Information requirement for efficient decentralized screening

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Abstract

We establish new efficiency results for decentralized markets with quality uncertainty. Buyers observe a constant flow of passing trade opportunities and related asset information, which allows buyers to screen asset quality by conditioning pricing on informative signals. We link key equilibrium properties with the intensity of screening and provide a new measure for the information required for efficient trade in an extensive class of frictional markets. Trade dynamics can be *standard* or *reversed.* Limit payoffs exceed those in a static one-price model.

Keywords: *Decentralized lemons market; Buyer signals; Trade dynamics; Screening; Efficient equilibrium; Equilibrium existence.* **JEL-codes:** D82, D83.

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1 Introduction

Asset markets are large and global. Trades are regularly executed *over-the-counter* in multiple *decentralized* exchanges. Some assets are clearly "lemons" as defined by Akerlof (1970), e.g., a firm might have issues with information security or customer management, just waiting to surface. However, even these assets often generate positive value for their owner, new trading opportunities arrive constantly, and buyers can inspect assets before trading. Indeed, the law requires *due diligence* in acquisitions and *caveat emptor* applies. How do such decentralized markets with informative signals fare? Will the market remain inefficient, as in Moreno and Wooders (2010) without signals? Or, will the lemons problem resolve on its own with time and the market settle to an efficient equilibrium? Which dynamic trade patterns, as characterized by Kaya and Kim (2018), are sustained in the long run? What is the role of frictions and information?

In this article, we reply to these questions by investigating the effect of signals in a decentralized lemons market, where (i) traders are small, numerous and anonymous, (ii) trade frictions are vanishingly small, and (iii) trading has settled to a steady-state.[1] The setup adheres loosely to the seminal model of dynamic trade by Moreno and Wooders (2010): asset sellers enter the market with different asset qualities, meet a sequence of random buyers, and exit the market upon trading. To incorporate asset information in this model, we introduce the assumption that a buyer can obtain a signal of a seller's asset quality before making the seller a price offer. Our setup emulates information-rich financial markets. This provides an extended version of canonical models for decentralized trade[2] where traders face not only a constant flow of trade opportunities, as in the previous literature, but also an incessant flow of asset information.

We establish new efficiency results for this formerly neglected class of markets that has more recently garnered great interest from financial economists.[3] In particular, this article observes that all key properties of an equilibrium – existence, efficiency, and dynamics – derive from the screening intensities of different asset qualities, i.e., the difficulty of obtaining a high price offer for low quality *versus* high quality. In the model, signal distributions differ between sellers, i.e., a lower signal suggests a lower asset quality. As a result, it is possible for buyers to screen the quality of assets by offering high prices only for high enough signals. Furthermore, assuming that signals are sufficiently informative relative to frictions of trading, low quality can be screened more strongly than high as frictions become negligible. To equate the costs of waiting with those of paying too much, a buyer could thus make obtaining a high price offer, e.g., either equally hard for both

[1] This case is particularly interesting as a decentralized counterpart of the *Walrasian* equilibrium; Gale (1986a,b, 1987); Rubinstein and Wolinsky (1985) and Binmore and Herrero (1988).

[2] See Wolinsky (1990); Serrano and Yosha (1993, 1996); Blouin and Serrano (2001); Blouin (2003)

[3] For examples of recent high impact work, see Rostek and Yoon (2021) for *imperfectly competitive trade* and Azevedo and Gottlieb (2017) for *perfect competition and adverse selection.*

qualities, or infinitely harder for low quality. This insight permits us to characterize steady-state equilibria by focusing on screening.

Our first main result is that a market settles to an efficient steady-state equilibrium for an extensive range of parameter values as trade frictions disappear. The range is partly characterized by *the severity of the lemons problem* and partly by *the relative trade surpluses* among different asset qualities, which is novel. Specifically, we show that an efficient limit equilibrium exists (i) if the trade surplus of low quality is larger, i.e., if $\Delta_l \geq \Delta_h$, or (ii) if the static lemons problem is not severe, i.e., if $\Delta_h \geq \Delta_g$; Δ_l (Δ_h) denotes the surplus of trading low (high) quality assets and Δ_g the gap between the value of buying low quality and selling high quality. As it turns out, efficiency hinges on adjusting screening to market conditions: In the former case (i), trade dynamics are standard (low quality trades faster) and the screening intensity of low quality is strong enough to make the seller accept a low price and not wait for a high signal. In the latter case (ii), trade dynamics are reversed (high quality trades faster) and the screening intensities of both qualities stay relatively low, encouraging a low quality seller to wait for a high price offer. Our efficiency results contrast with the persistence of trading problems in the literature (e.g., Blouin and Serrano (2001); Camargo and Lester (2014); Guerrieri and Shimer (2014); Moreno and Wooders (2010)).[4]

The analysis admits to quantify neatly the information requirements of efficient trading, basically by inverting the related screening mapping to uncover the information needs associated with the frictions. As our second key literature contribution, we can thus demonstrate that our findings, derived in a model with highly informative signals, immediately transfer to any markets where signals are *sufficiently informative relative to the prevailing trade frictions*. In general, the information required to separate assets elevates as frictions decrease because the low quality sellers' costs of waiting become smaller. This shows that our finding of efficient decentralized screening relies jointly on (i) the existence of small positive trade frictions and (ii) the availability of sufficiently informative signals. This revises and verifies Moreno and Wooders (2010)'s hypothesis that "decentralized trade mitigates the lemons problem".

Our third major result is the observation that, if there exists no *efficient* equilibrium, there exists *no* equilibrium in the market. This occurs when trading high quality is both more difficult (i.e., the lemons problem is severe) and more valuable (i.e., the trade surplus is larger), that is, for $\Delta_l < \Delta_h < \Delta_g$. This finding derives basically from a discrepancy between the required trade dynamics and the presumed trade surpluses. On one hand, we can show that, when the static lemons problem is severe, only standard trade dynamics

[4] Camargo et al. (2020) find that non-steady-state equilibria with aggregate uncertainty become efficient as frictions vanish. For other positive efficiency results in decentralized markets, see Golosov et al. (2014) for divisible assets and aggregate uncertainty and Asriyan et al. (2017) for correlated values and information spillovers. In our model, all learning happens through private quality screening.

may prevail.[5] This alleviates the lemons problem by increasing the average quality of assets. On the other hand, elevating asset quality and vanishing trade frictions also mean that the opportunity cost of trading increases. This intensifies screening and boosts the quality of unsold assets. Thereby, we find that buyers only offer high prices when they are almost certain about high asset quality, which increases their payoffs up to Δ_h.[6] However, this implies that buyers cannot agree on a price with low quality sellers under lower expectations, because the trade surplus is smaller Δ_l – contradicting the assumed standard dynamics. The existence and efficiency of an equilibrium thus depend not only on the *severity of the lemons problem*, as known since Akerlof (1970), but also on the *relative trade surpluses* across traded assets.[7]

This article contributes to the rapidly growing literature that studies adverse selection in decentralized market environments with random sequential search. There is also a large literature about dynamic trading with incomplete information in directed search markets, e.g., Inderst and Müller (2002); Inderst (2005); Guerrieri et al. (2010); Camargo and Lester (2014), and in competitive lemons markets, e.g., Janssen and Roy (2002, 2004); Daley and Green (2012); Fuchs and Skrzypacz (2019).

A voluminous literature studies whether decentralized trade results in equal payoffs as its centralized counterpart if trade frictions are small. Gale (1986a,b, 1987) and Binmore and Herrero (1988) investigate the question under complete information, finding efficient payoffs. Moreno and Wooders (2010) extend the analysis to markets with a lemons problem where no efficient one-price equilibrium may exist. They find that payoff limits remain as the highest payoffs in a static one-price model, i.e., inefficient *if and only if* the lemons problem is severe.[8] Unlike our current case, buyers can only separate sellers by randomizing between different prices, which leaves the surplus to low quality sellers and screens all assets with the same intensity – thereby fostering inefficient outcomes.

Our work contributes to this literature by showing that efficient decentralized screening can outperform inefficient centralized trade.[9] Previously, efficient trade mechanisms in a lemons market have been related to sorting. In Hendel et al. (2005), observed asset

[5]Otherwise, buyers should only offer high prices and only trade for high signals but, then, average asset quality decreases so much that buyers only offer low prices – a contradiction.

[6]Here, buyers obtain positive trade surplus for at least the highest signals, unlike in Moreno and Wooders (2010), where buyers mix between high and low prices and receive no payoffs.

[7]As stressed already by Wilson (1980), different stable equilibria can exist. For example, if $\Delta_l \geq \Delta_h \geq \Delta_g$, an inefficient limit equilibrium with standard trade dynamics exists in addition to the two previously described efficient ones. In this case, intensive screening of both assets erodes payoffs, however, the payoffs may exceed the static one-price model payoffs.

[8]For inefficient outcomes or non-Walrasian payoffs, see Rubinstein and Wolinsky (1985, 1990); De Fraja and Sakovics (2001); Blouin and Serrano (2001); Serrano (2002); Blouin (2003); Shneyerov and Wong (2010). As the key differences, our model allows buyers to choose their prices, has constant exogenous entry, and does not rely on coordinated punishments.

[9]When frictions remain positive, Moreno and Wooders (2010) also demonstrate that the surplus created by trade can be higher in the decentralized equilibrium than in the centralized equilibrium. However, the described payoffs remain generally inefficient. Moreover, as noted by Kim (2017), the result does not survive extension to continuous time trading.

vintages allow the establishment of approximately efficient rental markets for all assets. In Inderst and Müller (2002), different assets are traded in separate markets with distinct prices and liquidity conditions. Interestingly, in Inderst and Müller (2002) the expected quality in markets adjusts to support the Riley separating equilibrium outcome whereas, here, only the cutoff signal adjusts to support efficient trading while prices remain semi-pooling as in Moreno and Wooders (2010) or Cho and Matsui (2018).[10]

Another impressive literature considers dynamic trading with adverse selection. Due to the different time preferences of high and low quality sellers, standard dynamics are derived in almost all articles in the literature. The few exceptions that feature reversed dynamics (Taylor, 1999; Zhu, 2012; Kaya and Kim, 2018; Palazzo, 2017; Martel, 2018; Hwang, 2018; Martel et al., 2022) are characterized by a non-steady-state setup and observable time-on-market. Kaya and Kim (2018) explore a dynamic model where an asset seller meets a sequence of buyers who offer prices after observing the marketing time and a private quality signal of the asset. Trade dynamics depend on exogenous prior beliefs. If the prior is low, dynamics are standard. However, reversed dynamics prevail when buyers have inflated prior beliefs about quality, which alleviates screening to the point that no seller accepts low prices. Our article changes the setup by focusing on markets where the average asset quality is endogenous and constant.[11,12] Because assets exit the market upon trading, reversed dynamics mean that low quality remains in the market longer, thus decreasing the average market quality and buyers' quality expectations. As a consequence, because quality expectations at the cutoff are bounded above by prior, we can show that dynamics only reverse when the lemons problem is non-severe; this follows immediately from simple application of the monotone likelihood ratio property. Our work connects trade dynamics to efficient asset screening and delivers a measure of the information required in efficient trading for given trade frictions. Previous work remains mute about the relationship between efficiency and dynamics and the complementary roles of information and frictions in mitigating the lemons problem.

The article is organized as follows. The model is outlined in Section 2 and its basic features in Section 3. Section 4 describes limit equilibria first with unbounded information and later with bounded information. Section 5 concludes by discussing the effects of alternative model assumptions and the role of commitment granted by signal information. Proofs are relegated to the Appendix.

[10] Contrary to what we have, Inderst and Müller (2002) assume that buyers outnumber sellers, which erases buyers' payoffs fostering therefore an efficient outcome.

[11] In the sequential adverse selection experiment of Araujo et al. (2021) the majority of players applied stationary responses in contrast to the optimal time varying ones.

[12] We also dispense with the assumption in Kaya and Kim (2018) that time-on-market is observable, our focus being on decentralized markets where assets sell quietly.

2 Model

The model closely follows that of Moreno and Wooders (2010) except for the added buyer signals. The general setup emulates modern information-rich over-the-counter markets, where buyers face a steady flow of new trade opportunities and asset information.

Time t is discrete and horizon infinite. A unit mass of buyers and a unit mass of sellers enter the market in each period t. Thereafter, all buyers and sellers in the market are randomly matched in pairs in order to trade. A buyer and a seller who trade will exit the market. If there is no trade, the match dissolves and the buyer and the seller will return to the market, where both have an opportunity to trade with someone else in the next period.

Buyers and sellers discount future payoffs by the common discount factor $\delta < 1$. This discount factor captures trade frictions by showing how much payoffs are reduced if opportunities for trading are delayed. We will be focused on the limit $\delta \to 1$ where frictions of trade disappear.

Every seller holds an indivisible asset whose quality $\theta = h, l$ is the seller's private information. The payoff from an asset of quality θ to the seller is denoted as C_θ and the payoff to the buyer as U_θ. The buyer's payoff exceeds the seller's payoff, and gains from trade therefore arise: $U_\theta > C_\theta$.

We assume that one half of the entering sellers have a high-quality asset ($\theta = h$) and the rest a low-quality asset ($\theta = l$). The assumption is innocuous. It delivers a tractable parametrization that will help to highlight the drivers of our results. We relax it later without substantial changes.

We assign the following magnitudes to the payoffs, which allow for the presence of a lemons problem.

$$U_h > C_h > U_l > C_l$$

Properties of equilibria will depend on the relative trade surpluses of assets and the "gap", defined as follows

$$\begin{aligned}
\Delta_h &:= U_h - C_h, \\
\Delta_l &:= U_l - C_l, \\
\Delta_g &:= C_h - U_l.
\end{aligned}$$

Note that, although high-quality assets are always more valuable to both buyers and sellers, the low trade surplus Δ_l can still exceed the high Δ_h if the spread between a buyer's and a seller's payoffs is higher.

The gap $\Delta_g = C_h - U_l$ represents the temptation of low quality sellers to trade for a high price $p \geq C_h$ in stead of a low price $p \leq U_l$. The minimum price a high quality seller accepts is C_h; the maximum price a buyer is willing to pay for a low-quality asset is U_l. In a static one-price model, a lemons problem always arises if a buyer's payoff of buying a random asset, $\frac{U_h+U_l}{2}$, remains below a high quality seller's payoff of holding his asset, C_h, which gives

$$\begin{aligned}\overline{U} := \frac{U_h + U_l}{2} &< C_h,\\ U_h + U_l &< 2C_h,\\ U_h - C_h &< C_h - U_l,\\ \Delta_h &< \Delta_g.\end{aligned}$$

We can thus see that only low-quality assets can be traded in a static one-price model if the gap exceeds the high trade surplus. However, in the gap remains smaller, $\Delta_h \geq \Delta_g$, a lemons problem may not arise. In that case, the static one-price model has both an efficient equilibrium where trade occurs at $p \geq C_h$ (all assets are traded) and an inefficient equilibrium where trade occurs at $p \leq U_l$ (low quality is traded).

Our dynamic model extends the static one-price model in that (i) there could be trade at different prices for different signals in a meeting between a buyer and a seller, and (ii) trade could be postponed if the terms of trade in the ongoing meeting are not sufficiently attractive. Furthermore, unlike many papers which presume that the static lemons condition holds, our paper studies markets with both $\Delta_g \geq \Delta_h$ ("severe lemons problem") and $\Delta_h > \Delta_g$ ("non-severe lemons problem"). We also allow for both $\Delta_h \geq \Delta_l$ and $\Delta_l > \Delta_g$. These cases lead to different equilibria.

Assets are traded "over-the-counter" in meetings with a buyer and a seller. After a buyer and a seller are randomly matched, the buyer obtains a signal s of the seller's asset quality and, thereafter, makes the seller a take-it-or-leave-it-offer p about the price. If the seller accepts the price p, the asset is traded to the buyer, and both traders exit the market. Otherwise, the buyer and seller separate and wait until the next trade opportunity arises in the following period with someone else. The market is so large that the same buyer and seller are almost never matched again.

To investigate how much information is required for efficient decentralized screening, we allow the informativeness of signals span all values from uninformative to revealing. Signals s are distributed according to distribution functions $F_\theta : [0,1] \to [0,1]$, which are continuous and supported on the unit interval $[0,1] = \text{cl}\{s | f_\theta(s) > 0\}$, where f_θ denotes the density function related to F_θ.[13] For simplicity, we assume that higher signals indicate higher quality. Extreme signals at the limits of $[0,1]$ approach being perfectly revealing. [14] Assumption 1 captures these ideas.

[13]The set closure cl$A$ is the smallest closed set which contains the original set A.
[14]As F_θ are continuous, the likelihood of observing a revealing signal is almost zero.

Assumption 1

$$\frac{f_h(s)}{f_l(s)} \in (0,\infty)\,,\ \textit{for all } s \in (0,1)\,,$$
$$\frac{\partial}{\partial s}\frac{f_h(s)}{f_l(s)} \in (0,\infty)\,,\ \textit{for all } s \in (0,1)\,,$$
$$\lim_{s\to 0}\frac{f_h(s)}{f_l(s)} = 0,$$
$$\lim_{s\to 1}\frac{f_h(s)}{f_l(s)} = \infty.$$

The first two lines just state that signals $s \in (0,1)$ satisfy the standard monotone likelihood ratio property (MLRP). The two latter lines entail more specifically that any likelihood ratio $\frac{f_h(s)}{f_l(s)} \in (0,\infty)$ is attainable for an appropriate signal $s \in (0,1)$.

To focus on decentralized environments and simple trading strategies, we further assume that (i) the signals and actions in a pairwise meeting are not observable by outsiders, and (ii) strategies do not condition on the signals observed in earlier meetings.

We study simple steady-state equilibria in behavioral strategies $\boldsymbol{\sigma} = (p, a_h, a_l)$. The strategy of a buyer is a function $p : [0,1] \to \Delta\,[0,\infty)$ mapping a signal s to the probability distribution $G(s)$ of offers $p(s)$. The strategy of a seller is a function $a_\theta : \mathbb{R} \to [0,1]$ that maps a price p to the probability of acceptance $a_\theta(p)$.

The employed solution concept is a perfect Bayesian equilibrium (PBE). A PBE is a pair $(\boldsymbol{\sigma}, \boldsymbol{\pi})$ consisting of a strategy profile $\boldsymbol{\sigma}$ and a belief system $\boldsymbol{\pi}$ such that (i) the strategy profile $\boldsymbol{\sigma}$ is optimal given the beliefs $\boldsymbol{\pi}$, and (ii) the belief system $\boldsymbol{\pi}$ is derived from the profile $\boldsymbol{\sigma}$ with Bayes' rule whenever possible.

Our focus on a steady-state market, maintaining constant proportions of high and low-quality assets, enables us to endogenize buyers' expectations of average asset quality. Subsequently, we find that equilibrium trading strategies impose significant restrictions on market quality and future payoffs, mitigating a buyer's tendency for excessive asset screening. This finding is pivotal for supporting efficient limit equilibria.

The existence of an equilibrium is not immediately evident. In general, low frictions render buyers selective, to the point where they might only offer low prices accepted by low-quality sellers. In contrast, the implied surge in asset quality suggests they should only offer high prices - raising the possibility of a contradiction. Our analysis demonstrates how this contradiction can be avoided by adjusting asset screening.

Intuitively, as costs of waiting disappear, buyers become more selective and only offer a high price when almost certain about high quality. In a steady-state market, non-traded assets accumulate, and a buyer thus expects to trade both assets with equal probabilities in future matches. Therefore, as the benefit of waiting is bounded, a buyer becomes more willing to make a high offer, leading to the discovery of an equilibrium.

In the upcoming sections, we link the properties of equilibria to asset screening, that is, the time cost of obtaining a high price for the asset. Details of the analysis depend on whether $\Delta_g > \Delta_h$ and $\Delta_l > \Delta_h$.

3 Preliminaries

Any equilibrium defines continuation values, V_b for a buyer and V_θ for a seller.[15] Sequential rationality requires that the strategies of a buyer and a seller in a meeting are optimal given V_b and V_θ.

After observing a buyer's price offer p, a seller chooses whether to accept it. The optimal choice satisfies the Bellman equation:

$$V_\theta(p) = \max_{a_\theta} \quad a_\theta(p - C_\theta) + (1 - a_\theta)\delta V_\theta. \tag{1}$$

By accepting the offer, the seller obtains the price p but loses the value of holding the asset C_θ. Instead, by rejecting the price, the seller keeps the asset and retains the value of selling it later, δV_θ. The problem of the seller does not depend on whether the seller can observe the signal.

We can see immediately that the optimal strategy of a seller is a cutoff strategy: a seller accepts any price above a cutoff but rejects smaller offers. The cutoff equals the sum of a seller's reservation value and continuation value $C_\theta + \delta V_\theta$, denoting the opportunity cost of accepting the price.

Lemma 1 (Seller's cutoffs) *For any V_θ, the optimal strategy of a seller is a cutoff strategy, defined as follows for a seller of quality $\theta = h, l$*

$$a_\theta(p) = \begin{cases} 1, & \text{if } p \geq C_\theta + \delta V_\theta, \\ 0, & \text{if } p < C_\theta + \delta V_\theta. \end{cases}$$

Conditional on observing the quality signal s, a buyer offers the seller a price. The optimal price offer satisfies the Bellman equation:

$$\begin{aligned} V_b(s) = \max_p \quad & q(s)a_h(p)(U_h - p) + (1 - q(s))a_l(p)(U_l - p) + \\ & (q(s)(1 - a_h(p)) + (1 - q(s))(1 - a_h(p)))\delta V_b, \end{aligned} \tag{2}$$

where $q(s)$ denotes the probability conditional on signal s that the asset has high quality. If asset quality is high and the price is accepted, the buyer obtains $U_h - p$, but if the price is accepted by a low-quality seller, the buyer's payoff is $U_l - p$. Otherwise, the buyer

[15] These continuation values are derived in the Appendix.

returns to the market, obtaining the value of δV_b.

Knowing that a seller of quality θ accepts any price above a cutoff, a buyer who targets this seller never offers more than $C_\theta + \delta V_\theta$. In general, a buyer either offers i. a high price p_h that targets a high-quality seller, ii. a low price p_l that targets a low-quality seller, or iii. an even lower price p_0 that neither seller accepts.

Lemma 2 (Buyer's cutoffs) *For any (V_b, V_l), there is a cutoff signal $y \in [0,1]$ that allows to express the optimal strategy of a buyer as follows*

$$p(s) = \begin{cases} p_h, & \text{if } s \geq y, \\ p_l, & \text{if } s < y \text{ and } \Delta_l \geq \delta(V_l + V_b). \\ p_0, & \text{if } s < y \text{ and } \Delta_l < \delta(V_l + V_b), \end{cases}$$

where $p_0 < p_l = C_l + \delta V_l \leq p_h = C_h + \delta V_h = C_h$.

The optimal price strategy of a buyer is subtle as it depends on the endogenous valuations V_b and V_l. Without showing the existence of an equilibrium and without knowing the exact values of V_b and V_l – which come later – we can show that a buyer will offer p_h for signals that exceed a cutoff y, when the buyer is sufficiently certain of high quality. Instead, for signals below the cutoff y, a buyer offers lower prices p_l or p_0.

Which of these offers is made for low signals depends on the continuation values of a buyer and a low-quality seller, $\delta(V_l + V_b)$. In a static one-price setup, a buyer can offer a low price p_l and trade low quality if the expected asset quality is low. However, in our dynamic setting, both buyers and low-quality sellers can also wait for higher signals that suggest high quality and allow for trade at a high price p_h. This prospect can increase the continuation values $\delta(V_l + V_b)$ to the point where they exceed the gains from trade Δ_l. When this is so, it is impossible for a buyer and a low-quality seller to agree on a low price $p < U_l$ that would cover δV_b to the buyer and $C_l + \delta V_l$ to the seller.

It can be shown that the offer to a low-quality seller, p_l, lies below the offer to the high-quality seller, p_h, because the seller's reservation value, C_l, is lower. Additionally, we observe that the offer that targets the high-quality seller, p_h, cannot exceed C_h. This is because a holdup problem arises in a pairwise meeting, allowing the buyer to reduce the offer from p_h to $(1-\delta)C_h + \delta p_h$ unless p_h equals the seller's reservation value C_h. This entails that the continuation value of a high-quality seller must be zero. The gains from trade are therefore shared by buyers and low-quality sellers. The non-accepted price offer p_0 is indeterminate, but it has to lie below the low price cutoff $p_l (\leq p_h)$.

3.1 Expected quality

Because gains from trade are positive with both qualities, buyers are willing to pay higher prices for higher quality, but are reluctant to do so if the expected quality remains low. Buyers' optimal price strategies hence depend on their beliefs. Specifically, a buyer will offer a high price $p_h = C_h$ which both sellers will accept if and only if the probability $q(s)$ that the seller has a high-quality asset reaches a cutoff, i.e., $q(s) \geq q(y)$. The cutoff $q(y)$ solves the following equality, requiring that a buyer is indifferent between offering a high price, p_h, and either p_l or p_0 – whichever provides a higher buyer payoff:

$$q(y) \underbrace{(U_h - C_h)}_{>0} + (1 - q(y)) \underbrace{(U_l - C_h)}_{<0} = \max \{(1 - q(y)) (U_l - p_l) + q(y)\delta V_b, \delta V_b\} . \quad (3)$$

If a buyer offers a high price $p_h = C_h$, the buyer's payoff is positive $U_h - C_h = \Delta_h$ if the seller has a high-quality asset but, if the seller has a low-quality asset, the buyer's payoff is negative $U_l - C_h = -\Delta_g$. Instead, the payoff for offering a low price p_l is $(1 - q(y)) (U_l - p_l) + q(y)\delta V_b$ (only low quality sellers accept the offer) and the payoff for offering a low price p_0 is δV_b (neither of the sellers accepts this offer). If the seller does not accept a price, the buyer's continuation value is δV_b.

Buyer beliefs about asset quality $q(s)$ are shaped by both market composition and signal information. First, buyers take into account the endogenous market composition, that is, how many assets of each quality circulate in the market. We call these buyers' prior beliefs, that only condition on the equilibrium trade probabilities of assets, *unconditional* beliefs q_u. Second, buyers consider information conveyed by the signal they obtain in the current meeting. These *conditional* beliefs are denoted by $q_c(s)$.

Because sellers enter the market in equal proportions and exit the market upon trading, the market composition is determined solely by the sellers' relative trading probabilities. In a steady-state, the mass of assets of quality θ in the market remains constant, denoted as M_θ, and the inflow of each quality to the market has to equal outflow:

$$1/2 = M_\theta(1 - G_\theta(p_\theta -)).$$

On the left-hand side (lhs), 1/2 denotes the entry of assets of each quality in the market. In each time period, a unit mass of assets enters, half of each quality. The right-hand side (rhs) represents asset exits, with M_θ assets of each quality in the market. Each asset trades with a probability of $1 - G_\theta(p_\theta -)$, of a buyer offering at least p_θ.[16]

Solving for the measures M_θ and using Bayes' rule, q_u and $q_c(s)$ can be derived as

[16]Technically, $G_\theta(p_\theta -) = \lim_{p \to p_\theta} G_\theta(p)$ denotes the left derivative of a buyer's unconditional (marginal) offer distribution G_θ to sellers of quality θ at x.

follows

$$q_u = \frac{M_h}{M_h + M_l} = \frac{1}{1 + \frac{1-G_h(x_h-)}{1-G_l(x_l-)}}, \tag{4}$$

$$q_c(s) = \frac{M_h f_h(s)}{M_h f_h(s) + M_l f_l(s)} = \frac{1}{1 + \frac{1-G_h(x_h-)}{1-G_l(x_l-)} \frac{f_l(s)}{f_h(s)}}, \tag{5}$$

where $q_c(s)$ is derived from q_u by incorporating the information about the likelihood ratio $\frac{f_l(s)}{f_h(s)}$ of receiving the observed signal s from a low-quality asset *versus* high.

We observe that buyers' beliefs about asset quality increase under three conditions: when low-quality assets trade faster, when high-quality assets trade slower, or when the observed signal increases. Namely, if one asset quality is traded more slowly than the other asset quality, it amasses in the market in relative terms, increasing a buyer's expectation of meeting a seller with this quality. Further, because the likelihood ratio $f_h(s)/f_l(s)$ is by assumption increasing in s, buyers' conditional beliefs $q_c(s)$ are clearly increasing in the observed signal. Consequently, a buyer will offer a high price if and only if the signal is above the cutoff, i.e., $q(s) \geq q(y)$ *iff* $s \geq y$.

Our framework deviates from most earlier approaches in that buyers observe continuous signals with variable information content. By conditioning pricing on signals, equilibria in pure strategies are sustained, allowing signals to act as a purification device, as proposed by (Harsanyi, 1973).[17] Mixing between higher and lower prices occurs, for example, in a model without signals by Moreno and Wooders (2010) and in a model with binary signals by Kaya and Kim (2018). Similar to signals, mixing allows for the adjustment of screening, albeit less finely than signals. Unlike here, limit equilibria are hence inefficient in Moreno and Wooders (2010).

3.2 Trading dynamics

Whether trade dynamics are standard, reversed, or what we call "knife-edge" depends on the endogenous valuations V_b and V_l.

Lemma 3 *Feasible equilibrium dynamics can be classified into the following patterns:*

1. *If $\Delta_l \geq \delta(V_l + V_b)$, trade dynamics are standard and low-quality assets trade faster: $p(s) = C_h$ for $s \geq y$, $p(s) = p_l$ for $s < y$, and $q_u = \frac{1}{1+(1-F_h(y))} \geq 1/2$.*
2. *If $\Delta_l < \delta(V_l + V_b)$, trade dynamics are reversed and high-quality assets trade faster: $p(s) = C_h$ for $s \geq y$, $p(s) = p_0$ for $s < y$, and $q_u = \frac{1}{1+\frac{1-F_h(y)}{1-F_l(y)}} \leq 1/2$.*

[17]Later-defined "knife-edge" dynamics allow for both pure and mixed price strategies as buyers can either randomize between p_0 and p_l for $s < y$ or offer p_0 for $s \in [0, z)$ and p_l for $s \in [z, y)$. Any equilibrium payoffs can be sustained by pure strategies.

3. *If* $\Delta_l = \delta(V_l + V_b)$, *"knife-edge" trade dynamics arise:* $p(s) = C_h$ *for* $s \geq y$, $p(s) = p_l$ *for* $s \in [z, y)$ *and* $p(s) = p_0$ *for* $s \in [0, z)$, *and* $q_u = \frac{1}{1+\frac{1-F_h(y)}{1-F_l(z)}} \lesseqgtr 1/2$.

We focus on standard and reversed trade dynamics in the main text; the analysis of knife-edge dynamics is delegated to the Appendix. We can show immediately that reversed dynamics cannot arise under a severe lemons problem due to the deteriorating market quality.

Lemma 4 *A necessary condition for reversed dynamics is* $\Delta_h \geq \Delta_g$.

Proof. Consider the beliefs of a buyer who has observed the cutoff signal y. This buyer must be indifferent between offering prices p_h and p_0. As both assets are only traded for high signals, low quality with probability $1 - F_l(y)$ and high quality with probability $1 - F_h(y)$, the beliefs of the buyer are given as follows.

$$q_c(y) = \frac{1}{1 + \frac{1-F_h(y)}{1-F_l(y)} \frac{f_l(y)}{f_h(y)}} < q_0 = \frac{1}{2},$$

But now, our assumption of MLRP implies increasing hazard rate, $\frac{1-F_h(y)}{1-F_l(y)} \frac{f_l(y)}{f_h(y)} > 1$, indicating that the asset is more likely of low quality. Also, if $\Delta_g > \Delta_h$, the buyer is not willing to offer a high price if both qualities are equally likely – because the lemons problem is severe.

Thus, if $\Delta_g > \Delta_h$, the buyer is not willing to offer a high price after observing the cutoff signal – at which expectations are worse. This contradicts the assumption that a buyer is willing to offer p_h, demonstrating that reversed dynamics cannot arise under $\Delta_g > \Delta_h$. □

The result shows that steady-state trade places new restrictions on equilibrium dynamics, absent from non-steady-state setups such as Kaya and Kim (2018), where reversed dynamics arise when the exogenous prior is inflated above the long run level. Our analysis in the following Section 4 shows that the steady-state market composition not only places limitations on the quality at the cutoff $q_c(y)$, in shown in the proof of Lemma 4, but also notably restricts the buyer continuation value V_b, with significant effects on search incentives.

4 Equilibrium

4.1 Positive frictions

To evaluate market welfare in a steady-state equilibrium, we use the measure applied by Moreno and Wooders (2010),

$$W = V_b + \frac{1}{2}V_h + \frac{1}{2}V_l = V_b + \frac{1}{2}V_l,$$

denoting the expected present discounted value of the trade surplus accruing to one entry cohort of buyers and sellers. The maximum trade surplus is given by the complete information benchmark, $\frac{\Delta_h+\Delta_l}{2}$, which is reached if all assets are traded in the period they enter the market. Lemma 5 shows that the maximum is generally unattainable due to positive asset screening ($y > 0$) and positive trade frictions ($\delta < 1$).

Lemma 5 $y > 0$ *for* $\delta < 1$.

According to Lemma 5, the cutoff y is positive in dynamic markets with signals. This indicates that, although the surplus of trading is positive with both qualities, some meetings are not conductive to trade as would be efficient. By Lemma 2, high-quality sellers only trade for high prices $p_h = C_h$, which a buyer offers to them with probability $1 - F_h(y)$. Because the cutoff y is positive, this probability is less than one.

The result is notable in showing that screening reduces efficiency even when the lemons problem is not severe in the market. In the absence of signals, all sellers could trade in their first meeting for a common high price C_h, maintaining high average asset quality if the lemons problem is non-severe. Trade would thus be efficient. However, Lemma 5 demonstrates that the pooling equilibrium becomes impossible to sustain when signals are introduced. By Assumption 1, for any $M_h, M_l, \epsilon > 0$, there is a positive probability $F_h(s(\epsilon)) > 0$ of observing such a low signal $s < s(\epsilon)$ that a buyer's beliefs in (5) collapse to $q_c(s) < \epsilon$. Almost certain about low asset quality, a buyer hence makes a low price offer, which a high-quality seller rejects. An endogenous lemons problem therefore arises.

Previously, Daley and Green (2012) observe in a model with news that trade could be delayed without a severe lemons problem because traders wait for news to accumulate in order to trade. The reason for trade delay is much like here, that information renders buyer beliefs noisy. This will make it harder for a buyer and a seller to agree on a price when the noise takes a buyer's belief about an asset far from its seller's belief.[18]

4.2 Vanishing frictions

We move to investigate markets where trade frictions are negligible. Decreasing trading frictions increase the information requirements of buyers. Buyers thus require exceeding quality confirmation before a high offer is made.

Lemma 6 $y \to 1$ *as* $\delta \to 1$.

Lemma 6 shows that the cutoff approaches its upper bound as frictions vanish. This

[18]This is akin to the so called Hirshleifer (1971) effect, which shows that information can destroy efficient pooling opportunities.

is because the buyer continuation value of waiting for higher signals increases as trading frictions decrease. A buyer thus needs to be more strongly convinced about high asset quality to terminate search by offering C_h.

The mechanism is mediated by the MLRP. Specifically, because faster trading assets accumulate in a steady-state, increasing hazard rate implies that the quality that the buyer expects to trade at the cutoff (proportional to $\Delta_h - (1 - F_h)\frac{f_l}{f_h}\Delta_g$ or $\Delta_h - \frac{1-F_h}{1-F_l}\frac{f_l}{f_h}\Delta_g$) cannot attain the quality that the buyer expects to trade in later meetings (proportional to $\Delta_h - (1 - F_l)\Delta_g$ or $\Delta_h - \Delta_g$), unless $y \to 1$ as $\delta \to 1$. An incentive to postpone trade thus arises for buyers, driving up the cutoff as trading frictions diminish.

While buyers become more selective as frictions disappear, the efficiency properties of limit equilibria are uncertain. On the one hand, buyers obtain cheaper information when frictions decrease since it costs less to wait for highly informative signals. On the other hand, buyers also become more selective, possibly foregoing valuable trades. Interestingly, we find that equilibrium properties are not governed by either of the limit properties alone but the *proportions* of δ and y in which the limit $(y, \delta) \to (1, 1)$ is approached.

In particular, we find that there are different paths satisfying $(y, \delta) \to (1, 1)$ that correspond with potential limit equilibria.

1. In the first tentative equilibrium, the odds ratio of high asset quality $\frac{f_h(y)}{f_l(y)}$ remains low with respect to discounting δ. The general ease of trading at high prices thus entails that dynamics are reversed and efficient pooling prevails.

2. In the second equilibrium candidate, $\frac{f_h(y)}{f_l(y)}$ increases relative to discounting δ. This guarantees that the cost of obtaining a high price p_h is much lower for high quality than low-quality assets. Dynamics are standard and screening efficient.

3. In the third possible equilibrium, $\frac{f_h(y)}{f_l(y)}$ is even higher with respect to discounting δ. All sellers thus face extremely high cost of waiting for a high price offer. This excessive screening is inefficient. Trade dynamics remain standard.

Equilibrium existence hinges on the severity of the lemons problem (whether $\Delta_g > \Delta_h$) and the relative gains from trade (whether $\Delta_l > \Delta_h$).

4.3 Screening with unbounded signal information

We proceed to describe conditions when each of these equilibrium candidates represents a steady-state limit equilibrium. This is done by partitioning the signal space by screening, i.e., the time cost of trading different assets at a high price. Lemma 7 formalizes our notion of screening.

Lemma 7 *For any* $M > 1$*, there exist signals* $0 < s_0 < s_l < s_h < 1$ *and functions* $\nu_h(y,\delta) < \nu_l(y,\delta)$ *such that*

$$\nu_h(y,\delta) := \frac{1-\delta F_h(y)}{1-F_h(y)},$$
$$\nu_l(y,\delta) := \frac{1-\delta F_l(y)}{1-F_l(y)},$$

$$\nu_l(s_0,\delta) = \nu_h(s_l,\delta) = \frac{1}{M} < M \leq \nu_l(s_l,\delta) = \nu_h(s_h,\delta),$$

and $s_0 \to 1$ *as* $M \to \infty$.

Lemma 7 introduces screening functions ν_θ, which quantify the difficulty associated with selling an asset of quality θ for a high price. The inverse of ν_θ denotes the probability of receiving a high price offer for the asset in either this period or in any future time period.

$$\left(\frac{1-\delta F_\theta(y)}{1-F_\theta(y)}\right)^{-1} = (1-F_\theta(y))(1+\delta F_\theta(y)+\delta^2 F_\theta(y)^2+\ldots)$$

Both ν_h and ν_l are increasing in y and decreasing in δ because waiting for a high price has a higher cost if either the discount factor δ (representing frictions) is lower or the cutoff signal y (representing screening) is higher. In general, the screening function of high quality ν_h always stays below that of low quality ν_l because higher signals $s \geq y$ are observed more frequently with high-quality assets. In addition, both screening functions are continuous in arguments (y,δ), approaching unity as $\delta \to 1$, for fixed $y \in (0,1)$, and approaching infinity as $y \to 1$, for fixed $\delta \in (0,1)$.

Lemma 7 shows that screening partitions the signal space in four regions: First, if the cutoff y belongs to $I_0 = [0, s_0]$, it is very easy for all assets to trade for p_h. Second, if $y \in I_l = (s_0, s_l)$, obtaining a high price for low quality becomes hard (i.e., as hard as we want) whereas receiving a high price for high quality remains easy (i.e., as easy as we want). Third, presuming the cutoff reaches higher levels, $y \in I_h = [s_l, s_h)$, screening also intensifies for high quality. Finally, for $y \in I_1 = [s_h, 1]$, it becomes very hard to sell high quality, which never settles for a low price p_l.

Figure 1 illustrates this partitioning by mapping ν_h and ν_l as functions of y and showing the cutoffs s_0 and s_l corresponding to $M = 4$ and $\delta = 0.95$; s_h is so close to unity that it is indiscernible. Similar to cutoffs s_0 and s_h, s_l increases if δ increases. Thus, to keep the relative screening of low quality above a certain level, $\frac{\nu_l}{\nu_h}(y,\delta) \geq M^2$, screening must intensify if frictions of trading are decreased.

Leveraging these basic properties, we can demonstrate existence and characterize equilibria by focusing on screening. This involves rewriting payoffs in terms of ν_h and ν_l. A powerful steady-state property we find is that, irrespective of which dynamics of trade prevail, screening cannot increase the likelihood of trading one quality over the other in

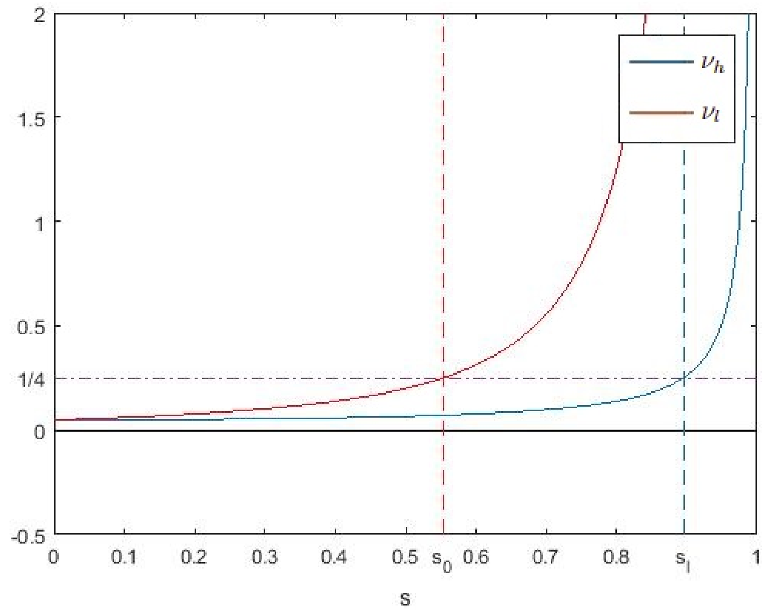


Figure 1: Illustration of Lemma 7.

the future. For example, if high quality is screened more strongly its concentration in market will elevate, entailing that a buyer will trade it equally often as before, despite stronger screening. As a result, because assets are assumed to enter the market in equal proportions, a buyer will expect to trade both assets with the same probability. This allows a simple expression of continuation values.

Proposition 3 shows that, under standard dynamics, the probability of trading high quality is $q_u(1-F_h) = \frac{1-F_h}{1+(1-F_h)}$ and the probability of trading low quality is $(1-q_u) = \frac{1-F_h}{1+(1-F_h)}$, which equal. We can thus show that the buyer continuation value is

$$V_b(y,\delta) = \frac{\Delta_h - (1-F_l(y))\Delta_g + F_l(y)(U_l - V_l)}{2+\nu_h(y,\delta)},$$

obtained by dividing buyer payoffs in (2) by the common trade probability of assets and reorganizing terms. Screening of high-quality assets ν_h reduces payoffs because a buyer is foregoing valuable trade opportunities of high-quality assets for low signals.

For reversed dynamics, Proposition 3 shows that a buyer expects to trade high quality with probability $q_u(1-F_h) = \frac{(1-F_l)(1-F_h)}{(1-F_l)+(1-F_h)}$ and expects to trade low quality with the same probability $(1-q_u)(1-F_l) = \frac{(1-F_l)(1-F_h)}{(1-F_l)+(1-F_h)}$. The buyer value is this case is hence

$$V_b(y,\delta) = \frac{\Delta_h - \Delta_g}{2+\nu_h(y,\delta)+\nu_l(y,\delta)},$$

derived as before by dividing buyer payoffs in (2) by the trade probability and reorganizing terms. The payoffs of a buyer are reduced by both ν_h and ν_l because assets only trade for high signals. A buyer is hence foregoing trades with both assets for low signals.

Similarly, screening also reduces the payoffs of a low quality seller in (1) who obtains rents from trading at high prices for high signals, which gives

$$V_l(y,\delta) = \frac{\Delta_g + \Delta_l}{1 + \nu_l(y,\delta)}.$$

Generally, an equilibrium with standard trade dynamics is given by y and (V_b, V_l) satisfying the following system[19]

$$q_c(s) = \frac{1}{1 + \frac{1-F_h(y)}{1}\frac{f_l(s)}{f_h(s)}}, \text{ for } s \in [0,1]$$

$$q_c(y)\,(U_h - C_h) + (1 - q_c(y))\,(U_l - C_h) = (1 - q_c(y))\,(U_l - V_l) + q_c(y)V_b, \tag{FP-s}$$

$$V_b + (V_l - C_l) \leq \Delta_l. \tag{IC-s}$$

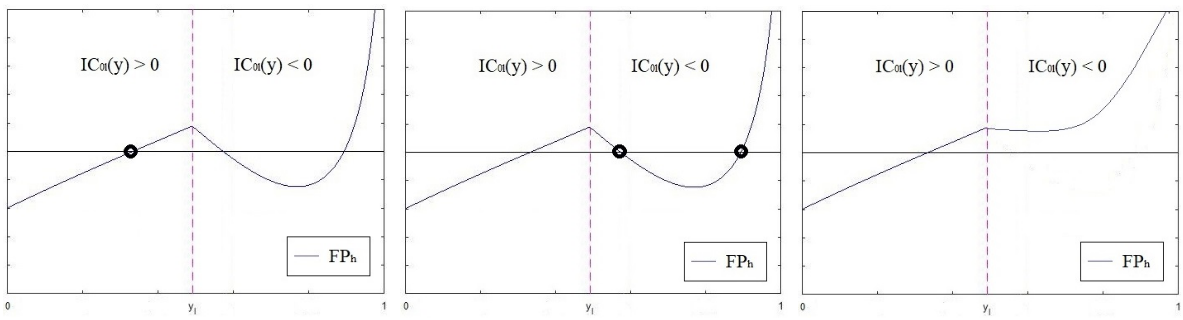
Similarly, an equilibrium with reversed trade dynamics is given by y and (V_b, V_l) satisfying the system of conditions

$$q_c(s) = \frac{1}{1 + \frac{1-F_h(y)}{1-F_l(y)}\frac{f_l(s)}{f_h(s)}}, \text{ for } s \in [0,1]$$

$$q_c(y)\,(U_h - C_h) + (1 - q_c(y))\,(U_l - C_h) = V_b, \tag{FP-r}$$

$$V_b + (V_l - C_l) > \Delta_l. \tag{IC-r}$$

In both systems the first line denotes buyer beliefs. The next line is a *fixed point condition* FP_h that defines the cutoff. The last line is an *incentive condition* IC_{0l} which ascertains that dynamics are as assumed. Because FP_h and IC_{0l} are continuous in y, we can demonstrate existence and characterize equilibria by locating for the roots of $FP_h(y)$ and $IC_{0l}(y)$ as frictions of trading are reduced. The roots that correspond to equilibria are shown as the black circles in Figure 2. We describe the conditions of equilibrium existence in the following Propositions 1-3.



(a) $\Delta_h \geq \Delta_g$ (b) $\Delta_l \geq \Delta_h$ (c) $\Delta_g > \Delta_h > \Delta_l$

Figure 2: FP_h and IC_{0l} for low frictions.

[19]See the Appendix for the details and additional commentary.

4.3.1 Reversed dynamics

The first equilibrium illustrated in Figure 2a has the most relaxed screening and thus reversed dynamics.

Proposition 1 (Reversed dynamics) *If $\Delta_h \geq \Delta_g$, there exists an efficient limit equilibrium where $\nu_h \leq \nu_l \to 0$,*

$$V_l \to \Delta_l + \Delta_g, V_b \to \frac{\Delta_h - \Delta_g}{2}$$
$$W = V_b + \frac{1}{2}V_l \to \frac{\Delta_h + \Delta_l}{2},$$

as $\delta \to 1$. The equilibrium features reversed dynamics and low average market quality with $q_u = 0$ and $q_c(y) = 1/2$.

We know from Lemma 4 that reversed dynamics cannot arise under a severe lemons problem. A necessary condition for equilibrium existence is thus $\Delta_h \geq \Delta_g$. We demonstrate next that this condition is sufficient as well.

By (FP-r), the cutoff signal under reversed dynamics satisfies the following fixed point condition

$$\frac{1}{1 + \frac{1-F_h(y)}{1-F_l(y)} \frac{f_l(y)}{f_h(y)}}(U_h - C_h) + \left(1 - \frac{1}{1 + \frac{1-F_h(y)}{1-F_l(y)} \frac{f_l(y)}{f_h(y)}}\right)(U_l - C_h) =$$
$$\frac{1}{2 + \nu_h(y,\delta) + \nu_l(y,\delta)}(U_h - C_h) + \frac{1}{2 + \nu_h(y,\delta) + \nu_l(y,\delta)}(U_h - C_h),$$

where the lhs captures the payoff for offering p_h, $E(u|y) - C_h$ and the rhs that of offering p_0, V_b.

The lhs spans from $U_l - C_h < 0$ to $\overline{U} - C_h > 0$ and the rhs spans from $\overline{U} - C_h > 0$ to 0 as y increases from 0 to 1. By the continuity of the lhs and rhs, we can thus see that a cutoff signal satisfying the fixed point condition (FP-r) exists.

Intuition is rather simple. For very low cutoffs, buyer is almost certain of low asset quality. On the other hand, buyer continuation value remains positive. A buyer will thus rather return to the market than obtain the negative payoff of trading low quality for a high price. As the cutoff is raised, buyer beliefs about the traded asset improve. At the same time, average asset quality in the market deteriorates under reversed dynamics because low-quality assets leave the market slower. Even for the highest cutoffs, a buyer only expects to trade assets with approximately the same probability. However, since there is no severe lemons problem, this gives the buyer a positive payoff. Further, due to stronger screening of assets, buyer continuation value decreases, gradually reaching a level at which a buyer is better off by making a high price offer for average asset quality than returning back to the market. Thus, although reduced frictions decrease the costs of waiting for higher signals, the benefit of waiting is limited as i. screening does not elevate the average traded quality, and ii. screening delays valuable future trading opportunities. If frictions are low but positive, the costs and benefits of waiting thus equal at a certain positive cutoff below s_l, given in Lemma 7.

After demonstrating the existence of a fixed point we characterize the limit equilibrium. Lemma 6 tells us that the cutoff increases to the limit $y \to 1$ as frictions decrease to the limit $\delta \to 1$. The fixed-point condition for equilibrium cutoff thus becomes

$$\frac{1}{2}\Delta_h + \frac{1}{2}(-\Delta_g) = V_b = \frac{\Delta_h - \Delta_g}{2 + \nu_h + \nu_l},$$

which shows that the screening of both qualities remains low in the limit equilibrium. It other words, we find that under reversed dynamics, the limit equilibrium is approached over a path for which both assets are screened very mildly: $\nu_h = \nu_h = 0$.

Lenient screening is important for reversed dynamics where both qualities only trade for high prices as it encourages low quality sellers to wait for high price signals. Lemma 7 is crucial in showing that it is indeed possible to approach the limit $(y, \delta) \to (1, 1)$ over a path which keeps both $\nu_h(y, \delta)$ and $\nu_l(y, \delta)$ as low as desired by keeping the cutoff below the value s_l. An limit equilibrium with the above properties thus exists. Because buyer payoff is $\frac{\Delta_h - \Delta_g}{2}$ and a low-quality seller's payoff is $\Delta_l + \Delta_g$, this equilibrium is also efficient; half the sellers hold a low-quality asset.

These are novel findings, extending reversed dynamics in markets with signals to efficient steady-state markets.[20] Kaya and Kim (2018) describe reversed dynamics in a non-stationary environment, of unknown efficiency properties. A significant caveat to practitioners arising from our research is that, although Kaya and Kim (2018) show that reversed dynamics arise under flexible conditions assuming the prior is above the steady-state beliefs, we observe instead that reversed dynamics cannot be sustained in the long run in steady-state markets under a severe lemons problem.

The restriction may seem unfortunate. There is ample evidence of reversed dynamics in different setups (Hendel et al., 2009; Lei, 2011; Tucker et al., 2013; Albertazzi et al., 2015; Jolivet et al., 2016; Aydin et al., 2019) whereas standard dynamics seem rare (Ghose, 2009). An explanation is suggested by our Lemma 5, which shows that an endogenous lemons problem arises in dynamic markets with informative signals irrespective of whether the lemons problem is *severe*, i.e., $\Delta_g > \Delta_h$. Hence, while the literature has concentrated on severe lemons problems, real-world applications might well be dominated by non-severe ones. The severity of the problem, is hardly known in practice.

[20] Without signals the standard dynamics in lemons markets derive straight from the *skimming property* (Fudenberg and Tirole, 1991), which states that all prices that are accepted by high quality sellers are also accepted by low quality sellers. If the same prices are offered to all sellers, this means that low quality is traded faster. By Lemmata 1 and 2, the skimming property holds also in this article. However, because signals enable buyers to target high prices to high quality sellers, as accurately as desirable, the property does not suffice to characterize trade dynamics with signals.

4.3.2 Standard dynamics

The remaining equilibria pictured in Figure 2b have more intensive screening and thus standard dynamics.

Proposition 2 (Standard dynamics) *If* $\Delta_l \geq \Delta_h$*, there exist both an efficient limit equilibrium where* $\nu_h \to 0 < \nu_l \to \frac{\Delta_g+\Delta_h}{\Delta_l-\Delta_h}$

$$\begin{aligned} V_l &\to \Delta_l - \Delta_h, V_b \to \Delta_h \\ W &= V_b + \frac{1}{2}V_l \to \frac{\Delta_h + \Delta_l}{2}, \end{aligned}$$

as $\delta \to 1$*, and an inefficient limit equilibrium where* $\nu_h \to \frac{\Delta_l-\Delta_h}{\Delta_h} < \nu_l \to \infty$

$$\begin{aligned} V_l &\to 0, V_b \to \Delta_h \\ W &= V_b + \frac{1}{2}V_l \to \Delta_h < \frac{\Delta_h + \Delta_l}{2}. \end{aligned}$$

as $\delta \to 1$*. These equilibria feature standard dynamics and high average quality with* $q_u = q_c(y) \to 1$*.*

By (FP-s) and (IC-s), the cutoff signal under standard dynamics satisfies the following fixed point condition.

$$\begin{aligned} &\frac{1}{1+\frac{1-F_h(y)}{1}\frac{f_l(y)}{f_h(y)}}(U_h - C_h) + \left(1 - \frac{1}{1+\frac{1-F_h(y)}{1}\frac{f_l(y)}{f_h(y)}}\right)(U_l - C_h) = \\ &\qquad \frac{1}{1+\frac{1-F_h(y)}{1}\frac{f_l(y)}{f_h(y)}}V_b + \left(1 - \frac{1}{1+\frac{1-F_h(y)}{1}\frac{f_l(y)}{f_h(y)}}\right)\max\{U_l - V_l, V_b\}, \end{aligned} \tag{FP-s$'$}$$

The lhs denotes the value of trading at a high price p_h and the rhs that of trading at a low price p_l.

The lhs spans from $U_l - C_h < 0$ to $U_h - C_h > 0$ and the rhs spans from V_b to $U_l - V_l$ as y increases from 0 to 1. By the continuity of the lhs and rhs, we can thus see that a cutoff signal satisfying the fixed point condition (FP-s′) exists. Indeed, there can exist several such cutoffs as in Figure 2b. The first one features too low screening to sustain standard dynamics. It satisfies (FP-s′) but not (IC-s). However, as the screening of low quality is increased, buyer rents from low quality trades increase. As these rents $U_l - V_l$ exceed the benefits of waiting for higher signals V_b, two cutoffs that satisfy (FP-s′) and (IC-s) are found for $\Delta_l \geq \Delta_h$. Intuition is that stronger screening decreases a low-quality seller's payoff from waiting for high signals. This allows buyers and low-quality sellers to trade at low prices for low signals, sustaining standard dynamics.

As $(\delta, y) \to (1, 1)$, the fixed point condition (FP-s) approaches the expression

$$\Delta_h = V_b = \frac{\Delta_h + \left(\Delta_l - \frac{\Delta_g + \Delta_l}{1+\nu_l}\right)}{2 + \nu_h}. \tag{6}$$

The lhs of (6) reinstates our earlier finding in Lemma 6 that buyers only offer high prices when almost sure about high asset quality as the costs of waiting vanish. Certainty of trading high quality is important as it shows that a buyer will capture the entire high trade surplus Δ_h since a buyer must be indifferent between offering a high price for Δ_h (lhs) and returning to the market for V_b (rhs) at $s = y$.[21]

On the other hand, as discussed, a buyer expects to trade both assets with equal probability if she returns to the market. The buyer continuation value V_b on the rhs of (6) thus lies below $\frac{\Delta_h+\Delta_l}{2}$. The screening of low-quality assets ν_l increases this benefit of waiting while the screening of high-quality assets ν_h decreases it, allowing to sustain equilibria where the costs and benefits of waiting are aligned.

Closer examination demonstrates that (6) has two solutions satisfying (FP-s) and (IC-s), an efficient one and an inefficient one. First, there is a solution where $\nu_h \to 0 < \nu_l \to \frac{\Delta_g+\Delta_h}{\Delta_l-\Delta_h}$ as $(y, \delta) \to (1, 1)$. Second, there is another solution with $\nu_h \to \frac{\Delta_l-\Delta_h}{\Delta_h} < \nu_l \to \infty$ as $(y, \delta) \to (1, 1)$. Lemma 7 shows that, presuming $\Delta_l > \Delta_h$, there are paths $(y, \delta) \to (1, 1)$ corresponding with such ν_h and ν_l.

The multiplicicity of equilibria originates from the strategic complementarity between the screening of low and high-quality assets:

$$\frac{\partial V_b}{\partial \nu_l} > 0, \frac{\partial V_b}{\partial \nu_h} < 0.$$

The effect of higher ν_l on V_b is positive because stronger screening reduces low quality sellers' payoff, permitting a buyer to capture a larger share of low quality trade surplus Δ_l if the signal is low. This contrasts starkly with the negative effect of ν_h on V_b. A higher ν_h implies both a higher average quality in the market and an increased buyer threshold for offering C_h. As a buyer expects to trade assets at the same rate in a steady-state market, more high-quality assets remain unsold although more meetings involve high-quality assets. Thus, exceeding numbers of meetings result in no trade, eroding buyer payoffs.

In general, standard dynamics increase the expected market quality so much that the (opportunity) cost of waiting $E(u|s) - p_h$ surpasses the benefit of waiting V_b without strong screening. To equate the cost and benefit, we can nevertheless increase the screening of low-quality assets, allowing a buyer to decrease the offer $p_l = V_l$ made to a low quality seller for low signals. The resulting increase in the buyer continuation value $V_b = \frac{\Delta_h + \left(\Delta_l - \frac{\Delta_g+\Delta_l}{1+\nu_l}\right)}{2+\nu_h}$

[21] In Moreno and Wooders (2010), the surplus of trading Δ_l is fully extracted by low quality sellers.

reaching up to Δ_h gives the first efficient equilibrium, where the screening of high-quality assets remains low.

Now, any additional increase in low quality screening and buyer continuation value implies that the benefit of waiting again exceeds the opportunity cost of trading high quality. To equate the benefit of waiting with the cost, it is thus necessary to increase the screening of high-quality assets, which restricts buyer continuation value by reducing trading frequency. The associated reduction in buyer continuation value $V_b = \frac{\Delta_h + \left(\Delta_l - \frac{\Delta_g + \Delta_l}{1+\nu_l}\right)}{2+\nu_h}$ until it meets Δ_h allows to sustain the inefficient equilibrium, with excessive screening of high-quality assets.

4.3.3 Non-existence

Figure 2c points out that a non-existence of an equilibrium is also a possibility. This is already suggested by our previous analysis, which shows that the maximum for market surplus and buyer continuation value is

$$V_b = \frac{\Delta_h + \Delta_l}{2}, \tag{7}$$

If high quality is more valuable, $\Delta_h > \Delta_l$, no equilibrium with standard dynamics, where $V_h = \Delta_h$, thereby exists. Adding to this, if the lemons problem is severe, $\Delta_g > \Delta_h$, no equilibrium with reverse dynamics exists either.

Proposition 3 (Non-existence of equilibrium) *If $\Delta_g > \Delta_h > \Delta_l$, there exists no steady-state limit equilibrium as $\delta \to 1$.*

The intuition for the non-existence of an equilibrium is given by a fundamental discrepancy between (i) the required screening to overcome a severe lemons problem and (ii) the higher payoff of trading high quality than low quality. In particular, although different qualities can in the limit be separated efficiently by signals, buyers cannot be indifferent between trading high quality, for the higher payoff of Δ_h, and low quality, for the lower payoff of Δ_l. The logic is quite simple. In equilibria with standard dynamics, negligible information costs and increasing market quality allow buyers to obtain the full surplus of high quality trade Δ_h. But this means that buyers are no longer interested in trading low quality for a lower trade surplus Δ_l, thus contradicting the assumed dynamics.

Figure 3 summarizes the existence conditions of different equilibria in terms of welfare and dynamics. A multiplicity of equilibria with different dynamics and efficiency properties arises when low quality is more valuable to trade while either a unique equilibrium or no equilirium exists if low quality has smaller trade surplus. The relatively neglected trade value of "lemons" thus determines trade possibilities.

The equilibrium set can be refined by focusing on, e.g., (i) *undefeated equilibria* with maximal payoffs (primarilty) to buyers and (secondarily) to sellers (Mailath et al., 1993)

	$\Delta_h > \Delta_l$	$\Delta_l > \Delta_h$
$\Delta_h > \Delta_g$	efficient, reversed	efficient, standard inefficient, standard efficient, reversed
$\Delta_g > \Delta_h$		efficient, standard inefficient, standard

Figure 3: Existence of equilibria for $r \to 0$.

or (ii) "simple" and "robust" equilibria. The former criterion advocates efficient standard dynamics, which yield the highest payoffs Δ_h to buyers and positive payoffs $\Delta_l - \Delta_h$ to sellers. However, the low information needs speak for efficient reversed dynamics.

Regarding comparative statics, we further observe that buyers' payoffs are increasing in Δ_h (and decreasing in Δ_g) while low quality sellers' payoffs are increasing in Δ_l (and decreasing in Δ_h and Δ_g). This arises because of two forces. The first force is that efficiency considerations combined with flexible screening possibilities allow buyers and sellers to enjoy the entire trade surplus $\frac{\Delta_l + \Delta_h}{2}$. The second novel force is that as frictions vanish optimal screening must keep a buyer indifferent between offering $\min\{p_0, p_l\}$ and offering p_h. As screening intensifies, the payoff of the former approximates V_b and the payoff of the latter approaches Δ_h. Buyers' payoffs will hence turn to $\frac{\Delta_h - \Delta_g}{2}$ under reversed dynamics and to Δ_h under standard dynamics. We are not aware of any counterpart to this result in the literature.

It is also noteworthy that, if there exists a steady-state equilibrium, there exists an *efficient* steady-state equilibrium. As discussed, efficient screening arises in our model because of two main reasons: (i) frictions of trading are vanishingly small and (ii) information in signals is sufficiently rich (e.g., in Moreno and Wooders (2010) no quality information is observed and in Kaya and Kim (2018) the observed information is coarse). However, a remaining problem that we have is that efficient trading only arises here in a steady-state equilibrium. Thereby, unless the market has reached an efficient steady-state equilibrium, the properties of the transition path are important for efficiency. This is left for future study. Non-steady-state dynamics may also play a key role when no steady-state equilibrum exists for $\Delta_g > \Delta_h > \Delta_l$.

4.4 Screening with bounded signal information

To demonstrate the usefulness of considering rich information structures, we next show how our analysis with unboundedly informative signals informs analyses with bounded signal information. To proceed, we thus suppose there exists an upper bound $B < \infty$ on the informativeness of quality signals s, i.e., $\frac{1}{B} \leq \frac{f_h}{f_h}(s) \leq B$.

To transport the idea immediately into our framework, we thus assume that all signals $\frac{f_h}{f_h}(s) < \frac{1}{B}$ and $\frac{f_h}{f_h}(s) > B$ are replaced by, respectively, the (lowest) signal $\underline{s}$ which gives $\frac{f_h}{f_h}(\underline{s}) = \frac{1}{B}$ and the (highest) signal $\overline{s}$ which gives $\frac{f_h}{f_h}(\overline{s}) = B$.[22] To retain the feature that high signals indicate high quality, we assume that $E[u|\overline{s}, q_0] > E[u|q_0] = \overline{U}$.

Our previous analysis permits us to derive limits on the information content of signals that suffices to sustain almost efficient trade with positive trade frictions. In other words, we obtain a new measure for bounded signal information $B < \infty$ needed for "constrained efficient screening" of assets with positive frictions $\delta < 1$.

Corollary 1 (of Propositions 1 and 2) *For any (small) $r >$ there exists (large) $B < \infty$ such that a steady-state equilibrium generates higher welfare than the static one-price model if $\Delta_g > \Delta_h$ and $\Delta_l \geq \Delta_h$ and almost equal payoff if $\Delta_h \geq \Delta_g$.*

In the limit $\delta \to 1$, equilibrium analysis can be conducted similarly as in the previous section. The upper bound on signal informativeness implies that the payoffs of offering p_h cannot exceed

$$E[u|\overline{s}, q_u] - C_h,$$

which gives Δ_h only when market quality is very high $q_c = 1$. Another novelty is that in the limit screening becomes ineffective with bounded signals, i.e., $\nu_\theta(\overline{s}, r) = \frac{1-\delta F_\theta(\overline{s})}{1-F_\theta(\overline{s})} \to 0$ as $\delta \to 1$ for any $\overline{s}$.

Still, if the lemons problem is not severe, the fixed point condition remains

$$\frac{1}{1 + \frac{1-F_h(y)}{1-F_l(y)} \frac{f_l(y)}{f_h(y)}} \Delta_h + \left(1 - \frac{1}{1 + \frac{1-F_h(y)}{1-F_l(y)} \frac{f_l(y)}{f_h(y)}}\right)(-\Delta_g) = \tag{8}$$
$$\frac{1}{2 + \nu_h(y,\delta) + \nu_l(y,\delta)} \Delta_h + \frac{1}{2 + \nu_h(y,\delta) + \nu_l(y,\delta)}(-\Delta_g),$$

as with unboundedly informative signals. Because $\Delta_h \geq \Delta_g$, we can easily see that a fixed point exists for low r as the utility of offering p_h on the lhs is $U_l - C_h$ at $y = 0$ and

[22] Because only the upper bound is binding, the lower information bound is redundant.

$E[u|\overline{s}, q_0] - C_h$ at $y = \overline{s}$ whereas the utility from p_0 on the rhs is $\overline{U} - C_h$ at $y = 0$ through $y = \overline{s}$. Payoffs thus remain as in Proposition 1 in the limit $\delta \to 1$.

Remark 1 *If $\Delta_g \leq \Delta_h$, an efficient equilibrium with bounded signals exists for $\delta \to 1$.*

This contrasts with cases where the lemons problem is severe. The ineffectiveness of screening low-quality assets then implies that a steady-state limit equilibrium cannot be sustained without mixing.

Remark 2 *If $\Delta_g > \Delta_h$, no pure equilibrium with bounded information exits for $\delta \to 1$.*

In other words, to make it unattractive for low quality sellers to wait for high prices, a buyer needs to randomize between offering p_l and p_h at $s = \overline{s}$, e.g., in proportions $r_l > 0$ and $r_h > 0$ with $r_l = 1 - r_h$. This mixing is optimal for a buyer at $\overline{s}$ provided

$$\frac{1}{1 + \frac{(1-F_h(\overline{s}))rg_h}{1-(1-F_l(\overline{s}))(1-r_h)} \frac{f_l(\overline{s})}{f_h(\overline{s})}}(\overline{s}) (U_h - C_h) + \left(1 - \frac{1}{1 + \frac{(1-F_h(\overline{s}))r_h}{1-(1-F_l(\overline{s}))(1-r_h)} \frac{f_l(\overline{s})}{f_h(\overline{s})}}\right)(U_l - C_h) =$$

$$\frac{1}{1 + \frac{(1-F_h(\overline{s}))r_h}{1-(1-F_l(\overline{s}))(1-r_h)} \frac{f_l(\overline{s})}{f_h(\overline{s})}} V_b + \left(1 - \frac{1}{1 + \frac{(1-F_h(\overline{s}))r_h}{1-(1-F_l(\overline{s}))(1-r_h)} \frac{f_l(\overline{s})}{f_h(\overline{s})}}\right)(U_l - V_l) ,$$

Again, sufficient screening of low quality requires $\frac{r}{(1-F_l(\overline{s}))r_h} \geq \frac{\Delta_g}{\Delta_l}$, which implies $r_h \to 0$ as $\delta \to 1$. Trading at high prices thus becomes very difficult at the limit, increasing the average market quality to the highest possible level, which gives a contradiction

$$\Delta_h = V_b = \frac{\Delta_h}{1 + \frac{r}{(1-F_h(\overline{s}))r_h}}.$$

Because $(\delta, r_h) \to (1, 0)$, $\frac{r}{(1-F_l(\overline{s}))r_h} \geq \frac{\Delta_g}{\Delta_l}$ is incompatible with $\frac{r}{(1-F_h(\overline{s}))p_h} \to 0$. To reduce average market quality, low quality must trade less often; knife-edge dynamics represent a possibility.

Thereby, if the lemons problem is severe and signals bounded, we need mixing both in high price offers and in low price offers, much like previously in Moreno and Wooders (2010) and Kaya and Kim (2018). Complete analysis lies beyond the scope of this article. However, a fact of life remains that highly informative signals about assets are observed with positive probability, albeit perhaps small, if waiting is costless. Characterizing the equilibrium in these natural circumstances is hence crucial to understanding market performance.

5 Conclusion

The main lessons from our analysis for practical market design are the following.

1. Large enough trade surpluses $\Delta_h > \Delta_g$, for high quality, or $\Delta_l > \Delta_h$, for low quality,

are sufficient to guarantee (almost) efficient trade in markets with signals.

2. Information requirements supporting (almost) efficient trade are negligible for vanishing frictions if $\Delta_h > \Delta_g$ but increase proportional to $\delta\frac{\Delta_g+\Delta_h}{\Delta_l-\Delta_h}$ if $\Delta_l > \Delta_h$.

3. With sufficient information, trading problems thus persist only in markets infested, at the same time, by (i) assets with high value differences (high Δ_g) and (ii) assets with low gains from trade (low Δ_h and Δ_l). Sorting out the assets with negative contribution to market performance, e.g., by a fixed entry cost as in Heinsalu (2020) or by splitting the markets as in Inderst and Müller (2002), can then help to restore efficient trading incentives in the market.

Our research has significant policy implications. We find that all efficient screening patterns, regardless of trading dynamics, exhibit lenient screening of high-quality assets. This characteristic of efficient screening is pivotal for designing screening protocols and tests to ensure high quality. We identify the potential for sustaining an inefficient equilibrium where excessively stringent screening, while preventing low-quality assets from trading at high prices, simultaneously severely impedes the trade of high-quality assets. Our results suggest that such screening protocols should be avoided. Optimal screening entails the lenient screening of high-quality assets, highlighting the importance of designs exhibiting this characteristic.

We close by discussing some extensions and alternative modeling frameworks.

Coasian payoffs

Because uninformed buyers are given full bargaining power over informed asset sellers, it is also interesting to study whether payoffs become *Coasian* as frictions disappear, i.e., whether buyers lose all commitment power to low prices and there will be efficient trade in the limit. Fuchs and Skrzypacz (2022) argue that a form of the Coase conjecture often survives even if trading is delayed. According to Fuchs and Skrzypacz (2022), translated to our case a generalized Coase conjecture could also mean that buyers trade at prices equal to (i) the highest seller valuation (i.e., here C_h) or (ii) the marginal buyer utility (i.e., here $E[U|s]$).

Indeed, when dynamics are reversed, we do find that trade only occurs for high prices C_h which both high and low quality sellers can accept. However, when dynamics are standard, a buyer will price at marginal utility for $s = y$ only when indifferent between offering p_l and p_h. In other words, in our model buyers are not always (i) pricing at the highest seller valuation C_h nor (ii) obtaining only the marginal buyer utility V_b. Thus, payoffs are not *Coasian* even when they are efficient.[23]

[23]Yet, a mathematical fact remains that buyers obtain positive payoffs in our model only if there is common knowledge of positive gains from trading high quality, i.e., $V_b \to 0$ as $\Delta_h \to 0$.

Here payoffs are non-*Coasian* in the limit under standard dynamics, in short, because signals grant the buyer an additional degree of commitment power, which is absent from models where no information is available to a buyer. Buyers know that, by waiting for a high signal, they can trade high quality with high certainty whereas, if they prefer not to wait, they also have a chance to buy low quality for low prices. Thus, low quality only obtains a payoff of $\Delta_l - \Delta_h > 0$ under standard dynamics whereas buyers obtain the payoff of $\Delta_h > 0$ if they trade high quality for p_h and $\Delta_l - (\Delta_l - \Delta_h) > 0$ if they trade low quality for p_l.[24]

Sellers offer prices

The signaling version of our model is studied more closely in Hämäläinen (2015). Focusing on seller-optimal equilibria, this article observes that, if $\Delta_l = \lambda$ is rather high relative to $\Delta_h = 1 - \lambda$, a steady-state equilibrium with standard dynamics exists for $\lambda \geq \underline{\lambda}$ but, if $\Delta_h = 1 - \lambda$ is instead high relative to $\Delta_l = \lambda$, a steady-state equilibrium with reversed dynamics exists for $\lambda \leq \overline{\lambda}$. In between, for $\lambda \in (\underline{\lambda}, \overline{\lambda})$ both kinds of dynamics can be supported in a steady-state equilibrium.

Standard dynamics arise in an equilibrium where sellers are pooling for high signals and separating for low signals. Reversed dynamics arise in an equilibrium where sellers pool for high signals but return to the market for low signals.[25] Seller-optimal prices leave no surplus to buyers, i.e., $V_b = 0$: Pooling prices thus equal $p(s) = E[U|s]$ whereas separating prices are $p_h = U_h$ for high quality and $p_l = U_l$ for low quality. In the seller-optimal case, $p(s)$ and p_l are accepted by buyers with probability one but, to prevent low quality from mimicking high, p_h can only be accepted with probability $\frac{p_l - V_l}{p_h - V_l} < 1$.

Efficiency properties of equilibria are not analyzed for vanishing trade frictions in Hämäläinen (2015). Reasonably, one would think it possible to employ the same cutoffs as in this article, e.g., screen low quality much harder than high quality under a severe lemons problem. A key question then is whether this would allow high quality trade almost certainly for high prices $p(s)$ (or p_h) and low quality trade almost certainly for low prices p_l, implementing therefore an efficient equilibrium where $V_l \to \Delta_l$ and $V_h \to \Delta_h$ as $\delta \to 1$. Lemma 7 suggests this is possible under signaling as well.[26]

[24] In the efficient equilibrium, the likelihood of the events adjusts so that buyers' payoffs will be given by $\Delta_h/2 + (\Delta_l - (\Delta_l - \Delta_h))/2 = \Delta_h > 0$.

[25] In a so called semi-pooling equilibrium, bridging the pooling and separating cases, low quality sellers mix between offering p_l and p_0 for $s < y$.

[26] The cutoff signal y and the associated screening, $\nu_l(y) \to n_l >> \nu_h(y) \to 0$, should yield $E[U|y] \geq V_b + V_h(\to U_h)$ as $\delta \to 1$ (high quality sellers offer pooling prices $p(y) = E[U|y]$ for $s = y$) and $U_l \geq V_b + V_l(\to U_l)$ as $\delta \to 1$ (low quality sellers offer separating prices $p_l = U_l$ for $s < y$); this may require giving at least a small payoff to buyers $V_b \to 0$ as $\delta \to 1$ to prevent low quality sellers from obtaining more than $U_l - C_l$ for $\delta \to 1$.

Different entry rates

Different entry flows e_h for high quality and $e_l = 1 - e_h$ for low quality, alter the steady-state market composition through the following equilibrium condition

$$e_\theta = M_\theta(1 - G_-(x_\theta)).$$

Because buyers' expectations q_u and $q_c(s)$ of sellers assets thus change, the fixed point condition under standard dynamics will transform into

$$\Delta_h - \frac{1-F_h(y)}{1}\frac{f_l(s)}{f_h(s)}\frac{e_l}{e_h}\Delta_g = \delta V_b' + \frac{1-F_h(y)}{1}\frac{f_l(s)}{f_h(s)}\frac{e_l}{e_h}\left(\Delta_l - \delta V_l'\right)$$

where

$$\begin{aligned} V_b' &= \frac{\Delta_h - (1-F_l)\frac{e_l}{e_h}\Delta_g + F_l\frac{e_l}{e_h}(\Delta_l - V_l')}{1+\frac{e_l}{e_h}+r\frac{e_l}{e_h}+\nu_h}, \\ V_l' &= \frac{1}{1+\nu_l}\left(\Delta_g + \Delta_l\right). \end{aligned}$$

The fixed point condition hence turns into

$$\Delta_h = \frac{\Delta_h + \frac{e_l}{e_h}\Delta_l - \frac{e_l}{e_h}\frac{1}{1+\nu_l}\left(\Delta_g + \Delta_l\right)}{1+\frac{e_l}{e_h}}.$$

for $\delta \to 1, y \to 1$ and $\nu_h \to 0$ and

$$\Delta_h = \frac{\Delta_h + \frac{e_l}{e_h}\Delta_l}{1+\frac{e_l}{e_h}+\nu_h}.$$

for $\delta \to 1, y \to 1$ and $\nu_l \to \infty$.

We can thus see that the existence condition and properties of equilibria for standard dynamics are unchanged. The payoffs are in the efficient equilibrium

$$V_b + e_l V_l = \Delta_h + e_l(\Delta_l - \Delta_h) = e_h\Delta_h + e_l\Delta_l$$

and in the inefficient equilibrium $V_l = \Delta_h$ and $V_l = 0$. An equilibrium with reversed dynamics exists if the following static lemons condition holds

$$\Delta_h \geq \frac{e_l}{e_h}\Delta_g.$$

Ergo, our assumption that different asset qualities enter the market at equal rates is innocuous.

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Appendix

The paper is under revision. Below proofs relate to en earlier version, set up in a continuous-time environment. Discrete time analysis is given by our earlier paper version. The associated proofs will be transferred here later.

Proof of Lemma 1

We denote by $V_\theta \geq C_\theta$ the continuation value of a seller with quality θ, which gives the opportunity cost of selling an asset of quality θ in the current meeting. Optimally, a seller accepts any price p that is higher or equals V_θ, i.e., $p \geq V_\theta$. Knowing this buyers offer either V_h (to target sellers with $V_\theta \leq V_h$) or V_l (to target sellers with $V_\theta \leq V_l$) or $p_0 < \min_\theta V_\theta$ to pass the meeting without trading. Especially, offering a strictly higher price $p > V_\theta$ to target a seller with quality θ is dominated by lowering the price until it equals the highest continuation value that lies below the offer.

Proof of Lemma 2

Denote by V_b the continuation valuation of a buyer and by $q(s)$ a buyer's belief after seeing signal s, i.e., the probability that the buyer assigns to the random event that the signal comes from a high-quality asset. We show in the main text that $q(s)$ is increasing in s. We can therefore conjecture that, if there is a signal y such that

$$V_b = q(y)\Delta_h + (1 - q(y))(-\Delta_g),$$

then the buyer optimally offers V_h for $s \geq y$ and $\min\{V_l, U_l - V_b\}$ for $s < y$. Otherwise, $y = 0$ or $y = 1$. Note that the maximal price a buyer offers for the asset quality θ is $\max\{U_\theta - V_b, 0\}$ because acquiring the asset now gives U_θ but purchasing another asset later yields V_b. It is assured that $V_h < U_h - V_b$ because $V_h = C_h$ but not certain that $V_l < U_l - V_b$.

If a seller expects to trade for price p with the next buyer, the continuation value of the seller solves the Bellman equation

$$\begin{aligned}
V_\theta(t) &= c_\theta dt + pdt + (1 - dt)(1 - rdt) V_\theta(t + dt) \\
V_\theta(t) &= (c_\theta + p)\, dt + \left(1 - (1 + r)dt + r(dt)^2\right) V_\theta(t + dt) \\
V_\theta(t + dt) &= \frac{c_\theta + p}{r + 1} + \frac{1}{r + 1} \frac{V_\theta(t + dt) - V_\theta(t)}{dt} + \frac{rdt}{r + 1} V_\theta(t + dt) \\
V_\theta(t) &= \frac{c_\theta + p}{r + 1} + \frac{1}{r + 1} V'_\theta(t) \\
V_\theta(t) &= \frac{rC_\theta + p}{r + 1} + \frac{1}{r + 1} V'_\theta(t)
\end{aligned}$$

as $dt \to 0$. During the interval dt the seller receives a dividend with probability $1dt$ and meets a buyer with probability $1dt$. If the seller does not meet a buyer, time goes on and the seller obtains the continuation value $(1 - rdt)V_\theta(t + dt)$, where $(1 - rdt) \approx e^{-rdt}$ when dt takes a small enough value. This implies that, in a steady-state equilibrium, $V_\theta(t) = \frac{rC_\theta + p}{r+1}$ as $V'_\theta(t) = 0$. $V_\theta(t)$ is therefore a weighted average between C_θ and p. The optimality of accepting the price p requires that $p \geq V_\theta(t)$.

This shows that a high quality seller has a higher continuation value than a low quality seller, $V_h > V_l$, because the dividend yield is higher $c_h > c_l$ even in cases where the prices p remain intact.[27]

We next show that the highest price offered by a buyer is $V_h = C_h$. The reasoning follows Diamond's paradox kind of logic. Suppose instead that the highest price is strictly larger $p' > C_h$. Thus,

$$V_h \leq \frac{rC_h + p'}{r+1} < p'.$$

But then the buyer can lower the offer to $p'' \in (V_h, p')$, which the seller would still accept with certainty. The original price offer $p' > C_h$ is thereby not optimal. The contradiction proves the result. □

Proof of Lemma 3

Lemma 3 follows from Lemmata 1–2 and the following analysis in the text once we note that a buyer strictly prefers offering p_l to p_0 (p_0 to p_l) if $V_b + (V_l - C_l) < \Delta_l$ ($V_b + (V_l - C_l) > \Delta_l$) but remains indifferent between p_l and p_0 if $V_b + (V_l - C_l) = \Delta_l$. Because $p_l = V_l$ and $p_0 = \max\{U_l - V_b, 0\}$, we have that $V_b + (V_l - C_l) = \Delta_l \Longleftrightarrow U_l - V_l = V_b \Longleftrightarrow U_l - p_l = V_b \Longleftrightarrow U_l - p_0 = V_b$.

Buyers' conditional beliefs (5) are obtained directly from (4) by Bayesian updating. We only have to consider the fact that, when buyers offer p_l for $s < y$, high quality trades with probability $1 - F_h(y)$ and low quality with probability 1 but, when buyers offer p_0 for $s < y$, high quality trades with probability $1 - F_h(y)$ and low quality with probability $1 - F_l(y)$ in a meeting.

If buyers instead offer p_0 for $s < z$, p_l for $s \in (z, y)$ and p_l for $s > y$, high quality trades with probability $1 - F_h(y)$ and low quality with probability $1 - F_l(z)$. □

Proof of Lemma 4

Consider a steady-state equilibrium with reversed dynamics in a market with a severe lemons problem $\Delta_h < \Delta_g$.

Under reversed dynamics, buyers' conditional beliefs at the cutoff signal $s = y$ are given by $q_c(y) = \left(1 + \frac{1-F_h(y)}{1-F_l(y)} \frac{f_l(y)}{f_h(y)}\right)^{-1}$.

By MLRP, $q_c(y) < 1/2$ such that $q_c(y)(U_h - C_h) + (1 - q_c(y))(U_l - C_h) < 0$ in cases where $\Delta_h < \Delta_g$.

But this implies that a buyer is not willing to make a high price offer $p_h = C_h$ at the cutoff signal $s = y$, which contradicts the assumption that a steady-state equilibrium with reversed dynamics can exist for $\Delta_h < \Delta_g$. □

[27] Here prices are higher when a seller has a higher quality asset because buyers offer higher prices for higher signals.

Derivation of value functions

Continuation values, V_l and V_b, are derived by dynamic programming, by defining the value functions (Bellman equations) related to buyers and low quality sellers' optimal stopping problems.[28]

In what follows, $U_b(s \geq y)$ and $U_b(s < y)$ denote the expected flow valuations of a buyer associated with observing a high signal $s \geq y$ and a low signal $s < y$, respectively,

$$\begin{aligned} U_b(s \geq y) :=& q_u \left(1 - F_h(y)\right)\left(U_h - C_h\right) + \\ & \left(1 - q_u\right)\left(1 - F_l(y)\right)\left(U_l - C_h\right), \\ U_b(s < y) :=& (1 - q_u)F_l(y) \min\left\{\left(U_l - p_l\right), p_0\right\}. \end{aligned}$$

A buyer meets sellers at a rate equal to unity. The probability of meeting a high quality seller and obtaining a high signal is $q_u \left(1 - F_h(y)\right)$ whereas that of meeting a low quality seller and receiving a high signal is $\left(1 - q_u\right)\left(1 - F_l(y)\right)$. If the signal is above c, the buyer offers p_h, which both sellers accept. The probability of observing a low signal when meeting with a high quality seller is $q_u F_h(y)$ and that of detecting a low signal when meeting a low quality seller is $(1 - q_u)F_l(y)$. If the signal is below c, the buyer offers the minimum of p_l (accepted by low quality) and p_0 (rejected by all sellers).

Under standard dynamics, a buyer's value function can be written as follows

$$\begin{aligned} V_b(t) &= dt\left(U_b(s \geq y) + U_b(s < y)\right) + dt q_u F_h(y) V_b(t) + \left(1 - (1 + r)dt\right) V_b(t + dt) \\ (1 + r)V_b(t + dt) - q_u F_h(y)V_b(t) &= U_b(s \geq y) + U_b(s < y) + \frac{V_b(t + dt) - V_b(t)}{dt} \\ V_b(t) &= \frac{U_b(s \geq y) + U_b(s < y)}{1 - q_u F_h(y) + r} + \frac{1}{1 - q_u F_h(y) + r} V_b'(t), \end{aligned} \tag{9}$$

as $dt \to 0$; the second order terms $(dt)^2$ are negligible and can thereby be ignored. Intuitively, a buyer trades under a high signal at rate $U_b(s \geq y)$ and under a low signal at rate $U_b(s < y)$. A buyer continues searching in the market either (i) if the buyer does not meet any seller in the market, which will occur with probability $1 - dt$, or (ii) if the buyer does not trade with a matched seller, happening with probability $q_c F_h(y)dt$. There is no trade in a meeting if the buyer's signal is low but the seller's quality high.

Under reversed dynamics, a buyer's value function can be defined instead as

$$\begin{aligned} V_b(t) &= U_b(s \geq y)dt + \left(1 - (1 - q_c F_h(y) - (1 - q_c)F_l(y))dt\right)\left(1 - r dt\right) V_b(t + dt), \\ &= \frac{U_b(s \geq y)}{1 - q_u F_h(y) - (1 - q_u)F_l(y) + r} + \frac{1}{1 - q_u F_h(y) - (1 - q_u)F_l(y) + r} V_b'(t). \end{aligned} \tag{10}$$

[28]We omit here the maximization over the strategy space because we have already described the optimal strategies in Lemmata 1–2 and 3.

as $dt \to 0$; the second order terms $(dt)^2$ are negligible and can thereby be ignored. In this case, there is no trade in a meeting if the signal is low, irrespective of the quality of the seller's asset. As before, a buyer will thus trade under a high signal at rate $U_b(s \geq y)$ but, when the signal is low, the trade rate is zero. As a result, a buyer continues searching in the market either (i) if there is no meeting with a seller, with probability $1 - dt$, or (ii) if there is no trade in a meeting, with probability $(q_c F_h(y) - (1 - q_c)F_l(y))dt$.

The ordinary differential equations (9) and (10) describe the evolution of V_b under different equilibrium trade dynamics. In a steady-state equilibrium, $V_b'(t) = 0$ for all t, because the evolution dynamics of V_b have then reached a steady-state.

In a steady-state equilibrium with standard dynamics, we thus obtain that a buyer's continuation value is

$$V_b = \frac{U_b(s \geq y) + U_b(s < y)}{1 - q_u F_h(y) + r},$$

whereas a buyer's continuation value in a steady-state equilibrium with reversed dynamics can be expressed as

$$V_b = \frac{U_b(s \geq y)}{1 - q_u F_h(y) - (1 - q_u)F_l(y) + r}.$$

Moving on to sellers, the continuation value of holding a high-quality asset is fixed at $V_h = C_h$ whereas the seller's continuation value of keeping a low-quality asset is given by

$$\begin{aligned} V_l(t) &= dt c_l + dt\left((1 - F_l(y))\,C_h + F_l(y)V_l(t)\right) + (1 - (1 + r)dt)\,V_l(t + dt) \\ (1 + r)V_l(t + dt) - F_l(y)V_l(t) &= c_l + (1 - F_l(y))\,C_h + \frac{V_l(t + dt) - V_l(t)}{dt} \\ V_l(t) &= \frac{rC_l + (1 - F_l(y))\,C_h}{1 - F_l(y) + r} + \frac{1}{1 - F_l(y) + r}V_l'(t), \end{aligned} \tag{11}$$

as $dt \to 0$; the second order terms $(dt)^2$ are negligible and can thereby be ignored.

Note that two events may happen to low quality sellers at each time point: (i) the seller's asset may generate a new dividend with payoff c_l, or (ii) the seller may encounter a new potential buyer with signal s. Both events follow a Poisson process with the rate equal to unity. If the buyer's signal is high, with probability $1 - F_l(y)$, the seller obtains $C_h - V_l$ whereas, if the buyer's signal is low, with probability $F_l(y)$, the seller receives V_l, irrespective of which trade dynamics prevail (i.e., with standard trade dynamics, a buyer offers $p_l = V_l$, which the seller accepts, and, with reversed trade dynamics, the buyer offers, $p_0 < V_l$, which the seller rejects). However, if neither event occurs, the seller continues searching in the market, which gives the seller the continuation value, $V_l(t+dt)$. In a steady-state equilibrium, V_l is hence given by

$$V_l(t) = \frac{(1 - F_l(y))\,C_h + rC_l}{1 - F_l(y) + r} = \frac{C_h - C_l}{1 + \frac{r}{1 - F_l(y)}} + C_l.$$

The first term captures the value of trading low quality for a high price whereas the second term denotes the valuation of dividends.

Value functions for screening

The screening intensities

$$\nu_h : \nu_h(y,\delta) = \frac{r}{1-F_h(y)},$$
$$\nu_l : \nu_l(y,\delta) = \frac{r}{1-F_l(y)},$$

are increasing in r and y.

Rearranging (11) gives

$$\begin{aligned} V_l &= \frac{C_h+\nu_l C_l}{1+\nu_l} = \frac{C_h+\nu_l C_l}{1+\nu_l} = \frac{C_h-C_l+(1+\nu_l)C_l}{1+\nu_l}, \\ &= C_h - \frac{\nu_l}{1+\nu_l}\left(C_h-C_l\right) = C_h - \frac{\nu_l}{1+\nu_l}\left(\Delta_g+\Delta_l\right), \\ &= C_l + \frac{1}{1+\nu_l}\left(C_h-C_l\right) = C_l + \frac{1}{1+\nu_l}\left(\Delta_g+\Delta_l\right), \end{aligned} \tag{12}$$

which shows that, as a function of y, $V_l(y) = V_l(\nu_l(y,\delta))$ is continuous and decreasing. Note also that $V_l(y)$ attains any value in between $V_l(0) = (C_h+rC_l)/(1+r)$ and $V_l(1) = C_l$ at some unique signal cutoff $y \in (0,1)$.

Assuming standard dynamics, (9) gives

$$\begin{aligned} V_b &= \frac{q_u(1-F_h)\Delta_h - (1-q_u)(1-F_l)\Delta_g + (1-q_u)F_l(U_l-V_l)}{1-F_hq_u+r} \\ &= \frac{q_u(1-F_h)\Delta_h - (1-q_u)(1-F_l)\Delta_g + (1-q_u)F_l(U_l-V_l)}{q_u(1-F_h)+(1-q_u)(1-F_l)+(1-q_u)F_l+r} \\ &= \frac{\Delta_h-(1-F_l)\Delta_g+F_l(U_l-V_l)}{2+r\frac{1}{q_u(1-F_h)}} = \frac{\Delta_h-(1-F_l)\Delta_g+F_l(U_l-V_l)}{2+r\frac{2-F_h}{1-F_h}} \\ &= \frac{\Delta_h-(1-F_l)\Delta_g+F_l(U_l-V_l)}{2+r\left(1+\frac{1}{1-F_h}\right)} = \frac{\Delta_h-(1-F_l)\Delta_g+F_l(U_l-V_l)}{2+r+\nu_h} \end{aligned} \tag{13}$$

if $V_b + (V_l - C_l) < \Delta_l$ and $V_b \geq 0$, and

$$\begin{aligned} V_b &= \frac{q_u(1-F_h)\Delta_h - (1-q_u)(1-F_l)\Delta_g + (1-q_u)F_lV_b}{1-F_hq_u+r} \\ &= \frac{\Delta_h-(1-F_l)\Delta_g+F_lV_b}{2+r+\nu_h} = \frac{\Delta_h-(1-F_l)\Delta_g}{2-F_l+r+\nu_h} \end{aligned} \tag{14}$$

if $V_b + (V_l - C_l) \geq \Delta_l$ and $V_b \geq 0$.[29]

Above, we have thus expressed average quality q_u in terms of y and F_l, F_h. To derive the first lines, we have used the fact that, assuming standard dynamics,

$$\begin{aligned} q_u(1 - F_h) &= \frac{1 - F_h}{2 - F_h}, \\ (1 - q_u)(1 - F_l) &= (1 - F_l)\frac{1 - F_h}{2 - F_h}, \\ (1 - q_u)F_l &= F_l\frac{1 - F_h}{2 - F_h}. \end{aligned}$$

Assuming reversed dynamics, (10) gives

$$\begin{aligned} V_b &= \frac{q_u(1 - F_h)\Delta_h - (1 - q_u)(1 - F_l)\Delta_g}{1 - q_u F_h(y) - (1 - q_u)F_l + r} \\ &= \frac{q_u(1 - F_h)\Delta_h - (1 - q_u)(1 - F_l)\Delta_g}{q_u(1 - F_h) + (1 - q_u)(1 - F_l) + r} \\ &= \frac{\Delta_h - \Delta_g}{2 + r\frac{1}{q_u(1-F_h)}} = \frac{\Delta_h - \Delta_g}{2 + r\frac{2-F_h-F_l}{(1-F_l)(1-F_h)}} \\ &= \frac{\Delta_h - \Delta_g}{2 + r\left(\frac{1}{1-F_h} + \frac{1}{1-F_h}\right)} = \frac{\Delta_h - \Delta_g}{2 + \nu_h + \nu_l} \end{aligned} \tag{15}$$

for $V_b + (V_l - C_l) > \Delta_l$ and $V_b \geq 0$.

To obtain the first lines, we have used the fact that, assuming reversed dynamics,

$$\begin{aligned} q_u(1 - F_h) &= (1 - F_h)\frac{1 - F_l}{2 - F_h - F_l}, \\ (1 - q_u)(1 - F_l) &= (1 - F_l)\frac{1 - F_h}{2 - F_h - F_l}. \end{aligned}$$

Independent of which trade dynamics prevail, we can thus see that $V_b(y)$ is continuous for all $y \in [0, 1]$, and first increasing in y and later decreasing in y. With standard dynamics, $V_b(0) = \max\left\{0, \frac{\Delta_h-\Delta_g}{2+\nu_h}\right\}$ and, with reversed dynamics, $V_b(0) = \max\left\{0, \frac{\Delta_h-\Delta_g}{2+\nu_h+\nu_l}\right\}$. In both cases, $V_b(y) \to 0$ as $y \to 1$.

Proof of Lemma 5

Suppose trade takes place with probability one. By Lemmata 1–2 and 3 this implies that $y = 0$. As a result, buyers offer C_h, which all sellers accept, for all signals s.

By the continuity of f_l and f_h, the buyers' conditional beliefs (5) are continuous and, according to Assumption 1, $q_c(0) = 0$. The continuity of beliefs q_c in signals s entails that

[29] To make sure the later given fixed point correspondences FP and incentive condition correspondences IC are everywhere continuous, we need both payoffs in our fixed point analysis for standard dynamics, although only the former ones are consistent with the assumed dynamics.

for any number $\epsilon > 0$ there exists a number $\delta(\epsilon) > 0$ such that $q_c(s) < \epsilon$ for all $s < \delta$.

Accordingly, if we choose the number $\epsilon = \frac{C_h - U_l}{U_h - U_l} = \frac{\Delta_g}{\Delta_h + \Delta_g} > 0$, then for all signals $s < \delta(\epsilon)$, beliefs $q_c(s)$ are so low that a buyer's expected valuation for offering a high price is negative,

$$E(u|s) - C_h < \epsilon U_h + (1-\epsilon)U_l - C_h = \epsilon \Delta_h - (1-\epsilon)\Delta_g < 0,$$

which shows that a buyer strictly prefers offering $\min\{p_0, p_l\}$ to offering C_h for $s < \delta(\epsilon)$. This contradicts the assumption that trade takes place with probability one for all signals. □

Proof of Lemma 6

Note first that for any $y < 1$

$$V_l(r) = C_l + \frac{1}{1+\nu_l(r)}(C_h - C_l) \to C_h,$$

as $r \to 0$, which means that

$$V_l + V_b > V_l > U_l$$

for low values of r because $C_h > U_l$. Thus, $y \to 1$ as $r \to 0$ is a necessary condition for the existence of a limit equilibrium under standard trade dynamics.

To provide a general proof, note that, according to (3), the cutoff is the signal y solving the following fixed point condition,

$$U_h - C_h + \frac{1 - G_-(p_h)}{1 - G_-(p_l)} \frac{f_l(y)}{f_h(y)} (U_l - C_h) = V_b + \frac{1 - G_-(p_h)}{1 - G_-(p_l)} \frac{f_l(y)}{f_h(y)} \max\{U_l - p_l, V_b\}. \quad (16)$$

This condition is obtained from (3) by rewriting the equation in terms of $q_c(y)$ and suppressing the denominators $1 + \frac{1-G_-(p_h)}{1-G_-(p_l)} \frac{f_l(y)}{f_h(y)}$.

As a preliminary observation, note that for the lowest signal values $s \in (0,1)$, the lhs of (16) is smaller than the rhs of (16). The lhs is negative for low enough s by Assumption 1. The rhs is non-negative because V_b is non-negative by definition. Instead, for the highest signal values $s \in (0,1)$, the lhs of (16) is larger than the rhs of (16) because the lhs is positive for low enough s whereas the rhs approaches zero by Assumption 1 for $V_b \to 0$ as $y \to 1$. By the continuity of the condition (16) a fixed point y will thus exist.

The smallest fixed point corresponds to a cutoff signal at which the sides of (16) are larger than zero. To see why satisfying (16) as $0 = 0$ is impossible, consider signals $y < 1$ for which the lhs of (16) is equal to zero. This entails both (i) zero buyers' payoff for offering p_h at $s = y$ and (ii) positive buyers' payoff for offering p_h for $s > y$. As a result, we can see from (9) and (10) that $V_b > 0$ because $U(s < y) > V_b$ (irrespective of whether

$V_b \geq U_l - p_l$ for which $U(s < y) = V_b$ or $V_b < U_l - p_l$ for which $U(s < y) > V_b$).

Lemma 8 shows later in more detail that, there is both a (higher) signal $y < 1$ which satisfies (16) as

$$U_h - C_h + \frac{1 - G_h(p_h -)}{1 - G_l(p_l -)} \frac{f_l(y)}{f_h(y)} (U_l - C_h) = V_b + \frac{1 - G_-(p_h)}{1 - G_-(p_l)} \frac{f_l(y)}{f_h(y)} (U_l - p_l),$$

and a (lower) signal $y > 0$ which satisfies (16) as

$$U_h - C_h + \frac{1 - G_h(p_h -)}{1 - G_l(p_l -)} \frac{f_l(y)}{f_h(y)} (U_l - C_h) = \left(1 + \frac{1 - G_h(p_h -)}{1 - G_l(p_l -)} \frac{f_l(y)}{f_h(y)}\right) V_b.$$

For standard dynamics, the latter condition results in[30]

$$\frac{\Delta_h - (1 - F_h) \frac{f_l}{f_h} \Delta_g}{1 + (1 - F_h) \frac{f_l}{f_h}} = \frac{\Delta_h - (1 - F_l) \Delta_g}{2 + \nu_h + r - F_l} = V_b, \tag{17}$$

whereas, for reversed dynamics, the same condition implies

$$\frac{\Delta_h - \frac{1 - F_h}{1 - F_l} \frac{f_l}{f_h} \Delta_g}{1 + \frac{1 - F_h}{1 - F_l} \frac{f_l}{f_h}} = \frac{\Delta_h - \Delta_g}{2 + \nu_h + \nu_l} = V_b. \tag{18}$$

We consider these cases one by one next. For (weakly) standard dynamics, observe that (17) can be written as

$$\alpha \Delta_h - (1 - \alpha) \Delta_g = \alpha' \Delta_h - \frac{1 - F_l}{1 - F_l + r + \nu_h} (1 - \alpha') \Delta_g,$$

where

$$\alpha = \frac{1}{1 + (1 - F_h(y)) \frac{f_l(y)}{f_h(y)}},$$
$$\alpha' = \frac{1}{2 + \nu_h + (r - F_l(y))}.$$

For any $y < 1$, $\frac{1 - F_l}{1 - F_l + r + \nu_h} \to 1$ as $r \to 0$. To satisfy (17), we thus need either $y \to 1$ as $r \to 0$ or $\alpha \to \alpha'$ as $r \to 0$ (or both).

[30] We suppress the arguments of $f_\theta(y)$'s, $F_\theta(y)$'s and $\nu_\theta(y, \delta)$'s to abbreviate the expressions.

Setting $\alpha = \alpha'$ results in

$$\begin{aligned}
(1 - F_h(y))\frac{f_l(y)}{f_h(y)} &= r + \nu_h + (1 - F_l(y)) \\
(1 - F_h(y))\frac{f_l(y)}{f_h(y)} - (1 - F_l(y)) &= r\frac{2 - F_h}{1 - F_h} \\
\frac{1}{2 - F_h}\left[(1 - F_h(y))^2 \frac{f_l(y)}{f_h(y)} - (1 - F_h(y))(1 - F_l(y))\right] &= r, \qquad (19)
\end{aligned}$$

where $\frac{1}{2-F_h} > 1/2$ whereas the function inside the square brackets is strictly positive by MLRP for all $y < 1$. By L'Hopital's rule, $(1 - F_l(y))/(1 - F_h(y)) \to f_l(y)/f_h(y)$ as $y \to 1$, which shows that the inside of the brackets approaches zero as y approaches one (but generally not otherwise). Thus, $\alpha \to \alpha'$ as $r \to 0$ cannot hold unless $y \to 1$ as $r \to 0$.

For (weakly) reversed dynamics, notice that (18) can be written as

$$\beta\Delta_h - (1 - \beta)\,\Delta_g = \beta'\Delta_h - \frac{1}{1 + \nu_h + \nu_l}\,(1 - \beta')\,\Delta_g,$$

where

$$\begin{aligned}
\beta &= \frac{1}{1 + \frac{1-F_h(y)}{1-F_l(y)}\frac{f_l(y)}{f_h(y)}}, \\
\beta' &= \frac{1}{2 + \nu_h + \nu_l}.
\end{aligned}$$

For any $y < 1$, $\frac{1}{1+\nu_h+\nu_l} \to 1$ as $r \to 0$. To satisfy (18), we thus need either $y \to 1$ as $r \to 0$ or $\beta \to \beta'$ as $r \to 0$ (or both).

Setting $\beta = \beta'$ results in

$$\begin{aligned}
\frac{1 - F_h(y)}{1 - F_l(y)}\frac{f_l(y)}{f_h(y)} &= 1 + \frac{r}{1 - F_h(y)} + \frac{r}{1 - F_l(y)} \\
\frac{1 - F_h(y)}{1 - F_l(y)}\frac{f_l(y)}{f_h(y)} - 1 &= r\left(\frac{1}{1 - F_h(y)} + \frac{1}{1 - F_l(y)}\right) \\
\frac{1 - F_h(y) - F_l(y) - F_h(y)F_l(y)}{2 - F_h(y) - F_l(y)}\left[\frac{1 - F_h(y)}{1 - F_l(y)}\frac{f_l(y)}{f_h(y)} - 1\right] &= r \qquad (20)
\end{aligned}$$

Note that $\frac{1-F_h(y)-F_l(y)-F_h(y)F_l(y)}{2-F_h(y)-F_l(y)} > 0$ whereas the function inside the square brackets is strictly positive by MLRP for all $y < 1$.

By L'Hopital's rule, $(1 - F_l(y))/(1 - F_h(y)) \to f_l(y)/f_h(y)$ as $y \to 1$, which shows that the function in the brackets approaches zero as y approaches one (but generally not otherwise). Thus, $\beta \to \beta'$ as $r \to 0$ cannot hold unless $y \to 1$ as $r \to 0$. $\square$

Proof of Lemma 7

We start by rewriting $\nu_l(r,s)$ as

$$\underbrace{\frac{r}{1-F_l(s)}}_{\nu_l(s,r)} = \underbrace{\frac{r}{1-F_h(s)}}_{\nu_h(s,r)} \frac{1-F_h(s)}{1-F_l(s)},$$

and study the limit as $s \to 1$. Because $F_\theta(s) \to 1$ as $s \to 1$, we can use L'Hopital's rule to have

$$\frac{1-F_h(s)}{1-F_l(s)} \to \frac{f_h(s)}{f_l(s)} \to \infty,$$

as $s \to 1$. This equals saying that for any $M > 1$ there exists a signal $s_l < 1$ such that

$$\nu_h(s_l, r)M^2 < \nu_l(s_l, r) \tag{21}$$

for any r and for $s > s_l$. We then turn to

$$\nu_\theta(s_0, r) = \frac{r}{1-F_\theta(s_0)},$$

which clearly approach zero as $r \to 0$ and infinity as $r \to \infty$. For any $M > 1$ and for $s_l < 1$ we can thus find $r_0 \in (0,\infty)$ and $s_0 \in (0, s_l)$ such that

$$\nu_h(s_l, r_0) = \frac{r_0}{1-F_h(s_l)} = 1/M, \tag{22}$$

$$\nu_l(s_0, r_0) = \frac{r_0}{1-F_l(s_0)} = 1/M. \tag{23}$$

The result follows from (21), (22) and (23). □

Lemma 8 *For any large enough $M > 1$, $0 < y_0(r) < y_l(r) < 1$ for all $r = 1/M$.*

1. *Under standard dynamics, $y(r) \geq y_l(r)$ where $y_l^1(r^1), y_l^2(r^2), ... \to 1$ as $r^1, r^2, ... \to 0$ along a sequence for which $\nu_l(y_l^1(r^1)), \nu_l(y_l^2(r^2)), ... \to n_l \geq \Delta_g/\Delta_l$.*

2. *Under reversed dynamics, $y(r) = y_0(r)$ where $y_0^1(r^1), y_0^2(r^2), ... \to 1$ as $r^1, r^2, ... \to 0$ along a sequence for which $\nu_l(y_0^1(r^1)), \nu_l(y_0^2(r^2)), ... \to n_0 = 0$.*

Proof of Lemma 8

To search for the lowest y_0, we continue the analysis from (19) and (20), which gives

$$\frac{1}{2-F_h}\left[(1-F_h(y))\frac{f_l(y)}{f_h(y)} - (1-F_l(y))\right] =: \nu_h', \tag{24}$$

$$\frac{1-F_h(y)-F_l(y)-F_h(y)F_l(y)}{2-F_h(y)-F_l(y)}\left[\frac{1}{1-F_l(y)}\frac{f_l(y)}{f_h(y)} - \frac{1}{1-F_h(y)}\right] =: \nu_h'.. \tag{25}$$

We consider signal cutoffs y' for which buyers would be indifferent between offering p_0 and p_h at $s = y'$.

Under (weakly) standard dynamics, we can employ (24) and express $\nu_l' := \nu_l(y')$ as

$$\begin{aligned}
\nu_l' &= \frac{1-F_h(y')}{1-F_l(y')}\nu_h' \\
\nu_l' &= \frac{1-F_h(y')}{1-F_l(y')}\frac{1}{2-F_h}\left[(1-F_h(y))\frac{f_l(y)}{f_h(y)} - (1-F_l(y))\right] \\
\nu_l' &= \frac{1-F_h(y')}{2-F_h(y')}\left[\frac{1-F_h(y)}{1-F_l(y')}\frac{f_l(y)}{f_h(y)} - 1\right] \to 0, \text{ as } y \to 1.
\end{aligned}$$

Under (weakly) reversed dynamics, we can apply (25) and rewrite $\nu_l' := \nu_l(y')$ as

$$\begin{aligned}
\nu_l' &= \frac{1-F_h(y')}{1-F_l(y')}\nu_h' \\
\nu_l' &= \frac{1-F_h(y')}{1-F_l(y')}\frac{1-F_h(y)-F_l(y)-F_h(y)F_l(y)}{2-F_h(y)-F_l(y)}\left[\frac{1}{1-F_l(y)}\frac{f_l(y)}{f_h(y)} - \frac{1}{1-F_h(y)}\right] \\
\nu_l' &= \frac{1-F_h(y)}{(1-F_h(y))+(1-F_l(y))}\left[\frac{1-F_h(y')}{1-F_l(y)}\frac{f_l(y)}{f_h(y)} - 1\right] \to 0, \text{ as } y \to 1.
\end{aligned}$$

This shows that, irrespective of which trade dynamics prevail, $y' \to 1$ as $r \to 0$ along a sequence for which $\nu_l(y')$ stays close to zero.

Turning to incentive conditions, note that $(V_l - C_l) \leq \Delta_l$ is a necessary condition for $V_b + (V_l - C_l) \leq \Delta_l$.

$$\begin{aligned}
(V_l - C_l) &\leq \Delta_l \\
\frac{1}{1+\nu_l}(\Delta_g + \Delta_l) &\leq \Delta_l \\
\Delta_g + \Delta_l &\leq (1+\nu_l)\Delta_l \\
\Delta_g &\leq \nu_l \Delta_l \\
\frac{\Delta_g}{\Delta_l} &\leq \nu_l
\end{aligned} \tag{26}$$

Next, consider y'' defined by (26) as

$$\nu_l'' := \nu_l(y'') = \frac{r}{1-F_l(y'')} = \frac{\Delta_g}{\Delta_l}.$$

We can see that $y'' \to 1$ as $r \to 0$ along a sequence for which $\nu_l(y'')$ remains bounded away from zero.[31]

Comparing ν_l' to ν_l'' we thus find that $y_0(r) < y_l(r)$ for low r because $y_0 \to y'$ as $r \to 0$

[31] The bound is not tight. Indeed, it is straightforward to show that $\nu_l(y_l) \to n_l = \frac{\Delta_g + \Delta_h}{\Delta_l - \Delta_h} > \frac{\Delta_g}{\Delta_l}$ as $r \to 0$.

and $y_l \geq y''$.

To complete the proof, we also need to show that $0 < y_0(r)$ and that $y_l(r) < 1$. This is easy. First, consider the incentives for offering p_0 as opposed to p_l, i.e., the roots of

$$IC_{0l} : IC_{0l}(y) = V_b(y) + (V_l(y) - C_l) - \Delta_l.$$

We have shown above that V_l is decreasing in y and $(V_l(0) - C_l) = \Delta_g + \Delta_l$, $(V_l(1) - C_l) = 0$, and $V_b + (V_l - C_l) \to 0$ as $y \to 1$, which shows by continuity that $y_l \in (0, 1)$.

Second, consider the fixed point condition that defines the cutoff y,

$$\Delta_h + (1 - F_l(y))\frac{f_l(y)}{f_h(y)}(-\Delta_g) = V_b + (1 - F_l(y))\frac{f_l(y)}{f_h(y)} \max\{U_l - V_l, V_b\} \geq 0,$$

with standard dynamics and

$$\Delta_h + \frac{1 - F_h(y)}{1 - F_l(y)}\frac{f_l(y)}{f_h(y)}(-\Delta_g) = V_b + \frac{1 - F_h(y)}{1 - F_l(y)}\frac{f_l(y)}{f_h(y)} \max\{U_l - V_l, V_b\} \geq 0,$$

with reversed dynamics.

Clearly, because $\frac{f_l(y)}{f_h(y)} \to \infty$ as $y \to 0$, the lhs is (strictly) negative and the rhs is (weakly) positive in any sufficiently small neighborhood of $y = 0$. Similarly, as $\frac{f_l(y)}{f_h(y)} \to 0$ as $y \to 1$, and $V_b(y) \to 0, V_l(y) \to C_l$ as $y \to 1$, the lhs is positive and the rhs is negative in a neighborhood of $y = 1$. Thus, the result $y_0 \in (0, 1)$ obtains by continuity. □

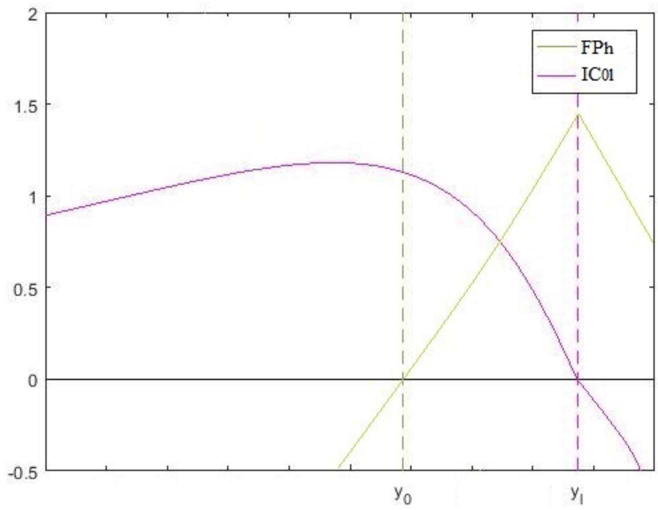


Figure 4: Illustration of Lemma 8.

Figure 4 illustrates the effect of different tentative cutoffs y on incentives to offer different prices p_0, p_l and p_h. The green line depicts a function $y \mapsto FP_h(y)$ (for "fixed point condition") which is positive for cutoffs for which a buyer prefers offering p_h over $\min\{p_0, p_l\}$. $FP_h(y)$ will thus cross the s-axis at y_0. The pink line defines a function

$y \mapsto IC_{0l}(y)$ (for "incentive condition") which is positive for cutoffs for which a buyer prefers offering p_0 over p_l for $s < y$. $IC_{0l}(y)$ therefore crosses the s-axis at y_l. Both of these lines are generally non-monotone.

For IC_{0l}, this is because the effect of screening on a buyer's payoff is first positive and later negative whereas the effect on a low quality seller's payoff is negative. $IC_{0l} = V_b + V_l - \Delta_l$ can hence be first increasing (as V_b increases faster than V_l decreases) and thereafter decreasing (when both V_b and V_l are decreasing). Indeed, we find that a *single-crossing property* must hold for incentives of offering p_0 over p_l. Thereby, the pink IC_{0l} curve crosses the s-axis once from above and divides the cutoffs into (i) lower ones $y < y_l$ for which dynamics would be reversed and (ii) higher ones $y > y_l$ for which dynamics would be standard.

Instead, the green FP_h curve is increasing for low screening, $y < y_l$, crossing the s-axis therefore at $y_0 < y_l$ already for relatively low screening. As detailed in Proposition 1, this low root of FP_h may for suitable parameters correspond to an equilibrium with reversed dynamics. However, as the benefit of trading low quality for a low price p_l begins to affect the payoff of a buyer V_b, FP_h generally decreases for $y \in (y_l, s_h)$ and ultimately increases for $y \in [s_h, 1]$, possibly thus crossing the s-axis for a second and a third time. These higher roots of FP_h would then represent equilibria with standard dynamics, outlined in Proposition 2.

We proceed by proving Proposition 2 first and then move to Propositions 1 and 3.

Proof of Proposition 2

Step I. Screening over different cutoff sequences $(y(r^i))_{r^i \to 0}$

By Lemma 7, for any $M > 1$ there exist $s_0 < s_l < s_h$ and $r < 1/M$ such that

$$\nu_l(s_0) = \nu_h(s_l) = 1/M < M \leq \nu_l(s_l) = \nu_h(s_h).$$

By Lemma 8, for any $M > \Delta_l/\Delta_g$ and $r < 1/M$ there exist y_0 and y_l such that

$$\nu_h(y_0) < \nu_l(y_0) < \nu_h(y_l) \leq 1/M < \Delta_g/\Delta_l \leq \nu_l(y_l),$$

where the cutoffs are defined by Lemma 8 in such a way that, if $y = y_0$, a buyer is indifferent between p_0 and p_h and, if $y = y_l$, a buyer is indifferent between p_0 and p_l.

For any sequence $r^1, r^2, \ldots \to 0$ we thus obtain five related cutoff sequences $(s_0(r^i), s_l(r^i), s_h(r^i), y_0(r^i), y_l(r^i))_{i=1,2,\ldots}$ with different associated screening intensities.

Because $s_0 \to 1$ and $y_0 \to 1$ as $r \to 0$, we also know that

$$\begin{aligned} \frac{f_l}{f_h}(y(r)) &< 1/N_f && \text{for all } r < 1/M, y = y_0, y_l, s_0, s_l, s_h, \\ 1 - F_\theta(y(r)) &< 1/N_F && \text{for all } r < 1/M, y = y_0, y_l, s_0, s_l, s_h, \end{aligned}$$

where $N_f \to \infty$ and $N_F \to \infty$ as $M \to \infty$.

Step II. Existence of fixed point sequences $(y(r^i))_{r^i \to 0}$

According to (16), y satisfies the following fixed point condition under standard trade dynamics

$$FP_h : FP_h = \frac{\Delta_h - (1 - F_h)\frac{f_l}{f_h}\Delta_g}{1 + (1 - F_h)\frac{f_l}{f_h}} - \frac{V_b + (1 - F_h)\frac{f_l}{f_h} \max\{U_l - V_l, V_b\}}{1 + (1 - F_h)\frac{f_l}{f_h}} = 0, \tag{27}$$

where V_b is defined by (13) for $V_b + V_l \leq U_l$ and by (14) for $V_b + V_l > U_l$ and V_l is defined by (12). Under these assumptions, FP_h is continuous in y and in r.

We proceed by proving that (27) is satisfied at some $y^1(r) \in (y_l(r), s_l(r))$ and at some $y^2(r) \in (y^1(r), s_h(r))$ for all low enough (fixed) values of r. As FP_h is continuous in y, it suffices to show that

$$FP_h(y, \delta) > 0, \qquad \text{for all } y \in (y_0(r), y_l(r)), \tag{28}$$

$$FP_h(y, \delta) < 0, \qquad \text{at } y = s_l(r), \tag{29}$$

$$FP_h(y, \delta) > 0, \qquad \text{at } y = s_h(r). \tag{30}$$

Case 1. To show that (28) holds, we consider (17) satisfied by y_0

$$\alpha\Delta_h - (1 - \alpha)\Delta_g - \alpha'\Delta_h + \frac{1 - F_l}{1 - F_l + r + \nu_h}(1 - \alpha')\Delta_g = 0, \tag{31}$$

where

$$\begin{aligned} \alpha(y) &= \frac{1}{1 + (1 - F_h(y))\frac{f_l(y)}{f_h(y)}}, \\ \alpha'(y) &= \frac{1}{1 + (1 - F_l(y)) + \nu_h(y, \delta) + r}. \end{aligned}$$

Lemma 6 shows that $y_0(r) \to 1$ as $r \to 0$. Lemma 8 proves that $\alpha(y_0) \to \alpha'(y_0)$ as $r \to 0$. Further, the terms multiplied by $(1 - \alpha)$ or $(1 - \alpha')$ become negligible for low r because $f_l/f_h(y) \to 0$, $1 - F_l(y(r)) \to 0$, and $\nu_h(y(r)) \to 0$ as $r \to 0$ for all $y \in (y_0, y_l)$. To sign the lhs of (31) for low values of r, we can hence focus on

$$\alpha - \alpha',$$

or, equivalently, on the difference between the numerators

$$(1 + (1 - F_h)\frac{f_l}{f_h}) - (1 + (1 - F_l) + r + \nu_h).$$

Differentiating this expression with respect to y results in

$$-f_h\frac{f_l}{f_h} + (1 - F_h)\frac{\partial}{\partial y}\frac{f_l}{f_h} + f_l - \frac{\partial}{\partial y}\nu_h = (1 - F_h)\underbrace{\frac{\partial}{\partial y}\frac{f_l}{f_h}}_{<0} - \underbrace{\frac{\partial}{\partial y}\nu_h}_{>0} < 0,$$

which shows that $\alpha - \alpha'$ is increasing in y for $y \in (y_0, y_l)$.

To show that (29) and (30) also hold true, we proceed by proving that

$$\Delta_h - (1 - F_h)\frac{f_l}{f_h}\Delta_g < V_b + (1 - F_h)\frac{f_l}{f_h}\max\{U_l - V_l, V_b\} \text{ at } y = s_l \tag{32}$$

$$\Delta_h - (1 - F_h)\frac{f_l}{f_h}\Delta_g > V_b + (1 - F_h)\frac{f_l}{f_h}\max\{U_l - V_l, V_b\} \text{ at } y = s_h. \tag{33}$$

Case 2. We can see from above that (32) is satisfied providing that

$$V_b(s_l) = \frac{\Delta_h - (1 - F_l(s_l))\Delta_g + F_l(s_0)(U_l - V_l(s_l))}{2 + r + \nu_h(s_l)} > \Delta_h.$$

Given the assumptions in Step I, $V_b(s_l)$ is clearly larger than

$$\underline{V}_b(s_l(M)) = \frac{\Delta_h - 1/N_F\Delta_g + (1 - 1/N_F)(\Delta_l - \frac{\Delta_g+\Delta_l}{1+M})}{2 + 1/M + 1/M}.$$

Taking the limit as $M \to \infty$, we thus observe as required that

$$\underline{V}_b(s_l(M)) \to \frac{\Delta_h + \Delta_l}{2} > \Delta_h$$

and

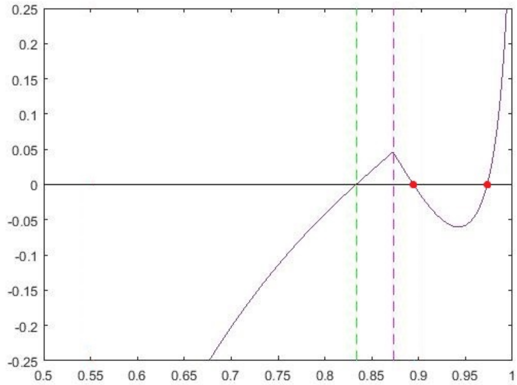
$$\underline{V}_b(s_l(M)) + (\underline{V}_l(s_l(M)) - C_l) \to \frac{\Delta_h + \Delta_l}{2} < \Delta_l.$$

Case 3. Additionally, we can see that (33) is satisfied at $y = s_h$ for any high enough values of M because the assumptions made in Step I imply here that

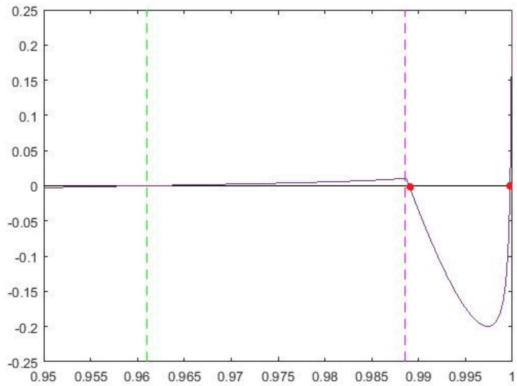
$$\Delta_h - 1/(M^2)\Delta_g > \frac{\Delta_h - (1 - F_l(s_h))\Delta_g + F_l(s_h)\Delta_l}{2 + M + M} + 1/(M^2)(\Delta_l - \frac{\Delta_g + \Delta_l}{1 + M})$$

$$\Delta_h > \underbrace{\frac{\Delta_h - (1 - F_l(s_h))\Delta_g + F_l(s_h)\Delta_l}{2 + M + M}}_{\to 0,\text{ as } M\to\infty} + \underbrace{\frac{1}{M(1 + M)}(\Delta_l + \Delta_g)}_{\to 0,\text{ as } M\to\infty}.$$

Figure 5 shows the graph of $FP_h(y)$ for $r = 0.05$ (Figure 5a) and $r = 0.005$ (Figure 5a) illustrating how decreasing r affects the position of fixed points y^1 and y^2 ('red dots')



(a) $r = 0.05$



(b) $r = 0.0005$

Figure 5: FP_h for $(\Delta_h, \Delta_g, \Delta_l) = (0.5, 1.5, 1)$.

with respect to y_0 ('green line') and y_l ('pink line'). Especially, the gap between y_0 and $y_l > y_0$ remains whereas the gap between y_l and $y^1 > y_l$ vanishes as $r \to 0$.

Details of equilibria with $(\Delta_h, \Delta_g, \Delta_l) = (0.5, 1.5, 1)$ and $r = 0.05$ (as in Figure 5a):

1st equilibrium at $y^1 \approx 0.894$: $V_b(0.894) \approx 0.450$ and $V_l(0.894) \approx 0.466$ such that $V_b + 0.5(V_l - C_l) \approx 0.683 < 0.5(\Delta_h + \Delta_l) = 0.75$,

2nd equilibrium at $y^2 \approx 0.974$: $V_b(0.974) \approx 0.498$ and $V_l(0.974) \approx 0.034$ such that $V_b + 0.5(V_l - C_l) \approx 0.515 < 0.5(\Delta_h + \Delta_l) = 0.75$.

Step III. Price offers at y^1 and y^2

One might still wonder whether the solutions to (27) that we have identified correspond to equilibria with standard dynamics or whether the preference for offering p_l over p_0 (as with standard dynamics) might change again to a preference for offering p_l over p_0 (as with reversed dynamics). However, if that was the case, (32) and (33) imply a buyer is indifferent between offering p_h and p_0 at y^1 and at y^2. This means that (31) holds at $y^1 \in (y_l, s_l)$ and at $y^2 \in (y^1, s_h)$.

But then our assumption made in Step I will imply that $\nu_h(y^1) < 1/M$ and $\nu_h(y^1) < 1/M$. As we can see following the proof of Lemma 7, $\nu_h(y^1) < 1/M$ and $\nu_h(y^1) < 1/M$

implies first $\alpha \to \alpha'$ as $M \to \infty$ which implies $\nu_l(y^1) \to 0$ and $\nu_l(y^2) \to 0$ as $M \to \infty$. This contradicts our assumptions in Step I. As a result, we can conclude that the cutoffs y^1 and y^2 must be such that a buyer prefers to offer p_l over p_0 at each of them as required by standard trade dynamics.

Step IV. Limit payoffs at y^1 and y^2

It remains to calculate the payoffs in the limit equilibria where $y = y^1 \in (y_l, s_h)$ and $y = y^2 \in (y^1, s_h)$ satisfy (27). These limit payoffs depend on $\nu_l(y, \delta)$ and $\nu_h(y, \delta)$, which assume different values for $(y, \delta) = (y^1(r), r)$ and $(y, \delta) = (y^2(r), r)$ for low r, although both $y^1(r) \to 1$ and $y^2(r) \to 1$ as $r \to 0$.

Case 1. The payoffs at $y^1(r)$ for $r \to 0$.

We have shown that $\nu_h(y^1) < 1/M$ but $\nu_l(y^1) \in (\frac{\Delta_g}{\Delta_l}, M)$. Applying (13) and (27), the equation defining y^1 thus becomes for high values of M, approximately,

$$\Delta_h - \frac{1}{N_f N_F}\Delta_g = \frac{\Delta_h - 1/N_F \Delta_g + (1 - 1/N_F)(\Delta_l - \frac{\Delta_g + \Delta_l}{1+\nu_l(y^1)})}{2 + 2/M} + \frac{1}{N_f N_F}\left(\Delta_l - \frac{\Delta_g + \Delta_l}{1 + \nu_l(y^1)}\right).$$

We are interested in the limiting payoffs as $M \to \infty$, which gives us

$$\begin{aligned}
\Delta_h &= \frac{\Delta_h + \Delta_l - \frac{\Delta_g + \Delta_l}{1+\nu_l(y^1)}}{2} \\
\Longrightarrow \nu_l(y^1) &= \frac{\Delta_g + \Delta_h}{\Delta_l - \Delta_h} > 0 \\
\Longrightarrow V_l - C_l &= \frac{\Delta_l + \Delta_g}{1 + \nu_l(y^1)} = \Delta_l - \Delta_h \\
\Longrightarrow V_b &= \frac{\Delta_h + \Delta_l - (\Delta_l - \Delta_h)}{2} = \Delta_h \\
\Longrightarrow V_b + V_l - C_l &= \Delta_h + \Delta_l - \Delta_h = \Delta_l \\
\Longrightarrow W &= V_b + \frac{1}{2}(V_l - C_l) = \Delta_h + \frac{1}{2}(\Delta_l - \Delta_h) = \frac{\Delta_h + \Delta_l}{2}.
\end{aligned}$$

Two results are notable. First, the limiting equilibrium payoffs will approach from below the payoffs at which a buyer is indifferent between offering p_0 and p_l for $s < y$. Second, the limiting equilibrium payoffs are efficient, equaling the payoffs from immediate trading. Moreover, the limiting payoffs are also higher than the payoffs $\Delta_l/2$ in the static one-price model.

Case 2. The payoffs at $y^2(r)$ for $r \to 0$.

We first make a guess that $y^2 \in (s_l, s_h)$ such that $\nu_l(y^2) > M$ but $\nu_h(y^2) \in (1/M, M)$.[32]

[32] The other possibility would be $\nu_l(y^2) > M$ and $\nu_h(y^2) > M$, which would result in lower limit payoffs.

The equation defining y^2 thus becomes for high values of M, approximately,

$$\Delta_h - \frac{1}{N_f N_F}\Delta_g = \frac{\Delta_h - 1/N_F \Delta_g + (1-1/N_F)(\Delta_l - \frac{\Delta_g+\Delta_l}{1+M})}{2 + 1/M + \nu_h(y^2)} + \frac{1}{N_f N_F}\left(\Delta_l - \frac{\Delta_g + \Delta_l}{1+M}\right)$$

We concentrate on the limiting payoffs as $M \to \infty$, which here gives us

$$\begin{aligned}
\Delta_h &= \frac{\Delta_h + \Delta_l}{2 + \nu_h(y^2)} \\
\Longrightarrow \nu_h(y^1) &= \frac{\Delta_l - \Delta_h}{\Delta_h} > 0 \\
\Longrightarrow V_l - C_l &= 0 \\
\Longrightarrow V_b &= \frac{\Delta_h + \Delta_l}{1 + \frac{\Delta_l}{\Delta_h}} = \Delta_h \\
\Longrightarrow V_b + V_l - C_l &= \Delta_h < \Delta_l \\
\Longrightarrow W = V_b + \frac{1}{2}(V_l - C_l) &= \Delta_h < \frac{\Delta_h + \Delta_l}{2}.
\end{aligned}$$

Now, the limiting payoffs are inefficient, below the payoffs from immediate trading. However, the limiting equilibrium payoffs may still exceed the static one-price model payoffs $\Delta_l/2$.

r	y^1	V_b	V_l	y^2	V_b	V_l
0.01	0.953	0.488	0.468	0.995	0.526	0.009
0.001	0.985	0.487	0.510	0.999	0.599	0.003
0.0001	0.995	0.501	0.494	1.000	0.529	0.000

Table 1: Comparison of equilibria with $(\Delta_h, \Delta_g, \Delta_l) = (0.5, 1.5, 1)$.

Table 1 compares equilibrium payoffs for different $r \leq 0.01$.

Proof of Proposition 1

Under reversed trade dynamics, the cutoff signal y satisfies the following fixed point condition and incentive condition

$$\begin{aligned}
FP_h : FP_h &= \frac{\Delta_h - (1-F_h)\frac{f_l}{f_h}\Delta_g}{1 + (1-F_h)\frac{f_l}{f_h}} - V_b = 0, \\
IC_{0l} : IC_{0l}(y) &= V_b(y) + (V_l(y) - C_l) - \Delta_l \geq 0.
\end{aligned}$$

In this case, the existence of equilibrium follows directly from Lemma 8, which proves that there exists y_0 at which a buyer is indifferent between p_h and p_0 and prefers offering p_0 over p_l.

Equilibrium uniqueness is given by the proof of Proposition 2 (Step II, Case 1), which shows that the fixed point y of F_{0h} is unique because F_{0h} is increasing when $IC_{0l} \geq 0$ is satisfied.

We can derive payoff limits as before. We know from Lemma 8 that $y_0 \to 1$ along a path such that $\nu_h < \nu_l < 1/M$, where M is a large value:

$$\Longrightarrow V_l - C_l = \frac{\Delta_l + \Delta_g}{1 + 1/M} \to \Delta_l + \Delta_g$$
$$\Longrightarrow V_b = \frac{\Delta_h - \Delta_g}{2 + 1/M + 1/M} \to \frac{\Delta_h - \Delta_g}{2}$$
$$\Longrightarrow W = V_b + \frac{1}{2}(V_l - C_l) = \frac{\Delta_h - \Delta_g}{2} + \frac{\Delta_l + \Delta_g}{2} = \frac{\Delta_h + \Delta_l}{2}.$$

The limiting payoffs are thereby efficient and equal the static one-price model payoffs.

Proof of Proposition 3

By Lemma 4, we know that an equilibrium cannot feature reversed dynamics if the static lemons problem is severe. Because we assume in this case that $\Delta_g > \Delta_h$, an equilibrium must thus feature standard dynamics. Under standard trade dynamics, the cutoff signal y satisfies the following fixed point condition and incentive condition

$$FP_h : FP_h = \Delta_h - \frac{1}{N_f N_F}\Delta_g - V_b - \frac{1}{N_f N_F}\max\{U_l - V_l, V_b\} = 0, \tag{34}$$
$$IC_{0l} : IC_{0l}(y) = V_b(y) + (V_l(y) - C_l) - \Delta_l \leq 0. \tag{35}$$

However, satisfying both conditions at the same time is impossible for sufficiently low r if $\Delta_h > \Delta_l$.

To see why, note that, by Lemma 8, satisfying the incentive condition $IC_{0l} \leq 0$ for low enough r requires high enough $y > y_l(r) > y_0(r)$, where $y_0(r) \to 1$ as $r \to 0$.

As a result, applying the notation in the proof of Proposition 2 (Step I), the fixed point condition $FP_h = 0$ for $y > y_0$ and $r < 1/M$ can be approximated by

$$\Delta_h - \underbrace{\left(\frac{1}{N_f N_F}\right)}_{\to 0}\Delta_g - V_b - \underbrace{\left(\frac{1}{N_f N_F}\right)}_{\to 0}\max\{U_l - V_l, V_b\} = 0,$$

where $1/N_f \to 0$, $1/N_F \to 0$ as $M \to \infty$.

We also show in the proof of Proposition 2 (Step II Case 1.) that $FP_h > 0$ for $y \in (y_0, y_l)$ and $r < 1/M$. By the continuity of $FP_h(y)$, it is thus easy to see that satisfying $FP_h(y) = 0$ for $y \in (y_0, y_l)$ and $r < 1/M$ is impossible without assuming that $V_b \geq \Delta_h$, which would violate $IC_{0l}(y) \leq 0$.

This also covers the case of weak incentives $IC_{0l} = 0$. However, Proposition 3 may mislead some readers into thinking that the above shows that we cannot obtain an equilibrium where buyers offer two prices p_h and p_l but does not preclude the existence of an equilibrium with three price offers p_h, p_l and p_0.

To convince all readers, we thus show that offering three prices would result in a contradiction as y_0 and y_l require different screening intensities $\nu_l(y,\delta)$ for low r, as shown in Lemma 8. So, let us try to construct an equilibrium where $y = y_0 = y_l$ such that a buyer would be willing to propose p_0 or p_l for $s < y_0 = y_l$ (i.e., p_0 for $s \in (0, z)$ and p_l for $s \in (z, y)$) and p_h for $s > y_0 = y_l$.

The probability of trading for low quality is given by $1 - F_l(z)$, which we also use in the following notation $\nu_0 = \frac{r}{1-F_l(z)} < \frac{r}{1-F_h(y)} = \nu_h(y)$. We can thus express the valuation of buyers as

$$V_b = \frac{q_u(y,z)(1-F_h(y))\Delta_h - (1-q_u(y,z))(1-F_l(y))\Delta_g}{1 - q_u(y,z)F_h(y) - (1-q_u(y,z))F_l(y) + r}$$
$$\frac{\Delta_h - \frac{\nu_0(z)}{\nu_l(y)}\Delta_g}{1 + \frac{\nu_0(z)}{\nu_l(y)} + r\frac{1}{q_u(y,z)(1-F_h(y))}}$$
$$\frac{\Delta_h - \frac{\nu_0(z)}{\nu_l(y)}\Delta_g}{1 + \frac{\nu_0(z)}{\nu_l(y)} + r\left(\frac{1-F_l(z)}{(1-F_l(z))(1-F_h(y))} + \frac{1-F_h(y)}{(1-F_l(z))(1-F_h(y))}\right)}$$
$$\frac{\Delta_h - \frac{\nu_0(z)}{\nu_l(y)}\Delta_g}{1 + \frac{\nu_0(z)}{\nu_l(y)} + \nu_0(z) + \nu_h(y)} \tag{36}$$

where

$$q_u(y,z) = \frac{1}{1 + \frac{1-F_h(y)}{1-F_l(z)}}.$$

By definition, y_l satisfies the incentive condition $IC_{0l}(y) = 0$, which can be written as

$$V_b = \frac{\Delta_l - \frac{1}{\nu_l(y)}\Delta_g}{1 + \frac{1}{\nu_l(y)}},$$

whereas y_0 satisfies the fixed point condition $FP_h(y) = 0$, which can be expressed as

$$V_b = \frac{\Delta_h - \frac{\nu_0(z)}{\nu_l(y)}m(y)\Delta_g}{1 + \frac{\nu_0(z)}{\nu_l(y)}m(y)}.$$

Above,

$$m(y) = \frac{f_l(y)}{f_h(y)}\frac{1-F_h(y)}{1-F_l(y)}.$$

Note that $m(y)$ is larger than unity by MLRP but approaches one as $y \to 1$. As shown by Lemma 8, satisfying the incentive condition requires that $y \to 1$. Thus, there exists no cutoff $y = y_0 = y_l$ that satisfies all the conditions for low r. $\square$

Documentation for Figures 1, 4 and 5

Figures 1, 4 and 5 are plotted using the function forms as follows:

$$f_h(s) = 2s, F_h(s) = s^2, f_l(s) = 2 - 2s, F_l(s) = 2s - s^2,$$

$$\nu_h(y, \delta) = \frac{r}{1 - F_h(y)} \qquad \text{'blue line'}$$

$$\nu_l(y, \delta) = \frac{r}{1 - F_l(y)} \qquad \text{'red line'}$$

$$y = y_0(r) \qquad \text{'green line'}$$

$$y = y_l(r) \qquad \text{'pink line'}$$

Proof of Corollary 1

Propositions 1 and 2 demonstrate that any (sufficiently large) information bound $B < \infty$ is associated with $y = \bar{s} < 1$ and $\delta < 1$ on the path $(y, \delta) \to (1, 0)$ to the efficient limit equilibrium.

If $\Delta_g > \Delta_h$ and $\Delta_l \geq \Delta_h$, the payoffs in the static one-price model equal $\Delta_l/2$ and those in the steady-state equilibrium with small frictions

$$V_b = \frac{\Delta_h - (1 - F_l(y))\Delta_g + F_l(y)(U_l - V_l)}{2 + r + \nu_h(y, \delta)} +$$
$$(V_l - C_l)/2 = \frac{\Delta_g + \Delta_l}{2 + 2\nu_l(y, \delta)}.$$

Instead, if $\Delta_g \leq \Delta_h$, the payoffs in the static one-price model equal $\Delta_h/2 + \Delta_l/2$ and those in the steady-state equilibrium with small frictions

$$V_b = V_b(y) = \frac{\Delta_h - \Delta_g}{2 + \nu_h(y, \delta) + \nu_l(y, \delta)} +$$
$$(V_l - C_l)/2 = \frac{\Delta_g + \Delta_l}{2 + 2\nu_l(y, \delta)}. \quad \square$$

Knife-edge dynamics

An equilibrium with knife-edge dynamics for vanishing frictions $r \to 0$ is given by y, $z < y$, and (V_b, V_l) satisfying the following system

$$\begin{aligned} q_c(s) &= \frac{1}{1 + \frac{1-F_h(y)}{1-F_l(z)}\frac{f_l(s)}{f_h(s)}}, \text{ for } s \in [0,1] \\ q_c(y)\,(U_h - C_h) + (1 - q_c(y))\,(U_l - C_h) &= V_b, \qquad (37) \\ V_b &= \frac{\Delta_h}{1 + \nu_h(y,\delta)}, \\ V_l &= C_l + \frac{\Delta_g + \Delta_l}{1 + \nu_l(y,\delta)}, \\ V_b + (V_l - C_l) &= \Delta_l. \qquad (38) \end{aligned}$$

Note that we can set $y \to 1$ because it otherwise becomes impossible to satisfy Eq. (38). Eq. (38) further implies that $\frac{\Delta_h}{1+\nu_h(y,\delta)} + \frac{\Delta_g+\Delta_l}{1+\nu_l(y,\delta)} = \Delta_l$. Joining Eqs. (37) and (38), we thus obtain that

$$q_c(y)\Delta_h + (1 - q_c(y))\,(-\Delta_g) = \frac{\Delta_h}{1 + \nu_h(y,\delta)} = \Delta_l - \frac{\Delta_g + \Delta_l}{1 + \nu_l(y,\delta)}. \qquad (39)$$

As $(y, \delta) \to (1, 0)$, market quality depends on the evolution of $z(y, \delta)$ as frictions disappear. In principle, $z(y, \delta)$ could assume any values between 0 and $y \to 1$. Depending on $z(y, \delta)$, buyers' beliefs

$$q_c(y) = \frac{1}{1 + \frac{1-F_h(y)}{1-F_l(z)}\frac{f_l(s)}{f_h(s)}}$$

thus span all the values from 1/2 (attained by letting $z(y, \delta) \to y$ as $(y, \delta) \to (1, 0)$) to 1 (attained by letting $z(y, \delta) \to 0$ as $(y, \delta) \to (1, 0)$). This allows some leeway in equilibrium construction because any triplet (q_c, ν_h, ν_l) satisfying Eq. (39) for $1/2 \leq q_c \leq 1$ and $0 \leq \nu_h \leq \nu_l \leq \infty$ defines a steady-state limit equilibrium for $(y, \delta) \to (1, 0)$.

$$\begin{aligned} \frac{\Delta_h}{1 + \nu_h} &= q_c\Delta_h + (1 - q_c)(-\Delta_g) \\ \frac{\Delta_h}{1 + \nu_h} &= \Delta_l - \frac{\Delta_g + \Delta_l}{1 + \nu_l}. \end{aligned}$$

For example, by letting $\nu_l \to \infty$, we immediately find an equilibrium with knife-edge dynamics given by

$$\nu_h \to \frac{\Delta_h}{\Delta_l} - 1, \text{ and } q_c \to \frac{\Delta_l + \Delta_g}{\Delta_h + \Delta_g}, \text{ as } (y, \delta) \to (1, 0),$$

for cases $\Delta_g > \Delta_h > \Delta_l$ where Proposition 3 shows that no equilibrium with standard or reversed dynamics exists.

This equilibrium is inefficient as $V_b + (V_l - C_l)/2 = \Delta_l < \frac{\Delta_l + \Delta_h}{2}$. To derive an efficient equilibrium, we thus require that

$$V_b = \frac{\Delta_l + \Delta_h}{2} - \frac{\Delta_g + \Delta_l}{(1+\nu_l)2}$$

$$\frac{\Delta_l + \Delta_h}{2} - \frac{\Delta_g + \Delta_l}{(1+\nu_l)2} = \Delta_l - \frac{\Delta_g + \Delta_l}{1+\nu_l},$$

which gives $\frac{1}{1+\nu_l}$ as a solution to the second-order equation

$$-\left(\Delta_g + \Delta_l\right)\left(\frac{1}{1+\nu_l}\right)^2 + \left(\Delta_g + \Delta_l\right)\frac{1}{1+\nu_l} + \frac{\Delta_h - \Delta_l}{2} = 0.$$

Presuming $\Delta_h > \Delta_l$, the equation has a positive solution

$$\frac{1}{1+\nu_l} = \frac{-\left(\Delta_g + \Delta_l\right) + \sqrt{\left(\Delta_g + \Delta_l\right)^2 + 2\left(\Delta_g + \Delta_l\right)\left(\Delta_h - \Delta_l\right)}}{2\left(\Delta_g + \Delta_l\right)} = \frac{-1 + \sqrt{1 + 2\frac{\Delta_h - \Delta_l}{\Delta_g + \Delta_l}}}{2},$$

Inserting this solution into Eqs. (39) gives

$$\nu_l = \frac{\sqrt{1 + 2\frac{\Delta_h - \Delta_l}{\Delta_g + \Delta_l}} + 3}{\sqrt{1 + 2\frac{\Delta_h - \Delta_l}{\Delta_g + \Delta_l}} - 1}$$

$$\nu_h = \frac{\Delta_h - \Delta_l + \frac{\Delta_g + \Delta_l}{2}\left(\sqrt{1 + 2\frac{\Delta_h - \Delta_l}{\Delta_g + \Delta_l}} - 1\right)}{\Delta_l - \frac{\Delta_g + \Delta_l}{2}\left(\sqrt{1 + 2\frac{\Delta_h - \Delta_l}{\Delta_g + \Delta_l}} - 1\right)}$$

$$q_c = \frac{\Delta_h}{\Delta_h + \Delta_g}\frac{1}{1+\nu_h} + \frac{\Delta_g}{\Delta_h + \Delta_g} \in (1/2, 1)$$

The existence of efficient knife-edge dynamics requires that $\nu_h < \nu_h$ and $q_c \in (1/2, 1)$. The latter condition is clearly satisfied for all $\nu_h \geq 0$ if $\Delta_g > \Delta_h$.

It is also easy to confirm that the former one is satisfied, e.g., if $\Delta_g = 3 > \Delta_h = 2 > \Delta_l = 1$ for which $\nu_l \approx 19.2 > \nu_h \approx 2.6$ and $q_c \approx 0.71 \in (1/2, 1)$.

More general analysis demonstrates that $\nu_h < \nu_l$ is equivalent to

$$\frac{y+4}{y} > \frac{\Delta_h - \Delta_l + \frac{\Delta_g + \Delta_l}{2}y}{\Delta_l - \frac{\Delta_g + \Delta_l}{2}y} \Longleftrightarrow -\left(\Delta_g + \Delta_l\right)y^2 - \left(2\Delta_g + \Delta_h\right)y + 4\Delta_l > 0$$

where $y = \left(\sqrt{1 + 2\frac{\Delta_h - \Delta_l}{\Delta_g + \Delta_l}} - 1\right)$. The lhs of the inequality represents a downward sloping parabola, with both a negative root and a positive root, which transforms our conditions for $\nu_l < \nu_h$ into

$$-\sqrt{\left(\frac{\Delta_g + \Delta_h/2}{\Delta_g + \Delta_l}\right)^2 + 4\Delta_l} - \frac{\Delta_g + \Delta_h/2}{\Delta_g + \Delta_l} < \sqrt{1 + 2\frac{\Delta_h - \Delta_l}{\Delta_g + \Delta_l}} - 1 < \sqrt{\left(\frac{\Delta_g + \Delta_h/2}{\Delta_g + \Delta_l}\right)^2 + 4\Delta_l} - \frac{\Delta_g + \Delta_h/2}{\Delta_g + \Delta_l}.$$