buy price $B \in \left[\frac{\omega}{2}, +\infty\right)$. On the contrary to the case in which the winner of an auction makes a take-it-or-leave-it offer, a bidder whose valuation is not less than a buy price submits a bid except the buy price at an equilibrium.

Finally, we consider the case in which bidders' valuations are not perfectly revealed. We adopt perfect Bayesian equilibrium as a solution concept. We have a corresponding proposition to Proposition 5.

Proposition 10. *Consider the case in which bidders' valuations are not perfectly revealed. Suppose that an initial seller conducts second-price sealed-bid auctions with a buy price $B \in \left(0, \frac{\omega}{4}\right]$. Then, there exists an equilibrium in which bidders submit a bid as follows:*

$$\beta_i^{\flat}(t_i) = B \text{ if } 0 \leq t_i \leq \omega.$$

Proof. See Appendix.

When we examine the case in which bidders' valuations are perfectly revealed before resale, there does not exist an equilibrium in which bidders who have a lower valuation than a buy price submit the buy price. On the other hand, such an equilibrium does exist when the case in which bidders' valuations are not perfectly revealed before resale.

6 Conclusion

We have only considered whether a certain bidding behavior can be observed at an equilibrium. To investigate the model from a more actual point of view, we need to examine other bidding behavior much further. Also, we assume that bidders' types are uniformly distributed. This assumption is quite reasonable to derive the results, but makes a situation uncomplicated. Therefore, we need to analyze the model in a more general framework.

Appendix

Proof of Proposition 1

We examine optimal behavior of bidder i and bidder j at the second stage. Suppose that bidder j loses an auction. Optimal behavior of loser j whose valuation is t_j is:

$$\begin{array}{ll} \text{accept} & \text{if } t_j \geq x, \\ \text{reject} & \text{otherwise.} \end{array}$$

Even if loser j accepts the offer x that is greater than his own valuation, he only obtains a negative payoff.

Next, we derive winner i's optimal behavior. Given loser j's optimal behavior, optimal behavior of winner i whose valuation is t_i is:

$$\begin{array}{ll} x = t_j & \text{if } t_i \leq t_j, \\ x \in (t_j, +\infty) & \text{otherwise.} \end{array}$$

When a bidder wins against a relatively lower valuation bidder, the former bidder consumes an item by himself rather than selling it.

Now, we examine optimal behavior of bidders at the first stage. For all $t_i \in [0, \bar{b}]$,

$$\int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \geq \int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j + \int_{t_i}^{\bar{b}} (t_j - t_j)\frac{1}{\omega}dt_j + \frac{1}{2}\int_B^{t^*} (t_j - \bar{b})\frac{1}{\omega}dt_j \qquad (1)$$

must hold. R.H.S. of (1) is an expected payoff obtained by submitting the bid $\bar{b}$. Clearly, (1) does not hold.

Proof of Proposition 2

Optimal behavior of winner i and loser j at the second stage is the same as Proposition 1. Now, we examine optimal behavior of bidders at the first stage. For all $t_i \in [0, \bar{b}]$,

$$\int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \geq \int_0^{t_i} (t_i - B)\frac{1}{\omega}dt_j + \int_{t_i}^{\bar{b}} (t_j - B)\frac{1}{\omega}dt_j + \frac{1}{2}\int_B^{\omega} (t_j - B)\frac{1}{\omega}dt_j \qquad (2)$$

must hold. Also, for all $t_i \in [B, \omega]$,

$$\begin{aligned}\int_0^{\bar{b}} (t_i - B)\frac{1}{\omega}dt_j + \frac{1}{2}\left\{\int_B^{t_i} (t_i - B)\frac{1}{\omega}dt_j + \int_{t_i}^{\omega} (t_j - B)\frac{1}{\omega}dt_j\right\} + \frac{1}{2}\int_B^{t_i} (t_i - t_i)\frac{1}{\omega}dt_j \\ \geq \int_0^{\bar{b}} (t_i - t_j)\frac{1}{\omega}dt_j + \int_B^{t_i} (t_i - t_i)\frac{1}{\omega}dt_j\end{aligned} \tag{3}$$

must hold.

Calculating (2), we have

$$\frac{1}{4\omega}(-B^2 - 2\bar{b}^2 + 4B\bar{b} + 2B\omega - \omega^2) \geq 0.$$

Evaluating $\bar{b}$ at B, we have

$$\frac{1}{4\omega}(B^2 + 2B\omega - \omega^2) \geq 0.$$

Thus, (2) holds if

$$(\sqrt{2} - 1)\omega \leq B.$$

Similarly, calculating (3), we have

$$\frac{1}{4\omega}(t_i^2 - 2Bt_i + 2B^2 - 4B\bar{b} + 2\bar{b}^2 - 2B\omega + \omega^2) \geq 0.$$

Evaluating $\bar{b}$ at B, we have

$$\frac{1}{4\omega}(t_i^2 - 2Bt_i - 2B\omega + \omega^2) \geq 0.$$

Here let

$$\gamma_i^\sharp(t_i) := t_i^2 - 2Bt_i - 2B\omega + \omega^2.$$

For $t_i \in [B, \omega]$, the function $\gamma_i^\sharp(\cdot)$ is increasing with respect to t_i. It is sufficient to show that (3) holds for $t_i = B$. That is,

$$-2B^2 - 2B\omega + \omega^2 \geq 0.$$

Thus, (3) holds if

$$0 \leq B \leq (\sqrt{2} - 1)\omega.$$

From the above argument, both (2) and (3) hold if and only if

$$B = (\sqrt{2} - 1)\omega.$$

Proof of Proposition 3

Optimal behavior of winner i and loser j at the second stage is the same as Proposition 1. Now, we examine optimal behavior of bidders at the first stage. For all $t_i \in [0, \omega]$,

$$\frac{1}{2}\left\{\int_B^{t_i} (t_i - B)\frac{1}{\omega}dt_j + \int_{t_i}^{\omega} (t_j - B)\frac{1}{\omega}dt_j\right\} + \frac{1}{2}\int_0^{t_i} (t_i - t_i)\frac{1}{\omega}dt_j \geq \int_0^{t_i} (t_i - t_i)\frac{1}{\omega}dt_j \tag{4}$$

must hold.

Calculating (4),

$$\frac{1}{4\omega}(t_i^2 - 2B\omega + \omega^2) \geq 0.$$

Here let

$$\gamma_i^\flat(t_i) := t_i^2 - 2B\omega + \omega^2.$$

For $t_i \in [B, \omega]$, the function $\gamma_i^\flat(\cdot)$ is increasing with respect to t_i. It is sufficient to show that (4) holds for $t_i = 0$. Thus, (4) holds if

$$B \leq \frac{\omega}{2}.$$

Proof of Proposition 4

Optimal behavior of winner i and loser j at the second stage is the same as Proposition 1. Now, we examine optimal behavior of bidders at the first stage. For all $t_i \in [0, \bar{b}]$,

$$\int_0^{t_i}(t_i - t_j)\frac{1}{\omega}dt_j \geq \int_0^{t_i}(t_i - t_j)\frac{1}{\omega}dt_j + \int_{t_i}^{\bar{b}}(t_j - t_j)\frac{1}{\omega}dt_j + \frac{1}{2}\int_B^{\omega}(t_j - \bar{b})\frac{1}{\omega}dt_j \tag{5}$$

must hold. Clearly, (5) does not hold.

Proof of Proposition 5

We examine optimal behavior of bidder i and bidder j at the second stage. Suppose that bidder j loses an auction. Optimal behavior of loser j whose valuation is t_j is:

$$\begin{array}{ll} \text{accept} & \text{if } t_j \geq x, \\ \text{reject} & \text{otherwise.} \end{array}$$

Next, we derive winner i's optimal behavior. On the on-equilibrium path, winner i expects that loser j's valuation is uniformly distributed on the interval $[0, \omega]$. Thus, winner i whose valuation is t_i makes an offer x to maximize

$$\int_0^{x}(t_i - B)\frac{1}{\omega}dt_j + \int_x^{\omega}(x - B)\frac{1}{\omega}dt_j. \tag{6}$$

Calculating (6), we have

$$\frac{1}{\omega}(-x^2 + t_i x + \omega x - B\omega).$$

Thus, winner i whose valuation is t_i makes an optimal offer

$$x^* = \frac{t_i + \omega}{2}.$$

On the on-equilibrium path, loser j also expects that winner i's valuation is uniformly distributed on the interval $[0, \omega]$. Note that loser j whose valuation is less than an optimal offer does not accept it.

To summarize, by submitting a buy price, bidder i whose valuation is t_i obtains an expected payoff

$$\frac{1}{2}\left[\int_0^{\frac{t_i+\omega}{2}}(t_i-B)\frac{1}{\omega}dt_j+\int_{\frac{t_i+\omega}{2}}^{\omega}\left\{\left(\frac{t_i+\omega}{2}\right)-B\right\}\frac{1}{\omega}dt_j\right]$$

for $t_i \in [0, \frac{\omega}{2}]$, and

$$\frac{1}{2}\left[\int_0^{\frac{t_i+\omega}{2}}(t_i-B)\frac{1}{\omega}dt_j+\int_{\frac{t_i+\omega}{2}}^{\omega}\left\{\left(\frac{t_i+\omega}{2}\right)-B\right\}\frac{1}{\omega}dt_j\right]+\frac{1}{2}\left[\int_0^{2t_i-\omega}\left\{t_i-\frac{(t_j+\omega)}{2}\right\}\frac{1}{\omega}dt_j\right]$$

for $t_i \in (\frac{\omega}{2}, \omega]$.

Now, we look for a profitable deviation. Suppose that loser j submits a bid $b \in [0, \bar{b}]$. On the off-equilibrium path, winner i expects that loser j's valuation is uniformly distributed on the interval $[b, \omega]$. Thus, winner i whose valuation is t_i makes an offer x to maximize

$$\int_b^x (t_i-B)\frac{1}{\omega-b}dt_j+\int_x^{\omega}(x-B)\frac{1}{\omega-b}dt_j. \tag{7}$$

Calculating (7), we have

$$\frac{1}{\omega-b}(-x^2+t_ix+\omega x-bt_i+bB-B\omega).$$

Thus, winner i whose valuation is t_i makes an optimal offer

$$x^{\dagger}=\frac{t_i+\omega}{2}.$$

On the off-equilibrium path, loser j also expects that winner i's valuation is uniformly distributed on the interval $[b, \omega]$. Then, loser j whose valuation is t_j obtains an expected payoff

$$\int_0^{2t_j-\omega}\left\{t_j-\left(\frac{t_i+\omega}{2}\right)\right\}\frac{1}{\omega}dt_i.$$

For all $t_i \in [0, \frac{\omega}{2}]$,

$$\frac{1}{2}\left[\int_0^{\frac{t_i+\omega}{2}}(t_i-B)\frac{1}{\omega}dt_j+\int_{\frac{t_i+\omega}{2}}^{\omega}\left\{\left(\frac{t_i+\omega}{2}\right)-B\right\}\frac{1}{\omega}dt_j\right]\geq 0 \tag{8}$$

must hold. For all $t_i \in (\frac{\omega}{2}, \omega]$,

$$\begin{aligned}&\frac{1}{2}\left[\int_0^{\frac{t_i+\omega}{2}}(t_i-B)\frac{1}{\omega}dt_j+\int_{\frac{t_i+\omega}{2}}^{\omega}\left\{\left(\frac{t_i+\omega}{2}\right)-B\right\}\frac{1}{\omega}dt_j\right]\\&\quad+\frac{1}{2}\left[\int_0^{2t_i-\omega}\left\{t_i-\left(\frac{t_j+\omega}{2}\right)\right\}\frac{1}{\omega}dt_j\right]\geq\int_0^{2t_i-\omega}\left\{t_i-\left(\frac{t_j+\omega}{2}\right)\right\}\frac{1}{\omega}dt_j\end{aligned} \tag{9}$$

must hold. Calculating (8), we have

$$\frac{1}{8\omega}(t_i^2+2\omega t_i+\omega^2-2B\omega)\geq 0.$$

Here let

$$\iota^{\flat}_{iL}(t_i) \coloneqq t_i^2 + 2\omega t_i + \omega^2 - 2B.$$

For $t_i \in [0, \frac{\omega}{2}]$, the function $\iota^{\flat}_{iL}(\cdot)$ is increasing with respect to t_i. It is sufficient to show that (8) holds for $t_i = 0$. Then, we have

$$B \le \frac{\omega}{2}.$$

Calculating (9), we have

$$\frac{1}{8\omega}(-3t_i^2 + 6\omega t_i - 4B\omega) \ge 0.$$

Here let

$$\iota^{\flat}_{iH}(t_i) \coloneqq -3t_i^2 + 6\omega t_i - 4B\omega.$$

For $t_i \in (\frac{\omega}{2}, \omega]$, the function $\iota^{\flat}_{iH}(\cdot)$ is increasing with respect to t_i. It is sufficient to show that (9) holds for $t_i = \frac{\omega}{2}$. Then, we have

$$B \le \frac{9\omega}{16}.$$

From the above argument, both (8) and (9) hold for a buy price $B \in (0, \frac{\omega}{2}]$.

Proof of Proposition 6

We examine optimal behavior of bidder i and bidder j at the second stage. Suppose that bidder j wins an auction. Optimal behavior of winner j whose valuation is t_j is:

$$\begin{array}{ll} \text{accept} & \text{if } t_j \le x, \\ \text{reject} & \text{otherwise.} \end{array}$$

When an offer x is less than winner j's valuation, he consumes an item rather than accepting the offer.

Next, we derive loser i's optimal behavior. Given winner j's optimal behavior, optimal behavior of loser i whose valuation is t_i is:

$$\begin{array}{ll} x \in [0, t_i] & \text{if } t_i \le t_j, \\ x = t_j & \text{otherwise.} \end{array}$$

When a bidder loses against a relatively higher valuation bidder, the former bidder's offer is not accepted.

Now, we examine optimal behavior at the first stage. For all $t_i \in [0, \bar{b}]$,

$$\int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \ge \int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j + \int_{t_i}^{\bar{b}} (t_i - t_j)\frac{1}{\omega}dt_j + \frac{1}{2}\int_B^{t^*} (t_i - \bar{b})\frac{1}{\omega}dt_j \qquad (10)$$

must hold. For all $t_i \in [B, t^*]$,

$$\begin{aligned} &\int_0^{\bar{b}} (t_i - t_j)\frac{1}{\omega}dt_j + \frac{1}{2}\left\{\int_B^{t_i} (t_i - \bar{b})\frac{1}{\omega}dt_j + \int_{t_i}^{t^*} (t_i - \bar{b})\frac{1}{\omega}dt_j\right\} + \frac{1}{2}\int_B^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \\ &\quad \ge \int_0^{\bar{b}} (t_i - B)\frac{1}{\omega}dt_j + \frac{1}{2}\left\{\int_B^{t_i} (t_i - B)\frac{1}{\omega}dt_j + \int_{t_i}^{t^*} (t_i - B)\frac{1}{\omega}dt_j\right\} + \frac{1}{2}\int_{t^*}^{\omega} (t_i - B)\frac{1}{\omega}dt_j \end{aligned} \qquad (11)$$

must hold. Moreover, for all $t_i \in (t^*, \omega]$,

$$\int_0^{\bar{b}} (t_i - B)\frac{1}{\omega}dt_j + \int_B^{t^*} (t_i - B)\frac{1}{\omega}dt_j + \frac{1}{2}\left\{\int_{t^*}^{t_i} (t_i - B)\frac{1}{\omega}dt_j + \int_{t_i}^{\omega} (t_i - t_j)\frac{1}{\omega}dt_j\right\}$$
$$+ \frac{1}{2}\int_{t^*}^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j$$
$$\geq \int_0^{\bar{b}} (t_i - t_j)\frac{1}{\omega}dt_j + \frac{1}{2}\int_B^{t^*} (t_i - \bar{b})\frac{1}{\omega}dt_j + \frac{1}{2}\int_B^{t^*} (t_i - t_j)\frac{1}{\omega}dt_j + \frac{1}{2}\int_{t^*}^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \tag{12}$$

must hold.

Clearly, (10) holds. We then consider the systems of inequalities (11) and (12). In practice, it is sufficient to show that for $t_i = t^*$,

$$\int_0^{\bar{b}} (t_i - t_j)\frac{1}{\omega}dt_j + \frac{1}{2}\left\{\int_B^{t_i} (t_i - \bar{b})\frac{1}{\omega}dt_j + \int_{t_i}^{t^*} (t_i - \bar{b})\frac{1}{\omega}dt_j\right\} + \frac{1}{2}\int_B^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j$$
$$= \int_0^{\bar{b}} (t_i - B)\frac{1}{\omega}dt_j + \frac{1}{2}\left\{\int_B^{t_i} (t_i - B)\frac{1}{\omega}dt_j + \int_{t_i}^{t^*} (t_i - B)\frac{1}{\omega}dt_j\right\} + \frac{1}{2}\int_{t^*}^{\omega} (t_i - B)\frac{1}{\omega}dt_j \tag{13}$$

holds. Calculating (13), we have

$$\frac{1}{4\omega}\{(t^*)^2 - 2\omega t^* - 2\bar{b}t^* + 2Bt^* - 2\bar{b}^2 + 6B\bar{b} - 3B^2 + 2B\omega\} = 0.$$

Evaluating $\bar{b}$ at B, we have

$$\frac{1}{4\omega}\{(t^*)^2 - 2\omega t^* + B^2 + 2B\omega\} = 0.$$

Thus, (13) holds if

$$t^* = \omega - \sqrt{-B^2 - 2B\omega + \omega^2}.$$

To be well-defined for t^*,

$$0 < \sqrt{-B^2 - 2B\omega + \omega^2} < B.$$

must hold. Thus, we have

$$\frac{(\sqrt{3}-1)\omega}{2} < B < (\sqrt{2}-1)\omega.$$

Proof of Proposition 7

Optimal behavior of loser i and winner j at the second stage is the same as Proposition 6. Now, we examine optimal behavior at the first stage. For all $t_i \in [0, \bar{b}]$,

$$\int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \geq \int_0^{t_i} (t_i - B)\frac{1}{\omega}dt_j + \int_{t_i}^{\bar{b}} (t_i - B)\frac{1}{\omega}dt_j + \frac{1}{2}\int_B^{\omega} (t_i - B)\frac{1}{\omega}dt_j \tag{14}$$

must hold. For all $t_i \in [B, \omega]$,

$$\begin{aligned}&\int_0^{\bar{b}} (t_i - B)\frac{1}{\omega}dt_j + \frac{1}{2}\left\{\int_B^{t_i} (t_i - B)\frac{1}{\omega}dt_j + \int_{t_i}^{\omega} (t_i - B)\frac{1}{\omega}dt_j\right\} + \frac{1}{2}\int_B^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \\ &\quad \geq \int_0^{\bar{b}} (t_i - B)\frac{1}{\omega}dt_j + \int_B^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j\end{aligned} \tag{15}$$

must hold.

Clearly, (14) holds. Calculating (15), we have

$$\frac{1}{4\omega}(-t_i^2 + 2\omega t_i + B^2 - 4B\bar{b} + 2\bar{b}^2 - 2B\omega) \geq 0.$$

Evaluating $\bar{b}$ at B, we have

$$\frac{1}{4\omega}(-t_i^2 + 2\omega t_i - B^2 - 2B\omega) \geq 0.$$

Here let

$$\eta_i^\sharp(t_i) := -t_i^2 + 2\omega t_i - B^2 - 2B\omega.$$

For $t_i \in [B, \omega]$, the function $\eta_i^\sharp(\cdot)$ is increasing with respect to t_i. It is sufficient to show that (15) holds for $t_i = B$. However,

$$-2B^2 \geq 0$$

does not hold for a buy price $B \in (0, +\infty)$.

Proof of Proposition 8

Optimal behavior of loser i and winner j at the second stage is the same as Proposition 6. Now, we examine optimal behavior at the first stage. For all $t_i \in [0, \omega]$,

$$\frac{1}{2}\left\{\int_0^{t_i} (t_i - B)\frac{1}{\omega}dt_j + \int_{t_i}^{\omega} (t_i - B)\frac{1}{\omega}dt_j\right\} + \frac{1}{2}\int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \geq \int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \tag{16}$$

must hold.

Calculating (16), we have

$$\frac{1}{4\omega}(-t_i^2 + 2\omega t_i - 2B\omega) \geq 0.$$

Here let

$$\eta_i^\flat(t_i) := -t_i^2 + 2\omega t_i - 2B\omega.$$

For $t_i \in [0, \omega]$, the function $\eta_i^\flat(\cdot)$ is increasing with respect to t_i. It is sufficient to show that (16) holds for $t_i = 0$. However,

$$-2B\omega \geq 0$$

does not hold for a buy price $B \in (0, +\infty)$.

Proof of Proposition 9

Optimal behavior of loser i and winner j at the second stage is the same as Proposition 6. Now, we examine optimal behavior at the first stage. For all $t_i \in [0, \bar{b}]$,

$$\int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \geq \int_0^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j + \int_{t_i}^{\bar{b}} (t_i - t_j)\frac{1}{\omega}dt_j + \frac{1}{2}\int_B^{\omega} (t_i - \bar{b})\frac{1}{\omega}dt_j \tag{17}$$

must hold. For all $t_i \in [B, \omega]$,

$$\begin{aligned}&\int_0^{\bar{b}} (t_i - t_j)\frac{1}{\omega}dt_j + \frac{1}{2}\left\{\int_B^{t_i} (t_i - \bar{b})\frac{1}{\omega}dt_j + \int_{t_i}^{\omega} (t_i - \bar{b})\frac{1}{\omega}dt_j\right\} + \frac{1}{2}\int_B^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j \\ &\geq \int_0^{\bar{b}} (t_i - B)\frac{1}{\omega}dt_j + \int_B^{t_i} (t_i - t_j)\frac{1}{\omega}dt_j\end{aligned} \tag{18}$$

must hold.

Clearly, (17) holds. Calculating (18), we have

$$\frac{1}{4\omega}(-t_i^2 + 2\omega t_i + 2B\bar{b} - B^2 - 2\bar{b}\omega) \geq 0.$$

Evaluating $\bar{b}$ at B, we have

$$\frac{1}{4\omega}(-t_i^2 + 2\omega t_i + B^2 - 2B\omega) \geq 0.$$

Here let

$$\eta_i^{\natural}(t_i) := -t_i^2 + 2\omega t_i + B^2 - 2B\omega.$$

For $t_i \in [0, \omega]$, the function $\eta_i^{\natural}(\cdot)$ is increasing with respect to t_i. It is sufficient to show that (18) holds for $t_i = B$. Then, we have

$$0 \geq 0.$$

Thus, (18) holds for a buy price $B \in (0, +\infty)$.

Proof of Proposition 10

We examine optimal behavior of bidder i and bidder j at the second stage. Suppose that bidder j wins an auction. Optimal behavior of winner j whose valuation is t_j is:

$$\begin{array}{ll}\text{accept} & \text{if } t_j \leq x, \\ \text{reject} & \text{otherwise.}\end{array}$$

Next, we derive loser i's optimal behavior. On the on-equilibrium path, loser i expects that winner j's valuation is uniformly distributed on the interval $[0, \omega]$. Thus, loser i whose valuation is t_i makes an offer x to maximize

$$\int_0^x (t_i - x)\frac{1}{\omega}dt_j. \tag{19}$$

Calculating (19), we have

$$\frac{1}{\omega}(t_i - x)x.$$

Thus, loser i whose valuation is t_i makes an optimal offer

$$x^* = \frac{t_i}{2}.$$

On the on-equilibrium path, winner j also expects that loser i's valuation is uniformly distributed on the interval $[0, \omega]$. Note that winner j whose valuation is not less than an optimal offer does not accept it.

To summarize, by submitting a buy price, bidder i whose valuation is t_i obtains an expected payoff

$$\frac{1}{2}\left[\int_0^{2t_i}(t_i - B)\frac{1}{\omega}dt_j + \int_{2t_i}^{\omega}\left\{\left(\frac{t_j}{2}\right) - B\right\}\frac{1}{\omega}dt_j\right] + \frac{1}{2}\int_0^{\frac{t_i}{2}}\left\{t_i - \left(\frac{t_i}{2}\right)\right\}\frac{1}{\omega}dt_j$$

for $t_i \in [0, \frac{\omega}{2}]$, and

$$\frac{1}{2}\int_0^{\omega}(t_i - B)\frac{1}{\omega}dt_j + \frac{1}{2}\int_0^{\frac{t_i}{2}}\left\{t_i - \left(\frac{t_i}{2}\right)\right\}\frac{1}{\omega}dt_j$$

for $t_i \in (\frac{\omega}{2}, \omega]$.

Now, we look for a profitable deviation. Suppose that loser i submits a bid $b \in [0, \bar{b}]$. On the off-equilibrium path, loser i expects that winner j's valuation is uniformly distributed on the interval $[b, \omega]$. Thus, loser i whose valuation is t_i makes an offer x to maximize

$$\int_b^x (t_i - x)\frac{1}{\omega - b}dt_j. \tag{20}$$

Calculating (20), we have

$$\frac{1}{\omega - b}(t_i - x)(x - b).$$

Thus, loser i whose valuation is t_i makes an optimal offer

$$x^\dagger = \frac{t_i + b}{2}.$$

On the off-equilibrium path, loser i also expects that winner j's valuation is uniformly distributed on the interval $[b, \omega]$. Then, loser i whose valuation is t_i obtains an expected payoff

$$\int_0^{\frac{t_i+b}{2}}\left\{t_i - \left(\frac{t_i + b}{2}\right)\right\}\frac{1}{\omega}dt_j. \tag{21}$$

Differentiating (21) with respect to b, we have

$$-\frac{b}{2\omega}.$$

Thus, (21) is decreasing with respect to b. In other words,

$$\int_0^{\frac{t_i}{2}}\left\{t_i - \left(\frac{t_i}{2}\right)\right\}\frac{1}{\omega}dt_i$$

is the highest expected payoff by submitting a bid except a buy price.

On the off-equilibrium path, winner j also expects that loser i's valuation is uniformly distributed on the interval $[b, \omega]$.

For all $t_i \in [0, \frac{\omega}{2}]$,

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^{2t_i}(t_i - B)\frac{1}{\omega}dt_j + \int_{2t_i}^{\omega}\left\{\left(\frac{t_j}{2}\right) - B\right\}\frac{1}{\omega}dt_j\right] + \frac{1}{2}\int_0^{\frac{t_i}{2}}\left\{t_i - \left(\frac{t_i}{2}\right)\right\}\frac{1}{\omega}dt_j \\
&\quad \geq \int_0^{\frac{t_i}{2}}\left\{t_i - \left(\frac{t_i}{2}\right)\right\}\frac{1}{\omega}dt_i
\end{aligned} \tag{22}$$

must hold. For all $t_i \in (\frac{\omega}{2}, \omega]$,

$$\begin{aligned}
&\frac{1}{2}\int_0^{\omega}(t_i - B)\frac{1}{\omega}dt_j + \frac{1}{2}\int_0^{\frac{t_i}{2}}\left\{t_i - \left(\frac{t_i}{2}\right)\right\}\frac{1}{\omega}dt_j \\
&\quad \geq \int_0^{\frac{t_i}{2}}\left\{t_i - \left(\frac{t_i}{2}\right)\right\}\frac{1}{\omega}dt_i
\end{aligned} \tag{23}$$

must hold. Calculating (22), we have

$$\frac{1}{8\omega}(3t_i^2 + \omega^2 - 4B\omega) \geq 0.$$

Here let

$$\lambda_{iL}^{\flat}(t_i) := 3t_i^2 + \omega^2 - 4B\omega.$$

For $t_i \in [0, \frac{\omega}{2}]$, the function $\lambda_{iL}^{\flat}(\cdot)$ is increasing with respect to t_i. It is sufficient to show that (22) holds for $t_i = 0$. Then, we have

$$B \leq \frac{\omega}{4}.$$

Calculating (23), we have

$$\frac{1}{8\omega}(t_i^2 - 4(t_i - B)\omega) \leq 0.$$

Here let

$$\lambda_{iH}^{\flat}(t_i) := t_i^2 - 4(t_i - B)\omega.$$

For $t_i \in (\frac{\omega}{2}, \omega]$, the function $\lambda_{iH}^{\flat}(\cdot)$ is decreasing with respect to t_i. It is sufficient to show that (23) holds for $t_i = \frac{\omega}{2}$. Then, we have

$$B \leq \frac{7\omega}{16}.$$

From the above argument, both (22) and (23) hold for a buy price $B \in (0, \frac{\omega}{4}]$.

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