Innovation size, markups, and Schumpeterian growth dynamics with Variable Elasticity of Substitution

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Abstract

I extend the static analysis of monopolistic competition under variable elasticity of substitution (VES) preferences proposed by Zhelobodko et al. (2012, "Monopolistic competition in general equilibrium: beyond the CES", *Econometrica*) along two lines by (1) adding costly product quality improvements - that is vertical innovation, and (2) nesting this market structure within the dynamic general equilibrium framework, that is the canonical two-sector-R&D growth model proposed by Young (1998, *JPE*). Thereby, this study presents the first analysis of Schumpeterian growth dynamics under VES preferences. I show that (a) The relation between innovation size and markups may be non-monotonic (b) for sufficiently low innovation cost there is a balanced growth path of drastic innovation determined by the population growth rate. However, for sufficiently high innovation cost, or non-positive population growth, the model economy converges to the limit values of demand elasticity, and (c) the efficient balanced-growth rate is also defined by population growth rate but differ from the market equilibrium path in the ratio between per-variety consumption level and product variety span.

JEL Classification: O-30, O-40

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1 Introduction

This study provides first analysis of Schumpeterian growth under Variable Elasticity of Substitution (VES) preferences. In the Schumpeterian growth process with VES preferences, the ongoing product-quality improvements drive corresponding changes in demand elasticity, which give rise to a set of balanced-growth characteristics that are eliminated under the traditional Constant Elasticity of Substitution (CES) specification. I introduce the VES preferences and the notion of Relative Love for Variety (RLV), proposed by Zhelobodko et al. (2012) in their static analysis of monopolistic competition, into Young's (1998) Two-Sector-R&D model of vertical innovation with endogenous

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product-variety span. The analysis incorporates both drastic and non-drastic innovation, that are usually studied separately in the literature.[1]

Within this framework I show that for sufficiently low innovation cost, the size of drastic innovations along the balanced-growth rate is determined by the population growth rate. However, for a sufficiently high innovation cost, or for constant or shrinking population, the economy converges to the limit values of the VES preferences with non-drastic innovation. That is the model economy converges to the CES benchmark. The ratio between innovation size and product variety span along the steady growth path in the decentralized economy can be either higher or lower than the welfare maximizing one. Furthermore, I show that equilibrium markups and innovation size may be positively or negatively related and may not uniquely determined - a given innovation size may be supported by more than one mark-up (that is uniform across varieties). By comparison, under CES markup and innovation size change in opposite directions with exogenous change in the elasticity parameter.

This study contributes to the ongoing effort to expand existing core analyses in Industrial Organization, and their implications to the study of international trade and R&D-based growth, beyond the CES specification of preferences or technology, which has been dominating the literature since introduced by Dixit and Stiglitz (1977). Although highly tractable and convenient, the CES specification lacks flexibility and yields some results that are at odds with empirical finding[2]. Consequently, over the last decade, a growing body of research proposed alternative VES specifications, in a static monopolistic-competition frameworks; see for example Zhelobodko et al. (2012) and Bertoletti and Etro (2015, 2016, 2021), Matsuyama and Ushchev (2022).

A sequence of recent papers implemented proposed VES specifications in the Romer's (1990) framework of R&D-based growth with horizontal innovation (variety expansion); see Bucci and Matveenko (2017), Boucekkine et al. (2017), and Etro (2018, 2019), Latzer Matsuyama and Parenti (2019). To the best of my knowledge this study is the first to incorporates VES preferences into a Schumpeterian growth model.

The pioneering Schumpeterian growth models of vertical innovation by Gorossman and Helpman (1991) and Aghion Howitt (1992), had a fixed number of product lines, and abstracted from the horizontal price competition across product lines.[3] The current analysis is carried within a two-sector R&D model with both horizontal and vertical innovation, as in Peretto (1998), Young (1998), Dinopoulos and Thompson (1998), Howitt (1999), and Segrestrom (2000). This line of models was designed to remove the scale-effect that presents in the aforementioned first-generation models,

[1]With drastic innovation the market leader can keep sellers of inferiror products out of the market while charging a monopolistic price, based on demand elasticity. Under non-drastic innovation, prices are set through vertical competition between the entrant and the incumbent of each product line.

[2]See Zhelobodko et al. (2012), page 2765.

[3]In Grossman and Helpman (1991) there is a unit mass of different product lines the price of each product is set through vertical price competition between the entrant that developed the highest- quality product, and the incumbent. In Aghion and Howitt (1992) there is a single product line and the price is set by the market leader either through vertical price competition, in case of non-drastic innovation, or as an unconstrained monopolistic price in the case of drastic innovation.

which did not align with empirical findings[4]. These two-sector R&D models rely on the CES specification,[5] and confine attention to drastic innovation, to conclude that (a) innovation size is determined by the CES parameter and the innovation technology parameters (b) markups are also pinned down by the CES, independently of innovation size (c) Innovation size falls below the socially optimal rate, and (d) population size and population growth rate do not affect innovation size, but only the size and growth rate of product-variety span, respectively. This work shows how all these characteristics of Shcumpeterian growth change once departing from the CES specification.

Our results on the effect of population growth on innovation size relate to the debate over the potential effectiveness of industrial-policy (e.g. R&D subsidies and the design of patents) in promoting R&D based growth, that is known in the literature as the debate on whether R&D-based growth is fully endogenous or semi-endogenous[6]. In a model where the exogenous population growth rate determines the growth rate, growth is semi-endogenous and industrial-policy interventions are futile. Our results imply that with the departure from the CES, growth may changes the nature of growth in the model economy - from fully endogenous to semi endogenous[7] - depending on the population growth rate and the productivity of innovation technology. Finally, all aforementioned studies focus on growth dynamics under drastic innovation and horizontal price competition, whereas the current analysis includes also the non-drastic innovation and vertical price competition, as the innovation size and corresponding pricing regime are determined endogenously in the model economy[8].

In the working paper where he first presented his Two-sector R&D growth model, Young (1995) considers the possibility of a variable elasticity of demand with respect to quality provision[9]. However, he does that based on different micro foundations - he builds on Salop's circular market model where the equilibrium prices (and markups) are independent of quality provision - and he confines attention to changes in population size (scale effect) on innovation size abstracting from population growth (strong scale effect)[10]. The current analysis provides a direct generalization of Young's (1998) CES model to the case of VES preferences or technology and shows that in this standard framework, population growth is necessary to maintain VES specification viable.

The remainder of the paper is organized as follows. Section 2 presents the model and the static equilibrium outcomes are derived. Section 3 characterizes the growth dynamics for the decentralized

[4] See Jones (1995a,b) and Jones (1999).

[5] Either CES preferences over a variety of consumption product varieties, or CES production technology of a final good with variety of intermediate goods.

[6] If long term growth is fully endogenous it can be affected by policy interventions and if it is semi-endogenous it is determined by parameter that are usually taken as exogenous, such as the population growth rate.

See Cozzi (2017a,b) for recent concise summary of the topic and proposed synthesis based resolutions of this debate.

[7] Li (2000) already showed that growth endogeneity in this class of models relies on two Knife edge assumptions regarding. Our results show that it derives also from the CES assumption.

[8] Sorek (2021) studied the difference between the drastic and non-drastic innovation regimes in Young's (1998) model with CES and their implications for industrial policy.

[9] See Section IV (p.18) there.

[10] When referring to the possibility of variable demand elasticity with respect to quality in the journal article, Young (1998) still abstracts from the possibility that markups may depend on innovation size and that demand elasticity may as well change with consumed quantity - see equations (11) and (25) and the following discussions, there.

economy and the welfare maximizing growth paths. Section 4 summarizes and concludes this study.

2 The Model

To maintain consistency with Zhelobodko et al. (2012), we replace Young's (1998) production function of a single final good that uses intermediate goods with an instantaneous utility function over variety of consumption goods. These two representations are equivalent for all the purposes of the current analysis[11]. We then extend Young's (1998) model by replacing the instantaneous CES utility function with a more general specification, which allows demand elasticity and the elasticity of substitution to vary with consumption level, that is a Variable Elasticity of Substitution - VES. All other specifications of Young's (1998) model economy remain, and therefore all the results presented below coincide with his once demand elasticity is re-assumed to be constant. Time is discrete and, in each period t the economy is populated with L_t infinitely lived agents, and population size grows at a constant rate $n \equiv \frac{\triangle L_{t+1}}{L_t}$. Each worker supplies one unit of labor, so within each period population size equals aggregate labor supply.

2.1 Preferences and consumer's optimization

The lifetime utility of the representative consumer is given by

$$U = \sum_{t=0}^{\infty} \beta^t \ln(c_t) \tag{1}$$

where $\beta \in (0,1)$ is the time preference parameter. The per-capita utilization level of consumption in (1), denoted c, is derived from M differentiated products ("varieties"), subject to the VES instantaneous utility presented in Zhelobodko et al. (2012)[12]:

$$c_t = \int_0^{M_t} u\left(c_{i,t}\right) di \tag{2}$$

The utility function $u(\bullet)$ is concave and at least thrice differentiable, and the consumption stream derived from each, c_i, is given by $c_i = q_i x_i$, where x_i and q_i designate the utilized quantity and quality, respectively. For the instantaneous utility function (13), Zhelobodko et al. (2012) defines the Relative Love for Variety, "RLV", as

$$RLV_i \equiv -\frac{c_i u''\left(c_i\right)}{u'\left(c_i\right)} > 0 \tag{3}$$

This measure of lover for variety, corresponds the Prath-Arrow measure of relative risk aversion, and it is inverse to the demand elasticity

[11] Sorek (2021) present this equivalence for drastic innovation and add the analysis of non-drastic innovation in this framework.

[12] in utility function (2) is equivalently to a final good production function that take M intermediate goods as inputs, use by perfectly competitive firms, as in Young's (1998).

$$s_i(c_i) = \frac{1}{RLV_i} \tag{4}$$

For equal consumption levels from all product varieties, equation (14a) defines also the elasticity of substitution for each pair of varieties. To sustain the monopolistic competition framework under study, it is required to assume that demand elasticity is greater than one and finite $s_i(c_i) \in (1, \infty) \iff RLV_i \in (0, 1)$. The static analysis of Zhelobodko et al. (2012) focuses on the relation between changes in consumed quantities, RLV changes, and market equilibrium outcomes. Here, the focus is on the relation between on-going quality changes, RLV changes, market outcomes, and market dynamics.

2.2 Technologies

Labor is the sole input for production and innovation, and the wage rate is normalized to one. One unit of labor produces one consumption good (regardless of its variety and quality). The two latter assumptions imply a unit marginal cost of production. Innovation is certain and is subject to the following cost function

$$f(q_{i,t+1}, \overline{q}_t) = \exp\left(\phi \frac{q_{i,t+1}}{\overline{q}_t}\right) \tag{5}$$

The innovation cost in sector i is increasing with the rate of quality improvement over the existing quality frontier – denoted $\overline{q}_t$, which is the highest quality already attained in the economy. I denote the rate of quality improvements $\kappa \equiv \frac{q_{i,t+1}}{\overline{q}_t}$ to rewrite the innovation cost

$$f_{i,t+1} \equiv f(\kappa_{i,t+1}) = \exp\left(\phi \kappa_{i,t+1}\right) \tag{6}$$

Due to the assumed certain outcome of R&D investments, innovation takes exactly one period, and therefore the effective market lifetime of each quality improvement is one period as well (as each and every period improved products take over the market lead).

3 General Equilibrium

3.1 Consumers' optimization

Lifetime utility (1) is maximized under the standard inter-temporal budget constraint

$$a_{t+1} = \frac{1 + r_{t+1}}{1 + n} a_t + w_t - e_t \tag{7}$$

where a denotes the consumer's assets in form of patents' ownership, $1 + r_{t+1}$ is the interest rate earned between periods t and $t+1$, and w is labor income. The maximization of (1) under the composite consumption stream specification (2) and the dynamic constraint (6), assuming cross-variety symmetry, yields the following modified Euler condition over the consumer's inter-temporal spending path[13]

$$\frac{e_{t+1}}{e_t}\frac{\sigma_{u,c_t}}{\sigma_{u,c_t+1}} = \frac{\beta\left(1+r_{t+1}\right)}{1+n} \tag{8}$$

where $\sigma_{u,c_t} \equiv \frac{c_t u'(c_t)}{u(c_t)}$, is the elasticity of the instantaneous utility from each variety with respect to the consumption level $c_{i,t}$. Under the assumed consumers' homogeneity, condition (16) holds also for the aggregate consumers' spending levels

$$\frac{E_{t+1}}{E_t}\frac{\sigma_{u,c_t}}{\sigma_{u,c_t+1}} = \beta\left(1+r_{t+1}\right) \tag{9}$$

Within each period, consumers maximize the instantaneous utility function (2), by allocating their spending subject to $e_t \leq \int\limits_0^{M_t}(p_{i,t}\cdot c_{i,t})di$, as to . Solving this standard static optimization problem, applying the Largrangian $\mathcal{L} = \int\limits_0^{M_t} u\left(c_{i,t}\right)di + \lambda_t\left[e_t - \int\limits_0^{M_t}(p_{i,t}\cdot c_{i,t})di\right]$, yields the per-variety inverse demand function

$$p_i = \frac{q_{i,t}u'\left(q_{i,t}\cdot x_{i,t}\right)}{\lambda_t} \tag{10}$$

3.2 Profit maximization

Following Young (1998), the present analysis assumes complete lagging-breadth patent protection and no leading breadth protection (minimal patentability requirement). Under these assumptions, the price set by the entrant must not exceed its innovation size, that is $p_{i,t} \leq \kappa_{i,t}$, in order to push the former product-line leader out of the market. If the profit maximizing price is strictly smaller than the innovation price the innovation is drastic, as the potential vertical competition with the previous product lines leader (the incumbent) is not binding. If the product price is set equal to innovation size the innovation is non-drastic. With CES preferences the given value of demand elasticity (relative to the innovation cost parameter), determines whether innovation size is drastic or non-drastic. For sufficiently high elasticity innovation is drastic. The size and price of non-drastic innovation are set independently: the price is decreasing with demand elasticity and innovation size is increasing with demand elasticity. Below a certain threshold value of demand elasticity, innovation becomes non-drastic and its size is independent of demand elasticity (and the price is equal to innovation size). The following analysis shows how the profit-maximizing innovation-size and price are jointly determined under VES preferences, along with demand elasticity. It starts with the case of Non-drastic innovation and proceeds to drastic innovation.

[13] An equivalent modified Euler condition was derived by Boucekkine et al. (2017), for continuous time. However, in their analysis of the horizontal innovation model, the elasticity of utility with respect to the consumption level is affected by the consumed quantities only, whereas in this analysis it is affected also by the quality of the consumed products.

3.2.1 Non-Drastic Innovation

With non-drastic innovation the product price is equal to the innovation size: $p_{i,t} = \kappa_{i,t}$. Therefore, each innovating firm maximizes the following present value profit:

$$\Pi_{i,t} = \frac{(\kappa_{i,t+1} - 1)\, x_{i,t+1}\,(\kappa_{i,t+1})\, L_{t+1}}{1 + r_{t+1}} - f_{i,t+1} \tag{11}$$

Under symmetric equilibrium, aggregate demand for each variety is $X_t^d = \frac{E_t}{pM_t}, \forall_i$. Substituting the latter expression into (10), we write the free-entry (zero-profit) condition

$$\frac{(\kappa_{i,t+1} - 1)\,\frac{E_{t+1}}{M_{t+1}\kappa_{i,t+1}}}{1 + r_{t+1}} = f_{i,t+1} \tag{10a}$$

The first order condition for maximizing (10) is[14]

$$\frac{\left(\frac{1}{\kappa_{i,t+1}^{*}-1} + \frac{x_{i,t+1}{}'\left(\kappa_{i,t+1}^{*}\right)}{x_{i,t+1}\left(\kappa_{i,t+1}^{*}\right)}\right)\left(\kappa_{i,t+1}^{*} - 1\right) x_{i,t+1}\left(\kappa_{i,t+1}^{*}\right) L_{t+1}}{1 + r_{t+1}} = \phi f_{i,t+1} \tag{12}$$

Imposing $p_{i,t} = \kappa_{i,t}$ in (4), we rewrite the inverse demand function as

$$q_{i,t-1}\kappa_{i,t}^{*} u'\left(q_{i,t-1}\kappa_{i,t}^{*} \cdot x_{i,t}\right) - \lambda_t \kappa_{i,t}^{*} = 0 \tag{13}$$

Applying the implicit function theorem to (12) yields

$$\frac{dx_{i,t}}{d\kappa_{i,t}^{*}} = -\frac{q_{i,t-1}u'\left(q_{i,t}\cdot x_{i,t}\right) + q_{i,t}q_{t-1}x_{i,t}u''\left(q_{i,t}\cdot x_{i,t}\right) - \lambda_t}{q_{i,t}q_{i,t}u''\left(q_{i,t}\cdot x_{i,t}\right)} \tag{12a}$$

Then, substituting $\lambda_t = \frac{q_{i,t}u'(q_{i,t}\cdot x_{i,t})}{\kappa_t^{*}}$ from (12) into (12a) yields

$$\frac{\frac{dx_{i,t}}{d\kappa_{i,t}^{*}}}{x_{i,t}} = -\frac{1}{\kappa_{i,t}^{*}} \tag{12b}$$

Plugging the latter result back into (11) and imposing the zero-profit (free-entry) condition, reveals that the size of non-drastic innovation is independent of demand elasticity, and is time invariant[15]:

$$\forall t, i : \frac{1}{\kappa^{e}\left(\kappa^{e} - 1\right)} = \phi \tag{14}$$

[14] The asterisk superscript denotes maximizers of individual value functions. The superscript "e" denotes equilibrium values.

[15] The implicit expression of κ_v in (13) defines a quadratic function with the positive root: $\kappa_v = \frac{1+\sqrt{1+\frac{4}{\phi}}}{2} > 1$. The same result was derived in Sorek (2021) for the CES preferences.

3.2.2 Drastic innovation

For drastic innovation the profit maximizing price is lower than innovation size. Given the inverse demand function (4), each innovating firm maximizes the following present value profit

$$\Pi_{i,t} = \frac{\left(\frac{q_{i,t+1}u'(q_{i,t+1}\cdot x_{i,t+1})}{\lambda_{t+1}} - 1\right) x_{i,t+1}L_{t+1}}{1+r_{t+1}} - f(\kappa_{t+1}) \tag{15}$$

The first order condition for maximizing (16) with respect to $x_{i,t}$ reads

$$q_{i,t}u'\left(q_{i,t}\cdot x_{i,t}\right) + x_{i,t}q_{i,t}^2 u''\left(q_{i,t}\cdot x_{i,t}\right) = \lambda_t \tag{16}$$

Condition (17), which defines the surplus (and profit) maximizing quantity, can be also written as

$$\frac{q_{i,t}u'\left(c_{i,t}\right)\left(1-RLV_i\right)}{\lambda_t} = 1 \tag{15a}$$

The left side of (17a) is the marginal revenue associated with the sale of each product unit and the right side is the marginal cost. Under the assumption $RLV_i < 1$, the marginal revenue is positive for any output level. For an increasing RLV the marginal revenue is guaranteed to decrease with output level, and assuming $\frac{c_{i,t}u'''(c_{i,t})}{u''(c_{i,t})} > -2$ guarantees that the marginal revenue is decreasing with output level also for a decreasing RLV. This condition over the elasticity of the second derivative of the utility function with respect to consumption was required also in Zhelobodko et al. (2012) to ensure the existence of profit maximizing prices. As the RLV corresponds the RRA measure, the expression $\frac{c_{i,t}u'''(c_{i,t})}{u''(c_{i,t})}$ also has a corresponding term in the literature on preference for risk, that is the Relative Prudence ("RP") measure which was first defined by Kimball (1990). Combining (17) with the price equation (4) yields the following profit-maximizing price

$$p_{i,t}^* = \frac{q_{i,t}u'\left(q_{i,t}\cdot x_{i,t}\right)}{q_{i,t}u'\left(q_{i,t}\cdot x_{i,t}\right) + q_{i,t}xq_{i,t}u''\left(q_{i,t}\cdot x_{i,t}\right)} = \frac{1}{1-RLV_{i,t}} \tag{17}$$

By equation (3a), the profit-maximizing price in (18) can be written also in the familiar form

$$p_{i,t}^* = \frac{s(c_{i,t})}{s(c_{i,t})-1} \equiv \varepsilon_{i,t} \tag{16a}$$

Equations (16)-(16a) are implicit expressions of the price as their right side changes with per-variety consumption level, which, in turn, depends negatively on the price. In the case of increasing (decreasing) RLV, the right side of (16) decreases (increases) with p_i^*: The optimal price is higher than the marginal cost, $p_i^* > 1$. Therefore, the left side of (16) is bounded to $\frac{1}{p_i^*} \in (0,1)$. The right side of (15) is smaller than one, and we assume $RLV < 1$ to ensure it is positive. Moreover, the per-variety consumption level depends also on product quality (positively), and on product variety span (negatively), implying that for increasing (decreasing) RLV, the profit maximizing price (and markup) increases decreases with innovation size. Plugging the price from (18) back into (16), and maximizing the profit for $\kappa_{i,t}$ under the free-entry condition, yields the following equilibrium rate of drastic quality improvements

$$\kappa_{i,t}^{e}=\frac{1}{\phi}\left[\frac{c_{i,t}u'''\left(c_{i,t}\right)}{u''\left(c_{i,t}\right)}-\frac{c_{i,t}u''\left(c_{i,t}\right)}{u'\left(c_{i,t}\right)}-\frac{\frac{u'(c_{i,t})}{c_{i,t}u''(c_{i,t})}+1}{2+\frac{c_{i,t}u'''(c_{i,t})}{u''(c_{i,t})}}\right] \tag{18}$$

The size of drastic innovation is determined by the innovation-cost parameter and curvature of the consumption utility function that is defined by the RLV and the elasticity of its second derivative $\frac{c_{i,t}u'''(c_{i,t})}{u''(c_{i,t})}$ (which was already discussed the analysis of the optimal price above). As consumption level itself depends on innovation size, equation (17) is an implicit expression of $\kappa_{i,t}^{*}$. Apparently, it defines potential nontrivial relations between the size of drastic innovation and consumption level[16], as will be demonstrated in the next section.

3.3 Product variety span

For each and every period the aggregate resources-uses constraint, that is also the labor market clearing condition), requires

$$L_t=\frac{E_t}{p_t}+M_{t+1}f_{t+1} \tag{19}$$

Equation (18) implies that aggregate labor supply is fully employed in production (the first addend) and R&D activity (the second addend). Applying the zero-profit condition along with the Euler condition (9) to (18) yields

$$L_t=\frac{E_t}{p^e}+(1-\frac{1}{p^e})\beta E_t\frac{\sigma_{u,c_t+1}}{\sigma_{u,c_t}} \tag{18a}$$

Then, solving (18a) for E_t, and plugging it back into (18) - with the relevant price expression - yields the following equilibrium variety span for non-drastic and drastic innovation, respectively[17]

$$\begin{aligned} M_{t+1}^{e} &= \frac{L_t}{f^e\left[\frac{1}{(\kappa^e-1)\beta\frac{\sigma_{u,c_{t+1}}}{\sigma_{u,c_t}}}+1\right]} \\ M_{t+1}^{e} &= \frac{L_t}{f_{t+1}^{e}\left[\frac{1}{\varepsilon_t(1-\frac{1}{\varepsilon_{t+1}})\beta\frac{\sigma_{u,c_{t+1}}}{\sigma_{u,c_t}}}+1\right]} \end{aligned} \tag{20}$$

[16] Under the CES specification equation (17) boild down to $\kappa_{i,t}^{*}=\frac{s-1}{\phi}$, as in Young (1998).

[17] For constant utility elasticity condition (15) coincides with the one derived for the CES preferences presented in Sorek (2021).

4 Growth Dynamics

Combining all the results derived thus far, this section characterizes growth under VES preferences in the model economy, which is then compared with the welfare maximizing balanced growth. This section concludes with an example of concrete of VES preferences, that is used to illustrate some implication of the results derived for the general VES preferences.

4.1 Endogenous and semi-endogenous growth

The growth dynamics combine vertical innovation - i.e. the quality improvements defined in equations (13) and (17), along with changes in product variety span at the rate $m_t \equiv \frac{\triangle M_{t+1}}{M_t}$. Recall that under symmetric equilibrium per variety consumption is $c_{i,t} = \frac{e_t q_t}{p_t M_t}$. With CES preferences, the innovation size and the price are necessarily time invariant and determined independently by the given demand elasticity, and the product variety span expands at the same rate as population growth. Consequently, the growth of variety consumption is $1 + g_{c_i} = \frac{\kappa^*}{1+m^*}$ and total consumption grows at the rate κ^*.

With VES preferences, there cannot be a steady growth that is not a balanced growth path along which quality and variety span grow at equal rates, i.e. $\kappa^* = 1 + m^*$. This is because if the product variety span changes at a rate that is different than the rate of quality improvements, per variety consumption and the RLV keep changing over time.[18]However, equations (19) imply that under stationary variety consumption, the rate of product variety span expansion equals to the population growth rate, that is $m = n$. Therefore, along a balanced growth path the innovation size must be also equal to the population growth rate. In this case growth is semi endogenous. However, the latter analysis implies that for a constant or shrinking population, there is no such balanced growth path and as consumption per variety keeps increasing permanently while driving the RLV and the different moments of the utility function to their limit CES values, which result in fully endogenous growth.

Nonetheless, such fully endogenous growth would prevail also with a growing population, if the relevant innovation size, $\kappa = 1 + n$, is not drastic: if demand elasticity that correspond this innovation size implies a profit maximizing price that is to high to deter vertical competition. In this case the innovation size defined by equation (13) sustains, while the product variety span grows at the rate $m = n$, thereby decreasing per variety consumption and driving demand elasticity to its limits CES value. As the size of drastic innovation decreases with the innovation cost parameter, while the corresponding price depends only on demand elasticity, a higher innovation cost increases the range of demand elasticity (and variety consumption level) for which innovation is drastic the range of population growth rates that can support a balanced growth path of semi-endogenous growth. The latter results are summarized in the following proposition that characterizing the balanced growth.

[18] For $1 + m > \kappa$ ($1 + m < \kappa$) per-variety consumption is declining (increasing) over time.

Proposition 1 *For sufficiently low (high) innovation cost growth is fully (semi) endogenous, and the balanced growth rate is determined by population growth rate (by the limit CES values).*

4.2 Welfare analysis

The allocation of labor over R&D and production activity that maximizes the lifetime utility (1), under the VES preferences (2), defines the socially-optimal rate of quality improvements and product variety span, where the optimal growth path is subject to the aggregate resources uses constraint (18). Substituting (18) into (1) we write the welfare maximization objective function[19]:

$$
\begin{aligned}
U &= \sum_{t=0}^{\infty} \beta^t \ln[\int_0^M u\left(q_{i,t} x_{i,t}\right)] = \sum_{t=0}^{\infty} \beta^t \ln M_t u \left(q_{t-1} \kappa_t \left[\underbrace{\frac{L_t - M_{t+1} f\left(\kappa_{t+1}\right)}{M_t}}_{x_t} \right] \right) = \\
&= \sum_{t=0}^{\infty} \beta^t \ln M_t + \sum_{t=0}^{\infty} \beta^t \ln u \left(q_{t-1} \kappa_t \left(\underbrace{\frac{L_t - M_{t+1} f\left(\kappa_{t+1}\right)}{M_t}}_{x_t} \right) \right)
\end{aligned}
\tag{21}
$$

The first order conditions for maximizing the above expression, with respect to κ_t and M_t, satisfy[20]

$$
\begin{aligned}
\kappa_t^{**} &= \frac{\sigma_t + \sum\limits_{j=0}^{\infty} \beta^{j+1} \sigma_{t+j}}{\phi\left(1-\sigma_t\right)} \\
M_t^{**} &= \frac{L_t}{f\left(\kappa_t^{**}\right)\left[\frac{\sigma_{t-1}}{(1-\sigma_t)\beta} + 1\right]}
\end{aligned}
\tag{22}
$$

The welfare maximizing innovation size and product variety presented in (21) and the ones presented in equations (13),(17) and (19) for the market equilibrium outcomes depend on different endogenous variables: the welfare maximizing expressions depend on the utility elasticity whereas the market equilibrium ones depend on the on the RLV and the RP[21]. by comparison, under the CES specification for which all the above moments of the utility function coincides, the welfare maximizing innovation size is $\kappa^{**} = \frac{s-1}{\phi(1-\beta)}$, that is always larger than the equilibrium drastic innovation size, given by $\kappa^e = \frac{s-1}{\phi}$, but may be greater than the equilibrium non-drastic innovation size (See Sorek 2021).Therefore, in general, the innovation size and variety span along any transitional

[19] It is assumed that the transversality condition, $\underset{t \longrightarrow \infty}{Lim} \beta^t u(c_t) = 0$, holds.

[20] The socially-optimal values are denoted with double asterisk super script.

[21] In the CES case the welfare maximizing innovation size is $\kappa^{**} = \frac{s-1}{\phi(1-\beta)}$, that is larger than the equilibirum drastic innovation size, given by $\kappa^e = \frac{s-1}{\phi}$. Sorek (2021) shows that small non-drastic innovations exceed the welfare maximizing size.

path may be greater or small than the welfare maximizing ones. For a balanced growth path the above efficiency conditions read

$$
\begin{aligned}
\kappa^{**} &= \frac{\sigma^{**}}{\phi\left(1-\sigma^{**}\right)\left(1-\beta\right)} \\
M^{**} &= \frac{L_t}{f\left(\kappa^{**}\right)\left[\frac{\sigma^{**}}{\left(1-\sigma^{**}\right)\beta}+1\right]} = \frac{L_t}{f\left(\kappa^{**}\right)\left[\frac{(1-\beta)\phi\kappa^{**}}{\beta}+1\right]}
\end{aligned} \tag{21a}
$$

The welfare maximizing product variety span is decreasing with the welfare maximizing innovation size. As in the decentralized economy, a balanced growth path requires a stationary VES, that is a stationary variety consumption level, implying $\kappa_{ss} = m = n$. For the two above conditions to jointly hold, there variety span defined in (21a) needs to support the a variety consumption level , $c_i = \frac{e^{**}q_t}{p^{**}M_t}$, that also satisfies $\kappa^{**} = 1 + n$. The variable that should adjust along the transitional dynamics to enable both conditions to be jointly satisfies is q_t. The BGP under the market equilibrium and the welfare maximizing BGP differ in their product variety span and in the stationary variety consumption levels. If there is no variety consumption level to satisfy $\kappa_{ss}^{**} = n$, the efficiency conditions in (21a) can be still satisfied for the limit levels of σ, that is at CES limit case.

Proposition 2 *Innovation size along the welfare maximizing BGP may be larger or smaller than the innovation size along the decentralized BGP, depending on the relative size of the utility-elasticity and the RLV.*

5 Example

#TO BE COMPLETED#

For the illustration of some possible properties of the general results derived thus far, consider the following example instantaneous utility function

$$
u\left(c_{i,t}\right) = \frac{\left(A + Bc_{i,t}\right)^{\frac{B-1}{B}}}{B-1} \tag{23}
$$

For $B \neq 1$ and $A = 0$, the utility function (22) boils down to the familiar *CES* specification. With $B \neq 1$ and $A \neq 0$, we have

$$
\begin{aligned}
\sigma_{u,c_t} &\equiv \frac{c_{i,t}u'\left(c_{i,t}\right)}{u\left(c_{i,t}\right)} = \frac{B-1}{\frac{A}{c_{i,t}}+B}, \\
RLV_i &\equiv -\frac{c_{i,t}u''\left(c_{i,t}\right)}{u'\left(c_{i,t}\right)} = \frac{1}{\frac{A}{c_{i,t}}+B} \\
RP_i &= -\frac{c_{i,t}u'''\left(c_{i,t}\right)}{u''\left(c_{i,t}\right)} = \frac{1+B}{\frac{A}{c_{i,t}}+B}
\end{aligned} \tag{22a}
$$

Notice that as consumption level approaches infinity all the measures presented above converge to their CES specification value, defined by B. Applying (22a) to equations (16a) and (17) yields the following expressions for the equilibrium price, drastic innovation size, and product variety span[22]

$$\begin{aligned} p_{i,t} &= \frac{1}{1-\frac{1}{\frac{A}{c_{i,t}}+B}} \\ \kappa_{i,t} &= \frac{1}{\phi}\left[\frac{\left(\frac{A}{c_{i,t}}+B\right)\left(\frac{A}{c_{i,t}}+B-1\right)}{2\frac{A}{c_{i,t}}+B-1}-\frac{B}{\frac{A}{c_{i,t}}+B}\right] \\ M^*_{t+1} &= \frac{L_t}{f^*_{t+1}\left(\frac{\frac{A}{c_{i,t}}+B-1}{\beta}+1\right)} \end{aligned} \tag{24}$$

Consider the case where $B > 1$ and $A > 0$, for which demand elasticity decreases with consumption level (that is increasing RLV), ranging from B to infinity. In this case, if $\frac{B}{B-1} < \frac{B-1}{\phi} < 1+n$, there exists a per variety consumption level that supports a balanced growth path for which $\kappa = 1+m = 1+n$.

If the latter conditions do not hold, the steady state growth is defined by the limit (CES) value of the utility function: for $\frac{B}{B-1} < 1+n < \frac{B-1}{\phi}$ innovation is drastic, $\kappa = \frac{B-1}{\phi}$, and for $\frac{B-1}{\phi}, 1+n < \frac{B}{B-1}$ innovation is non drastic, $\frac{1}{\kappa^e(\kappa^e-1)} = \phi$. In both case the product variety span and population size expand at the same rate, $m = n$.

The welfare maximizing innovation size and product variety span are given by

$$\begin{aligned} \kappa^{**}_t &= \frac{B-1}{\phi\left(\frac{A}{c^{**}_{i,t}}+1\right)(1-\beta)} \\ M^{**}_t &= \frac{L_t}{f^{**}\left[\frac{B-1}{\left(\frac{A}{c_{i,t}}+1\right)\beta}+1\right]} = \frac{L_t}{f^{**}\left[\frac{\kappa^{**}_t\phi(1-\beta)}{\beta}+1\right]} \end{aligned} \tag{25}$$

Recall that for the two efficiency conditions in (25) to hold, the variety span needs Differentiating the innovation size in (23) with respect to s_i yields

$$\frac{d\kappa_{i,t}}{ds_{i,t}} = \frac{1}{\phi}\left[\frac{2\frac{A}{c_{i,t}}(\frac{A}{c_{i,t}}+B-1)-(B-1)}{\left(2\frac{A}{c_{i,t}}+B-1\right)^2}+\frac{B}{\left(\frac{A}{c_{i,t}}+B\right)^2}\right] \tag{26}$$

The derivative is decreasing with $\frac{A}{c_{i,t}}$. At the CES limit, where c_i approaches infinity (i.e. $\frac{A}{c_i}$

[22] The innovation size can be also written in terms of demand elasticity: $\kappa_{i,t} = \frac{1}{\phi}\left(\frac{s_i(c_i)-1}{2-\frac{1+B}{s_i(c_i)}} - \frac{B}{s_i(c_i)}\right)$.

approaches zero and $s_i(c_i)$ approaches B) the derivative is negative ($\frac{1}{B} - \frac{1}{B-1} < 0$). However, for $c_i = A$ it is positive ($\frac{1+2B}{(1+B)^2} > 0$). This implies a non-monotonic relation between the elasticity of substitution and the profit maximizing innovation size. Such non-monotonic relation between innovation size and demand elasticity, implies that there may be two RLV levels that supports the same rate of balanced growth and differ only in the markup and variety span.

For $B = -1$ the utility function (22) takes the quadratic form $u(c_{i,t}) = -(A - c_{i,t})^2$, and for $B = 1$ it converges to the logarithm form $u(c_{i,t}) = \ln(A + c_{i,t})$. The for the latter and get similar dynamics, as for the CARA specification that yields the drastic innovation size $\kappa_{i,t} = \frac{1}{\phi}\left(\frac{\frac{A}{c}+1}{2} - \frac{1}{\frac{A}{c}+1}\right)$ decreases or increases with c

$$\begin{aligned}
\sigma_{u,c_t} &\equiv \frac{c_{i,t}u'(c_{i,t})}{u(c_{i,t})} = \frac{\frac{1}{\frac{A}{cB}+1}}{\ln(A + Bc_{i,t})}, \\
RLV_i &\equiv -\frac{c_{i,t}u''(c_{i,t})}{u'(c_{i,t})} = \frac{1}{\frac{A}{c}+1} \\
RP_i &= -\frac{c_{i,t}u'''(c_{i,t})}{u''(c_{i,t})} = \frac{2}{\frac{A}{c}+1}
\end{aligned}$$

#TO BE COMPLETED#

6 Discussion and Conclusion

#TBA#

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