# AMBIGUITY AVERSION AND THE DECLINING PRICE ANOMALY: THEORY AND ESTIMATION

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ABSTRACT. Using a unique data set from the sale of train tickets in Sweden, we study the "declining price anomaly" in sequential auctions theoretically and empirically. First, our reduced form analysis suggests that a model with ambiguity averse bidders fits the observed bidder behavior in these auctions better than other existing models. Motivated by this, we study sequential second-price auctions that closely resemble the auction mechanism for train tickets and assume bidders have *maxmin* expected utilities over multiple priors. In the unique symmetric equilibrium, bidders use their worst-case conditional beliefs to evaluate their payoffs. The equilibrium generates declining prices due to an underestimation of future payoffs that is brought on by ambiguity aversion. We also provide a new revenue ranking between some common multi-unit auction formats and show that ambiguity raises the seller's revenue despite declining prices.

Finally, we non-parametrically estimate both the true distribution of valuations *and* the worst-case beliefs using a novel identification technique that exploits bidders' inter-temporal first order conditions. Our estimation uncovers a first-order stochastic dominance relationship between beliefs and the true distribution, which is consistent with ambiguity aversion. Our counterfactuals show that, while ambiguity increases the seller's revenue by at least 18% compared to the common prior case, switching to sequential first-price auctions would further increase revenue by at least 11%.

Keywords: declining price anomaly, ambiguity, estimation, train ticket auctions

JEL classification: C11, C44, C57, D44

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## 1. INTRODUCTION

Sequential auctions are one of the oldest and most commonly used auction formats for selling multiple units of a good. In a canonical version of this format, goods are sold sequentially in auctions to the highest bidder in each round. In much the same way, in 2010-11 the Swedish rail road company (SJ) sold train tickets in over 6,500 online sequences of auctions where tickets sold in a sequence were identical. Prices in these auctions on average declined 8.5% between consecutive rounds which is a curious, but common, pattern for sequential auctions with identical objects being sold. In fact, following the seminal work on wine auctions by Ashenfelter (1989), many studies have documented instances of such "declining price anomalies," which contradict the theoretical predictions in standard sequential auctions (Milgrom and Weber (2000)).[1]

In this paper, we first empirically evaluate the reduced form implications of the existing formalized theoretical explanations for this puzzle using our data set, and then develop and structurally estimate a sequential auction model with ambiguity averse bidders, which best fits the reduced form evidence. Specifically, there are three "preference-based" explanations of the declining price anomaly in the literature: *risk aversion* (McAfee and Vincent (1993) and Mezzetti (2011)), *loss aversion* (Rosato (2019)) and *ambiguity aversion* (Ghosh and Liu (2021)). While all three models predict declining prices, they have different implications regarding the relationship between bids and previous round prices in an independent private value (i.p.v.) paradigm: risk aversion predicts history independence, loss aversion predicts negative history dependence and ambiguity aversion predicts history dependence that can be positive or negative (depending on the set of bidders' priors). We show that bidding in the train ticket auctions has a slight positive history dependence: controlling for all other factors, a one percent increase in the price in round $k-1$ corresponds to $0.04-0.08$ percent increase in the bids in round $k$. Only a model with ambiguity averse bidders is consistent with this pattern.

Given the reduced form results, we study a theoretical model of sequential second-price auctions (sSPAs), which closely resembles the auction mechanism in our data set, with ambiguity averse bidders in order to develop the basis of a structural analysis. Given the dynamic nature of a bidder's problem, ambiguity adds additional complications of dynamic inconsistency, which we resolve by using a solution concept that generalizes weak perfect Bayesian equilibrium. We prove the existence and uniqueness of a symmetric equilibrium in which bidders in each round use their worst-case conditional beliefs to calculate their probability of winning. Thus, bidding can be history-dependent, as worst case beliefs can change round to round depending on the right truncation of the valuation support which is precisely the conditioning bidders do in sequential auctions with a monotone equilibrium.

[1] See Ashenfelter and Genovese (1992) (condominiums), McAfee and Vincent (1993) (wines), Beggs and Graddy (1997) (art), Van den Berg et al. (2001) (roses), Lambson and Thurston (2006) (fur), and Snir (2006) (computers).

Furthermore, we show that under simple conditions prices decline in equilibrium due to bidders' inter-temporal pessimism resulting from ambiguity aversion. Since bidders use their worst-case beliefs, they underestimate their 'option value' of participating in future rounds. Hence they bid more aggressively in the current round causing prices to decline in future rounds as lower valuation bidders, who are relatively less aggressive than the current round winner, will win future rounds in a monotone equilibrium. We also show that sSPAs generate higher revenues compared to uniform price auctions but lower revenues compared to sequential first price auctions (sFPAs) under a consistency condition. Also, the revenue generated in a uniform price auction with ambiguity is the same revenue generated in a model of sequential auctions (SPA or FPA) without ambiguity. Thus, somewhat surprisingly, declining prices do not lead to lower revenues compared to a model where prices are constant.

When bidders are ambiguity averse, the distribution of bids is affected by two distributions: the true distribution of values *and* the distribution of values bidders use to calculate their payoffs, i.e. their worst-case belief. It is not possible to identify both of these from auction data of a single-unit auction without referring to exogenous variations as in Aryal et al. (2018). However, dynamic bidding data from sequential auctions can be used to identify both the distributions without any additional restrictions other than that of dynamic consistency, as was shown by Ghosh and Liu (2021) (GL from now on) for sFPAs. The idea is to use first order conditions from sequential rounds to map distributions of bids from multiple rounds to distributions of valuations and beliefs, thus extending the idea of Guerre et al. (2000) (GPV henceforth) to sequential auctions with ambiguity. We apply this to our setting of sSPAs and prove an identification result that allows us to back out the model's primitives. Specifically, for simplicity consider a two round sSPA. Since it is weakly dominant to bid one's valuation in the final round, it is straightforward to identify the true distribution of values from final round bids. However, bids in the first round depend on the bidder's beliefs as well as her expected payoff, and therefore her bid, in the final round. Thus, we can identify bidders (worst-case) beliefs using bidders' bids from *both* rounds and the inter-temporal first-order condition from the first round, which gives a tight connection between the two bids and bidders' beliefs.

Applying the above methodology to the train ticket auctions, our estimation recovers the true distribution of valuations as well as bidders' beliefs. Importantly the distribution of beliefs first order stochastically dominates the true distribution of valuations, thus confirming the presence of ambiguity from the bidders' point of view. Using the recovered distributions we find that ambiguity greatly contributes to the seller's revenue: removing ambiguity or switching to a uniform price auction would have decreased the revenue by 18% to 21%. Thus we find that declining prices may be synonymous with higher revenues compared to the no ambiguity case in sequential auctions. Finally, we also carry out an exercise where we replace the selling mechanism with sFPAs and find that this would have increased revenue by 11% to 15%. We also perform various other robustness checks to check the validity of our approach.

This paper is related to a few strands of literature starting with price anomalies in auctions. As mentioned previously, there is considerable evidence in support of declining prices in sequential auctions. In addition Keser and Olson (1996) and Neugebauer and Pezanis-Christou (2007) document the existence of this phenomenon in experimental settings. Chanel et al. (1996) and Deltas and Kosmopoulou (2003) found evidence of increasing prices, all though the occurrence of declining prices seems to be more common (Ashenfelter and Graddy (2003), Ashenfelter and Graddy (2011)). While all of these studies document the evidence of price anomalies, to the best of our knowledge, the current paper is the first to empirically investigate the declining prices using a structural econometrics approach as well as empirically test the existing theories that can account for this anomaly.

Since the finding in Ashenfelter (1989), several explanations for puzzle have been offered. One set of explanations suggest that specifics of the sale mechanism, such as winner's option to buy remaining units at the same price (Ashenfelter (1989)), absentee bidding (Ginsburgh (1998)), participation fees (Menezes and Monteiro (1997)) and supply side uncertainty (Jeitschko (1999)), can account for the anomaly. Another set suggests that the specific features of the goods being sold can lead to declining prices. For example Bernhardt and Scoones (1994), Engelbrecht-Wiggans (1994), Gale and Hausch (1994) and Kittsteiner et al. (2004) found that heterogeneity between the goods can lead to declining prices. In a non-auction setting Sweeting (2012) showed that the selling price of perishable goods declines as one gets closer to the expiry date.[2] All these explanations, while important, are specific to particular settings. However, declining prices have been observed across a wide variety of formats, goods, and settings. Thus, while our setting may share some features with the aforementioned papers, we evaluate more 'general' explanations of the anomaly so that our structural methodology may be applicable to other data sets where this phenomenon has been observed. Furthermore, the suitability of one explanation over another is not obvious since few papers have compared the various approaches, as we do in our paper.

Our paper is also connected to the the literature on auctions with ambiguity. In various settings Salo and Weber (1995), Lo (1998), Levin and Ozdenoren (2004), Chen et al. (2007) and Laohakunakorn et al. (2019) study single-unit first and second price auctions with ambiguity. In the presence of ambiguity Bose and Daripa (2009) and Auster and Kellner (2020) show that a dutch

[2]Given that we are studying perishable goods as well, this explanation may seem particularly relevant. However there are some important differences. The sequential auctions for train tickets end within one hour of each other (so less likelihood of entry and exit of bidders and the travel date is not that much closer to the final sequential auction compared to the first) and the bidders are strategic. Thus prices are set by the bidders by competing against each other and not by the seller who may have an incentive to lower prices as the travel date approaches. All these aspects differentiate our setting from the baseball tickets environment studied in Sweeting (2012).

auction can perform better than static auctions. Bose et al. (2006) and Bodoh-Creed (2012) study optimal auctions under ambiguity. To the best of our knowledge, our paper and GL are the only papers that study sequential auctions where multiple units are sold in an environment with ambiguity. We also provide a new revenue ranking for multi-unit auctions with ambiguity.

There is a rich literature on the estimation of variables of interest from auction data.[3] Most papers in this literature require the assumption of a common prior to identify and estimate the valuation distributions. A notable exception is Aryal et al. (2018) who identify and estimate the distribution of valuations in static auctions in the presence of ambiguity using variation in the number of bidders. Complimenting this work, our identification result shows that considering data from *dynamic auctions* can provide techniques to ascertain the presence of ambiguity and correctly estimate the variables of interest.[4]

Jofre-Bonet and Pesendorfer (2003), Donald et al. (2006), Groeger (2014), Donna and Espín-Sánchez (2018) and Kong (2021) study dynamic auctions using models with capacity constraints, multi-unit demand, learning by doing, complementarities and synergies and affiliation respectively. Much like our paper, in these papers bidding in different rounds is linked. However all these papers are in the common prior framework. Furthermore, to the best of our knowledge, ours is the only paper that tackles the declining price anomaly in sequential auctions using a structural approach.

Finally, our paper is also related to the literature on identification of games of incomplete information. While much of this literature assumes that players beliefs are correct in equilibrium, Aguirregabiria and Magesan (2020) study dynamic discrete games with rational players who may have incorrect beliefs. They show that an exclusion restriction, typically used to identify games of incomplete information, provides testable nonparametric restrictions of the null hypothesis of equilibrium beliefs as well as a function that only depends on a players beliefs which can be used to estimate players beliefs. Complimenting this approach we show that data on players actions at multiple points in time (bids in different rounds) can also be used to estimate biased (i.e. worst-case) beliefs without appealing to an exclusion restriction.

The rest of the paper is organized as follows. Section 2 describes the auction mechanism in our data set and provides summary statistics, including the pattern of declining prices. In this section we also evaluate the existing models in a reduced form analysis and provide evidence in support of the ambiguity aversion model. In Section 3 we first argue that the auction mechanism used for train tickets closely resembles sSPAs. This section then provides the main theoretical

[3]See Athey and Haile (2007) and Hickman et al. (2012) for surveys. See Hortaçsu and McAdams (2018) for a survey on multi-unit auctions.

[4]There are several papers that measure the effect of ambiguity on bidding in static auctions in experimental settings, as in Chen et al. (2007). Beyond auction settings, most papers that carry out estimation exercises in the presence of ambiguity consider experimental (laboratory or field) settings. See Cabantous (2007), Abdellaoui et al. (2011), and Ahn et al. (2014) among others.

results regarding equilibrium existence and uniqueness, price path, and revenue rankings. Section 4 presents the empirical analysis, including the non-parametric estimation of model primitives and counterfactual exercises. In Section F, we split the data into two time periods and provide further evidence to support the ambiguity-based approach. Section 5 concludes.

## 2. AUCTION MECHANISM, DATA, AND REDUCED FORM ANALYSIS

2.1. **Auction Mechanism.** The data set we use consists of bids made in auctions for train tickets sold by the Swedish rail road company (SJ). The auctions were executed online using Tradera's website, at the time a subsidiary of eBay. There were often multiple tickets sold in a group of auctions, where the winner of one of the auctions within the group received one ticket. Within a group, all tickets are observationally identical. Specifically, tickets within a group are for the same route, same type of train, same class, and same departure time. All other features such as seat number, aisle vs. window seat, and direction of the seat were not known until a winner of an auction actually went to the train station and picked up the ticket on the day of departure.

Auctions within a group ran parallel for some time, before ending within a one hour time span. The exact closing time of each auction differed within that hour. The difference in closing times gave the auctions within a group a distinctive sequential nature. We therefore refer to a group of auctions selling identical tickets as a *sequence* from here on. The closing time of an auction was made public to the bidders at the same time as the auction was posted.

Figure 1 illustrates a sequence of three auctions. All auctions within a sequence start within one hour of each other. Then all auctions in the sequence are active in parallel for about two days. Lastly, all auctions within the sequence end, in a sequential manner, between 9 and 10 pm two days prior to the departure of the train. We index auctions in a sequence by the ending order, where auction 1 is the auction that ends first in a sequence, and so on. By this logic, the last auction is auction $K$, where $K$ is the total number of tickets in a sequence. We will often refer to an auction within a sequence as a *round*.

FIGURE 1. Illustration of a sequence with 3 tickets

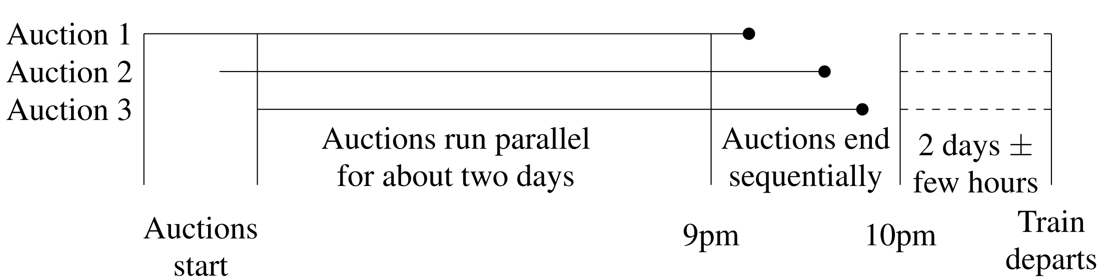


FIGURE 2. Auction screen shot

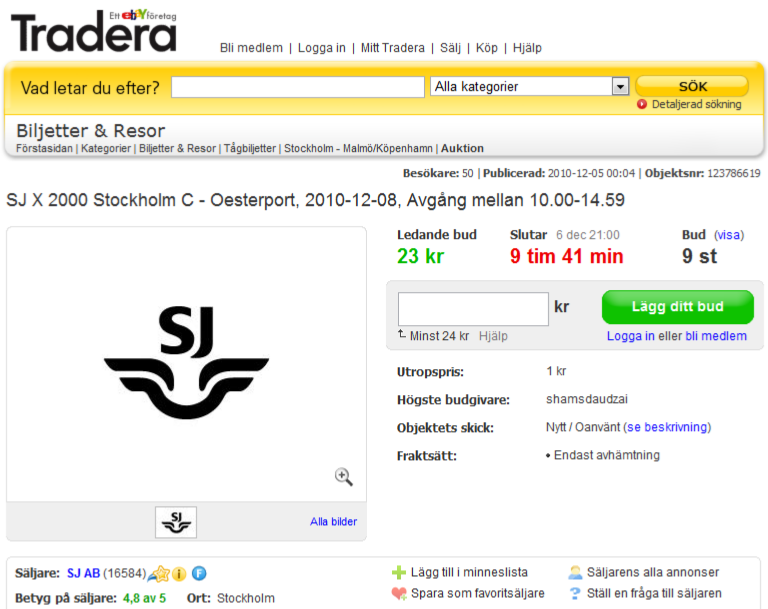



Figure 2 shows the screen faced by a bidder who is about to place a bid in an auction. Each auction within a sequence is executed using an incremental bidding mechanism, which is often referred to as the "proxy" bidding mechanism. In this mechanism, each auction is an ascending price auction, where bidders can choose to actively participate, or they can choose to record a maximum willingness to pay (MWTP). Active bidders bid the smallest amount necessary to become leaders of the auction, and can then raise their bid if a higher bid comes in. The smallest amount necessary is equal to the (current leading bid + an increment).[5] These bidders are often referred to as incremental bidders because they only raise the leading bid by one increment at a time. On the other hand, if a bidder records an MWTP, then the website will place bids in increments (so called " proxy" bids) on the bidder's behalf when new bids are placed by other bidders. The website will do so until another bidder records a bid that is higher than the first bidder's MWTP. As a result, a bidder who wins an auction having placed an MWTP only has to pay the second highest bid plus an increment, or, if the second highest bid is within an increment of the MWTP, her own bid. Thus this auction format is a hybrid between second price and first price auction (see Hickman (2010) for a formal analysis of this mechanism).

To illustrate how proxy bidding works consider the following example. If a bidder records an MWTP, call it $bid_1$, that is higher than the current highest recorded MWTP of some other bidder, call it $bid_2$, then the new leading bid becomes $lead = min\{bid_2 + increment, bid_1\}$. Now, if a third

[5]The increments in these auctions were: 1 Swedish Krona (SEK) if the leading bid is in the interval 1–99 SEK, 5 SEK for 100–249 SEK, 10 SEK for 250–999 SEK, 25 SEK for 1000–2499, 50 SEK for 2500–4999 SEK and 100 SEK for 5000 SEK and up.

bidder records a bid, call it $bid_3$, that is higher than $lead$, but lower (or equal) to $bid_1$. Then the website will keep the bidder who placed $bid_1$ as the leader of the auction, and raise the leading bid on behalf of this bidder to $lead' = min\{bid_3 + increment, bid_1\}$. The bidder who placed $bid_1$ will win the auction if no more bids are placed, and she will pay the price $lead'$.

2.2. **Data Description.** There were 42,007 tickets (auctions) grouped into 7,202 sequences that were conducted between November 10, 2010 and June 6, 2011. In the reduced form analysis, we consider sequences of 15 tickets since this includes over ninety five percent of the data.[6] There are 35,157 tickets grouped into 6,874 sequences with 15 tickets or less. The ticket information contained in the data includes departure-destination pairs, departure time and date, and the type of train (i.e. fast train or regional train).[7]

The data set contains all bids made in the auctions for train tickets. A bidder who engaged in incremental bidding could have recorded multiple bids in the same auction. We therefore treat the highest bid that a bidder records in an auction as the bidder's revealed bidding strategy. A further complication is that we cannot treat most winning bids as revealed strategies. That is due to proxy bidding and the fact that most auctions are settled using the second price rule (see table 1). Thus, we treat the winning bids as prices only, and treat non-winning bids as revealed bidding strategies. In addition to the bids, the data also contains bidder identifiers and the date, hour, and minute that the bid was placed.

In the following sections we describe some features of the data. In many cases we divide the data into two categories. One is the set of all bids that were submitted. The other is a subset of bids that were placed in an auction *after* the previous round of the sequence ended. These bids capture the sequential nature of the auctions.

2.2.1. *Summary statistics.* As we discussed before, the bidding mechanism implies that winners of an auction on occasion have to pay their own bid. This happens about 9 percent of the time (see table (1)). If one considers auctions where the winning price was greater than 99 SEK, the share increases to 14 percent.

Another feature of the proxy bidding mechanism is that the increment that a bidder must raise the current leading bid by increases as the current leading bid increases. This has implications for the available bidding range to a bidder. As can be seen in table 2, at the time of placing their highest bid, 72 percent of bidders were free to place any bid as long as it was higher than the

[6]In the entire data set there were sequences of up to 30 tickets. Even when the the entire data set is considered the declining price pattern can still be observed.

[7]See Andersson et al. (2012) and Andersson and Andersson (2017) for additional information about the data.

TABLE 1. Pricing Rule

| | All Auctions | | Price> 99 | |
|---|---|---|---|---|
| | Number | Percent | Number | Percent |
| price=$2^{nd}$ highest bid + increment | 31,941 | 91 | 14,995 | 86 |
| price=highest bid | 3,216 | 9 | 2,364 | 14 |
| Total | 35,157 | 100 | 17,379 | 100 |

current leading bid. 20 percent of bidders were restricted to place a bid that surpassed the current leading bid by at least 5 SEK.

TABLE 2. Highest Bid by Increment Category

| Increment | Number | Percent |
|---|---|---|
| 1 SEK | 107,254 | 72 |
| 5 SEK | 29,926 | 20 |
| 10 SEK | 12,226 | 8 |
| 25 SEK | 6 | 0 |
| missing | 72 | 0 |
| Total | 149,484 | 100 |

TABLE 3. Summary Statistics: Auctions

| | Auctions=35157 | |
|---|---|---|
| | Mean | SD |
| Price | 131.90 | 128.43 |
| Bidders in auction | 4.25 | 2.31 |
| Bids | 11.02 | 8.79 |

On average 4 bidders participated in each auction and 11 in a sequence. The average sequence consisted of 5 tickets. Almost all tickets were for either intercity or fast train trips. There was a fairly even distribution of the weekday of train departure. The auctions were conducted in close succession with less than ten minutes between closing times. This suggests that there was no discounting for tickets sold in later rounds of a sequence. The average price of tickets was 131.90 SEK, not conditioning on train types. The average price decline between two auctions within a sequence was about 9%.

Turning to all the bids, the average bid in the auctions was about 90.69 SEK. Importantly this was below the price at which the bid increments changed. In addition, on average, bidders led the auction only once. That is bidders typically did not lead the auction multiple times which suggests that bidders were bidding as if they were in a sealed bid mechanism.

2.3. **Declining Prices.** To formally document that prices decline we estimate equation (1):

$$ln(price_{k,j}) = \beta_0 + \beta_1 k + \beta_2 k^2 + \beta_3 x_k + \theta_j + \varepsilon_{k,j} \quad (1)$$

where $price_{k,j}$ is the price of auction $k$ in sequence $j$, $x_k$ captures auction-specific covariates, and $\theta_j$ is a sequence fixed effect.

Figure 3 and Table 6 confirm the declining price path. The decline is stronger early on, and the price path evens out towards the end in sequences of many tickets. Our preferred specification for

TABLE 4. Summary Statistics: Sequences

| | Sequences=6874 | |
|---|---|---|
| | Mean | SD |
| $\ln(\text{Price}_k / \text{Price}_{k-1})$ | -0.09 | 1.05 |
| Bidders in sequence | 10.75 | 6.48 |
| Tickets in sequence | 5.11 | 2.82 |
| Traintype: Intercity | 0.48 | 0.50 |
| Traintype: Regional | 0.01 | 0.08 |
| Traintype: X2000 | 0.52 | 0.50 |
| Sunday | 0.07 | 0.26 |
| Monday | 0.15 | 0.36 |
| Tuesday | 0.16 | 0.36 |
| Wednesday | 0.18 | 0.38 |
| Thursday | 0.16 | 0.37 |
| Friday | 0.14 | 0.34 |
| Saturday | 0.14 | 0.35 |
| Minutes between auctions | 8.65 | 5.96 |

TABLE 5. Summary Statistics: Bids

| | All Bids Bids=387579 | | Highest Bids Bids=149484 | | Non-Winning Bids Bids=114327 | |
|---|---|---|---|---|---|---|
| | Mean | SD | Mean | SD | Mean | SD |
| Bid | 90.69 | 100.76 | 106.20 | 114.43 | 98.30 | 108.56 |
| Share incremental bid | 0.28 | 0.45 | 0.24 | 0.43 | 0.23 | 0.42 |
| Share auction elapsed | 89.36 | 22.43 | 87.58 | 24.51 | 85.94 | 25.74 |
| Share leading bid | 0.42 | 0.49 | 0.70 | 0.46 | 0.61 | 0.49 |
| Bids by bidder | 2.59 | 3.02 | . | . | . | |
| Leading bids by bidder | 1.08 | 0.95 | . | . | . | |
| Share ever leading | 0.78 | 0.41 | . | . | . | |
| Share leading and returning | 0.25 | 0.43 | . | . | . | |

documenting declining prices is column (2) of table 6 where the sequence fixed effects are included to deal with unobserved sequence heterogeneity. The estimates in column (3), which uses sequence observable characteristics as controls instead of sequence fixed effects, are not statistically different from the estimates in column (2). This suggests that auction heterogeneity is captured well in the variables observed in the data.[8] In Appendix D we show that the documented decline in prices is robust to alternative specifications as well as holding sequence length fixed.

[8]An important assumption in our estimation in section 4 is that auction heterogeneity is observed. The fact that the estimates in column 2 and 3 of table 6 are very similar is consistent with this assumption.

FIGURE 3. Visual - Declining Price

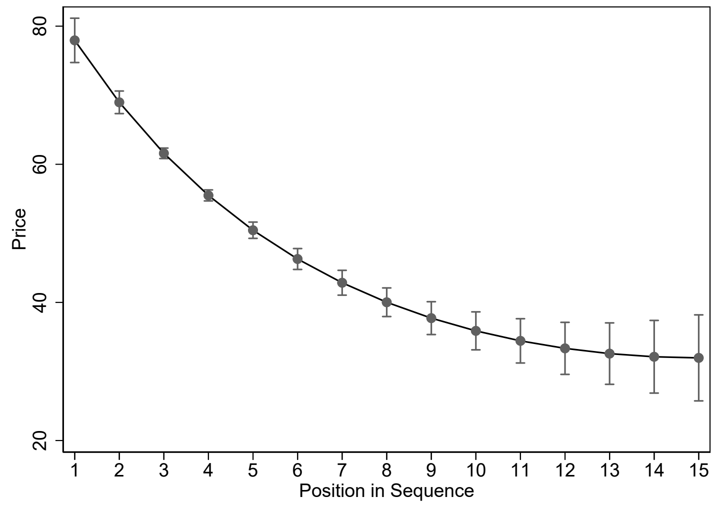



TABLE 6. Declining Price - $K \leq 15$

| VARIABLES | (1) ln Price | (2) ln Price | (3) ln Price |
|---|---|---|---|
| k | -0.141*** | -0.136*** | -0.170*** |
| | (0.00816) | (0.0124) | (0.0160) |
| k squared | 0.00546*** | 0.00451*** | 0.00560*** |
| | (0.000784) | (0.000939) | (0.00129) |
| Observations | 34,982 | 34,982 | 34,955 |
| Sequence FE | YES | YES | |
| Sequence Controls | | | YES |
| Auction Controls | | YES | YES |

Note: *** p<0.01, ** p<0.05, * p<0.1. Robust standard errors clustered at the sequence level. In column (3), $\theta_j$ is replaced by sequence specific observable variables $x_j$. The variables in $x_j$ are the same variables that we use to homogenize bids in section 4.

2.4. **Bidder Behavior.** The standard sequential auction model studied in Milgrom and Weber (2000) predicts constant prices in the i.p.v. setting.[9] Intuitively this occurs due to a ‘no-arbitrage’ condition which equates the demand and supply effect on prices as the auctions proceed. Another aspect of equilibrium bidding in the standard auction model is that of *history independence*: prices in previous rounds do not affect a bidders bid in a round. Under i.p.v. a bidder will bid more aggressively as the rounds progress due to shrinking relative supply but will not be affected by previous round prices, as prices only scale a bidders current round payoff.

[9]In an affiliated values setting prices increase. See pages 219-220 of Krishna (2009) for a full and concise treatment.

The price prediction of the standard model is clearly refuted by the data. As we discussed in the introduction, risk aversion, loss aversion and ambiguity aversion can theoretically explain such a price trend. However, the three models have different predictions regarding history dependence of bids on previous round prices in an i.p.v. framework. Specifically, risk aversion predicts history independence for the same reason as the standard model. Loss aversion predicts a negative relationship due to a 'discouragement effect'. And finally, with ambiguity aversion bidding can be history-dependent (positive or negative), since bidders use their worst-case beliefs *conditional* on previous round prices. A fuller description of the three models can be found in Appendix A.

We now document the relationship between various observable variables and bidding within a sequence and make a case for ambiguity aversion as a plausible explanation for declining prices, over other explanations. Some of the variables we consider include the number of bidders (demand), the number of items left (supply), and the price in the previous auction (history). The main comparative static on equilibrium bidding that distinguishes the preference based explanations from each others is the relationship between bids in a round and the price-history in the sequence. We aim to evaluate that relationship and estimate equation (2) to evaluate the relationships between bidding and observables:

$$\ln(bid_{i,k,j}) = \beta_0 + \beta_1 \ln(price_{k-1,j}) + \beta_2 n_{k,j} + \beta_3 (K_j - k) + \theta_{i,j} + \varepsilon_{i,k,j}. \quad (2)$$

Here $bid_{i,k,j}$ is the bid placed by bidder $i$ in auction $k$ of sequence $j$. $n_{k,j}$ is the number of bidders in auction $k$ of sequence $j$, and $K_j$ is the number of tickets for sale in sequence $j$. $\theta_{i,j}$ captures bidder-sequence fixed effects. This implies that identification comes from bidders who have participated in at least two of the auctions in sequence $j$. Furthermore, this implies that valuation for the object is held constant, given that we have included bidder specific fixed effects, when evaluating how the covariates of interest impact bidding behavior.

The main parameter of interest is $\beta_1$, which captures the potential history dependence of round $k$ bids on round $k-1$ prices. A complication is that auctions are not purely sequential but run parallel before ending sequentially. Hence, bids made while auctions run parallel are made when the history is unknown. For this reason, equation (2) will be estimated using bids made after auction $k-1$ has ended.

Before stating the results, it is important to note that it is possible that the effect of the number of bidders, i.e. the demand effect, is not identified when we include a sequence fixed effect. The reason being that, in theory, it is the total number of bidders in the sequence that matters. However, the auction mechanism did not require a one-to-one mapping between the number of items left and the total number of bidders in the sequence. Bidders were free to participate in whatever auction they wanted. In section 3 we document that the average number of bidders in a sequence decline throughout, but it is not as clear of a picture as a theory with single unit demand would predict.[10] An added benefit from this though is that we do have proper within sequence variation in the number of bidders that is distinguishable from the number of items left. It is not clear however that the best measure of the demand effect is the *total number of bidders in an auction*. The reason is that such a variable, which is measured at closing of the auction, is affected by the bidding itself. The proxy bidding mechanism is an ascending price auction, with the implication being that a potential bidder with a low valuation who finds the auction late might discover that the current leading bid is above what that bidder is willing to bid. This is especially true if bidders with high valuations have arrived early in the auction and thus driven up the leading bid.[11] To get around this challenge, we are using the *number of bidders at the time of the bid* in addition to the total number of bidders participating in the auction. The number of bidders who have participated at the time of the bid might also better reflect the information that a bidder has when placing the bid.[12]

TABLE 7. Bidder Behavior - $K \leq 15$

| VARIABLES | (1) ln bid | (2) ln bid |
|---|---|---|
| ln price(k-1) | 0.0759*** | 0.0487*** |
| | (0.0143) | (0.0101) |
| Bidders when Bidding | 0.0600*** | 0.0671*** |
| | (0.00416) | (0.00636) |
| (K-k) | -0.0776*** | -0.0422*** |
| | (0.00377) | (0.00266) |
| Bidders in current auction | | -0.0636*** |
| | | (0.00585) |
| ln Current Lead when Bidding | | 0.379*** |
| | | (0.0121) |
| Observations | 25,963 | 25,947 |
| BidderXSequence FE | YES | YES |
| BidderXRound Controls | YES | YES |

Note: *** p<0.01, ** p<0.05, * p<0.1. Robust standard errors that are clustered at bidder X sequence level. *Current Lead when Bidding* is the bid that must be surpassed when the bidder places the bid.

Table 7 reports the results from estimating equation (2). All estimates indicate that bids are history dependent. It can further be rejected the coefficient on $\ln price(k-1)$ is less than or equal to 0 ($p < 0.01$ when testing $H_0 : \beta_{\text{ln price(k-1)}} \leq 0$). Among the preference based explanations that

[10] With single unit demand and constant participation, the number of bidders and units should always decline by one after each round.

[11] In other words, this is a reverse causality issue: it can be argued that high bids leads to few observed bidders in an auction. In an effort to control for this, we also include the *current leading bid* when we use *total number of bidders at closing*.

[12] A regression without any bidder number variable returns similar estimates.

can account for the declining price pattern, only the ambiguity aversion model can account for the documented positive relationship between bids in a round and the price in the previous round in an i.p.v. paradigm. We show this is indeed the case in the theoretical model in the next section. This evidence suggests that neither risk nor loss aversion can *alone* account for the declining price anomaly as these preferences structures predict zero and negative history dependence respectively.

While being positive, the magnitude of the history dependence is small. A one percent increase in the price in round $k-1$ corresponds to 0.04-0.08 percent increase in the bids in round $k$. It also appears as if bidders bid more aggressively if there are many bidders present in the auction when they make their bid. Bidders also bid more aggressive when there are few tickets left. This captures the supply effect.

While the number of bidders at the time of bidding has a positive effect on bids, the number of bidders at the end of an auction seems to be negatively related to the bid. As we mentioned before, the source of this relationship could be the open bidding nature of the auction as late arriving low valued bidder do not submit bids if the current leading bid at the time is too high.

## 3. Model and Analysis

3.1. **Setting Up a Theoretical Model.** In this section we study a model of sequential auctions with ambiguity, adapted to the data on hand, and establish theoretical results that will aide our structural analysis. All proofs can be found in Appendix B. Here we first discuss the modeling assumption in light of the actual auction mechanism in the data.

3.1.1. *Second price auctions.* As we discussed before, the auction mechanism is a hybrid of first and second price auctions. However for a few reasons these auctions were closer to SPA. First, over 90 percent of the time winners don't pay their own bids as can be seen in table 1. Second, the bid increment is fairly small (1 SEK) and bidders are only allowed to bid in whole numbers. This makes the payment close to the bid of the second highest bidder. Over 70 percent of the highest bids by a bidder in an auction were made when the bid increment was still 1 SEK. (Table 2).

3.1.2. *Sealed bids.* An important aspect of the auction mechanism is its open nature. That is, bidders can observe the current and past bids in any given auction within a sequence. This calls to question whether bidders are bidding incrementally and learning from current bids within an auction. In an i.p.v. setting without ambiguity, of course sealed bid second price auctions are strategically equivalent to ascending price or English auctions. However, such equivalence does not hold in the presence of ambiguity as bidders can be learning and updating their worst-case beliefs within a round ( cf. Auster and Kellner (2020)). This can complicate matters as one has to account for updating of worst-case beliefs not only across rounds but also within a round. However, based on the data, it appears as if bidders are not bidding incrementally but rather as if they were in a sealed bid auction.

While on average bidders were submitting two to three bids per auction, on average bidders submitted a leading bids only once as can seen from table 5. Furthermore, as can be seen in table 8, about 80 percent of the time a bidder led the auction only once. This suggests that most of the time a bidder submitted a bid which was the maximum amount she was willing to bid in an auction. Once the price moved beyond her bid, she did not bid again in the auction. Thus, bidders seemed to be bidding as if they were in a sealed bid auction. One reason for this could be that all though the auction mechanism allows had an open nature, bidders can also record their highest bid and the mechanism will increase the price incrementally. This could end up saving the bidders time as they do not have to be actively involved.

TABLE 8. Leading Bids

| Leading bids by a bidder | Frequency | Percent | Cumulative |
|---|---|---|---|
| 0 | 32,241 | 21.57 | 21.57 |
| 1 | 88,972 | 59.52 | 81.09 |
| 2 | 18,567 | 12.42 | 93.51 |
| 3 | 5,908 | 3.95 | 97.46 |
| 4 | 2,160 | 1.44 | 98.91 |
| $\geq 5$ | 1,636 | 1.09 | 100 |

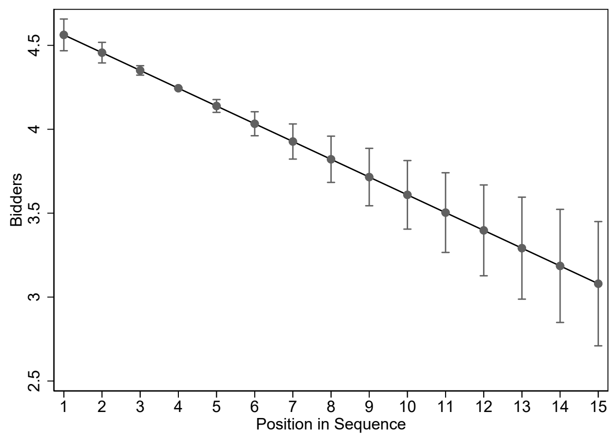
Note: The table reports the frequency of the counts that a given bidder is submitting a bid that at the point of submission is the leading bid.

3.1.3. *Single-unit demand.* A common theoretical assumption to guarantee the existence of an equilibrium in sequential auctions is single-unit demand. We believe this to be a suitable assumption for the train auctions for two reasons. First, since the goods were train tickets on a specific route on a specific day which can only be used by one person, a bidder would not have any use for more tickets unless buying for other passengers. Second, in the data, the magnitude of multiple wins for the same bidder is relatively small compared to single-unit wins. For all sequences with less than 15 tickets, 87% of tickets were sold to bidders who only won one unit.

3.1.4. *Fixed number of bidders.* In many empirical settings, the number of bidders is endogenous to the environment and mechanism. In a sequential auction setting modeling the number of bidders as endogenous or uncertain takes on an additional layer of complication as entry and exit can occur not only before a sequence but also between auction rounds. While allowing the number of bidders to be flexible may be relevant in many settings, for a few reasons that we state below we believe that it is reasonable to fix the number of bidders in the framework we study.

While the auctions ran for about two days, they end sequentially within a span of an hour, two days before the departure date. Such a mechanism does not allow for a lot of time for auction entry between rounds: if bidders intend on competing for a ticket they are likely to be participating in or be aware of all rounds. Further evidence for lack of entry is provided in figure 4. This figure is a result of regressing the number of bidders in an auction round on the position of the auction in the sequence. Since the average number of bidders declines, it suggests that no entry is taking place between rounds. Furthermore, since the slope of the decline is less than one, combined with our argument of single unit demand, it suggests that there is little or no exit between rounds.

FIGURE 4. Average number of bidders




Note: Since we combine across all sequences, the absolute number of bidders is not indicative of the number of bidders in a round conditional on sequence size.

While we have argued for lack for entry or exit of bidders between rounds, the number of bidders at the start of a sequence can still be endogenous. In a private value paradigm with zero entry fees (it was free for bidders to participate in the auctions) the only reason for a bidder to not participate would be that their valuation is below the reserve price which in the data set was quite small. Thus the likelihood of such an event is quite small. Since we can observe all bidders and their respective bids, the only remaining possibility is bidders who entered a sequence but never submitted a bid in any round which, while possible, may not be in the bidder's best interest.

3.1.5. *No reserve price.* The reserve price in each auction was set at 1 SEK. No bidder who values a ticket below this amount would participate in the auction. Thus there is no loss in generality in assuming that the lowest valuation in the support of values was equal to the reserve price.

3.1.6. *Continuous bidding space.* Bidders were only allowed to submit bids using a discrete bid space: bidders had to submit bids in multiples of 1 SEK. Solving a model with a discrete action space leads to bidders playing mixed strategies, which are not easy to handle in empirical work. As is the convention in structural estimation of auctions, we assume that the bidding space is a continuum. This allows the use of differential equations to solve for an equilibrium and also aids our identification strategy.

3.1.7. *Independent private values.* When a bidder won a train ticket in an auction the ticket information was sent to their telephones and could only be used by the owner of phone. As such it seems reasonable that the ticket was for personal consumption. Furthermore, most ticket winners won only one ticket. These observations point towards an i.p.v. paradigm. In addition since in our structural analysis we 'wash out' all trip specific fixed effects, it seems prudent to assume an i.p.v. in the theoretical model.

3.2. **Model.** $K \geq 2$ units of a good (identical tickets) are sold, one in each round, sequentially in sealed-bid second-price auctions with no reserve price. There are $N \geq K+1$ bidders. Each bidder $i$ has a unit demand and values the good at $v_i \in [\underline{v}, \overline{v}]$. Bidders draw their values from a common distribution function $F$ with a strictly positive density function $f$. The distribution $F$ belongs to a compact and convex set of atom-less and piecewise smooth distributions $\Delta$. Bidders do not know the distribution $F$; instead, their prior beliefs are described by the set $\Delta$ following the multiple-priors approach of Gilboa and Schmeidler (1989). Accordingly, we assume bidders have *maxmin expected utility* (MEU): in each round they maximize the minimum expected utility over the set of priors $\Delta$, conditional on the available information. Finally, we assume that bidders follow prior-by-prior Bayesian updating.

The timing of the game is as follows. In each round $k = 1, \ldots, K$, bidders submit sealed bids. The bidder who submits the highest bid wins the unit and pays the second highest bid. The winning bidder leaves the auction and the winning bid is announced.[13] The remaining bidders compete in the next round using the same procedure until all units are sold. Let $p_k$ be the winning bid in round $k$. A public history in round $k$ is a sequence of winning bids $\tilde{p}_{k-1} = (p_1, \ldots, p_{k-1})$. Finally, ties are broken via a fair coin-flip.[14]

We impose the following regularity condition on the set of priors, which allows us to obtain equilibria in closed form. For any $y \in [\underline{v}, \overline{v}]$, conditional on the event that a bidder's value is less than $y$, the set of posterior distributions, denote by $\Delta_y$, is

$$\Delta_y = \left\{ F_y(\cdot) = \frac{F(\cdot)}{F(y)} : [\underline{v}, y] \to [0,1],\ \forall F \in \Delta \right\}. \tag{3}$$

In addition, we need the following definitions that are slight variations of those in Topkis (2011). Let $\geq_{FOSD}$ be the first-order stochastic dominance partial order on $\Delta$, i.e., $F_1 \geq_{FOSD} F_2$ if and only

[13] Announcing the winning price in sSPA alters the game as the bidders know the highest valuation amongst the remaining bidders in the next round, making the game asymmetric and possibly precluding the existence of an equilibrium. Thus this is a common assumption in the study of sSPA. See Milgrom and Weber (2000). However, in our reduced form exercise we do not observe the winning bid and thus have to use prices. Despite this difference, note that history dependence between a highest winning and bids in the next round also implies history dependence between prices and bids in the next round due to correlation between the highest and second highest bids.

[14] The tie-breaking rule will be irrelevant since we consider monotone equilibria.

if $F_1(x) \leq F_2(x)$ for all $x \in [\underline{v}, \overline{v}]$. For any $F_1, F_2 \in \Delta$, let $F_1 \vee F_2$ be the *join* of $F_1$ and $F_2$.[15] Note that $F_1 \vee F_2 \geq_{FOSD} F_i$ for $i = 1,2$. If the join of every pair of distributions in $\Delta$ belongs to $\Delta$, then $\Delta$ is a *semi-lattice*. A semi-lattice $\Delta$ is *complete* if every nonempty subset of $\Delta$ has a join in $\Delta$.

**Assumption 3.1.** For each $y \in [\underline{v}, \overline{v}], \Delta_y$ is a complete join semi-lattice.

Let $\bar{F} \in \Delta$ be such that $\bar{F} \geq_{FOSD} F$ for all $F \in \Delta$. Furthermore, let $\bar{F}(\cdot|y) \in \Delta_y$ be such that $\bar{F}(\cdot|y) \geq_{FOSD} F(\cdot|y)$ for all $F \in \Delta$.[16] Finally, for some results, we will need the following stochastic order. Let $\geq_{rh}$ be the reverse-hazard rate stochastic dominance relation. From Shaked and Shanthikumar (2007) 1.B.41, $F_1 \geq_{rh} F_2$ if and only if $F_1(x)F_2(y) \leq F_1(y)F_2(x)$ for all $x \leq y$. Note that if $F_1 \geq_{rh} F_2$ then $F_1 \geq_{FOSD} F_2$.

Let $b_{i,k}$ denote bidder $i$'s bid in $k$-th auction. If bidder $i$ wins auction $k$ then $b_{i,k+l} = 0$ for all $l = 1, \ldots K-k$. A strategy for a player $i$, $\beta_i = \{\beta_{i,1}, \ldots, \beta_{i,K}\}$, is a sequence of bid functions, where $\beta_{i,k}(v_i, \tilde{p}_{k-1})$ is bidder $i$'s bid in auction $k$ given the history of winning bids. A strategy $\beta_i$ is *monotone* if for each $k = 1, \ldots, K$, $\beta_{i,k}$ is increasing in $v_i$ for all $\tilde{p}_{k-1}$. Denote $\boldsymbol{\beta}_k = \{\beta_{i,k}\}_{i=1}^N$ and $\boldsymbol{\beta} = \{\beta_k\}_{k=1}^K$. A strategy profile $\boldsymbol{\beta}$ is symmetric if $\beta_i = \beta_j \triangleq \beta$, for all $i, j = 1, \ldots, N$. As is common in auctions, we focus on monotone and symmetric strategies and drop the subscript with respect to bidders. To slightly abuse notations, we use $\beta$ to denote a monotone and symmetric strategy profile.

3.3. **Payoffs and Equilibrium Concept.** Observe that, since strategies are monotone, a bidder with the $k$-th highest value will win the $k$-th round. Therefore, the previous winning bids, $p_1, \ldots, p_{k-1}$, can be mapped back to the realized values, $y_1 \geq \ldots \geq y_{k-1}$, of the winners before round $k$. By induction backwards, the bidding functions can be rewritten as $\beta_k(v, y_{k-1})$, as in round $k$ a bidder believes that all other remaining bidders' values are bounded above by $y_{k-1}$.

It is well known that in dynamic decision problems under ambiguity optimal (sequentially rational) choices made by agents can violate dynamic consistency (Siniscalchi (2011)). In order to accommodate possible time inconsistency that can occur for MEU maximizing bidders, we follow the multiple-selves approach introduced by Strotz (1955) and use *consistent planning equilibrium* as our solution concept which we define below.[17]

In a SPA a bidders expected payment conditional on winning is the expected next highest bid. In addition a bidder's *continuation payoff* in any round of a sSPA with ambiguity is her next self's expected payoff (evaluated with her current self's worst-case belief). Thus, formally, a bidder's

[15] The join $F_1 \vee F_2$ is the point-wise lower envelope of $F_1$ and $F_2$.

[16] See example 4.3 in GL to see cases where $\bar{F}$ and $\bar{F}(\cdot|y)$ are different. That is $\bar{F}(\cdot|y) \neq \bar{F}(\cdot)/\bar{F}(y)$ for some y.

[17] Alternatively we can impose conditions on the set of priors to ensure dynamic consistency which we do in the next section to ensure identification.

payoff is given by

$$\Pi_K(v,z,y_{K-1}) = \min_{F\in\Delta}\left(\frac{F(z)}{F(y_{K-1})}\right)^{N-K}\int_{\underline{v}}^{z}(v-\beta_K(x,y_{K-1}))\,d\left(\frac{F(x)}{F(z)}\right)^{N-K},$$

and, for $k=1,\dots,K-1$,

$$\begin{aligned}\Pi_k(v,z,y_{k-1}) = {} & \min_{F\in\Delta}\left(\frac{F(z)}{F(y_{k-1})}\right)^{N-k}\int_{\underline{v}}^{z}(v-\beta_k(x,y_{k-1}))\,d\left(\frac{F(x)}{F(z)}\right)^{N-k} \\ & + \int_{z}^{y_{k-1}}\Gamma_{k+1}(v,x,F)\,d\left(\frac{F(x)}{F(y_{k-1})}\right)^{N-k}\end{aligned} \tag{4}$$

where the first term is the bidder's current round payoff and the second term is her continuation payoff if she bids as if her valuation was $z$ in the current round. In order to define the equilibrium we also define a term $\Gamma_{k+1}$ recursively as

$$\Gamma_{k+1}(v,x,F) = \left(\frac{F(v)}{F(x)}\right)^{N-k-1}\int_{\underline{v}}^{v}(v-\beta_{k+1}(z,v))\,d\left(\frac{F(z)}{F(v)}\right)^{N-k-1} + \int_{v}^{x}\Gamma_{k+2}(v,w,F)\,d\left(\frac{F(w)}{F(x)}\right)^{N-k-1} \tag{5}$$

and

$$\Gamma_K(v,x,F) = \left(\frac{F(v)}{F(x)}\right)^{N-K}\int_{\underline{v}}^{v}(v-\beta_K(z,x))\,d\left(\frac{F(z)}{F(v)}\right)^{N-K}.$$

In words, $\Gamma_{k+1}(v,x,F)$ is the payoff in round $k+1$ to a bidder with value $v$ who bids according to the strategy $\boldsymbol{\beta}$ in $k+1$ and all future rounds, given the value of round $k$'s winner, $x$, evaluated using some belief $F$. Importantly, the belief $F$ also enters the bidder's continuation payoff. Then we have the following definition.

**Definition 3.2.** A strategy profile $\beta = (\beta_k)_{k=1}^{K}$ is a *consistent planning equilibrium* if for each $k=1,\dots,K$, $v$, and $y_{k-1}$, we have

$$v \in \arg\max_{z}\Pi_k(v,z,y_{k-1}).$$

3.4. **Equilibrium and Implications.** With this definition in place, we now state our theoretical result on the equilibrium existence and uniqueness.

**Proposition 3.3.** *In the unique symmetric equilibrium bidders follow the strategy*

$$\beta_k(v,y_{k-1}) = \int_{\underline{v}}^{v}\beta_{k+1}(x,v)\,d\frac{\bar{F}(x|\,y_{k-1})^{N-k-1}}{\bar{F}(v|\,y_{k-1})^{N-k-1}},$$

*for* $k = 1,\ldots,K-1$, *and*

$$\beta_K(v, y_{K-1}) = v.$$

In the final round of a sSPA, it is weakly dominant for a bidder to bid her valuation. This is independent of what belief a bidder has about the valuations of others. However, as can be seen in equation (4), in a general round $k$, a bidder's optimal bid will depend on what belief she uses to calculate her payoff. From equation (4) we can see that a simple point-wise minimization to find the minimizer can not be done since we do not know how the payoff function in the next round behaves with respect to current round prices. Thus, the crucial step in the proof is establishing a monotonicity property of the function $\Gamma_{k+1}(v, x, F)$ in $x$, which we do recursively. That allows us to pin down a bidder's belief in a round. It turns out that $\Gamma_{k+1}(v, x, F)$ is weakly decreasing in $x$ and is less than $v - \beta_k(v, y_{k-1})$ for $x \geq v$. As a result, the integrand in the objective function of the minimization problem is monotone decreasing, which implies that the worst-case belief–the minimizer–is the lower envelope of the set of conditional distributions. This also explains the closed-form of the equilibrium strategies.

To understand the optimal bidding function, note that at the margin, a bidder with valuation $v$ is calculating her optimal bid conditional on the event that she is the highest valued bidder in the current round (first order condition in round $k$). Now, suppose the bidder contemplates bidding a little less. This deviation would only matter if there was another bidder with a valuation equal to hers. For this deviation to not be profitable, her current round payoff in this event must equal her next round payoff. That is her current round payment $\beta_k(v, y_{k-1})$ (if she won the second highest bid would be approximately equal to her bid since there is another bidder with valuation $v$) must equal what she would pay in expectation in the next round, the expected second highest bid amongst bidders with valuations lower than hers. Importantly the expectation is taken with respect to the bidder's worst case belief in the current round. From the closed form of the equilibrium strategies we can see that bidding in all rounds except the final round can be history-dependent which is a consequence of the dependence of worst-case conditional beliefs on the price history.

3.4.1. *Declining prices.* With regard to prices note that in sSPAs the price in a round is the second highest bid. Thus, in equilibrium the price in round $k$ will be the bid placed by the bidder with the second highest valuation in that round. That is,

$$p_k = \beta_k(y_{k+1}, y_{k-1}).$$

Bidders in the next round observe the winning bid and hence the valuation of the winner, $y_k$. Thus, the expected price in the next round calculated at some $F \in \Delta$ is given by the expected bid of the second highest valued bidder out of $N-k$ bidders conditional on the value being less than $y_{k-1}$.

$$\mathbb{E}\left[P_{k+1}|p_k\right] = \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x, y_k) dF_2^{(N-k)}(x|x \leq y_{k+1}),$$

where $F_2^{(N-k)}$ is the distribution of the second highest value out of $N-k$ draws. Our next result states a simple condition under which prices are declining in sSPAs.

**Proposition 3.4.** *If for any $y \in [\underline{v}, \overline{v}], \bar{F}(\cdot|y) \geq_{rh} F(\cdot)/F(y)$ then $p_{K-1} \geq \mathbb{E}_F\left[P_K|p_{K-1}\right]$. If in addition $\bar{F}(\cdot|y) \geq_{rh} \bar{F}(\cdot|y')/\bar{F}(y|y')$ for $y \leq y'$ then $p_k \geq \mathbb{E}_F\left[P_{k+1}|p_k\right]$.*

It is important to note that the above condition is only a sufficient condition for declining prices. While in general, as GL show, prices in sequential auctions with ambiguity can have more complicated trends based on the set of priors, declining prices can be theoretically observed under simple conditions as the one stated above. Furthermore, in the next section we show that we can also observe declining prices under the assumption of dynamic consistency.

3.4.2. *Dynamic consistency.* From the bidding functions, we can observe that in all but the last round, a bid depends on a bidder's worst-case belief. Furthermore, worst-case beliefs depends on previous prices and thus can change from round to round. This means that in a $K$ round auction there can be $K+1$ objects to recover: $K$ worst-case beliefs and one true distribution of values. Furthermore, the worst-case beliefs could be different for each previous round price observed in the data leading to non-identification of beliefs without further restrictions on the set of priors. To circumvent this issue GL showed that if the set of priors is *rectangular* then the worst-case belief is invariant to conditioning. Rectangularity essentially implies that the set of priors is closed under iterated expectations. This has a few implications. One, under rectangularity of priors Epstein and Schneider (2003) showed that the bidder's preferences are dynamically consistent. Two, it simplifies the equilibrium bidding in sSPAs and allows straightforward comparisons with other auction formats. And finally, it allows us to identify bidders' beliefs as well as the true data generating process from data. Thus, we can restate the equilibrium as follows.

**Corollary 3.5.** *If $\Delta$ is rectangular then in the unique symmetric equilibrium bidders follow the strategy*

$$\beta_k(v) = \int_{\underline{v}}^{v} \beta_{k+1}(x) d\left(\frac{\bar{F}(x)}{\bar{F}(v)}\right)^{N-k-1}, \text{ for } k \leq K-1, \text{ and } \beta_K(v) = v, \text{ where}$$

$$\bar{F}(x) = \min_{F \in \Delta} F(x).$$

*Furthermore, in this case $p_k \geq \mathbb{E}_F[P_{k+1}|p_k]$, that is prices are declining.*

In Lemma 5.1 of GL the authors show that under rectangularity the lower envelope of set of priors is invariant to conditioning (right truncation). That is $\bar{F}(\cdot|y)$ is equal to $\bar{F}(\cdot)/\bar{F}(y)$. Substituting in the equilibrium bid functions the bid functions in the above Corollary are clear. Furthermore, note that under dynamic consistency, we have

$$\begin{aligned} p_k &= \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x) d\frac{\bar{F}(x)^{N-k-1}}{\bar{F}(y_{k+1})^{N-k-1}} \geq \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x) d\frac{F(x)^{N-k-1}}{F(y_{k+1})^{N-k-1}} \\ &\geq \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x) dF_2^{(N-k)}(x|x \leq y_{k+1}) = E_F\left[P_{k+1}|p_k\right], \end{aligned}$$

where the first inequality is strict when $F \neq \bar{F}$, since the conditional distribution $\bar{F}(\cdot)/\bar{F}(y_k)$ first-order stochastically dominates $F(\cdot)/F(y_k)$ and $\beta_{k+1}$ is increasing. The second inequality follows from simple algebraic manipulation of the distributions.

GL showed that the lower envelope of the set of distributions is unique. Thus, from the above Corollary, we know that the empirical distributions of bids depend on two distributions: the true distribution of valuations, $F$, and bidders' worst case beliefs, $\bar{F}$.

3.5. **Revenue Comparisons.** The following results can found in Bougt et al. (2024). Now we compare the resulting revenues from two alternative auction formats, sFPAs and uniform price auction (UPA). This will guide the counter factual analysis that is performed in section 4.2.2. In the absence of ambiguity the *revenue equivalence theorem* states that all three formats will generate the same revenue in expectation. However, this is no longer true in the presence of ambiguity. In fact we will show that under *dynamic consistency*, there is a clear revenue ranking between the three formats.

GL proved the existence and uniqueness of an equilibrium in monotone strategies in sFPAs with ambiguity under assumption 3.1. In addition they proved the following result.

**Proposition 3.6.** *(Proposition 5.2 from GL) If $\Delta$ is rectangular, then the equilibrium in sFPAs is given by*

$$\beta_k^I(v) = \frac{1}{\bar{F}(v)^{N-k}} \int_{\underline{v}}^{v} \beta_{k+1}(x) d\bar{F}(x)^{N-k}, \ \forall k \leq K-1 \text{ and } \beta_K^I(v) = \frac{1}{\bar{F}(v)^{N-K}} \int_{\underline{v}}^{v} x d\bar{F}(x)^{N-K}.$$

From the above and Corollary 3.5 notice that bidding in round $k+1$ of sFPAs is the same as bidding in round $k$ of sSPAs. Thus we have the following.

**Corollary 3.7.** *Suppose $\beta_k^{II}$ is the equilibrium bidding function in round $k$ of sSPAs. Then under rectangularity, for $k < K, \beta_k^{II}(v) = \beta_{k+1}^I(v)$.*

A commonly used static multi-unit auction format is the UPA with the highest losing bid as the selling price. This auction format is the multi-unit analogue of a single-unit SPA and thus comparable to sSPAs. This auction also has a symmetric equilibrium in weakly dominant strategies. Bidders bid truthfully and bid their own valuations since their bid only affects their probability of winning and not the price they pay. Thus the price in equilibrium is the $K+1$-th highest valuation out of $N$ valuations. Then it is straightforward to show that the expected revenue calculated using some $F \in \Delta$ in this auction format is

$$R_F^{up} = K\mathbb{E}_F\left[V_{K+1}^{(N)}\right] = K\mathbb{E}_F\left[V_1^{(N-K)}\right],$$

where $V_l^{(m)}$ be the random variable denoting the $l$-th highest draw out of $m$ draws from some distribution $F$.

Now we compare the revenue across the two sequential auction formats and the UPA. Let $R_F^{format}$ be the expected revenue calculated using $F \in \Delta$ and $format \in \{I, II, up\}$. Since we will always calculate the revenue with respect to some $F$ across formats, from now on we drop the subscript.

The expected revenue in final round of sequential SPA is the second highest valuation of the remaining $N-K+1$ bidders. This is equal to

$$\mathbb{E}_F\left[V_1^{(N-K)}\right].$$

Thus the final round of sequential SPA generates $R^{up}/K$ expected revenue. Furthermore, due to declining prices we know that the expected revenue (in a round) progressively declines in sSPA. Thus sSPAs generate more revenue than UPA.

Under rectangularity we know from Corollary 3.7, $\beta_k^{II}(v) = \beta_{k+1}^{I}(v)$. Let $P_k^I$ and $P_k^{II}$ be the price in round $k$ in sFPAs and sSPAs respectively. Then, since the equilibrium is monotone, for $k < K$ we get,

$$\mathbb{E}_F\left[P_{k+1}^I\right] = \mathbb{E}_F\left[\beta_{k+1}^I\left(V_{k+1}^{(N)}\right)\right] = \mathbb{E}_F\left[\beta_k^{II}\left(V_{k+1}^{(N)}\right)\right] = \mathbb{E}_F\left[P_k^{II}\right].$$

Now, since prices are a supermartingale in sFPAs, as was shown in GL, we know that for $k < K$

$$\mathbb{E}_F\left[P_k^I\right] \geq \mathbb{E}_F\left[P_{k+1}^I\right] = \mathbb{E}_F\left[P_k^{II}\right],$$

and for $k = K$, since prices are a supermartingale in sSPAs we know that

$$\mathbb{E}_F\left[P_K^{II}\right] \leq \mathbb{E}_F\left[P_{K-1}^{II}\right] = \mathbb{E}_F\left[P_K^I\right].$$

Thus the revenue from sFPAs is higher than sSPAs. The above arguments are collected in the following result.

**Proposition 3.8.** *If $\Delta$ is rectangular then $R^I \geq R^{II} \geq R^{up}$ with a strict inequality if $\Delta$ is not a singleton.*

The intuition for the first revenue comparison in the above result is based on declining prices. For the second revenue comparison note the following. In the final round of sSPAs bidders bid their true valuation. Thus there is no difference between the expected revenue in the final round of a sSPA with ambiguity and without. That is pessimism does not affect final round bids. However, this is not true in the final round of sFPAs, as bidders use the lower envelope of the set of conditional distribution to calculate their bid. Thus, the bid in the final round of sFPAs is higher with ambiguity than without. This has two effects. First, it implies that revenue in the final round of sFPAs will be higher than the revenue in the final round of sSPAs. Second, the discrepancy caused by pessimism and the lack of it in the final round bids in sFPAs and sSPAs, respectively, implies that the continuation payoff in round $K-1$, from the bidder's point of view, is lower in sFPAs than sSPAs. Therefore, bidders in round $K-1$ shade their bids more in sSPAs than in sFPAs, leading to lower revenue in sSPAs than sFPAs in round $K-1$. The argument applies to remaining rounds.

## 4. Empirical Estimation

In a seminal paper, GPV showed that empirical distributions of bids in auctions can be mapped back to a distribution of valuations in an i.p.v. paradigm, under the assumption of a common prior. That is their methodology is applicable as long as the true data generating process is the same as the bidder's beliefs. As we have established in the previous section for sSPAs, and GL showed for sFPAs, this agreement between the truth and beliefs may not exist in the case of multiple-priors and ambiguity aversion. This makes the task of estimation arduous. However, GL showed that data from sFPAs where bidders bid in multiple rounds can be used to non-parametrically identify bidders valuations and estimate the true distribution of valuations as well as bidders' beliefs. The idea in that paper was to use first order conditions in multiple rounds to estimate the distribution of values along with the beliefs. In Appendix B.3 we formally show that the identification result from GL can be applied to our setting as well. However, for completeness we show how multi-round bidding in sSPAs can be used to estimate the true distribution of values as well as bidders beliefs.

**Remark 4.1.** As we stated in section 3.4.2, under general conditions identification and estimation in sSPAs with ambiguity may be impossible due to changes in worst-case beliefs. Thus, as we alluded to before, we will estimate the model under the assumption of dynamic consistency. Under this assumption the worst-case beliefs are invariant to prices, and thus bidding will be history independent as stated in Corollary 3.5. However, this is in contradiction to what we observe in the data as we showed in section 2.4. We showed that bidding is history-dependent and indeed this was the reason we argued that ambiguity aversion is the most suitable explanation for the declining prices observed in the auctions. However the history dependence was small, and thus our estimation procedure should not miss much. Furthermore we carry out robustness checks where we simulate the pricing data using our estimates of beliefs and valuation distribution and find it to be close to actual pricing data.

4.1. **Identification Strategy.** Let $G_k(b_k)$ represent the bid distribution in the $k$-th round of a $K$ round sSPAs, and $g_k(b_k)$ the corresponding bid densities. Let $\phi_k = \beta_k^{-1}$ be the inverse bidding strategy. In equilibrium, bidding strategies are monotone and increasing. Thus,

$$G_k(b_k) = F(\phi_k(b_k)) \text{ and } g_k(b_k) = f(\phi_k(b_k))\phi_k'(b_k).$$

The aim of the identification exercise is to back out the primitives of the model, $F$ and $\bar{F}$ from bidding data. Our main identification result states that the valuations and beliefs can be identified from the bid distributions, the number of bidders, and the the number of items in a sequence. It is summarized in the following proposition.

**Proposition 4.2.** *For any $K \geq 2$ unit sSPAs with single unit demand and any bidder $i$ who participated in at least one of the final two rounds, her valuation can be identified as*

$$v_i = \begin{cases} b_{i,K};\ \text{if } b_{i,K} > 0 \\ b_{i,K-1} + \left(\frac{N-K+1}{N-K}\right)(b_{i,K-1} - b_{i,K-2})\frac{g_{K-2}(b_{i,K-2})}{g_{K-1}(b_{i,K-1})};\ \text{if } K > 2\ \&\ b_{i,K-1} > 0 \end{cases}$$

*The worst case beliefs can be identified using*

$$\frac{\bar{f}(v)}{\bar{F}(v)} = \frac{d\log\bar{F}(v)}{dv} = \frac{1}{(N-k-1)(b_{k+1}-b_k)\phi_k'(b_k))} = \frac{f(v)}{(N-k-1)(b_{k+1}-b_k)g_k(b_k)}$$

*and the integrating constant given by $\bar{F}(\bar{v}) = 1$.*

The above states that $F(v)$ and $\bar{F}(v)$ can be recovered from data on (losing) bids and the number of participants from sequential auctions that are at least two rounds (i.e. they are sequential auctions). Note that it is sufficient to observe the bids of the "always" loser(s).

We know that with single unit demand a bidder will place a positive bid in the final round if and only if she has not won a unit yet. Given that bidders bid their valuation in the final round of sSPAs, we can recover their valuations simply from their bids. This would be the sole strategy if a data set only contained bids from sequences where $K = 2$. Further, when $K > 2$, then we can also recover *pseudo-valuations* of bidders who did not participate in the final round using their bids in consecutive rounds. The formula stated in the proposition follows from the first order conditions of a bidder's optimization problem in rounds $K-1$ and $K-2$. Finally, we find an expression for the reversed hazard rate of $\bar{F}(v)$ also using first order conditions. The details can be found in Appendix B.

The obvious benefit of being able to create pseudo-valuations using bidding in rounds other than the final round is that it has the potential of increasing power when estimating $F(v)$ and $\bar{F}(v)$.

However, there is an additional benefit to this result. In general, the estimation of (first-price) auction models rely on bidding being characterized by first order conditions.[18] While this has solid theoreitcal foundations, it is not without challenges. As Bajari and Hortasu (2005) write:

> ”Despite the introduction of powerful new methods to estimate these (auction) models, many applied researchers are not comfortable with the strict rationality assumptions imposed in the econometric analysis. This scepticism is not without merit. [...] our ability to verify or reject bidder rationality is imperfect in empirical auctions. Hence, in most applications, the strong rationality assumptions of structural auction models will be *identifying assumptions*. Moreover, nonparametric structural models of auctions are often just identified, since we can often perfectly rationalize observed bids with an appropriately constructed model...”

The method outlined here do, to some extent, address these concerns. We do also rely on strong assumptions implied by first order conditions to characterize bidding in early rounds and to recover bidders' worst case beliefs. However, we can test those assumptions by comparing recovered *pseudo-valuations*, which also rely on first order conditions, of bidders who participate in the last round, against the *same bidders' revealed valuations*. The latter can be recovered using a weaker assumption that bidding in the last round is characterized by a weakly dominant strategy. Thus, our model is in a sense over-identified, and we can therefore test if our different identifying assumptions are consistent with each other.

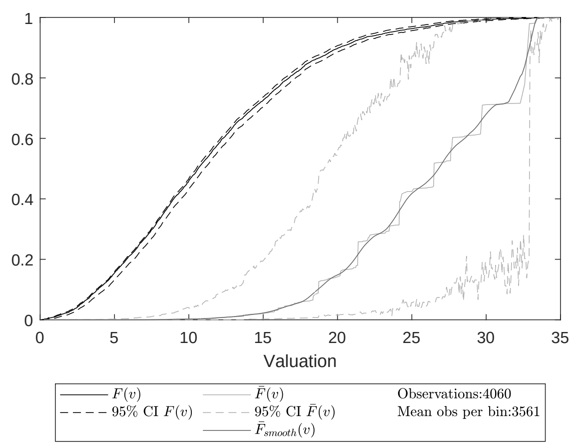
4.2. **Non-parametric Estimation.** While tickets within a sequence are for the same train, tickets in different sequences vary across dimensions such as type of train used, route, departure day of the week, and so on. Thus, in order to carry out a structural estimation of $F$ and $\bar{F}$ using the above identification procedure, we need to homogenize the bids for different sequences. Using the sequence specific covariates we homogenize bids by removing the effect of these variables that are common to all bidders in a sequence. In Appendix E we discuss this procedure which is based on a similar exercise in Shneyerov (2006).

While bidders rarely bid multiple times within an auction, as suggested by the fact that bidders lead auctions on average once, there are instances where bidders record multiple bids. This is due to the incremental bidding nature of the auction mechanism. For such bidders, we use the highest bid they placed in the auction as their revealed bidding strategy. In addition, we consider the bids for bidders who increase their bids between rounds. This is because any rational bidder should increase her bid between rounds due to diminishing number of tickets. This logic is true both in a common prior as well as in our framework with ambiguity and dynamic consistency.

[18] See Donald and Paarsch (1993); Guerre et al. (2000); Campo et al. (2011) for some examples.

4.2.1. *Results.* Figure 5 displays the recovered distributions. The distribution of valuations includes valuations recovered from multiple rounds. Visually one can see that $\hat{\hat{F}}(v) \geq_{FOSD} \hat{F}(v)$, indicating that the subset of bidders for whom the necessary objects are nicely identified do behave as if they are ambiguity averse. The confidence interval also suggest that the stochastic relation between $\hat{\hat{F}}(v)$ and $\hat{F}(v)$ is statistically significant. Further, table 9 lists the results from a Kolmogorov-Smirnov test. The first row of table 9 confirms that $\hat{F}(v) \geq \hat{\hat{F}}(v)$ for all $v$ and the second row confirms that the inequality is strict for some $v$.

FIGURE 5. $\hat{F}(v)$ and $\hat{\hat{F}}(v)$




Note: Recovered from bidders who increased their bids between auctions. Observations refer to the number of recovered valuations used in the estimations. Confidence intervals are constructed by a bootstrapping procedure where the distributions have been estimated 1,000 times after which the range of valuations have been divided into 500 bins of equal width. The 95% confidence interval is constructed by taking the 2.5 and 97.5 percentile of a each distribution from the re-sampled observation. The mean observations per bin is the average number of re-sampled observations within each bin.

TABLE 9. Kolmogorov Smirnov Test

| $H_0$ | Max Distance | P-value |
|---|---|---|
| $F(v) \geq \bar{F}(v)$ | 0.0003 | 0.999 |
| $F(v) \leq \bar{F}(v)$ | 0.7923 | 0.000 |
| $F(v) = \bar{F}(v)$ | 0.7923 | 0.000 |

Note: The test performed is a one sample Kolmogorov Smirnov test where $D_n = \sup_v |\hat{F}_n(v) - \hat{\hat{F}}(v)|$.

We ended section 4.1 by discussing how dynamic bidding can be used to test if the assumption that bidding in early rounds of a sequence can be characterized by first order conditions is consistent with the assumption that bidding in the last round is characterized by a weakly dominant strategy. To test this, we want to compare a bidder's revealed valuation to the same bidder's pseudo-valuation. As is outlined in Proposition 4.2, the bid in the last round of a sequence is treated as a bidder's revealed valuation, while the pseudo-valuation is recovered using the first order conditions that characterize optimal bidding in earlier rounds. We estimate equation (6) to test the hypothesis that these two methods recover the same valuation. Note,

$$\hat{v}_{im} = \gamma_1 D_{1,m} + \gamma_2 D_{2,m} + \theta_i + \varepsilon_{im}. \tag{6}$$

$\hat{v}_{im}$ is bidder $i$'s recovered valuation that was constructed using method $m$, where $m = 1$ indicates a *revealed valuation*, and $m = 2$ indicates a *pseudo-valuation*. $D_{j,m} = 1$ if $j = m$ and 0 otherwise.[19] We estimate equation (6) both with and without $\theta_i$, which is a bidder fixed effect. The inclusion of $\theta_i$ allows for a comparison of $\hat{v}_{i1}$ and $\hat{v}_{i2}$ that should be the same.

The estimation of equation (6) can be found in table 10. The results in column (2) is validating that there is no systematic difference between revealed valuations and constructed pseudo-valuations. It is evidence in favor of the hypothesis that bidding in earlier round is characterized by first order conditions as well as of our approach.

Column (1) reveals another reassuring pattern that is worth noting. That is, if the bidder fixed effect is not included, then recovered pseudo-valuations are slightly higher than revealed valuations. This should not be surprising as there should be selection of higher valuation bidders who do not have their valuations revealed through bidding in the last round. That is because bidders with higher valuations are more likely to win earlier rounds.

TABLE 10. Valuation Evaluation

| VARIABLES | (1) Valuation | (2) Valuation |
|---|---|---|
| Revealed valuation | 11.30*** | 11.53*** |
| | (0.11) | (0.06) |
| Pseudo valuation | 1.13*** | -0.10 |
| | (0.29) | (0.31) |
| Observations | 4,059 | 4,059 |
| Bidder FE | NO | YES |

Note: *** p<0.01, ** p<0.05, * p<0.1. Robust standard errors clustered at the bidder level. *Revealed valuations* are bids made for the last item in the sequence. *Pseudo-valuations* has been recovered using the second part of proposition 4.2.

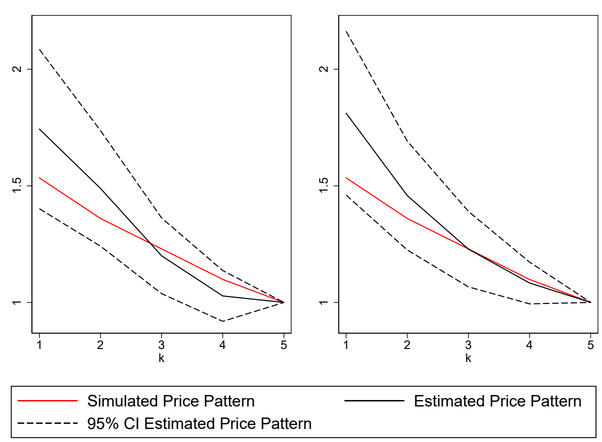
4.2.2. *Counter factual analysis.* We perform three counter factual experiments based on our recovered distributions of valuations and beliefs. First, is a robustness check of our method. We use our estimated model primitives to generate auction outcomes and compare them against the

[19] We really estimate $\hat{v}_{im} = \gamma_1 + \tilde{\gamma}_2 D_{2,m} + \theta_i + \varepsilon_{im}$ such that $\gamma_1$ can be interpreted as mean revealed valuation, and $\tilde{\gamma}_2$ can be interpreted as the difference in mean for pseudo-valuations relative to revealed valuations.

results from the reduced form analysis. Second, we check the revenue implications of ambiguity by calculating the revenue generated in a counterfactual model where the bidders beliefs were the same as the true (recovered) distribution of valuations. And finally, we calculate the revenue that would be generated in alternate auction formats to see if the seller could have done better/worse by using some other commonly used auction formats.

To offer evidence on the internal consistency of our approach we use the mean distributions, $\hat{F}(v)$ and $\hat{\hat{F}}(v)$, that we have estimated to simulate prices in sequences of auctions. In this exercise we hold $K$ fixed to be able to compare the price pattern in the simulated data to the price pattern estimated using equation (1) in the non-homogenized raw data. By holding $K$ fixed we can normalize both patterns by the average price in auction $K$. Hence, we can compare the price pattern in terms of price ratios, where the normalizing price for an object $k$ is the hypothetical average price that would have realized if no ambiguity were present. The result of this exercise for $K = 5$, which is the average length of a sequence in the data, can be seen in figure 6. The result suggests that the simulated price pattern falls within the 95% confidence interval of the price pattern estimated in the raw data.[20]

FIGURE 6. Price Pattern - Simulated vs. Reduced Form




Note: The simulated price pattern is generated using the mean distributions, $\hat{F}(v)$ and $\hat{\hat{F}}(v)$. The sequences in the simulated sample holds length fixed at $K = 5$, and uses $N \in \{6,...,21\}$. The fraction of the sample with a particular $N$ matches the fraction in the true data for $K = 5$. 87% of sequences where $K = 5$ have $N \in \{6,...,21\}$, 10% of sequences have $N \leq 5$, and the remainder has $N > 21$. The estimated price pattern is the normalized predictions from estimating equation (1) on the sub-sample where $K = 5$ and $N \in \{6,...,21\}$. The estimates are normalized by the predicted price in the last auction. The left panel predictions are based on a dummy regression: $ln(price_{kj}) = \sum_{i=1}^{K_j} \gamma_i D_{i,k} + \theta_j + \beta x_k + \varepsilon_{kj}$. The right panel is based on a prediction where a linear quadratic relationship is imposed: $ln(price_{kj}) = \beta_0 + \beta_1 k + \beta_2 k^2 + \beta_3 x_k + \theta_j + \varepsilon_{kj}$.

[20] A joint test fails to reject the null hypothesis that the price trend in the raw data is the same as for the simulated price trend (p-value of an F-test of the dummy regression is 0.22, and the p-value of and F-test of the linear quadratic regression is 0.15).

In the next part of the counter factual analysis we change some aspect of the environment to see what implications the change has for revenues. We first ask how much revenue would SJ lose if there was no uncertainty regarding $F(v)$?[21] To do this exercise we compute bidders' optimal bids as if they knew the true distribution of (pseudo) values, $\hat{F}$. This is done by calculating optimal bids as in Corollary 3.5 but replacing $\bar{F}$ with $\hat{F}$.

The second question we ask is how much revenue would SJ have gained by selling the tickets through sequences of first price auctions while maintaining the uncertainty regarding $F(v)$. From the estimation, since we know $\hat{F}, \hat{\hat{F}}$ and bidders' pseudo valuations we can do the above exercise by using the optimal bidding function for sFPAs given in Proposition 3.6. The results can be found in table 11. The findings suggest that SJ's revenues would decrease by something in the range of 18.6 and 21.2% if there was no uncertainty regarding $F(v)$.[22] On the other hand, if SJ had been able to change the selling mechanism to a first price auction instead, then their revenues would increase by something in the range of 11.5 to 15.4%.

TABLE 11. Revenue Changes

| | (1)<br>Mean %$\Delta$ in Revenues | (2)<br>CI |
|---|---|---|
| **Remove Uncertainty** | −19.7% | [−21.2%,−18.6%] |
| **Change to first price** | 13.5% | [11.5%,15.4%] |

Note: The changes in revenues are calculated using the revenue made in sequences of second price auctions when uncertainty is present as the baseline.

## 5. CONCLUSION

Using a data set comprised of bids placed in sequential train ticket auctions in Sweden, we studied various aspects of bidder behavior. Prices in these auctions show a declining trend, confirming the presence of the declining price anomaly. While there are many theoretical explanations for such price trends, we first argued using a reduced form analysis that a predictions from a model with ambiguity averse bidders match aspects of the data better than other models that also provide an explanation for the declining price anomaly. Then, we studied a theoretical model of sSPAs with ambiguity and showed that prices in this set up decline as a result of pessimism about future rounds. Finally, adapting the first-order approach from the seminal work of Guerre et al. (2000) to our setting with sequential auctions and ambiguity, we non-parametrically estimated the true data generating process as well as bidder's beliefs under dynamic consistency. We showed that

[21] Note that revenue equivalence holds between selling mechanisms when uncertainty is removed. We are aware that the mechanism used by Tradera (and eBay) can be considered a hybrid between first and second price auctions, but as we have motivated in section 3, we base our structural approach on the assumption that the data was generated through sequences of second price auctions.

[22] The mean is given by $\frac{1-1.245}{1.245}$, while the CI is given by $[\frac{1-1.229}{1.229}, \frac{1-1.269}{1.269}]$. These numbers can be found in table 12.

TABLE 12. Revenue Comparisons

| | (1) Average | (2) Lower Bound | (3) Upper Bound | (4) CI |
|---|---|---|---|---|
| **No Ambiguity** (distributions used) | $\hat{F}$ | $\hat{F}_{.975}$ | $\hat{F}_{.025}$ | |
| first price=second price=uniform price | 1.000 | 0.971 | 1.031 | |
| **Ambiguity** (distributions used) | $\hat{F}$ $\hat{\hat{F}}$ | $\hat{F}_{.975}$ $\hat{\hat{F}}_{.975}$ | $\hat{F}_{.025}$ $\hat{\hat{F}}_{.025}$ | |
| first price | 1.412 | 1.329 | 1.509 | [1.370, 1.464] |
| second price | 1.245 | 1.192 | 1.307 | [1.229, 1.269] |
| uniform price | 1.000 | 0.971 | 1.031 | |

Note: The revenues are from sequences with $(N,K)$ pairs that match the SJ data, where $1 \leq K \leq 10$ and $K < N \leq 20$. There are 5,279 sequences in the SJ data that satisfy this criteria. 1,000 sequences have been simulated for each $(N,K)$ pair, after which the average revenue has been calculated. A weighted average of these averages have then been used to get total revenues. The weights correspond to the frequency in the real data of the $(N,K)$ pair relative to the total number of sequences that satisfy the restriction above. Lastly, the revenues have been normalized by the revenues made in the hypothetical case of no ambiguity where valuations have been drawn from $\hat{F}(v)$ (column 1).

the estimated belief first-order stochastically dominates the estimated value distribution, further confirming the presence of ambiguity as the driving force behind declining prices. A future avenue of research might be using other variations, such as exogenous variations in the number of bidders, to relax dynamic consistency assumption and identify changing worst case beliefs.

## APPENDIX A. EXPLANATIONS OF DECLINING PRICES

*Risk aversion*: Informally, Ashenfelter (1989) argued that risk aversion could possibly explain the anomaly since, theoretically, prices in later rounds are more variable than the current round and hence risk-averse bidders may bid more in the current round to avoid future price variations.[23] This logic was formalized in McAfee and Vincent (1993) but the results required that the bidder's utility function satisfy non-decreasing absolute risk aversion (NDARA) in wealth which is an unconventional assumption.[24] Using an additively separable utility function with a convex cost function, which captures 'aversion to price risk', Mezzetti (2011) proved the existence of a monotone equilibrium that generated declining prices. Hence, NDARA preferences and aversion to price risk can account for the declining price anomaly. Importantly, the equilibria under both preference structures are characterized by history-independent bidding in the i.p.v. paradigm, which is a prediction that can be tested using data.

[23]For example, in Krishna (2009) (page 226):"[Even] though prices are expected to decline in the future, his greater aversion to risk offset the incentive to wait for a random future price, which is lower on average."

[24]McAfee and Vincent (1993) write that "Ashenfelter suggests that declining prices is consistent with risk averse bidders....We show that this intuition is not likely to be satisfied in practice."

*Loss aversion*: Rosato (2019) studied a sequential auction model where bidders have expectations based *reference dependent preferences* ((Köszegi and Rabin, 2006)), within an i.p.v. paradigm. The winning bid in a round is part of forming reference points for the bidders who remain in the next round. The equilibrium in this setting is also characterized by declining prices due to a 'discouragement effect': a higher winning bid in a round leads to less aggressive bidding in the next round, and since bidders choose bids conditional on being pivotal, they underestimate the discouragement effect. This model also predicts a negative relationship between winning bids in the a round and bids in the next, which can be tested empirically.

*Ambiguity aversion*: GL studied sequential First Price Auctions (sFPAs) with *ambiguity averse* bidders with maxmin expected utility (MEU) (Gilboa and Schmeidler, 1989). They showed that when bidders are ambiguity averse they use their worst-case conditional belief, which is given by the lower envelope of the set of conditional distributions, to calculate their expected payoff in each round, where the conditioning is with respect to the previous round prices. Since bidders use their worst-case beliefs, they underestimate their future payoffs and bid more aggressively in the current round causing prices to decline in the future rounds as lower valuation bidders, who are relatively less aggressive than the current round winner, will win future rounds in a monotone equilibrium. Since bidders use their worst-case beliefs *conditional* on previous round prices, bidding can be history-dependent, in contrast to the risk-aversion and the standard model.

## APPENDIX B. PROOFS

B.1. **Proof of Proposition 3.3.** We solve for the equilibrium strategies backward starting from the final round. In the final round bidders have a weakly dominant strategy to bid their valuation. Given this, their payoff in the final round is given by

$$\min_{F_{y_{K-1}} \in \Delta_{y_{K-1}}} F_{y_{K-1}}(v)^{N-K} v - \int_{\underline{v}}^{v} x dF_{y_{K-1}}(x)^{N-K} = \min_{F_{y_{K-1}} \in \Delta_{y_{K-1}}} \int_{\underline{v}}^{v} (v-x) dF_{y_{K-1}}(x)^{N-K}$$

Since $v-x$ is decreasing in $x$, the above payoff function is minimized by the lower envelope of $\Delta_{y_{K-1}}$ that is $\bar{F}(\cdot|y_{K-1})$. Now, define the following payoff function for round $K$.

$$\Gamma_K\left(v, y, F_{y_{K-2}}\right) = \int_{\underline{v}}^{v} (v-x) d\frac{F_{y_{K-2}}(x)^{N-K}}{F_{y_{K-2}}(y)^{N-K}}$$

This is the consistent planning bidder's evaluation of her round $K$ payoff in round $K-1$ given her 'belief' about the distribution of values in round $K-1$, $F_{y_{K-2}} \in \Delta_{y_{K-2}}$ and the round $K-1$ winner's valuation $y$. Clearly $\Gamma_K$ is increasing in $v$ and decreasing in $y$. Now consider the payoff in round $K-1$. Suppose bidders are following some strategy $\beta_{K-1}$ in this round that possibly depends on previous round prices and a bidder bids as if her valuation is $z \geq v$. Then,

(7)
$$\begin{aligned}\Pi_{K-1}(v,z,y_{K-2}) = \min_{F_{y_{K-2}}\in\Delta_{y_{K-2}}} & \int_{\underline{v}}^{z} (v-\beta_{K-1}(x,y_{K-2}))\,dF_{y_{K-2}}(x)^{N-K+1} \\ & + \int_{z}^{y_{K-2}} \Gamma_K(v,x,F_{y_{K-2}})dF_{y_{K-2}}(x)^{N-K+1}\end{aligned}$$

Let $\hat{F}_{y_{K-2}}$ minimize the above payoff. Taking first order conditions with respect $z$ and setting $z=v$,

$$v-\beta_{K-1}(v,y_{K-2}) = \Gamma_K(v,v,\hat{F}_{Y_{K-2}}) = v - \int_{\underline{v}}^{v} xd\frac{\hat{F}_{y_{K-2}}(x)^{N-K}}{\hat{F}_{y_{K-2}}(v)^{N-K}} \implies \beta_{K-1}(v,y_{K-2}) = \int_{\underline{v}}^{v}\beta_K(x)d\frac{\hat{F}_{Y_{K-2}}(x)^{N-K}}{\hat{F}_{y_{K-2}}(v)^{N-K}}.$$

Now, note that for $v\le y$,

$$v-\beta_{K-1}(x,y_{K-2}) \ge v-\beta_{K-1}(v,y_{K-2}) = \Gamma_K(v,v,\hat{F}_{Y_{K-2}}) \ge \Gamma_K(v,y,\hat{F}_{Y_{K-2}})$$

since $\beta_{K-1}(\cdot,y_{K-2})$ is increasing and $\Gamma_K$ is non-increasing in the second argument. Thus, inspecting (7) it is clear that in $\hat{F}_{Y_{K-2}}(\cdot) = \bar{F}(\cdot|y_{K-2})$. Thus

$$\beta_{K-1}(v,y_{K-2}) = \int_{\underline{v}}^{v}\beta_K(x)d\frac{\bar{F}(x|y_{K-2})^{N-K}}{\bar{F}(v|y_{K-2})^{N-K}}.$$

Next, we consider the consistent planning payoff function $\Gamma_{K-1}$.

$$\begin{aligned}\Gamma_{K-1}(v,y,F_{y_{K-3}}) &= \int_{\underline{v}}^{v}(v-\beta_{K-1}(x,y))\,d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}} + \int_{v}^{y}\Gamma_K(v,x,F_{y_{K-3}})d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}} \\ &= \int_{\underline{v}}^{v}\left(v-x+\int_{\underline{v}}^{x}\frac{\bar{F}(z|y)^{N-K}}{\bar{F}(x|y)^{N-K}}dz\right)d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}} \\ &\quad + \int_{v}^{y}\Gamma_K(v,x,F_{y_{K-3}})d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}}\end{aligned}$$

Now, note that the first term in the above is non-increasing in $y$ due to the Envelope Theorem. For the second term, note that the derivative with respect to $y$,

$$(N-K+1)\frac{f_{y_{K-3}}(y)}{F_{y_{K-3}}(y)}\left(\Gamma_K(v,y,F_{y_{K-3}}) - \int_{v}^{y}\Gamma_K(v,x,F_{y_{K-3}})d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}}\right)\le 0$$

since $\Gamma_K(v,y,F)$ is non-increasing in $y$ as we established in step 1 of the proof. Thus, $\Gamma_{K-1}(v,y,F_{y_{K-3}})$ is non increasing in $y$.

A bidder's payoff function in round $K-2$ is given by

$$\Pi_{K-2}(v,z,y_{K-3}) = \min_{F_{y_{K-3}} \in \Delta_{y_{K-3}}} \int_{\underline{v}}^{z} (v - \beta_{K-2}(x,y_{K-3}))\, F_{y_{K-3}}(x)^{N-K+2} + \int_{z}^{y_{K-3}} \Gamma_{K-1}(v,x,F_{y_{K-3}}) dF_{y_{K-3}}(x)^{N-K+2}.$$

Let $\hat{F}_{K-3}$ minimize the above payoff function. Then, first order conditions and $z = v$ imply

$$v - \beta_{K-2}(v,y_{K-3}) = \Gamma_{K-1}(v,v,\hat{F}_{y_{K-3}}) = \int_{\underline{v}}^{v} (v - \beta_{K-1}(x,v))\, d\frac{\hat{F}_{y_{K-3}}(x)^{N-K+1}}{\hat{F}_{y_{K-3}}(v)^{N-K+1}} \implies$$

$$\beta_{K-2}(v,y_{K-3}) = \int_{\underline{v}}^{v} \beta_{K-1}(x,v) d\frac{\hat{F}_{y_{K-3}}(x)^{N-K+1}}{\hat{F}_{y_{K-3}}(v)^{N-K+1}}$$

Now note that

$$\int_{\underline{v}}^{v} \beta_{K-1}(x,v) d\frac{\hat{F}_{y_{K-3}}(x)^{N-K+1}}{\hat{F}_{y_{K-3}}(v)^{N-K+1}} = \int_{\underline{v}}^{v} \left( x - \int_{\underline{v}}^{x} \frac{\bar{F}(z|v)^{N-K}}{\bar{F}(x|v)^{N-K}} dz \right) d\frac{\hat{F}_{y_{K-3}}(x)^{N-K+1}}{\hat{F}_{y_{K-3}}(v)^{N-K+1}}$$

Again, it is straightforward to show that the above is increasing in $v$ using the Envelope Theorem. Now note that

$$v - \beta_{K-2}(x,y_{K-3}) \geq v - \beta_{K-2}(v,y_{K-3}) = \Gamma_{K-1}(v,v,\hat{F}_{y_{K-3}}) \geq \Gamma_{K-1}(v,y,\hat{F}_{y_{K-3}})$$

where the second inequality follows monotonicity of $\beta_{K-2}(\cdot,y_{K-3})$ and the fourth inequality follows from the monotonicity of $\Gamma_{K-1}(v,\cdot,F)$. Thus $\hat{F}_{y_{K-3}} = \bar{F}(\cdot|y_{K-3})$ and

$$\beta_{K-2}(v,y_{K-3}) = \int_{\underline{v}}^{v} \beta_{K-1}(x,v) d\frac{\bar{F}(x|y_{K-3})^{N-K+1}}{\bar{F}(v|y_{K-3})^{N-K+1}}$$

The proof for the remaining rounds follows exactly the same procedure. The independence of $\bar{F}(\cdot|y)$ from the bidders valuation follows from the assumption that $\Delta$ is a semi-lattice. For a proof see Step 5 of the proof of Proposition 4.1 of Ghosh and Liu (2021).

B.2. **Proof of proposition 3.4.** Note,

$$\mathbb{E}[P_{k+1}|p_k] = \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x,y_k)dF_2^{(N-k)}(x|x \le y_{k+1})$$
$$= \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x,y_k)d\left((N-k)\left(\frac{F(x)}{F(y_{k+1})}\right)^{N-k-1} - (N-k-1)\left(\frac{F(x)}{F(y_{k+1})}\right)^{N-k}\right)$$
$$\le \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x,y_k)d\left((N-k)\left(\frac{\bar{F}(x|y_{k-1})}{\bar{F}(y_{k+1}|y_{k-1})}\right)^{N-k-1} - (N-k-1)\left(\frac{\bar{F}(x|y_{k-1})}{\bar{F}(y_{k+1}|y_{k-1})}\right)^{N-k}\right)$$
$$\le \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x,y_k)d\left(\frac{\bar{F}(x|y_{k-1})}{\bar{F}(y_{k+1}|y_{k-1})}\right)^{N-k}$$

where the first inequality follows from the monotonicity of $\beta_{k+1}(\cdot,y_k)$ and $\bar{F}(\cdot|y) \ge_{rh} F(\cdot)/F(y)$.[25] The second inequality follows from the monotonicity of $\beta_{k+1}(\cdot,y_k)$. Now, consider the case $k = K-1$. Then using the above inequality and the equilibrium strategy in the final round we can see that

$$\mathbb{E}[P_K|p_{K-1}] \le \int_{\underline{v}}^{y_K} \beta_K(x)d\left(\frac{\bar{F}(x|y_{K-2})}{\bar{F}(y_{k+1}|y_{K-2})}\right)^{N-K-1} \le \int_{\underline{v}}^{y_K} \beta_K(x)d\left(\frac{\bar{F}(x|y_{K-2})}{\bar{F}(y_{k+1}|y_{K-2})}\right)^{N-K} = p_{K-1}(y_K,y_{K-2})$$

Now, let $k = K-2$.

$$\mathbb{E}[P_{K-1}|p_{K-2}] \le \int_{\underline{v}}^{y_{K-1}} \beta_{K-1}(x,y_{K-2})d\left(\frac{\bar{F}(x|y_{K-3})}{\bar{F}(y_{K-1}|y_{K-3})}\right)^{N-K-1}$$
$$= \int_{\underline{v}}^{y_{K-1}}\int_{\underline{v}}^{x} \beta_K(z)d\frac{\bar{F}(z|y_{K-2})^{N-K}}{\bar{F}(x|y_{K-2})^{N-K}}d\left(\frac{\bar{F}(x|y_{K-3})}{\bar{F}(y_{K-1}|y_{K-3})}\right)^{N-K-1}$$
$$\le \int_{\underline{v}}^{y_{K-1}}\int_{\underline{v}}^{x} \beta_K(z)d\frac{\bar{F}(z|y_{K-3})^{N-K}}{\bar{F}(x|y_{K-3})^{N-K}}d\left(\frac{\bar{F}(x|y_{K-3})}{\bar{F}(y_{K-1}|y_{K-3})}\right)^{N-K-1}$$
$$= \int_{\underline{v}}^{y_{K-1}} \beta_{K-1}(x,y_{K-3})d\left(\frac{\bar{F}(x|y_{K-3})}{\bar{F}(y_{K-1}|y_{K-3})}\right)^{N-K-1}$$
$$= p_{K-2}$$

where the inequality follows from the supposition in the statement of the proposition. The remaining cases can be proved analogously.

[25]The final condition implies that $\frac{\bar{F}(x|y_{k-1})}{\bar{F}(y_{k+1}|y_{k-1})} \ge_{FOSD} \frac{F(y_{k+1}|y_{k-1})}{F(y_{k+1}|y_{k-1})} = \frac{F(y_{k+1})}{F(y_{k+1})}$ (From equation 1.B.43 of Shaked and Shanthikumar (2007), reverse hazard rate dominance is equivalent to conditional stochastic dominance, which implies $\bar{F}(\cdot|y_{k-1})/\bar{F}(y_{k+1}|y_{k-1}) \ge_{FOSD} F(\cdot)/F(y_{k+1})$). Finally, note that if $G \ge_{FOSD} H$ then $G_2^{(n)} \ge_{FOSD} H_2^{(n)}$

B.3. **Proof of proposition 4.2.** Let $G_k(b_k)$ represent the bid distribution in the $k$-th round of a $K$ round sSPAs, and $g_k(b_k)$ the corresponding bid densities. Let $\phi_k = \beta_k^{-1}$ be the inverse bidding strategy. In equilibrium, bidding strategies are monotone and increasing. Thus,

$$G_k(b_k) = F(\phi_k(b_k)) \text{ and } g_k(b_k) = f(\phi_k(b_k))\phi_k'(b_k). \tag{8}$$

In order to identify the true $F$ we need to back out the valuations of the bidders from their bids. This is straightforward for all bidders who participated in the final round since in the final round it is a weakly dominant strategy to bid truthfully. Thus, $v$ of any bidder in the final round is given by

$$v = b_K = \phi_K(b_K). \tag{9}$$

Having identified the bidders' values in the final round, we turn to the remaining rounds. To do so, we first calculate the equilibrium payoff in the final round. Rectangularity ensures that

$$\Pi_K(v,b_K,y_{K-1}) = \Pi(v,v,y_{K-1}) = \min_{F\in\Delta}\int_{\underline{v}}^{v}(v-x)d\left(\frac{F(x)}{F(y_{K-1})}\right)^{N-K} = \int_{\underline{v}}^{v}(v-x)d\left(\frac{\bar{F}(x)}{\bar{F}(y_{K-1})}\right)^{N-K}.$$

Now, consider the bidder'a payoff in round $K-1$. It is given by

$$\begin{aligned}\Pi_{K-1}(v,b_{K-1},y_{K-2}) = &\int_{\underline{v}}^{\phi_{K-1}(b_{K-1})}(v-\phi_{K-1}^{-1}(x))d\left(\frac{\bar{F}(x)}{\bar{F}(y_{K-2})}\right)^{N-K+1}\\ &+\int_{\phi_{K-1}(b_{K-1})}^{y_{K-2}}\Pi_K(v,v,x)d\left(\frac{\bar{F}(x)}{\bar{F}(y_{K-2})}\right)^{N-K+1}.\end{aligned}$$

First order conditions with respect to $b_{K-1}$ and the equilibrium condition, $\phi_{K-1}(b_{K-1}) = v$, imply

$$v - b_{K-1} = \Pi_K(v,v,\phi_{K-1}(b_{K-1})) = \int_{\underline{v}}^{v}(v-x)d\left(\frac{\bar{F}(x)}{\bar{F}(v)}\right)^{N-K}.$$

Now, note that the above can also be written as

$$b_{K-1}\bar{F}(\phi_{K-1}(b_{K-1}))^{N-K} = \int_{\underline{v}}^{\phi_{K-1}(b_{K-1})} xd\bar{F}(x)^{N-K}.$$

Taking derivative of the above with respect to $b_{K-1}$ we get

$$\begin{aligned}&\bar{F}(\phi_{K-1}(b_{K-1}))^{N-K} + b_{K-1}(N-K)\bar{F}(\phi_{K-1}(b_{K-1}))^{N-K-1}\bar{f}(\phi_{K-1}(b_{K-1}))\phi_{K-1}'(b_{K-1})\\ &= \phi_{K-1}(b_{K-1})(N-K)\bar{F}(\phi_{K-1}(b_{K-1}))^{N-K-1}\bar{f}(\phi_{K-1}(b_{K-1}))\phi_{K-1}'.\end{aligned}$$

Rearranging and using the fact that $\phi_k(b_k) = v$ we have

$$v = b_{K-1} + \frac{\bar{F}(\phi_{K-1}(b_{K-1}))}{(N-K)\bar{f}(\phi_{K-1}(b_{K-1}))\phi'_{K-1}(b_{K-1})} = b_{K-1} + \frac{\bar{F}(v)}{(N-K)\bar{f}(v)\phi'_{K-1}(b_{K-1})}. \tag{10}$$

Using equations (8) and (9) in (10) and rearranging we have

$$\frac{\bar{f}(v)}{\bar{F}(v)} = \frac{f(v)}{(N-K)(b_K - b_{K-1})g_{K-1}(b_{K-1})}. \tag{11}$$

Thus, if $K = 2$, we have valuations from observable bids given equation (9), and the above expression along with the fact that $\bar{F}(\bar{v}) = 1$ gives $\bar{F}$ in terms of observable bids, and distributions that can be estimated from bids. We can therefore conclude that $F$ and $\bar{F}$ is identified from bids placed in sequences of auctions with only 2 items for sale.

Next consider the round $K-2$. Let $\Pi_{K-1}(v,x)$ be the bidder's equilibrium expected payoff in the next round conditional on the current rounds winner's valuation $x$. Then, a bidder's expected payoff from bidding $b_{K-2}$ is

$$\Pi_{K-2}(v, b_{K-2}, y_{K-3}) = \int_{\underline{v}}^{\phi_{K-2}(b_{K-2})} (v - \phi_{K-2}^{-1}(x)) d\left(\frac{\bar{F}(x)}{\bar{F}(y_{K-3})}\right)^{N-K+2} + \int_{\phi_{K-2}(b_{K-2})}^{y_{K-3}} \Pi_{K-1}(v,x) d\bar{F}(x)^{N-K+2}.$$

First order conditions imply

$$v - b_{K-2} = \Pi_{K-1}(v, \phi_{K-2}(b_{K-2})) = \int_{\underline{v}}^{v} (v - \phi_{K-1}^{-1}(x)) d\left(\frac{\bar{F}(x)}{\bar{F}(\phi_{K-2}(b_{K-2}))}\right)^{N-K+1}$$

Note that the above can be written as

$$b_{K-2}\bar{F}(\phi_{K-2}(b_{K-2}))^{N-K+1} = \int_{\underline{v}}^{\phi_{K-2}(b_{K-2})} \phi_{K-1}^{-1}(x) d\bar{F}(x)^{N-K+1}$$

Taking the derivative of the above equation with respect to $b_{K-2}$ we get

$$\begin{aligned}&\bar{F}(\phi_{K-2}(b_{K-2}))^{N-K+1} + b_{K-2}(N-K+1)\bar{F}(\phi_{K-2}(b_{K-2}))^{N-K}\bar{f}(\phi_{K-2}(b_{K-2}))\phi'_{K-2}(b_{K-2})\\ &= \phi_{K-1}^{-1}(\phi_{K-2}(b_{K-2}))(N-K+1)\bar{F}(\phi_{K-2}(b_{K-2}))^{N-K}\bar{f}(\phi_{K-2}(b_{K-2}))\phi'_{K-2}(b_{K-2})\end{aligned}$$

Noting that $\phi_{K-2}(b_{K-2}) = v$, and hence, $\phi_{K-1}^{-1}(\phi_{K-2}(b_{K-2})) = b_{K-1}$. Thus the above simplifies to

$$b_{K-1} - b_{K-2} = \frac{\bar{F}(v)}{(N-K+1)\bar{f}(v)}\frac{1}{\phi'_{K-2}(b_{K-2})} \tag{12}$$

Now, using equations (10) and (12) to solve for $\frac{\bar{f}(v)}{\bar{F}(v)}$ we have

$$v = b_{K-1} + \left(\frac{N-K+1}{N-K}\right)(b_{K-1} - b_{K-2})\frac{\phi'_{K-2}(b_{K-2})}{\phi'_{K-1}(b_{K-1})}.$$

Lastly, using the mappings from distributions of bids to the distribution of valuation we get

$$v = b_{K-1} + \left(\frac{N-K+1}{N-K}\right)(b_{K-1} - b_{K-2})\frac{g_{K-2}(b_{K-2})}{g_{K-1}(b_{K-1})} \tag{13}$$

Equations (9) and (13) are the results for recovery of valuations.

Lastly, for the expression for $\frac{\bar{f}(v)}{\bar{F}(v)}$ for $K \geq 2$. Start with the payoff in an arbitrary round $k$

$$\Pi_k(v, b_k, y_{k-1}) = \int_{\underline{v}}^{\phi_k(b_k)} (v - \phi_k^{-1}(x))d\left(\frac{\bar{F}(x)}{\bar{F}(y_{k-1})}\right)^{N-k} + \int_{\phi_k(b_k)}^{y_{k-1}} \Pi_{k+1}(v, v, x)d\left(\frac{\bar{F}(x)}{\bar{F}(y_{k-1})}\right)^{N-k}.$$

First order conditions imply

$$b_k\bar{F}(\phi_k(b_k))^{N-k-1} = \int_{\underline{v}}^{\phi_k(b_k)} \phi_{k+1}^{-1}(x)d\bar{F}(x)^{N-k-1}$$

Differentiate the above with respect to $b_k$, rearranging, and using the mappings from distributions of bids to valuations gives us

$$\frac{\bar{f}(v)}{\bar{F}(v)} = \frac{d\log\bar{F}(v)}{dv} = \frac{1}{(N-k-1)(b_{k+1}-b_k)\phi'_k(b_k))} = \frac{f(v)}{(N-k-1)(b_{k+1}-b_k)g_k(b_k)}.$$

B.4. **Identification of $\bar{F}$ and $F$.** In this section we show that the result from GL can be applied in our setting as well under the assumption of rectangularity. As we discussed before in section 3.4.2 the advantage of this assumption is that under it bidders worst-case beliefs do not depend on their own valuation which limits of number of objects that need to be identified. Suppose $K = 2$. Let

$$v = b_2 + \left(\frac{N-1}{N-2}\right)(b_2 - b_1)\frac{g_1(b_1)}{g_2(b_2)} \equiv \zeta(b_1, b_2, G_1, G_2). \tag{14}$$

The following is the identification result from GL.

**Proposition B.1.** *(Proposition 5.5 in Ghosh and Liu (2021)) For any distributions $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, there exist distributions $F, \bar{F} : [\underline{v}, \bar{v}] \to [0,1]$ such that $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are the equilibrium joint distribution of bids in rounds 1 and 2 of a sequence of first-price auctions with ambiguity if and only if*

*C1 : The joint distribution of bids in any round is given by*

$$\boldsymbol{G}_k(b_1,\ldots,b_{N-k+1}) = \prod_{i=1}^{N-k+1} G_k(b_i).$$

*C2 :* $\underline{b}_1 = \underline{b}_2$ *and* $\overline{b}_1 < \overline{b}_2$*. Furthermore, there exists an increasing function* $\psi(\cdot)$ *such that* $\psi(b_{i,2}) = b_{i,1} \leq b_{i,2}$ *for* $i \in \{1,\ldots,N\}$*, and* $b_{i,k}$ *is bidder* $i$*'s bid in auction round* $k$*. Finally,* $G_2(b) \leq G_1(b)$ *for any* $b$*.*

*C3 : The function* $\zeta(\psi(b_2), b_2, G_1, G_2)$ *is strictly increasing on* $[\underline{b}_2, \overline{b}_2]$*.*

*Moreover when* $\hat{F}, \bar{F}$ *exist they are unique with support* $[\underline{v}, \overline{v}]$ *and satisfy the equations* $F(v) = G_2(\zeta^{-1}(v, G_1, G_2))$ *and (11).*

The above result shows how round specific bid distributions can be used to identify the distributions $F$ and $\bar{F}$ in sFPAS. Due to Corollary 3.7 this result from GL applies to our setting of sSPAs as well as long as $K \geq 3$.

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