Cross-moment interaction in multivariate semi-nonparametric densities for risk forecasting

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Abstract

This paper introduces the moment interaction between different assets in the semi-nonparametric modeling of the multivariate distribution. We analyze bivariate portfolios where skewness and kurtosis may interact between different assets showing that these new parameters may be significant pieces of information, particularly for risk measuring. Model performance for risk assessment is tested with backtesting techniques considering equally weighted portfolios of S&P 500 and Nasdaq 100 indices and major cryptocurrencies (Bitcoin and Ethereum), the latter with high frequency data. Results show an adequate performance in terms of value-at-risk and median shortfall, especially for high confidence levels.

Keywords: Gram-Charlier expansions, multivariate distributions, Cross-Skewness, Cross-Kurtosis, Risk management.

JEL classification: C14, C58, G17.

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1. Introduction

The existing literature devoted to analyzing financial risk in a multivariate context is relevant for portfolio choice. The most widely used approaches for this type of analysis have been the GARCH family with the Constant Conditional Correlation (Bollerslev, 1990) and the BEKK model (Engle and Kroner, 1995) in the 1990's or the DCC and DECO model (Engle and Sheppard, 2001; Engle, 2002; Engle and Kelly, 2012) from the beginning of the century. Due to the fact that multivariate distribution of financial assets is characterized by a non-Gaussian performance it is a challenge to find the best joint distribution of the GARCH model to fit the data. One interesting approach is the semi-nonparametric modeling introduced in finance by Mauleon (2003) and Perote (2004). Since then, it has been successfully applied in different papers that include: (i) Positive transformations (Del Brio et al., 2009; Ñíguez and Perote, 2012); (ii) Two-stage dynamic conditional correlation (DCC) (Del Brio et al., 2011; Del Brio et al., 2019); (iii) Alternative non-orthogonal expansions (Ñíguez and Perote, 2016); (iv) Inclusion of moment spillovers (Del Brio et al., 2017); (v) Risk quantification (Jiménez et al., 2020).

However, despite the advantages of multivariate modeling and the availability of stepwise procedures to provide consistent estimations, these techniques have still been scarcely used for risk quantification in a univariate setting. These methods, based on the Gram-Charlier (GC) and other related series of expansions, provide flexible structures to incorporate as many moments (parameters) as required to improve data fits (Kendall and Stuart, 1977). This implicitly acknowledges the importance of high-order moments, mostly skewness and kurtosis, to incorporate leptokurtic and wavy tails. The so-obtained densities are built upon the basis of the orthogonality property of the Hermite polynomials (HPs) linked to the derivatives of a standard Gaussian distribution. Indeed, such a property can be used for capturing the interaction between the moments, which are also relevant pieces of information for risk management since, for example, thick tails may be a consequence of both skewness and kurtosis altogether (Jiménez et al., 2022). The present paper generalizes the distribution in the latter paper to a multivariate context, but also incorporates all types of cross-moment interaction effects between different assets in a multivariate setting. We explore the potential significance of such parameters and apply backtesting model performance techniques to show to what extent these parameters may contribute to improving portfolio risk measures.

Our results applied to an equally weighted portfolio compounded by S&P 500 and Nasdaq 100 show that the cross-moment parameters have information content for capturing the interaction between high-order moments of different series. Our multivariate model shows a good performance with the two risk measures employed: Value at Risk and Median Shortfall, considering ups and downs mainly in the far end of the tail. In order to provide more information about this new methodology, we have selected another type of asset, a portfolio of the two main cryptocurrencies, Bitcoin and Ethereum, but this time with high-frequency (hourly) data to check the robustness of the model to higher volatility and frequency series. For these new series and high confidence levels results are as good as in the previous case.

The next section presents our methodology and the risk measures employed for assessing model performance. An empirical application is presented in Section 3 and, finally, Section 4 summarizes the main conclusions.

2. Methodology

2.1. The Multivariate semi-nonparametric approach

A natural approach to the multivariate semi-nonparametric modeling is by defining the Multivariate Gram–Charlier density (MGC) through the product of n independent (univariate) marginal GC, as defined next:

$$F(\boldsymbol{x}_t) = \frac{1}{n}\left[\prod_{i=1}^{n} \phi(x_{it})\right] \sum_{i=1}^{n} h_i(x_{it}) = = \frac{1}{n} \Phi(\boldsymbol{x}_t) \sum_{i=1}^{n} h_i(x_{it}) \qquad (1)$$

where $\phi(x_{it}) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}x_{it}^2\right)$ and $h_i(x_{it})$ is a m-order GC expansion that weighs the HP succession $\{H_s(x_{it})\}_{s=1}^{m}$by the cumulant/moment function, parametrized by $\{\delta_{si}\}_{s=1}^{m}$ series.

$$h_i(x_{it}) = 1 + \sum_{s=2}^{m} \delta_{si} H_s(x_{it}), \qquad (2)$$

$$H_s(x_{it}) = \frac{(-1)^s}{\phi(x_{it})} \frac{d^s \phi(x_{it})}{dx_{it}^s}. \tag{3}$$

Such a function is an asymptotic (as *m* tends to infinity) multivariate frequency function and even for a finite *m* is a density in a constrained domain – see Jondeau and Rockinger (2001) for the δ_{3i} and δ_{4i} (i.e. skewness and excess kurtosis) positive area and Lin and Zhang (2022) for a generalization theorem for expansions of any order. This is because the HPs constitutes an orthonormal basis satisfying:

$$\int H_s(x_{it}) H_m(x_{it}) \phi(x_{it})\, dx_{it} = 0 \;\; \forall s \neq m, \tag{4}$$

$$\int \mathrm{H_s}(x_{it})^2 \phi(x_{it})\, \mathrm{d}x_{it} = \mathrm{s}! \;\; \forall \mathrm{s} \geq 0. \tag{5}$$

This property makes $\int \dots \int F(x_t) dx_{1t} \dots dx_{nt} = 1$ but also the marginal densities behaving as univariate GC, i.e. $f(x_{it}) = \phi(x_{it}) h_i(x_{it})$. Furthermore, the $\{\delta_{si}\}_{s=1}^m$ parameters are directly related to the first *m* of moments the variable x_{it}, which allow to identify their relative importance in the fitting. Recently, Jiménez et al. (2022) have argued that by including the crossed HP terms in the GC expansion, e.g. δ_{sji} terms, one may capture the interaction between the moment *s* and *j*, which might be a valuable source of information for risk management. Therefore $h_i(x_{it})$ may be defined as follows:

$$h_i(x_{it}) = 1 + \sum_{s=2}^{m} \delta_{si} H_s(x_t) + \sum_{s=2}^{m} \sum_{j=1, j>s}^{m} \delta_{sji} H_s(x_{it}) H_j(x_{it}). \tag{6}$$

In particular, the authors consider the univariate case that only amounts for the effect of polynomials $H_3(x_{it}) = x_{it}^3 - 3x_{it}$ and $H_4(x_t) = x_{it}^4 - 6x_{it}^2 + 3$ and their interaction, thus incorporating the impact of δ_{3i} (skewness) δ_{4i} (excess kurtosis) and δ_{5i} (interaction between skewness and kurtosis). The rationale behind the incorporation of the latter parameter lies in the fact that both 'skewness' and 'kurtosis' altogether are jointly responsible for the effects of extreme outliers in the density fitting. In this paper we generalize the MGC by directly considering these moment interaction terms. For instance, the bivariate case whose marginal densities are those in Jiménez et al. (2022) are:

$$F(x_{1t}, x_{2t}) = \frac{1}{2} \phi(x_1) \phi(x_2) [2 + \delta_{31} H_3(x_{1t}) + \delta_{41} H_4(x_{1t}) + \delta_{51} H_3(x_{1t}) H_4(x_{1t})$$

$$+\delta_{32}H_3(x_{2t}) + \delta_{42}H_4(x_{2t}) + \delta_{52}H_3(x_{2t})H_4(x_{2t})]. \quad (7)$$

Furthermore, our modeling goes a step further by introducing the interaction between skewness and kurtosis (parameters γ_3 and γ_4) and also the cross-moment interaction (parameters γ_{34} and γ_{43}) of the different variables in the model. In particular, we explore potential gains in risk measures from considering the following density specification:

$$F(x_{1t}, x_{2t}) = \frac{1}{2}\phi(x_1)\phi(x_2)[2 + \delta_{31}H_3(x_{1t}) + \delta_{41}H_4(x_{1t}) + \delta_{51}H_3(x_{1t})H_4(x_{1t})$$

$$+\delta_{32}H_3(x_{2t}) + \delta_{42}H_4(x_{2t}) + \delta_{52}H_3(x_{2t})H_4(x_{2t})$$

$$+\gamma_3 H_3(x_{1t})H_3(x_{2t}) + \gamma_4 H_4(x_{1t})H_4(x_{2t}) + \gamma_{34}H_3(x_{1t})H_4(x_{2t}) + \gamma_{43}H_3(x_{2t})H_4(x_{1t}). \quad (8)$$

2.2. The semi-nonparametric dynamic conditional correlation model

The MGC–DCC model considers a dynamic correlation which is a richer time-varying structure for two pairs of assets providing more information about correlation. For this purpose, we applied an AR(1)-GARCH (1,1) model for the conditional mean and variance and thus the model can be parameterized as follows:

$$\boldsymbol{y}_t = \boldsymbol{\mu}_t + \boldsymbol{u}_t, \quad (9)$$

$$\boldsymbol{u}_t | \Omega_{t-1} \sim GC(0, \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t), \quad (10)$$

$$\boldsymbol{D}_t^2 = diag\{\lambda_i\} + diag\{\alpha_i\} \circ u_{t-1}u'_{t-1} + diag\{\beta_i\} \circ \boldsymbol{D}_{t-1}^2, \quad (11)$$

$$\boldsymbol{\varepsilon}_t = \boldsymbol{D}_t^{-1}\boldsymbol{u}_t, \quad (12)$$

$$\boldsymbol{Q}_t = \boldsymbol{S} \circ (\boldsymbol{ii}' - \boldsymbol{A} - \boldsymbol{B}) + \boldsymbol{A} \circ \boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}'_{t-1} + \boldsymbol{B} \circ \boldsymbol{Q}_{t-1}, \quad (13)$$

$$\boldsymbol{R}_t^{DCC} = \widetilde{\boldsymbol{Q}}_t^{-1/2}\boldsymbol{Q}_t\widetilde{\boldsymbol{Q}}_t^{-1/2}, \quad (14)$$

$$\rho_t = \frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1, j>i}^{n}\frac{q_{ijt}}{\sqrt{q_{iit}q_{jjt}}}, \quad (15)$$

where $\lambda_i > 0$, $\alpha_i > 0$ *and* $\beta_i > 0$ $\forall i = 1,2,\ldots,n;$ $\frac{-1}{n-1} < \rho_t < 1$; $\boldsymbol{S}$ is the unconditional correlation matrix; $\boldsymbol{i}$ is a vector of ones; $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{ii}' - \boldsymbol{A} - \boldsymbol{B}$ positive definite matrices; $\widetilde{\boldsymbol{Q}}_t = diag\{\boldsymbol{Q}_t\}$ (a diagonal matrix with the same diagonal as $\boldsymbol{Q}_t$) and $\circ$ the Hadamard product of two identically sized matrices (computed by element-by-element multiplication).

These models were originally defined for the Gaussian distribution. In this research, we assume a Gram-Charlier conditioned on the information set Ω_{t-1}, as stated in equation (10). This involves a non-trivial evaluation of the polynomial terms in equation (2) on $\boldsymbol{x}_t = \boldsymbol{R}_t^{-1/2}\boldsymbol{\varepsilon}_t$, which for the bivariate DCC model can be expressed as in Del Brio et al. (2011a):

$$x_{1t} = a_t\varepsilon_{1t} + b_t\varepsilon_{2t} \text{ and } x_{2t} = b_t\varepsilon_{1t} + a_t\varepsilon_{2t} \tag{16}$$

where $a_t = \frac{1}{2}\left(\frac{1}{\sqrt{1+\rho_t}} + \frac{1}{\sqrt{1-\rho_t}}\right)$ and $b_t = \frac{1}{2}\left(\frac{1}{\sqrt{1+\rho_t}} - \frac{1}{\sqrt{1-\rho_t}}\right)$.

Based on Del Brio et al. (2011) a stage-wise procedure is feasible:

(i) The conditional mean and variance are estimated by Quasi maximum likelihood (QML) for each asset independently, in this case with an AR(1)-GARCH(1,1) model.

(ii) Conditional correlation, as well as the Gram Charlier parameters are jointly estimated in a multivariate framework.

(iii) Risk measures are computed for the selected portfolio (in a univariate model), considering a given quantile – see Eq. (19) for a useful expression for computation purposes – and the mean, variance, and conditional correlation of the previous stages.

2.3. Risk measures

One of the best-known risk measures is Value at Risk (VaR). It represents the maximum loss for a given confidence level probability. It focuses on the tail of the distribution, and jointly with backtesting techniques, provides consistent information for model validation in risk management. Apart from VaR, we have also considered Median Shortfall (MS) (Kou and Peng, 2014), which is computed as a VaR but with a higher confidence level. From a statistical standpoint, the latter is a more robust measure since it is not sensitive to outliers. Both measures may be obtained as in Eqs. (17) and (18).

$$VaR_{\alpha,t+1} = \mu_{i,t+1} + \sigma_{i,t+1}q_{\alpha}, \tag{17}$$

$$MS_{\alpha,t+1} = VaR_{\frac{1+\alpha}{2},t+1}, \tag{18}$$

where μ_{it} and σ_{it}, are the conditional mean and variance, respectively, the confidence level is α and the time horizon *t*+1. In this specific case, we have to consider the interaction between skewness and kurtosis and also the cross-moment interaction of the different variables in the model for the α-quantile of the distribution, which is q_α and can be numerically computed as

$$\int_{-\infty}^{q_\alpha} g(x)\,dx = \int_{-\infty}^{q_\alpha} \phi(x)\,dx - \phi(x)[\delta_3 H_2(x) + \delta_4 H_3(x) + \delta_5 H_4(x) H_2(x) + 4\delta_5 H_3(x) H_1(x) + 12\delta_5 H_2(x) + 24\delta] \quad (19)$$

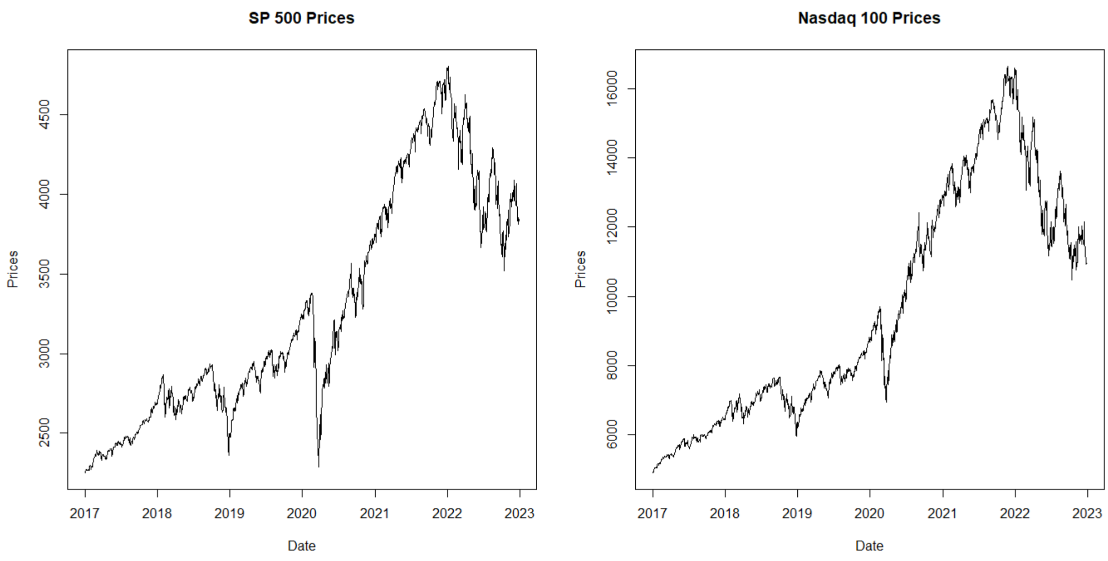
3. Empirical application

3.1. In-sample analysis

S&P500 and Nasdaq 100 daily prices (P_t) have been considered from January 3rd, 2017, to December 27th, 2022, with 1,507 observations, downloaded from www.investing.com. Daily percentage returns have been computed as

$$R_t = 100[ln(P_t) - ln(P_{t-1})]. \quad (20)$$

Fig. 1 show prices and returns for the two indices, for the whole sample.



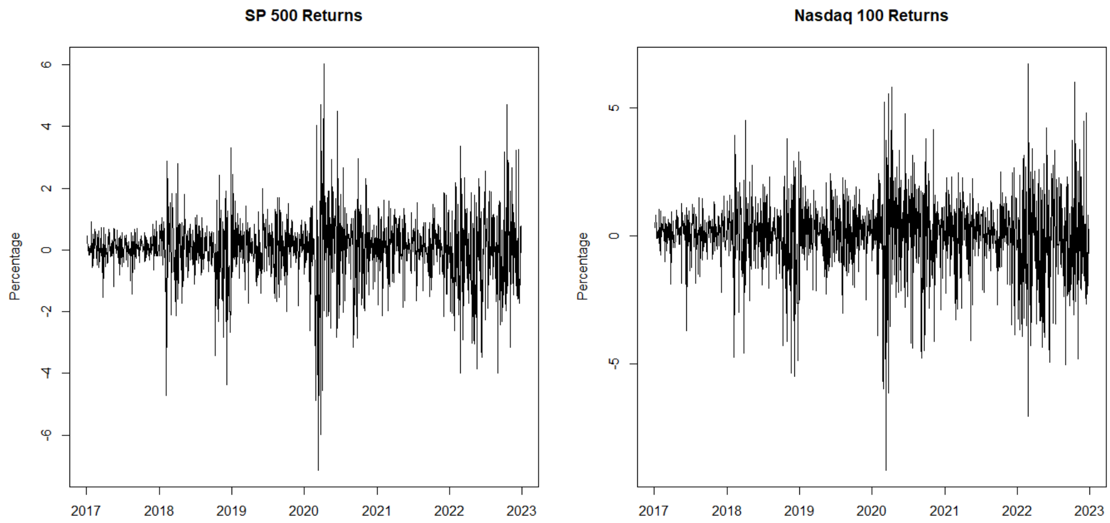


Fig. 1. SP &500 and Nasdaq 100 prices and returns. Daily prices from January 3rd, 2017, to December 27th, 2022.

For this purpose, one equally weighted portfolio has been formed and its descriptive statistics, as well as every series are shown in Table 1.

Table 1. Returns descriptive statistics

	Min.	Median	Mean	Max.	Stand. Dev.	Annual volatility	Ex. Kurtosis	Skewness
S&P 500	-9.143	0.151	0.053	6.722	1.482	23.435	3.384	-0.623
Nasdaq 100	-7.141	0.108	0.036	6.034	1.060	16.767	5.118	-0.613
Portfolio	-8.142	0.129	0.044	5.926	1.234	19.518	4.060	-0.663

S&P 500 index, Nasdaq 100 index and Portfolio (equally weighted) for returns from January 3rd, 2017, to December 27th, 2022, with 1,507 observations.

They present the typical features of financial assets with negatively skewed distribution, close to zero mean, and high volatility, both in terms of standard deviation and annual volatility. The leptokurtosis is shown in the column excess kurtosis for all the returns, whose values describe heavy tails. Portfolio returns, when equally weighted, have a middle-values except for the skewness, which is the highest.

The bivariate empirical histogram of Nasdaq 100 and S&P 500 is displayed in Fig. 2. It is noteworthy the presence of scattered areas at the distribution tails that have a non-negligible probability and that might depend on the moment interactions between both assets.

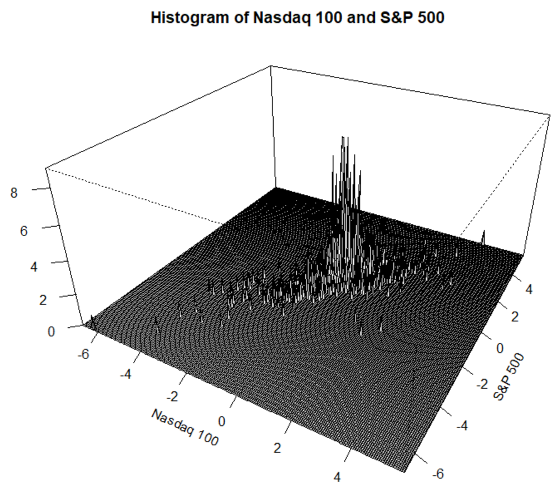


Fig. 2. Empirical bivariate histogram of S&P 500 and Nasdaq 100 returns. Daily prices from January 3rd, 2017, to December 27th, 2022.

The Maximum likelihood (ML) estimates for the MGC-DCC are shown in Table 2. The fitted values correspond to the first in-sample window of the backtesting with 1,006 observations where φ_0 and φ_1 are the conditional correlation parameters of the DCC model; The parameters of the HPs $H_3(x_{it})$ and $H_4(x_{it})$ are δ_{3i} and δ_{4i}, respectively; The other parameters δ_{51}, δ_{52}, γ_{34}, γ_{43}, γ_3 and γ_4 account for the HP interactions: $H_3(x_{1t})H_4(x_{1t})$, $H_3(x_{1t})H_4(x_{2t})$, $H_3(x_{2t})H_4(x_{i1t})$, $H_3(x_{1t})H_3(x_{1t})$ and $H_4(x_{1t})H_4(x_{2t})$, respectively.

The correlation parameters for the three models are significant, as well as the kurtosis parameters, δ_{4i}, for both assets (i=1,2) which are positive, reflecting the leptokurtic shape. The evidence of skewness is linked to parameter δ_{3i}, with negatives values according to the nature of the data and being mostly significant. One major focus of our research is the analysis of the performance of the new parameters, which proxy the moment interactions between different variables. For the specific case of δ_{5i}, parameters are all negative and significant, showing that the interaction of skewness and kurtosis is valuable to capture the thick tails of the marginal distributions. Interestingly, we find that there are also significant results for the cross-asset interaction terms, i.e. parameters linked to the interaction between $H_3(x_{it})$ and $H_4(x_{jt})$ for $i \neq j$. In particular, γ_4 is positive and significant revealing an interaction between kurtosis of both series being significant. Furthermore, γ_{34} and γ_{43} are both negative and significant (in the extended model with all the parameters) showing that skewness and kurtosis of different assets may also be interrelated.

Table 2. Parameter estimates

Parameter	MGC3(δ_{3i}, δ_{4i}, δ_{5i})	MGC4(δ_{3i}, δ_{4i}, δ_{5i},γ_{34}, γ_{43})	GC5(δ_{3i}, δ_{4i}, δ_{5i},γ_{34},γ_{43},γ_3,γ_4)
φ_0	0.869	0.858	0.836
	(0.000)	(0.000)	(0.000)
φ_1	0.085	0.086	0.100
	(0.000)	(0.000)	(0.004)
δ_{31}	-0.046	-0.063	-0.134
	(0.218)	(0.084)	(0.001)
δ_{41}	0.087	0.097	0.134
	(0.000)	(0.000)	(0.000)
δ_{32}	-0.065	-0.068	-0.047
	(0.073)	(0.063)	(0.218)
δ_{42}	0.107	0.097	0.090
	(0.000)	(0.000)	(0.000)
δ_{51}	-0.002	-0.001	-0.003
	(0.009)	(0.062)	(0.014)
δ_{52}	-0.002	-0.006	-0.014
	(0.001)	(0.004)	(0.000)
γ_{34}	-	-0.023	-0.039
	-	(0.006)	(0.000)
γ_{43}	-	-0.006	-0.024
	-	(0.465)	(0.005)
γ_3	-	-	-0.019
	-	-	(0.409)
γ_4	-	-	0.023
	-	-	(0.000)

Estimates for the first in-sample window. φ_0 and φ_1 are the parameters of the DCC model; δ_{3i} and δ_{4i} are the parameters of the basic HPs $H_3(x_{it})$ and $H_4(x_{it})$, respectively; δ_{5i} , γ_{34}, γ_{43}, γ_3 and γ_4 are the parameters of the HP interactions: $H_3(x_{it})H_4(x_{it})$, $H_3(x_{1t})H_4(x_{2t})$, $H_3(x_{2t})H_4(x_{1t})$,$H_3(x_{1t})H_3(x_{2t})$ and $H_4(x_{1t})H_4(x_{2t})$, respectively. P-values in parentheses.

Fig. 3 plots the bivariate estimates of models in Table 2. The pictures illustrate the ability of the densities to capture the probabilistic mass at different areas at the tails. For comparison purposes, in Fig. 3 (c) we depicted the MGC model with no interaction term (i.e only with the parameters δ_3 and δ_4), which seems to exhibit flatter and less wavy tails.

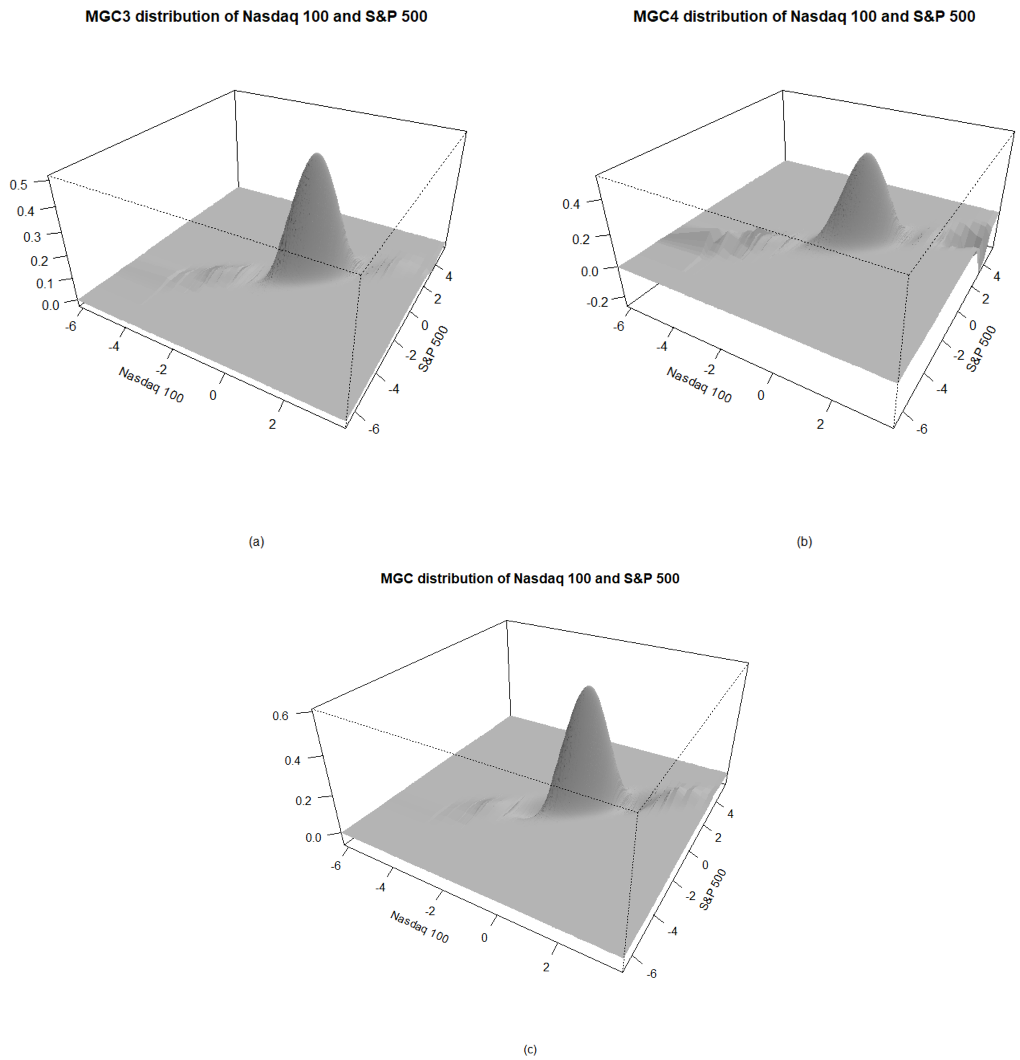


Fig. 3. Fitted bivariate MGC3 (a), MGC4 (b) and MGC (c) for the first in-sample window.

3.2. Out-of-sample analysis

The out-of-sample performance is tested through a backtesting technique with constant-sized estimation window of 1,006 observations and a testing window size of 500 observations for two risk measures: VaR and MS. Table 3 displays the results of different confidence levels assuming a univariate distribution for an equally weighted portfolio and different confidence levels varying from 97.5%-VaR to 99.5%-VaR (99%-MS). Model performance is assessed in terms of the Conditional Coverage Test (CC), Dynamic Quantile (DQ) test, and the actual over-expected ratio (AE).

Table 3. Backtesting VaR and MS for S&P 500 – Nasdaq 100

	E.E.	Exc.	CC	DQ	AE
97.5%-VaR	13	5	(0.048)	(0.564)	0.400
98.125%-VaR	9	5	(0.272)	(0.817)	0.530
98.75%-VaR (97.5%-MS)	6	5	(0.830)	(0.916)	0.800
99%-VaR	5	3	(0.613)	(0.991)	0.600
99.375%-VaR	3	2	(0.785)	(0.999)	0.640
99.5%-VaR (99%-MS)	3	2	(0.940)	(1.000)	0.800

E.E. (Exc.) stands for the expected (observed) number of exceptions. Conditional Coverage (CC) test assumes the null hypothesis of correct model specification, where exceptions satisfy the Unconditional Coverage and Independence test. Dynamic Quantile (DQ) (with 4 lags) test is under the null hypothesis of correct model specification. P-values in parentheses. For the actual over expected ratio (AE) ratio the closer to one the better model.

The results for the six confidence levels depict good results for all the tests. For the specific case of 97.5%-VaR, the Conditional Coverage test is almost 5%, the worst result. The improvement of the number of exceptions in comparison to expected exceptions, as well as the p-values according to the progressing confidence level is noteworthy, which means the best performance for the extremes of the tails, which supports the relevance of cross-moment interaction parameters to account for the waves at the end of the distribution. This analysis also corroborates, similarly to Jiménez et al. (2022), the good performance of the interaction between skewness and kurtosis even with a multivariate perspective.

3.3. Robustness check

To assess the model robustness, we extend the model performance analyses to different types of data. Particularly, we use cryptocurrencies, which exhibit typical stylized features similar to other financial assets, but with higher volatility and fatter tails. For this specific case, we have considered high-frequency (hourly) data for Bitcoin and Ethereum, downloaded from www.CryptoDataDownload.com. from January 1st, 2022, to May 31st, 2022, with 3,623 observations. The dynamics of these series in levels and returns are displayed in Fig. 4 and the bivariate histogram of the data is depicted in Fig. 5.

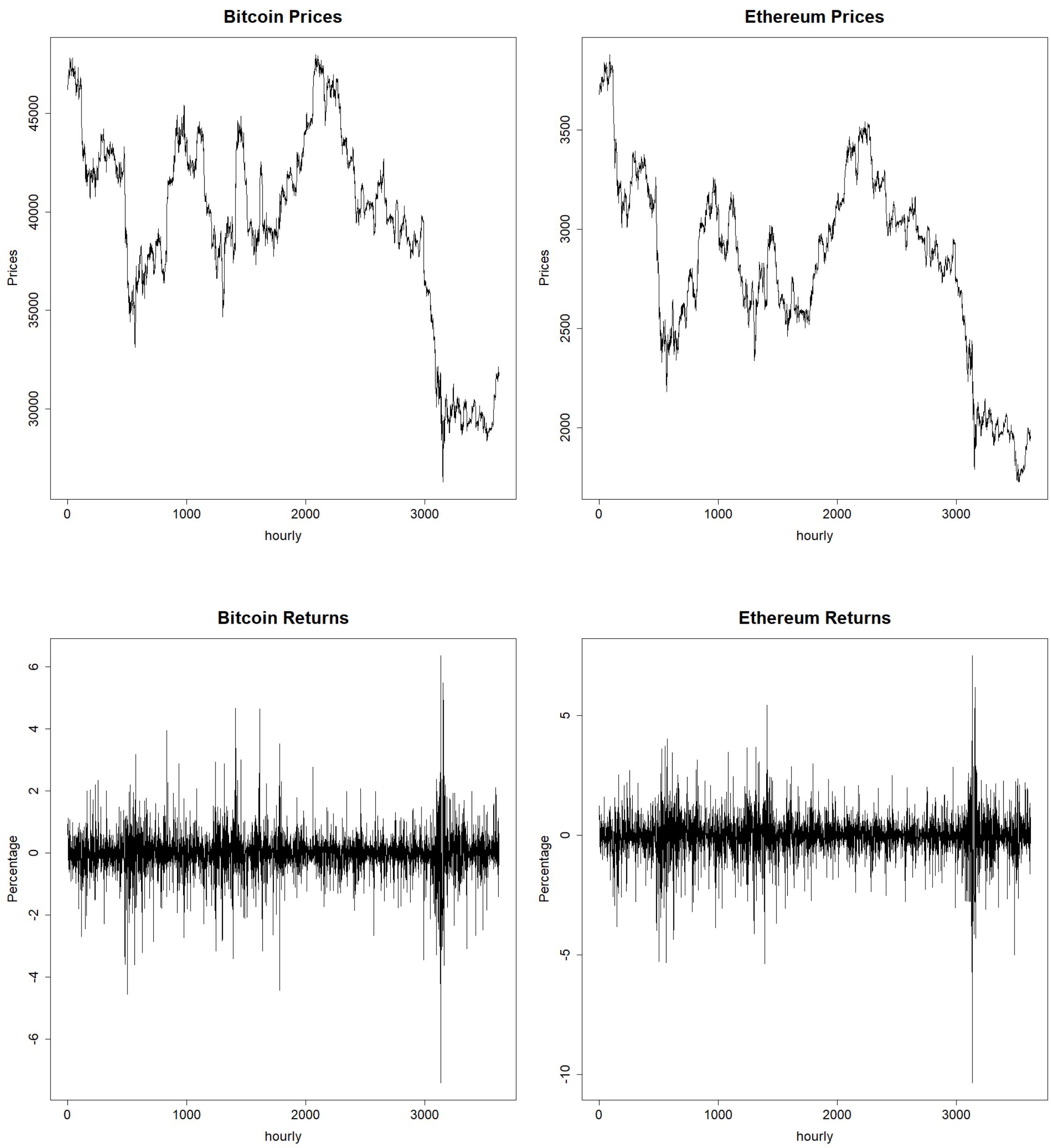


Fig. 4. Bitcoin and Ethereum prices and return. Hourly data from January 1st, 2022 at 0:00, to May 31st, 2022 at 23:00, with 3,623 observations.

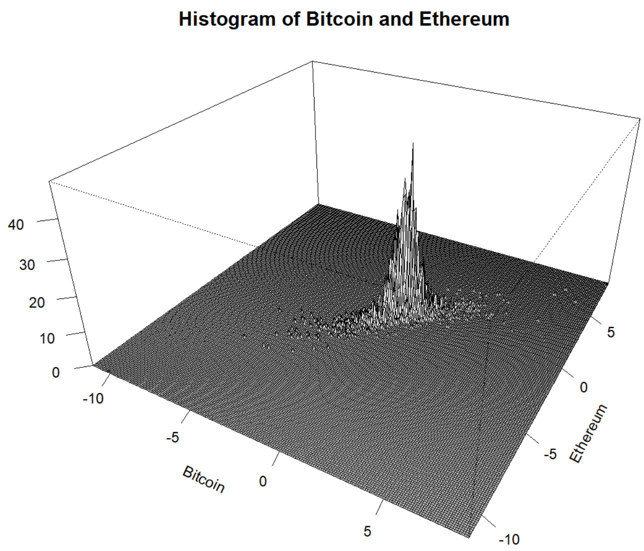
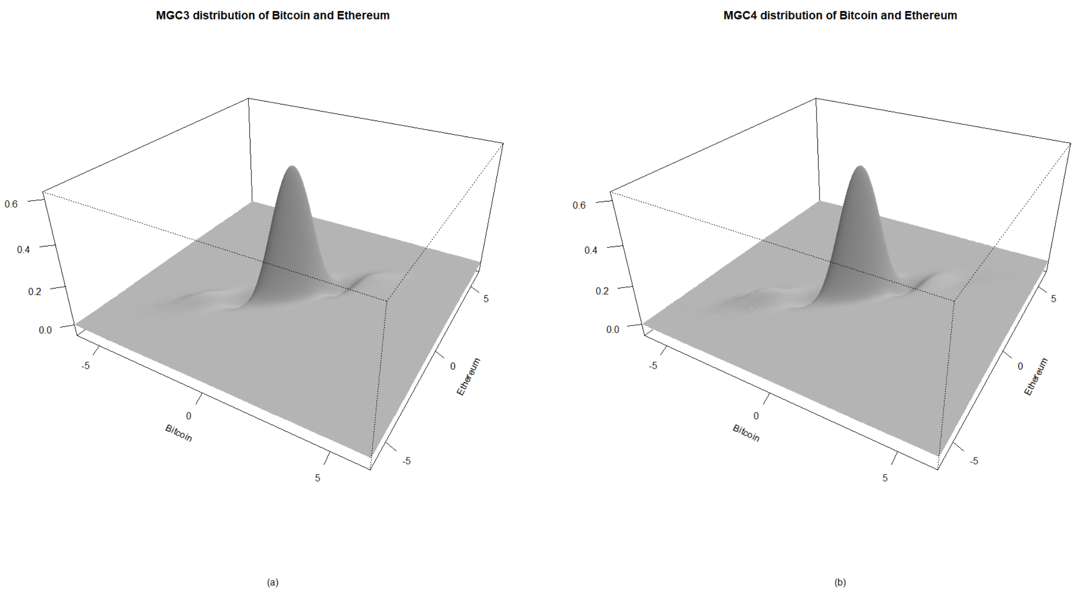


Fig. 5. Empirical histogram of Bitcoin and Ethereum returns. Hourly prices for the samples (January 1st, 2022, at 0:00, to May 31st, 2022 at 23:00, with 3,623 observations).

Similar to other financial assets the histogram not only shows the leptokurtosis but also the fat and wavy tails with scattered areas of non-zero probability at the extremes. In this case, the GC with moment interactions seems also capable of capturing these areas, as illustrated in Fig. 6.



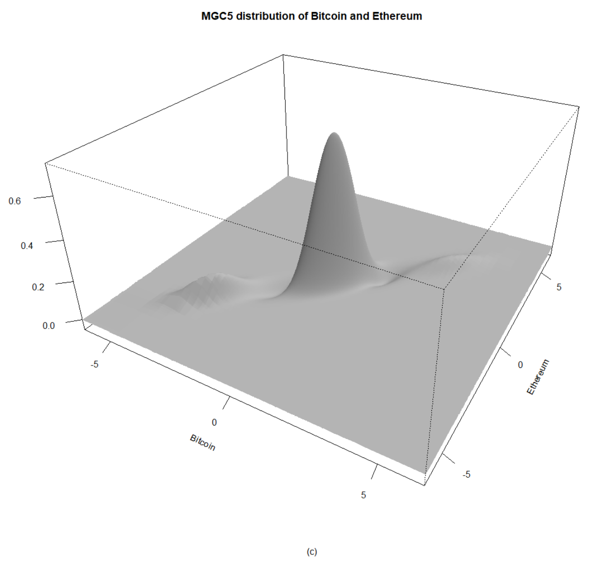


Fig. 6. Fitted bivariate MGC3 (a), MGC4 (b) and MGC5 (c) for the first in-sample window for Bitcoin - Ethereum.

Supporting these extended GC specifications, Table 4 shows their good performance a backtesting for all the confidence levels. As in Section 3.2, it is considered a testing window size of 500 observations for both VaR and MS with a constant- sized estimation window of 3,123 observations. It is remarkable the results for 97.5-MS and 99%-VaR where the number of expected exceptions and the observed exceptions are exactly the same. As in the portfolio of stock indices, it seems that the information gathered in the interaction parameters helps improve risk measures.

Table 4. Backtesting VaR and MS for Bitcoin and Ethereum

	E.E.	Exc.	CC	DQ	AE
97.5%-VaR	13	8	(0.339)	(0.868)	0.64
98.125%-VaR	9	7	(0.647)	(0.966)	0.75
98.75%-VaR (97.5%-MS)	6	6	(0.925)	(0.998)	0.96
99%-VaR	5	5	(0.951)	(0.999)	1.00
99.375%-VaR	3	5	(0.589)	(0.954)	1.60
99.5%-VaR (99%-MS)	3	5	(0.360)	(0.810)	2.00

E.E. (Exc.) stands for the expected (observed) number of exceptions. Conditional Coverage (CC) tests assume the null hypothesis of correct model specification, where exceptions satisfy the Unconditional Coverage and Independence test. Dynamic Quantile (DQ) (with 4 lags) tests are under the null hypothesis of correct model specification. P-values in parentheses. For the actual over expected ratio (AE) ratio the closer to one the better model.

4. Conclusions

This work focuses on providing a general multivariate methodology extending the Gram-Charlier distribution with dynamic conditional correlations to a more flexible model. Thanks to the orthogonality properties of this type of expansion we propose an easy and consistent method of estimation of the higher-order cross-moment interaction effects between assets. These interaction terms, introduced for the first time in a multivariate context, seem to be relevant pieces of information to improve risk measures. A very simple and accurate stepwise procedure is applied to obtain risk measures such as Value at Risk and Median Shortfall, along with backtesting techniques to assess the model performance.

We consider an equally weighted portfolio for daily returns for S&P 500 and Nasdaq 100 and six different confidence levels of VaR (97.5%, 98.125%, 98.75%, 99%, 99.375%, and 99.5%). Furthermore, we implement another risk measure, Median Shortfall, since it has been argued to be more robust to outliers and may be obtained as a simple VaR but with a higher level of confidence (98.75% VaR is equal to 97.5%- MS and 99.5%-VaR is equal to 99%-MS). The results show an accurate model performance, mostly for the ability of capturing ups and downs at the tails probability though the cross-moment parameter interactions.

Furthermore, in order to test the accuracy of the methodology we select series that that might exhibit extreme volatility even at high frequencies, particularly cryptocurrency with high-frequency (hourly) data. The results show again a satisfactory performance of the backtesting with similar results of the stock indices portfolio at daily frequency.

All in all, we suggest that this new semi-nonparametric approach that incorporates all types of parameter interaction in a multivariate setting, provides consistent and accurate results, mainly with a high confidence level of risk measures forecasting. Therefore, we propose it as a useful tool for risk management, especially with portfolios involving highly volatile assets such as cryptocurrencies.

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