Price Competition and Active or Inactive Consumer Search*

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Abstract

We propose a price-competition model in which prices are dispersed and a subset of consumers decide whether to make immediate purchases with no active price search or search sequentially. We formulate an incomplete-information model with production heterogeneity and information friction: firms' production cost types are drawn from an interval and privately observed. The model includes active or inactive consumer search as an equilibrium outcome while allowing a competition-induced switch between the two outcomes. We consider the market where competition becomes more intense with more firms, and study how firms and consumers interact in determining pricing and active-or-inactive search.

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1 Introduction

We propose a price-competition model in which prices are dispersed and a subset of consumers decide whether to make immediate purchases with no active price search or engage in sequential search. We formulate an incomplete-information model with production heterogeneity and information friction: firms' production cost types are drawn from an interval and privately observed. This approach brings two important consequences. First, we obtain price dispersion in a pure-strategy equilibrium where firms set prices conditional on their production cost types. Second, the model includes active or inactive consumer search as an equilibrium outcome while allowing a competition-induced switch between the two outcomes in the market with arbitrarily many firms. With these ingredients in place, the model can capture how firms and consumers interact in determining pricing and active-or-inactive search. We provide some answers to the questions that may arise when competition becomes more intense with more firms in the market. How does active or inactive search make a difference in pricing response to competition? How does pricing response to competition affect whether search is active or inactive?

In the model, firms have private information as to their production cost types before selecting prices, where production cost types are drawn from a continuum and independent and identically distributed (iid) across firms.[1] We follow Stahl (1989) and assume that a subset of consumers are uninformed about prices and have the option to incur a positive search cost and sequentially search for another firm. The remaining consumers observe prices and purchase at the lowest price in the market. We look for a symmetric pure-strategy equilibrium in which the strategy of firms is a price function that satisfies incentive compatibility constraints: no firm can gain by mimicking firms with other cost types on the function, or by selecting a price that deviates from the function. We establish its existence and uniqueness in two steps. First, we show that incentive compatibility of firms boils down to a set of price functions. Every function in the set is strictly increasing on the interval of possible production cost types. Second, for any search cost, we identify

[1] Private information as to production costs arises from the fact that there are firm-specific variations in technology and supply contracts while firms lack knowledge of firm-specific component of competitor costs. The current model can be modified such that production costs consist of two separate parts: (i) the firm-specific component that is private information and iid and (ii) the common component that is publicly observed. This modification shifts the distribution function of cost types with no qualitative changes in our results. Private information as to production costs is commonly found in the collusion literature, as in Athey and Bagwell (2001, 2008), Aoyagi (2003), Athey, Bagwell and Sanchirico (2004), Skryzpacz and Hopenhayn (2004), Lee (2010), and Bagwell and Lee (2010).

a unique price function from the set and verify the optimality of search decision.

The results show that any symmetric pure-strategy equilibrium can be classified into one of two types: (i) Random Equilibrium (RE) in which uninformed consumers make immediate purchases from the first firm they randomly select, and (ii) Search Equilibrium (SE) in which uninformed consumers engage in sequential search. More specifically, for any search cost, there is a unique symmetric pure-strategy equilibrium, and if the search cost is higher than (below) a cut-off value, the equilibrium is RE (SE). The cut-off value is unique and approaches zero if production heterogeneity decreases such that the interval of possible cost types becomes sufficiently narrow. In RE, firms select prices lower than the reservation price and consumer search is inactive. In SE, the market is divided into the lower-price and higher-price segments, in that each firm selects a price at one segment. The higher-price segment induces active search with its prices above the reservation price and captures the transactions that are made at one price after all prices are observed with none of them at the lower-price segment. The two equilibria have different implications for price dispersion. In RE, transaction prices are just as dispersed as posted prices: transactions can take place at any price of the price function. In SE, except in the event that all prices belong to the lower-price segment, transaction prices are less dispersed than posted prices: transactions can take place either at any price of the lower-price segment or at only one price of the higher-price segment. For any given number of firms, more active search must accompany a decrease in transaction price dispersion, since search can become more active only if the higher-price segment expands and includes more prices.

We are interested in how firms and consumers interact in determining pricing and active-or-inactive search when competition becomes more intense as the number of firms, N, increases in the market. We first show that whether search is active or inactive makes a difference in pricing response to competition. The difference is particularly pronounced in higher prices. If search is inactive (i.e., in RE), the expected price is higher in the market with more firms under the assumption that the interval of possible cost types is sufficiently narrow and demand is sufficiently inelastic. Two factors contribute to this result. One is that an increase in N has the price-increasing effect on a range of higher prices: firms with higher cost types raise prices since they focus on the profit from uninformed consumers with less prospect of winning informed consumers. The other is that the price-increasing effect is strong enough that the expected price rises.[2] In contrast, if search is active (i.e.,

[2]The assumption stated above on the interval of possible cost types and demand function works for strengthening the price-increasing effect by decreasing the pay-off advantage of having lower costs and

in SE), then an increase in N has the price-decreasing effect on the original higher-price segment except the highest possible (boundary) price: firms remaining at the higher-price segment decrease prices, and any firm's move to the lower-price segment causes its price decrease. A firm at the higher-price segment does not increase its price when there are more rivals. Intuitively, the firm's price increase can occur only if a range of rival prices also goes up at the segment, and the lowered demand for those rivals and the increase in N deter the firm's price increase by decreasing the information rents in its expected profit.

We next study how pricing response to competition affects whether search is active or inactive (i.e., whether the equilibrium is SE or RE). We show that a sufficiently large N has the search-inactivating effect: prices are distributed in favor of inactive search in the market with a sufficiently large N. In SE, an increase in N has the price-decreasing effect on the higher-price segment, and this effect generates the price dispersion above the reservation price. However, a more critical factor for active search is the price dispersion below the reservation price: if prices are not sufficiently distant from the reservation price in terms of consumer's expected utility, then the market fails to be segmented since firms at the lower-price segment can increase prices without inducing search. We find that a sufficiently large N has the price-increasing effect on the lower-price segment, and this effect discourages active search by causing the segment's prices to be more concentrated near the reservation price. We also find that a sufficiently large N does not allow a switch from inactive to active search: in any RE, prices remain sufficiently close to the boundary price in terms of consumer's expected utility, when the market has a sufficiently large N.

We finally analyze pricing in the environment where the search cost declines. If search is inactive and the original search cost is not too high, then a decrease in the search cost causes non-uniform price reductions: the price function shifts downward with less steep slope (i.e., with greater reductions in higher prices). Firms truthfully select prices on the new function since they are less tempted to mimic lower types when prices are lower, and consumer search remains inactive since prices are less distant from the new boundary price in terms of consumer's expected utility. On the other hand, if search is active, then a decrease in the search cost induces more active search with two opposing effects on prices: the higher-price segment expands and the lower-price segment includes a price-decreasing

setting lower prices. The price-increasing effect of competition has been shown in various contexts as in Stiglitz (1987), Schulz and Stahl (1996), Janssen and Moraga-González (2004), and Chen and Riordan (2007, 2008).

interval while it shrinks. Price reductions arise only from firms remaining at the lower-price segment. The difference in price reductions may lead to opposite impacts on the range between the highest and lowest possible prices: if search is inactive, the range may fall, but if search is active, the range may increase.

A large number of existing search models are built on the assumption that production costs of firms are common or perfectly correlated. A competition-driven interaction between pricing and active-or-inactive search has not been explored by papers with the assumption. In Varian (1980) and Stahl (1989), uninformed consumers are inactive in search in a mixed-strategy equilibrium. Janssen and Moraga-González (2004) consider a fixed-sample-size search and find equilibria in which uninformed consumers' search intensity declines with N. However, the equilibrium in which uninformed consumers search for only one price fails to exist in their model if N is large. Janssen et al. (2011) introduce common cost uncertainty in a sequential-search model, and find an equilibrium in which uninformed consumers purchase at the first visit despite their potential learning about the common cost.[3] Using a duopoly version of common cost uncertainty, Janssen et al. (2017) show that uninformed consumers search beyond the first visit in a non-reservation-price equilibrium.[4] Janssen and Shelegia (2020) introduce common cost uncertainty in a duopoly competition and show that consumer search and prices depend on the extent to which consumers attribute a price deviation to a monopolist upstream input supplier. Using a setting with one large seller and a continuum of remaining sellers, Menzio and Trachter (2015) include sequential search in their mixed-strategy equilibrium where the reservation price is conditional on the large seller's price.

There are search models assuming that production costs are heterogeneous.[5] In Reinganum (1979), there are a continuum of firms, and all consumers are uninformed but inactive in their search in the equilibrium. In Spulber (1995), all consumers are informed and purchase at the same lowest price. Bagwell and Lee (2014) show that an increase in N induces higher prices for higher-cost firms and lower prices for lower-cost firms and that uninformed consumers are harmed as a result. This result is based on the environment

[3] Their model introduces incomplete information with the assumption of common cost uncertainty: all firms draw the same production cost type, but this realization is unknown to consumers. This assumption holds if production costs are perfectly correlated but unknown to consumers. Dana (1994), Yang and Ye (2008) and Tappata (2009) similarly adopt incomplete information in their newspaper-search models.

[4] Janssen et al. (2017) extend this result to a model with imperfectly correlated production costs.

[5] The search literature allows heterogeneity in a labor-market context. Albrecht and Axell (1984) and Gaumont, Schindler, and Wright (2006) allow workers to have heterogeneous intrinsic values of outside options, and different match values for jobs, respectively.

where consumer search is inactive and has no interaction with pricing response to competition. Bénabou and Gertner (1993) and Fishman (1996) presents a model of search with learning in which consumers, after observing a price hike, resolve whether it is due to a firm-specific cost shock or a common inflationary factor. Focusing on a duopoly market, these papers highlight the role of consumer learning and search in how production cost shocks increase prices, while putting aside the impact of competition on the interaction between consumer search and pricing.

Some recent studies investigate price-directed search models in which consumers observe prices before search but learn their match values for products after search. In this line of research, consumers' search order is affected by prices, and complexity of search and demand analysis arises from different search paths consumers can follow. Armstrong and Zhou (2011), Shen (2015), and Haan et al. (2018) analyze a duopoly market and Choi et al. (2018) consider a market with N firms.[6] It is shown in the literature that sufficient heterogeneity of consumers' pre-search preference is crucial for the existence of a pure-strategy equilibrium with endogenous search order. In our model, a subset of consumers do not observe prices before search, and the heterogeneity of production costs is crucial for the existence of a pure-strategy equilibrium with active sequential search.

The remainder of this paper is organized as follows. Section 2 describes the model. Section 3 characterizes RE and verifies its existence. Section 4 characterizes SE and presents its existence. Section 5 investigates the interaction between pricing and active-or-inactive search in the market where competition becomes more intense with more firms, and analyzes pricing in the market where the search cost decreases. Section 6 concludes. Appendices provide the proofs.

2 Model

We propose a price-competition model in which prices are dispersed and a subset of consumers decide whether to make immediate purchases or search sequentially. The model uses an incomplete-information setting with production heterogeneity and information friction: firms' production cost types are drawn from an interval and privately observed.

[6] Choi et al. (2018) show that the demand of a firm can be derived without the need to keep track of different search paths that consumers can follow before they eventually purchase from the firm.

2.1 Basic Assumptions

There are $N \geq 2$ ex ante identical firms in a homogeneous-good market. Each firm i has private information about its unit production cost θ_i which is drawn from the support $[\underline{\theta}, \overline{\theta}]$ of a differentiable distribution function $F(\theta)$, where $\overline{\theta} > \underline{\theta} \geq 0$. The cost draws are iid across firms. The density $f(\theta) \equiv F'(\theta)$ is positive for all $\theta \in [\underline{\theta}, \overline{\theta}]$. The market contains a unit mass of consumers. Each consumer has a twice-continuously differentiable demand function $D(p)$ that satisfies $D(p) > 0 > D'(p)$ for all $p < r$, where $r \equiv \sup\{p : D(p) > 0\}$. We follow Stahl (1989) and assume that a fraction $U \in (0, 1)$ of consumers are uninformed about prices and have the option to incur a search cost, $s > 0$, and search sequentially. The remaining fraction $I = 1 - U$ of consumers observe prices and purchase at the lowest price in the market. The model maintains a standard assumption: uninformed consumers randomly pick an initial firm at no cost, and having perfect recall, they can come back to previously-visited firms at no additional cost.

A firm with cost type θ earns $\pi(p, \theta) = (p - \theta)D(p)$ when it sets price p and sells to the entire unit mass of consumers. We assume that $\pi(p, \theta)$ is strictly concave in p and has its maximum at monopoly price, $p^m(\theta) = \arg\max_p \pi(p, \theta)$, and that $r > p^m(\overline{\theta})$. It then follows that $p^m(\theta)$ is strictly increasing in θ with positive margin, $p^m(\theta) - \theta > 0$, for all θ. We also define a strictly decreasing function, $u(p) \equiv \int_p^r D(x)dx$, to represent consumer surplus at price p. In some part of Section 5.1, we consider inelastic demand, $D(p) = 1$ for $p \leq r$ and $D(p) = 0$ for $p > r$, under the assumption that $r > \overline{\theta}$.

2.2 Equilibrium Requirements

We analyze the game in which (i) firms learn their own cost types and simultaneously choose their prices, and (ii) each consumer chooses a firm to visit and makes desired purchases given the firm's price. We are interested in Perfect Bayesian Equilibrium which may be defined in terms of the following requirements: (i) each firm selects its price to maximize its expected profit given its cost type and the strategies of other firms, (ii) each consumer selects an initial firm to visit and any subsequent firm to visit in a way that maximizes expected utility at each point given the information that the consumer then has and the beliefs about prices at firms not yet visited, and (iii) where possible, consumers' beliefs are formed in a manner consistent with Bayes' rule given the equilibrium strategies of firms.

We impose two requirements on the equilibrium concept. First, firms use symmetric pure strategies. Second, informed consumers randomize equally over the firms with the same lowest price if there is more than one such firm, and uninformed consumers randomize equally over all "unvisited" firms whenever they select a firm to visit. The strategy that firm i uses is a price function $p(\theta_i)$ that maps from $[\underline{\theta}, \overline{\theta}]$ to $\mathbb{R}_+$. Let $\mathbf{p}(\theta_{-i})$ denote the vector of price selections made by firms other than i when their cost types are the $(N-1)$-tuple θ_{-i}. The vector of prices selected by firm i and its rivals determines the market share for firm i, $m(p(\theta_i), \mathbf{p}(\theta_{-i}))$, where m maps from $\mathbb{R}_+^N$ to $[0,1]$. We use the interim market share and interim profit for firm i to represent the expected market share and expected profit for firm i, respectively, when it has cost type θ_i, sets the price $p(\theta_i)$ and anticipates that its rivals employ the strategy p to determine their prices upon observing their cost types. Firm i has the interim market share

$$M(p(\theta_i); p) \equiv \mathbb{E}_{\theta_{-i}}[m(p(\theta_i), \mathbf{p}(\theta_{-i}))],$$

and earns the interim profit $\pi(p(\theta_i), \theta_i)M(p(\theta_i); p)$. We drop subscript i and write the interim profit in a direct form: if a firm with cost type θ picks the price $p(\widehat{\theta})$ when its rivals employ the strategy p to determine their prices, then its interim profit is

$$\Pi(\widehat{\theta}, \theta; p) \equiv \pi(p(\widehat{\theta}), \theta)M(p(\widehat{\theta}); p).$$

Any symmetric pure-strategy equilibrium has the price function $p(\theta)$ that satisfies incentive compatibility constraints: no firm can gain by mimicking firms with other cost types on the function, or by selecting a price that deviates from the function. Formally, the function $p(\theta)$ satisfies the on-schedule constraints,

$$\pi(p(\theta), \theta)M(p(\theta); p) \geq \pi(p(\widehat{\theta}), \theta)M(p(\widehat{\theta}); p) \text{ for all } \theta \text{ and } \widehat{\theta} \neq \theta, \tag{On-IC}$$

and the off-schedule constraints,

$$\pi(p(\theta), \theta)M(p(\theta); p) \geq \pi(\widehat{p}, \theta)M(\widehat{p}; p) \text{ for all } \theta \text{ and } \widehat{p} \notin p([\underline{\theta}, \overline{\theta}]). \tag{Off-IC}$$

In addition, it satisfies the participation constraints,

$$\pi(p(\theta), \theta)M(p(\theta); p) \geq 0 \text{ for all } \theta. \tag{IR}$$

These inequality constraints imply some basic requirements of equilibrium. We list them in the following lemma.

Lemma 1. *Suppose $p(\theta)$ satisfies On-IC, Off-IC, and IR. (i) The interim profit is positive, $\pi(p(\theta),\theta) > 0$ and $M(p(\theta);p) > 0$, for all $\theta < \overline{\theta}$. (ii) $p(\theta)$ is nondecreasing on $[\underline{\theta},\overline{\theta}]$ and strictly increasing on $(\underline{\theta},\overline{\theta})$ with the boundary value $p(\overline{\theta}) \geq \overline{\theta}$. (iii) If $M(p(\overline{\theta});p) > 0$, then $p(\theta) \leq p^m(\theta)$ for all θ, and if $M(p(\overline{\theta});p) = 0$, then $p(\theta) \leq \min\{\overline{\theta}, p^m(\theta)\}$ for all $\theta < \overline{\theta}$.*

Lemma 1 (i) shows that the interim profit cannot be zero for any firm with $\theta < \overline{\theta}$. If it is zero, this firm can increase it above zero by deviating to $\widehat{p} \in (\theta, \overline{\theta})$ given that all other firms have cost types above $\widehat{p}$ with probability $[1 - F(\widehat{p})]^{N-1} > 0$. For (ii), we use On-IC for any two types, θ_1 and θ_2, and derive an inequality,

$$[D(p(\theta_1))M(p(\theta_1);p) - D(p(\theta_2))M(p(\theta_2);p)][\theta_2 - \theta_1] \geq 0, \tag{1}$$

which shows that $D(p(\theta))M(p(\theta);p)$ is nonincreasing in θ. We note from (i) that $D(p(\theta)) > 0$ and $M(p(\theta);p) > 0$ for all $\theta < \overline{\theta}$. The inequality (1) thus implies that $p(\theta)$ is nondecreasing in θ on the entire interval $[\underline{\theta},\overline{\theta}]$. Moreover, $p(\theta)$ is not constant on any subinterval of $(\underline{\theta},\overline{\theta})$. If it has a constant subinterval, a firm can experience a discrete increase in its expected market share by decreasing its price by an infinitesimal amount. The boundary value $p(\overline{\theta})$ has the requirement, $p(\overline{\theta}) \geq \overline{\theta}$, since if $p(\overline{\theta}) < \overline{\theta}$, there is $\widehat{\theta} < \overline{\theta}$ such that $p(\widehat{\theta}) < \widehat{\theta}$, which contradicts (i). The result (iii) shows that any price above monopoly price $p^m(\theta)$ is dominated. This is obviously true for all θ if $M(p(\overline{\theta});p) > 0$. If $M(p(\overline{\theta});p) = 0$, the additional restriction, $p(\theta) \leq \overline{\theta}$ for all $\theta < \overline{\theta}$, is needed to prevent a firm with $\overline{\theta}$ from mimicking lower types: if $M(p(\overline{\theta});p) = 0$ and $p(\widehat{\theta}) > \overline{\theta}$ for any $\widehat{\theta} < \overline{\theta}$, then a firm with $\overline{\theta}$ can earn $\pi(p(\widehat{\theta}),\overline{\theta})M(p(\widehat{\theta});p) > 0$ by mimicking $\widehat{\theta}$ and selecting $p(\widehat{\theta})$.

A symmetric pure-strategy equilibrium consists of a price function and uninformed consumers' search decision. In the next two sections, we show its existence and uniqueness in two steps. First, we focus on incentive compatibility of firms and characterize the set of all price functions that satisfy the infinitely many inequality constraints, On-IC and Off-IC. Second, for any search cost, we identify a unique price function from the set and verify the optimality of search decision and the participation constraints, IR. We find two kinds of equilibria: (i) Random Equilibrium (RE) in which uninformed consumers purchase from the first firm they randomly select, and (ii) Search Equilibrium (SE) in which uninformed consumers engage in sequential search. Lemma 1 (ii) indicates that a firm with θ selects the lowest price in the market with probability $[1-F(\theta)]^{N-1}$ and thus, a firm with $\overline{\theta}$ has no chance of winning informed consumers, $[1 - F(\overline{\theta})]^{N-1} = 0$. In RE, a firm with $\overline{\theta}$ makes the profit from uninformed consumers by deterring their search. In SE, a firm with $\overline{\theta}$ induces their search, and despite having no chance of winning any consumers, $M(p(\overline{\theta}); p) = 0$, it does not mimic firms with $\theta < \overline{\theta}$ which set $p(\theta) \leq \overline{\theta}$ as shown in Lemma 1 (iii). The price function in SE includes a price range above the reservation price, ρ, and uninformed consumers find it optimal to purchase at the currently-observed price $p(\widehat{\theta})$ if $p(\widehat{\theta}) \leq \rho$ and search for another firm if $p(\widehat{\theta}) > \rho$. The reservation price ρ is endogenous but constant at any round of search given that production costs are privately informed and iid across firms.[7] As discussed in further detail below, our two-step analysis relies on an assumption on demand function.

Assumption 1. *The following inequality holds for all θ and $p \in (\theta, p^m(\theta))$:*

$$\frac{\pi(p,\theta)}{\pi_p(p,\theta)} \left(\frac{\pi_{pp}(p,\theta)}{\pi_p(p,\theta)} - \frac{D'(p)}{D(p)} \right) < 1,$$

where π_p and π_{pp} are the first-order and second-order derivatives, respectively.

A large family of demand functions satisfy this assumption, including any linear function and a constant-elasticity function, $D(p) = p^{-\epsilon}$ with $\epsilon > 1$. The assumption holds as long as $D(p)$ is not too convex.

3 Random Equilibrium

In this section, we characterize RE and establish its existence. This section focuses on the reservation price $\rho \geq \overline{\theta}$, since if $\rho < \overline{\theta}$, the boundary-value requirement, $p(\overline{\theta}) \geq \overline{\theta}$, in Lemma 1 (ii) shows that a firm with $\overline{\theta}$ cannot deter consumer search.

3.1 Characterization of RE

We begin by showing that if $\rho \geq \overline{\theta}$, then a firm with $\overline{\theta}$, having no chance of winning informed consumers, selects $p(\overline{\theta})$ to deter search and maximize the profit from uninformed consumers, $p(\overline{\theta}) = \min\{\rho, p^m(\overline{\theta})\}$, and uninformed consumers purchase from the first firm they randomly select. Thus, if $\rho \geq \overline{\theta}$, then the market-share function $M(p(\theta); p)$ can be represented by $M(\theta) = \frac{U}{N} + [1 - F(\theta)]^{N-1} I$, and the possible boundary prices range from

[7] Specifically, if an uninformed consumer has visited firm i and contemplates visiting another firm j, then the consumer's belief about the price at firm j is not altered by the price observed at firm i.

$p(\overline{\theta}) = p^m(\overline{\theta})$ to $p(\overline{\theta}) = \overline{\theta}$. Given the possible boundary prices, we next contemplate the set of price functions that solve

$$p'(\theta) = -\frac{\pi(p(\theta),\theta)[\partial M(\theta)/\partial\theta]}{\pi_p(p(\theta),\theta)M(\theta)} \text{ with } p(\overline{\theta}) \in [\overline{\theta}, p^m(\overline{\theta})], \tag{2}$$

where $M(\theta)$ is strictly decreasing, $\frac{\partial M(\theta)}{\partial\theta} = -(N-1)[1-F(\theta)]^{N-2}f(\theta)I < 0$, for all $\theta < \overline{\theta}$. The differential equation in (2) is the first-order condition, $\Pi_1(\widehat{\theta},\theta;p) = 0$ for $\widehat{\theta} = \theta$, which requires $p(\theta)$ to be strictly between marginal cost and monopoly price, $\theta < p(\theta) < p^m(\theta)$, for any $\theta \in (\underline{\theta}, \overline{\theta})$. Suppose $\mathcal{P}^D$ denotes the set of price functions that solve the differential equation for $\theta \in (\underline{\theta}, \overline{\theta})$ with the boundary values in (2) and no jump at $\{\underline{\theta}, \overline{\theta}\}$. Now, letting $\mathcal{P}^{IC}$ represent the set of all price functions that satisfy On-IC and Off-IC, the following lemma characterizes this set by showing that $\mathcal{P}^D$ and $\mathcal{P}^{IC}$ are equivalent.

Lemma 2. *If $\rho \geq \overline{\theta}$, then $\mathcal{P}^{IC}$ consists of the functions, $p(\theta) \in \mathcal{P}^D$ for $\theta \in (\underline{\theta}, \overline{\theta})$ with the boundary values, $p(\overline{\theta}) = \min\{\rho, p^m(\overline{\theta})\}$, and no jump at $\{\underline{\theta}, \overline{\theta}\}$.*

The key result in Lemma 2 is that if $\rho \geq \overline{\theta}$, the infinite number of inequality constraints, On-IC and Off-IC, boils down to (2). The boundary values, $p(\overline{\theta}) = \min\{\rho, p^m(\overline{\theta})\}$, and no jump at $\{\underline{\theta}, \overline{\theta}\}$ are necessary for incentive compatibility of $p(\overline{\theta})$ and $p(\underline{\theta})$, and the first-order condition, $\Pi_1(\widehat{\theta},\theta;p) = 0$ for $\widehat{\theta} = \theta$, is necessary for a local optimality under which no firm with $\theta \in (\underline{\theta}, \overline{\theta})$ gains by mimicking other types in the neighborhood of θ. These necessary conditions are also sufficient to ensure that On-IC and Off-IC hold. Intuitively, if a firm has no incentive to mimic its neighboring types along $p(\theta)$, then it does not mimic any other type beyond the neighborhood, given that the interim profit has the single-crossing property, $\Pi_{12} > 0$.[8] Moreover, if no firm gains by mimicking any other type along $p(\theta)$, then no firm gains by deviating to a price above $p(\overline{\theta})$ or below $p(\underline{\theta})$. Figure 1 illustrates an example of two price functions that solve (2) and have the highest and lowest possible boundary values, $p(\overline{\theta}) = p^m(\overline{\theta})$ and $p(\overline{\theta}) = \overline{\theta}$, respectively.

[8]The single-crossing property, $\Pi_{12} = -\frac{\partial}{\partial\theta}D(p(\theta))M(\theta) > 0$, means that the increase in $D(p(\theta))M(\theta)$ that accompanies a price reduction is more appealing for a firm with lower θ than with higher θ. For the results that given the boundary condition, the first-order condition is necessary and also sufficient for On-IC, we rely on Lemma 1 in Bagwell and Lee (2014).

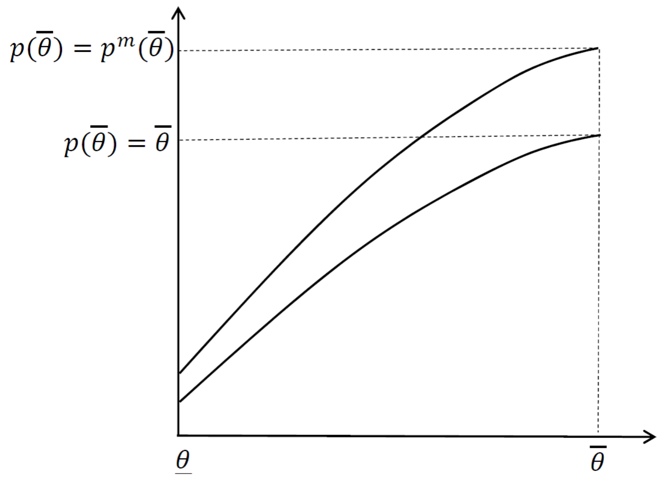

Figure 1: Illustrations of $p(\theta)$ with $p(\overline{\theta}) = p^m(\overline{\theta})$ and with $p(\overline{\theta}) = \overline{\theta}$.

3.2 Existence of RE

Lemma 2 shows that if $\rho \geq \overline{\theta}$, then incentive compatibility of firms restricts pricing strategy to $p(\theta) \in \mathcal{P}^D$ for $\theta \in (\underline{\theta}, \overline{\theta})$ with the boundary prices ranging from $p(\overline{\theta}) = p^m(\overline{\theta})$ to $p(\overline{\theta}) = \overline{\theta}$. Focusing on this set of functions, we introduce consumer search. To this end, we select a price function and define the expected utility gain from search at the boundary price $p(\overline{\theta})$. Specifically, we use the price function with $p(\overline{\theta}) = p^m(\overline{\theta})$ to define

$$\overline{v} \equiv \int_{\underline{\theta}}^{\overline{\theta}} [u(p(\theta)) - u(p^m(\overline{\theta}))] dF(\theta), \tag{3}$$

and use the price function with $p(\overline{\theta}) = \overline{\theta}$ to define

$$\underline{v} \equiv \int_{\underline{\theta}}^{\overline{\theta}} [u(p(\theta)) - u(\overline{\theta})] dF(\theta). \tag{4}$$

The values $\overline{v}$ and $\underline{v}$ are above zero. For any price function in the set, we can rewrite the expected utility gain from search at $p(\overline{\theta})$ as

$$\int_{\underline{\theta}}^{\overline{\theta}} [u(p(\theta)) - u(p(\overline{\theta}))] dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} \int_{p(\theta)}^{p(\overline{\theta})} D(x) dx dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} F(\theta) D(p(\theta)) p'(\theta) d\theta.$$

It follows from (2) that the term $D(p(\theta))p'(\theta)$ in the integrand becomes

$$D(p(\theta))p'(\theta) = \frac{\pi(p(\theta),\theta)D(p(\theta))}{\pi_p(p(\theta),\theta)}\left(-\frac{\partial M(\theta)/\partial\theta}{M(\theta)}\right).$$

Assumption 1 comes into play here to ensure that the price function with a higher $p(\overline{\theta})$ has a larger value of the term $\frac{\pi(p(\theta),\theta)D(p(\theta))}{\pi_p(p(\theta),\theta)}$ for each $\theta \in (\underline{\theta}, \overline{\theta})$ and thus generates a higher expected utility gain from search at $p(\overline{\theta})$. Intuitively, if a price function shifts upward, its slope becomes steeper since firms get more tempted to mimic lower types when prices go up. For the demand functions that satisfy Assumption 1, prices then become more distant from the new boundary value $p(\overline{\theta})$ in terms of consumer's expected utility.

We now use Lemma 2 and identify the price functions from $\mathcal{P}^{IC}$ for $s \geq \overline{v}$ and $s \in [\underline{v}, \overline{v})$, by showing that the boundary values that are conditional on $\rho \geq p^m(\overline{\theta})$ and $\rho \in [\overline{\theta}, p^m(\overline{\theta}))$ are determined by $s \geq \overline{v}$ and $s \in [\underline{v}, \overline{v})$, respectively. We can describe any reservation price $\rho \geq \overline{\theta}$ by an equation:

$$u(\rho) = \max\{\int_{\underline{\theta}}^{\overline{\theta}} u(p(\theta))dF(\theta) - s, 0\}. \tag{5}$$

The right-hand side represents the expected utility from incurring the search cost and finding a price below $p(\overline{\theta})$, including the scenario where $u(\rho) = 0$ and $\rho = r$ for s sufficiently large. The equation (5) indicates that ρ is determined by which function is used by firms in their price-setting given the search cost. If $s \geq \overline{v}$, then firms use the price function with the boundary value $p(\overline{\theta}) = p^m(\overline{\theta})$. This is true because we can use (5) and $s \geq \overline{v}$ to find $\rho \geq p^m(\overline{\theta})$, and given $\rho \geq p^m(\overline{\theta})$, Lemma 2 confirms that the function satisfies On-IC and Off-IC. If $s \in [\underline{v}, \overline{v})$, then firms use the price function with the boundary value $p(\overline{\theta}) \in [\overline{\theta}, p^m(\overline{\theta}))$ that satisfies

$$s = \int_{\underline{\theta}}^{\overline{\theta}} [u(p(\theta)) - u(p(\overline{\theta}))]dF(\theta). \tag{6}$$

We can use (5) and (6) to find $\rho = p(\overline{\theta})$ and thus $\rho \in [\overline{\theta}, p^m(\overline{\theta}))$. Lemma 2 confirms that the function satisfies On-IC and Off-IC. In Appendix A.1, we present the uniqueness result by further showing that the differential equation in (2) has a unique solution and that there is no alternative function that satisfies On-IC and Off-IC.

Lemma 3. *For any $s \geq \overline{v}$ and $s \in [\underline{v}, \overline{v})$, there is a unique price function that satisfies*

On-IC and Off-IC.

We clear the remaining requirements of equilibrium with two findings. First, for price functions with $p(\overline{\theta}) = p^m(\overline{\theta})$ and $p(\overline{\theta}) \in [\overline{\theta}, p^m(\overline{\theta}))$, purchasing from a first firm is optimal for uninformed consumers with $s \geq \overline{v}$ and $s \in [\underline{v}, \overline{v})$, respectively. Second, we apply the envelope theorem and derive the interim profit:[9]

$$\Pi(\theta, \theta; p) = \pi(p(\overline{\theta}), \overline{\theta})\frac{U}{N} + \int_{\theta}^{\overline{\theta}} D(p(x))M(x)dx. \tag{7}$$

The first term $\pi(p(\overline{\theta}), \overline{\theta})\frac{U}{N}$ is common for all θ and the second term is information rents that are greater for lower θ. Given $p(\overline{\theta}) \in [\overline{\theta}, p^m(\overline{\theta})]$, the participation constraints, IR, hold.

Proposition 1. *For any $s \geq \underline{v}$, there exists a unique symmetric pure-strategy equilibrium. If $s \geq \overline{v}$, then the equilibrium is RE with $\rho \geq p^m(\overline{\theta})$, and if $s \in [\underline{v}, \overline{v})$, then it is RE with $\rho \in [\overline{\theta}, p^m(\overline{\theta}))$.*

Proposition 1 shows the existence of a unique symmetric pure-strategy equilibrium in which search is inactive: firms select prices lower than ρ and uninformed consumers make immediate purchases. The equilibrium has some notable features. First, transaction prices are just as dispersed as posted prices: since search is inactive, transactions can take place at any posted price between $p(\underline{\theta})$ and $p(\overline{\theta})$. Second, the margin, $p(\theta) - \theta$, is non-monotonic with respect to θ: using (7), we can show that for any $N > 2$, the margin is strictly decreasing in θ for θ close to $\overline{\theta}$, but for N sufficiently large, there is a wide range of θ in which $p(\theta) - \theta > p(\underline{\theta}) - \underline{\theta}$. The intuition for this result is that the slope of $p(\theta)$ becomes flatter when θ gets close to $\overline{\theta}$, since firms are then little tempted to mimic lower cost types, but rather focus on the profit from uninformed consumers with little chance of winning informed consumers. As we show in Section 5, when N is large, firms have such tendency in a wide range of θ below $\overline{\theta}$, whereas they compete intensely for informed consumers for θ close to $\underline{\theta}$.

[9] We derive the interim profit by using $\frac{d\Pi(x,x;p)}{dx} = -D(p(x))M(x)$ given $\Pi_1(x, x; p) = 0$ and taking the integral on both sides. See Milgrom and Segal (2002) for more details about the envelope theorem.

4 Search Equilibrium

In this section, we characterize SE and establish its existence. Lemma 2 shows that if $\rho \geq \overline{\theta}$, then a firm with $\overline{\theta}$ selects $p(\overline{\theta})$ to deter consumer search. Thus, this section focuses on the reservation price $\rho < \overline{\theta}$, which implies $\rho < p(\overline{\theta})$ given the boundary-value requirement, $p(\overline{\theta}) \geq \overline{\theta}$, in Lemma 1 (ii).

4.1 Characterization of SE

If $\rho < \overline{\theta}$, then there is a cut-off type θ_c such that $p(\theta)$ is continuous on $[\underline{\theta}, \theta_c]$ and $(\theta_c, \overline{\theta}]$ with the boundary values, $p(\theta_c) = \rho$ and $p(\overline{\theta}) = \overline{\theta}$. The cut-off type is an interior point, $\theta_c \in (\underline{\theta}, \overline{\theta})$, given $s > 0$ and $\rho < p(\overline{\theta})$. If $p(\theta)$ has a jump at any point on the subintervals, or if $p(\theta_c) \neq \rho$, then there is a firm that can increase its price while keeping its original expected market share. The boundary condition $p(\overline{\theta}) = \overline{\theta}$ prevents a firm with $\overline{\theta}$ from mimicking firms with $\theta < \overline{\theta}$ which set $p(\theta) < p(\overline{\theta}) = \overline{\theta}$.

Lemma 4. *If $\rho < \overline{\theta}$, there exists a cut-off type $\theta_c \in (\underline{\theta}, \overline{\theta})$ such that $p(\theta)$ is continuous on $[\underline{\theta}, \theta_c]$ and $(\theta_c, \overline{\theta}]$ with $p(\theta_c) = \rho$ and $p(\overline{\theta}) = \overline{\theta}$.*

We can now describe the reservation price $p(\theta_c)$ by an equation:

$$u(p(\theta_c)) = [1 - F(\theta_c)]u(p(\theta_c)) + \int_{\underline{\theta}}^{\theta_c} u(p(\theta))dF(\theta) - s. \tag{8}$$

The left-hand side of (8) is the utility from purchasing at $p(\theta_c)$, and the right-hand side is the expected utility from incurring the search cost s and either reverting to $p(\theta_c)$ or finding a price below $p(\theta_c)$. We define an implicit function to rewrite (8) as

$$\Delta(\theta_c) \equiv \int_{\underline{\theta}}^{\theta_c} [u(p(\theta)) - u(p(\theta_c))]dF(\theta) - s = 0, \tag{9}$$

which shows that the expected utility gain from search at $p(\theta_c)$ equals the search cost.

We next derive the market-share function $M(\theta)$. A firm with $\theta \leq \theta_c$ sets its price $p(\theta) \leq p(\theta_c)$ and captures informed consumers if and only if it has the lowest cost type. The firm equally shares uninformed consumers with other $j \in \{0, 1, ..., N-1\}$ firms in the event that these j firms also have cost types lower than θ_c and the remaining $N - j - 1$ firms have cost types above θ_c. A firm with $\theta > \theta_c$ has its price above $p(\theta_c)$ and takes a positive expected market share if and only if it has the lowest cost type and capture the entire consumers.

Lemma 5. *(i) If $\rho < \overline{\theta}$, the interim market-share function is*

$$M(\theta) = \begin{cases} [1 - F(\theta)]^{N-1} I + \mu(\theta_c) U & \text{for } \theta \leq \theta_c \\ [1 - F(\theta)]^{N-1} & \text{for } \theta > \theta_c, \end{cases} \tag{10}$$

where

$$\mu(\theta_c) \equiv \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{1}{j+1} [F(\theta_c)]^j [1 - F(\theta_c)]^{N-j-1} = \frac{1 - [1 - F(\theta_c)]^N}{N \cdot F(\theta_c)}. \tag{11}$$

(ii) For all $\theta_c > \underline{\theta}$, $\mu(\theta_c) > [1 - F(\theta_c)]^{N-1}$ and $\mu(\theta_c)$ is strictly decreasing in θ_c.

A firm with $\theta \leq \theta_c$ has the expected market share $\mu(\theta_c)$ for uninformed consumers. In Appendix A.1, we simplify $\mu(\theta_c)$ to derive the last term in (11). Using this term, we can show that if $\theta_c \to \underline{\theta}$, then $\mu(\theta_c) \to 1$ and if $\theta_c \to \overline{\theta}$, then $\mu(\theta_c) \to \frac{1}{N}$. For all $\theta_c > \underline{\theta}$, we can also verify $\mu(\theta_c) > [1 - F(\theta_c)]^{N-1}$ and show how $\mu(\theta_c)$ decreases in θ_c,

$$\frac{\partial \mu(\theta_c)}{\partial \theta_c} = -\frac{f(\theta_c)}{F(\theta_c)} \left[\mu(\theta_c) - [1 - F(\theta_c)]^{N-1} \right]. \tag{12}$$

The market-share function $M(\theta)$ in (10) implies that

$$M(\theta_c) = [1 - F(\theta_c)]^{N-1} I + \mu(\theta_c) U \text{ and } M_+(\theta_c) = [1 - F(\theta_c)]^{N-1},$$

where $M_+(\theta_c)$ represents the limit from the right. Since $\mu(\theta_c) > [1 - F(\theta_c)]^{N-1}$ for $\theta_c > \underline{\theta}$, $M(\theta)$ has a discontinuity at θ_c, $M(\theta_c) > M_+(\theta_c)$. Consistent with this, the price function has a jump at θ_c, which can be described as a gap between $p(\theta_c) = \rho$ and $p_+(\theta_c)$, where $p_+(\theta_c)$ denotes the limit from the right. To prevent any deviation, the jump requires that a firm with θ_c should be indifferent between the two prices:

$$\pi(p(\theta_c), \theta_c) M(\theta_c) = \pi(p_+(\theta_c), \theta_c) M_+(\theta_c). \tag{13}$$

The jump at θ_c is thus determined by (13), and if $\theta_c \to \overline{\theta}$ or $\theta_c \to \underline{\theta}$, then it dissipates.[10]

[10] If $\theta_c \to \overline{\theta}$, then $p(\theta_c)$ and $p_+(\theta_c)$ converge to $p(\overline{\theta}) = \overline{\theta}$ since $M(\theta_c) \to \frac{U}{N}$ and $M_+(\theta_c) \to 0$. If $\theta_c \to \underline{\theta}$, then $p(\theta_c)$ and $p_+(\theta_c)$ approach the same price $p(\underline{\theta})$ since $M(\theta_c) \to 1$ and $M_+(\theta_c) \to 1$.

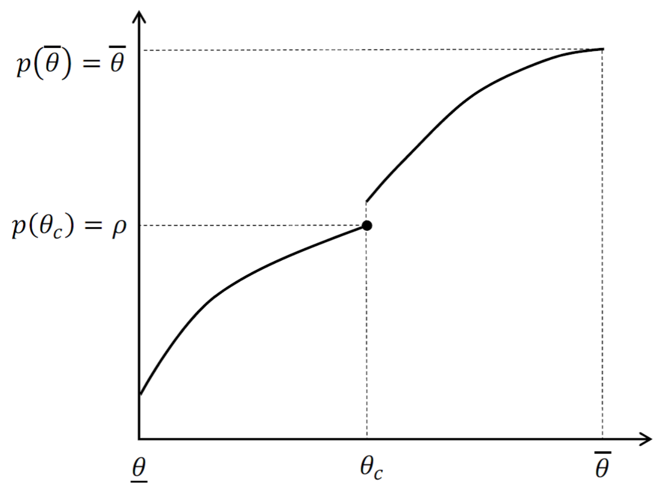

Figure 2: An illustration of $p(\theta)$ in SE.

In the following lemma, we characterize $\mathcal{P}^{IC}$ that represents the set of all price functions that satisfy On-IC and Off-IC. The boundary values, $p(\theta_c) = \rho$ and $p(\overline{\theta}) = \overline{\theta}$, and the jump at θ_c are necessary for incentive compatibility of $p(\theta)$. The first-order condition, $\Pi_1(\widehat{\theta}, \theta; p) = 0$ for $\widehat{\theta} = \theta$, is also necessary for a local optimality under which no firm with $\theta \in \{(\underline{\theta}, \theta_c), (\theta_c, \overline{\theta})\}$ gains by mimicking other types in the neighborhood of θ. This first-order condition corresponds to the differential equation in (2), but with $M(\theta)$ in (10). Suppose $\mathcal{P}^D$ represents the set of price functions that solve the differential equation for $\theta \in \{(\underline{\theta}, \theta_c), (\theta_c, \overline{\theta})\}$ with the boundary values and the only jump at θ_c. The following lemma shows that these necessary conditions are sufficient for incentive compatibility of firms and thus, $\mathcal{P}^D$ and $\mathcal{P}^{IC}$ are equivalent. Figure 2 illustrates an example of $p(\theta)$.

Lemma 6. *If $\rho < \overline{\theta}$, then $\mathcal{P}^{IC}$ consists of the functions, $p(\theta) \in \mathcal{P}^D$ for $\theta \in \{(\underline{\theta}, \theta_c), (\theta_c, \overline{\theta})\}$ with the boundary values, $p(\theta_c) = \rho$ and $p(\overline{\theta}) = \overline{\theta}$, and the only jump at θ_c that satisfies (13).*

4.2 Existence of SE

Although Lemma 6 simplifies our analysis, there still is a difficulty from the differential equation and jump. For additional convenience, we thus use the interim profits to analyze the price functions in Lemma 6. If $\rho < \overline{\theta}$, the interim profits consist of only information rents given $p(\overline{\theta}) = \overline{\theta}$ and $M(\overline{\theta}) = 0$. The interim profit for $\theta > \theta_c$ is independent of θ_c,

$$\pi(p(\theta), \theta)M(\theta) = \int_{\theta}^{\overline{\theta}} D(p(x))[1 - F(x)]^{N-1}dx, \tag{14}$$

which shows that θ_c determines the length of the interval $(\theta_c, \overline{\theta}]$, but not $p(\theta)$ for $\theta > \theta_c$. On the contrary, $p(\theta)$ for $\theta \leq \theta_c$ is determined by θ_c. If θ_c is determined, then $p(\theta_c)$ follows from the interim profit for θ_c,

$$\pi(p(\theta_c), \theta_c)M(\theta_c) = \int_{\theta_c}^{\overline{\theta}} D(p(x))[1 - F(x)]^{N-1}dx. \tag{15}$$

Note that this interim profit for θ_c takes the jump into account, since the right-hand side of (15) equals $\pi(p_+(\theta_c), \theta_c)M_+(\theta_c)$ in (13). If θ_c and $p(\theta_c)$ are determined, then $p(\theta)$ for $\theta < \theta_c$ is given by the interim profit for $\theta < \theta_c$,

$$\pi(p(\theta), \theta)M(\theta) = \int_{\theta}^{\theta_c} D(p(x))M(x)dx + \pi(p(\theta_c), \theta_c)M(\theta_c). \tag{16}$$

We now introduce consumer search and show that the price functions that are conditional on $\theta_c \in (\underline{\theta}, \overline{\theta})$ are determined by $s \in (0, \underline{v})$. The following lemma presents the monotonicity result that allows us to identify a unique $\theta_c \in (\underline{\theta}, \overline{\theta})$ for any $s \in (0, \underline{v})$.

Lemma 7. *If $\rho < \overline{\theta}$, then (i) $p(\theta_c)$ strictly increases in θ_c, (ii) an increase in s within the range of $(0, \underline{v})$ raises θ_c, and (iii) if $s \to 0$, then $\theta_c \to \underline{\theta}$, and if $s \to \underline{v}$, then $\theta_c \to \overline{\theta}$.*

Lemma 7 shows that if the search cost increases within the range of $(0, \underline{v})$, then search becomes less active with higher θ_c and $p(\theta_c)$. Our proof is structured by two main findings. We first use the interim profit for $\theta \leq \theta_c$ and find that an increase in θ_c raises $p(\theta_c)$ and entails a price-increasing interval below $p(\theta_c)$, and if the new price function crosses the original function, it does so only once.[11] We next use the implicit function (9), $\Delta(\theta_c) = 0$, and obtain the result (ii), $\frac{\partial \theta_c}{\partial s} > 0$, by showing that an increase in θ_c raises the expected utility gain from search at $p(\theta_c)$,

$$\frac{\partial}{\partial \theta_c} \int_{\underline{\theta}}^{\theta_c} [u(p(\theta)) - u(p(\theta_c))]dF(\theta) > 0. \tag{17}$$

[11] We allow that an increase in θ_c may entail a price-decreasing interval, since firms with θ close to $\underline{\theta}$ may compete more aggressively for winning informed consumers when the expected market share for uninformed consumers, $\mu(\theta_c)$, gets smaller with higher θ_c.

This inequality follows from how an increase in θ_c shifts $p(\theta)$ for $\theta \leq \theta_c$: the expected utility gain from search at $p(\theta_c)$ increases on the price-increasing interval for the demand functions that satisfy Assumption 1 and on the price-decreasing interval because search at $p(\theta_c)$ becomes more attractive with price reductions if such interval exists.

We finally identify the price function from $\mathcal{P}^{IC}$ for each $s \in (0, \underline{v})$. Lemma 7 makes this simple by showing that for any $s \in (0, \underline{v})$, there is a unique $\theta_c \in (\underline{\theta}, \overline{\theta})$ such that $\Delta(\theta_c) = 0$. Given $p(\overline{\theta}) = \overline{\theta}$, once θ_c is determined, the price function that satisfies On-IC and Off-IC can also be identified: we can obtain $p(\theta)$ for $\theta > \theta_c$ from (14), $p(\theta_c)$ and the jump from (15), and $p(\theta)$ for $\theta < \theta_c$ from (16). The uniqueness result follows since the differential equation in Lemma 6 has a unique solution for any $\theta_c \in (\underline{\theta}, \overline{\theta})$.[12] The remaining requirements of equilibrium are also cleared. First, the interim profits in (14)-(16) show that IR holds. Second, given the price function with θ_c, sequential search with the reservation price $p(\theta_c)$ is optimal for uninformed consumers whose search cost satisfies $\Delta(\theta_c) = 0$.

Proposition 2. *For any $s \in (0, \underline{v})$, there exists a unique symmetric pure-strategy equilibrium, and this equilibrium is SE with $\rho < \overline{\theta}$.*

Proposition 2 shows the existence of a unique symmetric pure-strategy equilibrium in which the market is divided into the lower-price and higher-price segments, in that each firm selects a price in one segment. The equilibrium has distinct features as follows. First, the higher-price segment induces active search and captures the transactions that are made at one price after all prices are observed with none of them at the lower-price segment. Second, consumer search is monotonic with respect to the search cost: if the search cost decreases within the range of $(0, \underline{v})$, then search tends to become more active, since the higher-price segment expands and likely includes more prices. Third, except in the event that all firms have $\theta \leq \theta_c$ and their prices belong to the lower-price segment, transaction prices are less dispersed than posted prices: transactions can take place either at any price of the lower-price segment or at only one price of the higher-price segment. For any given N, more active search must accompany a decrease in transaction price dispersion, since search can become more active only if the higher-price segment expands and includes more prices. Fourth, the margin, $p(\theta) - \theta$, is discrete and non-monotonic with respect to θ, since it shows a jump between the two segments and then approaches

[12] Once θ_c is determined, $M(\theta)$ in (10) is determined, and given $M(\theta)$, the conditions we used in Lemma 3 for the unique solution remain the same.

zero as $\theta \to \overline{\theta}$.

We conclude this section by noting that the heterogeneity of production costs and the search cost are critical in determining whether the equilibrium is RE or SE. For any $s > 0$, if the heterogeneity of production costs decreases such that the interval of possible cost types, $\overline{\theta} - \underline{\theta}$, is sufficiently narrow, then the equilibrium is RE. This is because if $\overline{\theta} - \underline{\theta}$ becomes sufficiently small, the cut-off value $\underline{v}$ approaches zero as the expected utility gain from search at the boundary price $p(\overline{\theta})$ approaches zero. However, for any length of the interval, $\overline{\theta} - \underline{\theta}$, if s is sufficiently small, then the equilibrium is SE since $s < \underline{v}$ is true.

5 Competition, Pricing, and Search

In this section, we investigate how firms and consumers interact in determining pricing and active-or-inactive search when competition becomes more intense with more firms in the market. We also analyze pricing in the environment where the search cost declines.

5.1 Price-Increasing and Price-Decreasing Competition

We characterize pricing in the market where the number of competing firms, N, increases. If ρ is higher than the highest possible boundary value, $\rho \geq p^m(\overline{\theta})$, then the price function with $p(\overline{\theta}) = p^m(\overline{\theta})$ conveys the same result as in Bagwell and Lee (2014). Specifically, an increase in N has the price-increasing and price-decreasing effects: if N rises, there is a unique $x^* \in (\underline{\theta}, \overline{\theta})$ such that $p(\theta)$ decreases for $\theta \in [\underline{\theta}, x^*)$ and increases for $\theta \in (x^*, \overline{\theta})$ while $p(x^*)$ and $p(\overline{\theta})$ are unchanged. With more rivals, firms with lower θ compete more aggressively for informed consumers, whereas firms with higher θ focus on the profit from uninformed consumers with less prospect of winning informed consumers. In this pricing, however, consumer search has no role.

If $\rho \in [\overline{\theta}, p^m(\overline{\theta}))$, then consumer search and pricing interact each other although search is inactive. The interaction can be found by using two implicit functions: how firms with $\theta < \overline{\theta}$ select prices given the boundary price $p(\overline{\theta}) = \rho$ is determined by

$$\pi(p(\theta), \theta)M(\theta) - \pi(\rho, \overline{\theta})\frac{U}{N} - \int_{\theta}^{\overline{\theta}} D(p(x))M(x)dx = 0, \tag{18}$$

and the level of ρ is determined by

$$\Delta(\overline{\theta}) =_{\underline{\theta}}^{\overline{\theta}} [u(p(\theta)) - u(\rho)]dF(\theta) - s = 0. \tag{19}$$

Using (18) and (19), we can identify two separate effects:

$$\frac{\partial p(\theta)}{\partial N} = \left.\frac{\partial p(\theta)}{\partial N}\right|_{d\rho=0} + \frac{\partial p(\theta)}{\partial \rho}\frac{\partial \rho}{\partial N}. \tag{20}$$

The first term in (20) represents the *competition effect* that captures how an increase in N affects prices when ρ is held constant. We use (18) and find that the competition effect preserves the same result as in Bagwell and Lee (2014). The second term in (20) represents the *search effect* that captures how an increase in N affects prices through the change in ρ. Our emphasis here is that inactive search may amplify the price-increasing force in the competition effect. To deliver this simply, we focus on $\rho > \overline{\theta}$ and inelastic demand.[13] We can then calculate the search effect:

$$\frac{\partial p(\theta)}{\partial \rho}\frac{\partial \rho}{\partial N} = \frac{U/N}{M(\theta)} \frac{\int_{\underline{\theta}}^{\overline{\theta}} \left.\frac{\partial p(\theta)}{\partial N}\right|_{d\rho=0} dF(\theta)}{1 - \int_{\underline{\theta}}^{\overline{\theta}} \frac{U/N}{M(\theta)} dF(\theta)}. \tag{21}$$

The term $\int_{\underline{\theta}}^{\overline{\theta}} \left.\frac{\partial p(\theta)}{\partial N}\right|_{d\rho=0} dF(\theta)$ represents the *expected competition effect*, and if the price-increasing force in the competition effect is sufficiently strong, this term is positive and so is the search effect. Given inelastic demand, we can also identify $p(\theta)$ from (18),

$$p(\theta) = \theta + \pi(\rho, \overline{\theta})\frac{U/N}{M(\theta)} + \int_{\theta}^{\overline{\theta}} \frac{M(x)}{M(\theta)} dx, \tag{22}$$

and integrate it to find the expected price, $\mathbb{E}_\theta p(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} p(\theta)dF(\theta)$. We now find that if $\overline{\theta} - \underline{\theta}$ is sufficiently small, then an increase in N raises the expected price beyond the expected competition effect:

$$\frac{\partial \mathbb{E}_\theta p(\theta)}{\partial N} = \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial p(\theta)}{\partial N} dF(\theta) > \int_{\underline{\theta}}^{\overline{\theta}} \left.\frac{\partial p(\theta)}{\partial N}\right|_{d\rho=0} dF(\theta) > 0. \tag{23}$$

The first inequality in (23) follows from the last inequality. This is because in response

[13] In later analysis, we obtain the results when demand is sufficiently inelastic.

to such positive expected competition effect, the reservation price is adjusted to increase, $\frac{\partial \rho}{\partial N} > 0$, and the search effect therefore causes an additional price increase beyond the competition effect in (20). To obtain the last inequality in (23), we note that the interim profit has the term $\pi(\rho, \overline{\theta})\frac{U}{N}$ that is common for all θ. Due to this common term, when $\rho > \overline{\theta}$ is held constant, the expected price $\mathbb{E}_\theta p(\theta)$ includes the term,

$$\pi(\rho, \overline{\theta}) \int_{\underline{\theta}}^{\overline{\theta}} \frac{U/N}{M(\theta)} dF(\theta) = \pi(\rho, \overline{\theta}) \int_0^1 \frac{1}{1 + \frac{I}{1-I} N x^{N-1}} dx,$$

which is independent of $\overline{\theta} - \underline{\theta}$ and strictly increasing in N. If $\overline{\theta} - \underline{\theta}$ is sufficiently small, this term is dominant in the expected competition effect. Similarly, in the previous case of $\rho \geq p^m(\overline{\theta})$ where consumer search has no role and the boundary price $p(\overline{\theta}) = p^m(\overline{\theta})$, if $\overline{\theta} - \underline{\theta}$ is sufficiently small, then an increase in N raises the expected price through the term $\pi(p^m(\overline{\theta}), \overline{\theta}) \int_{\underline{\theta}}^{\overline{\theta}} \frac{U/N}{M(\theta)} dF(\theta)$.

Taken together, if $\rho > \overline{\theta}$, the expected price is higher in the market with larger N under the assumption that $\overline{\theta} - \underline{\theta}$ is sufficiently small and demand is sufficiently inelastic. Two factors contribute to this result. One is that an increase in N has the price-increasing effect on a range of higher prices: firms with higher θ raise prices since they focus on the profit from uninformed consumers with less prospect of winning informed consumers. The other is that the price-increasing effect is strong enough that the expected price rises. The assumption on $\overline{\theta} - \underline{\theta}$ and demand function works for strengthening the price-increasing effect by increasing the proportion of the common term $\pi(p(\overline{\theta}), \overline{\theta})\frac{U}{N}$ relative to information rents in the interim profit. In addition, we note that the condition $\rho > \overline{\theta}$ is redundant for the result, because if $\overline{\theta} - \underline{\theta}$ is sufficiently small, then $s > \underline{v}$ is true and the equilibrium is RE with $\rho > \overline{\theta}$.

If $\rho < \overline{\theta}$, then pricing response to competition depends on whether firms belong to the lower-price or higher-price segment. Firms at the lower-price segment can focus on the profit from uninformed consumers, but a formal analysis of their pricing is elusive since their pricing also interacts with how the lower-price segment expands or shrinks with N.[14] On the contrary, we can clearly show that an increase in N has the price-decreasing effect on the original higher-price segment: firms remaining at the higher-price segment decrease prices except $p(\overline{\theta})$, and any firm's move to the lower-price segment causes its price decrease. Suppose a firm with $\widehat{\theta}$ at the higher-price segment increases its price $p(\widehat{\theta})$

[14] In the next subsection, their pricing is elaborated for N sufficiently large.

or keeps it the same when there are more rivals. We use the interim profit for $\theta \in (\theta_c, \overline{\theta})$ and find that the firm can do so only if a range of rival prices on $(\widehat{\theta}, \overline{\theta})$ also goes up. The lowered demand for those rivals and the increase in N decrease the information rents and profit-if-win for the firm,

$$\pi(p(\widehat{\theta}), \widehat{\theta}) = \int_{\widehat{\theta}}^{\overline{\theta}} D(p(x)) \left(\frac{1 - F(x)}{1 - F(\widehat{\theta})} \right)^{N-1} dx,$$

which contradicts the firm's choice to increase $p(\widehat{\theta})$ or keep it the same. We lastly note that the condition $\rho < \overline{\theta}$ is redundant if the search cost is sufficiently small, because $s < \underline{v}$ is then true and the equilibrium is SE with $\rho < \overline{\theta}$.

Proposition 3. *(i) If $\overline{\theta} - \underline{\theta}$ is sufficiently small and demand is sufficiently inelastic, then the expected price goes up when N increases. (ii) If the search cost is sufficiently small, then all prices at the original higher-price segment except $p(\overline{\theta})$ decrease when N increases.*

Proposition 3 considers the market where competition becomes more intense with more rivals, and shows that whether search is active or inactive makes a difference in pricing reaction to competition. The difference is particularly pronounced in higher prices in the market: if search is inactive (i.e., in RE), then a range of higher prices may go up to the extent that the expected price becomes higher, but if search is active (i.e., in SE), then prices go down at the original higher-price segment.

5.2 Search-Inactivating Competition

In this subsection, we show that pricing response to competition can inactivate search in the market with a sufficiently large N. We first characterize pricing in the limit where N goes to infinity. We find that in the limit, the price-increasing effect is dominant for all prices lower than ρ, except the lowest possible price $p(\underline{\theta})$ that approaches the marginal cost $\underline{\theta}$. If $\rho \in [\overline{\theta}, p^m(\overline{\theta}))$, then all prices are lower than ρ and $p(\theta)$ approaches $\min\{\rho, p^m(\theta)\}$ for $\theta > \underline{\theta}$ in the limit. If $\rho \geq p^m(\overline{\theta})$, then $p(\theta)$ converges to $p^m(\theta)$ for $\theta > \underline{\theta}$ in the limit. This approximation is a pure-strategy version of the mixed-strategy result in Rosenthal (1980) and Stahl (1989).[15] In contrast, if $\rho < \overline{\theta}$, then the price-decreasing and price-increasing effects coexist in the limit: $p(\theta)$ approaches marginal costs at the higher-price segment,

[15] The mixed-strategy equilibrium in Rosenthal (1980) and Stahl (1989) approaches the degenerate distribution at the monopoly price when N goes to infinity.

and $p(\theta)$ converges to $\min\{\theta_c, p^m(\theta)\}$ at the lower-price segment where firms can focus on the profit from uninformed consumers.[16]

Lemma 8. *Suppose $N \to \infty$. (i) If $\rho \geq p^m(\overline{\theta})$, then $p(\theta)$ converges to*

$$\widetilde{p}(\theta) = \begin{cases} p^m(\theta) & \text{for } \theta \in (\underline{\theta}, \overline{\theta}] \\ \underline{\theta} & \text{for } \theta = \underline{\theta}. \end{cases} \tag{24}$$

(ii) If $\rho \in [\overline{\theta}, p^m(\overline{\theta}))$, then $p(\theta)$ converges to

$$\widetilde{p}(\theta) = \begin{cases} \min\{\rho, p^m(\theta)\} & \text{for } \theta \in (\underline{\theta}, \overline{\theta}] \\ \underline{\theta} & \text{for } \theta = \underline{\theta}. \end{cases} \tag{25}$$

(iii) If $\rho < \overline{\theta}$, then $p(\theta)$ converges to

$$\widetilde{p}(\theta) = \begin{cases} \theta & \text{for } \theta \in (\theta_c, \overline{\theta}] \\ \min\{\theta_c, p^m(\theta)\} & \text{for } \theta \in (\underline{\theta}, \theta_c] \\ \underline{\theta} & \text{for } \theta = \underline{\theta}. \end{cases} \tag{26}$$

Lemma 8 shows that pricing response to extreme competition differs depending on whether search is active or inactive. If $\rho \geq \overline{\theta}$, then discounts are rare: a sharp price drop occurs only near the lowest possible price $p(\underline{\theta})$. If $\rho < \overline{\theta}$, then discounts are not necessarily rare: a sharp price drop also occurs at the higher-price segment. If $\rho \geq \overline{\theta}$, there is a wide range of $p(\theta) - \theta > p(\underline{\theta}) - \underline{\theta}$ below $p(\overline{\theta})$, but if $\rho < \overline{\theta}$, the margin drops quickly close to zero at the higher-price segment.

We proceed to show that a sufficiently large N does not allow a switch from inactive to active search: in RE, for any price function with the boundary price $p(\overline{\theta}) \in [\overline{\theta}, p^m(\overline{\theta})]$, prices remain sufficiently close to $p(\overline{\theta})$ in terms of consumer's expected utility, when the market has a sufficiently large N. On the contrary, in SE, there is a search-inactivating increase in N. Recall that active search arises when the market is segmented based on $p(\theta_c)$ at which the search cost equals the expected utility gain from search:

$$s = \int_{\underline{\theta}}^{\theta_c} [u(p(\theta)) - u(p(\theta_c))] dF(\theta). \tag{27}$$

[16] As the higher-price segment approaches marginal costs, the jump at θ_c dissipates and $p(\theta_c)$ approaches θ_c in the limit.

If $\theta_c \to \overline{\theta}$, then $p(\theta)$ for $\theta \leq \theta_c$ approaches the price function with the boundary value $p(\overline{\theta}) = \overline{\theta}$ in RE, and the monotonicity result in Lemma 7 shows that the right-hand side of (27) approaches its upper bound:

$$\underline{v} = \int_{\underline{\theta}}^{\overline{\theta}} [u(p(\theta)) - u(\overline{\theta})] dF(\theta).$$

Suppose the market has a sufficiently large N. If $p^m(\underline{\theta}) \geq \overline{\theta}$, then $\underline{v}$ approaches zero. This is because the price function is almost flat on $(\underline{\theta}, \theta_c]$ and thus prices on $(\underline{\theta}, \theta_c]$ are concentrated near $p(\theta_c)$ even when $\theta_c \to \overline{\theta}$. If $p^m(\underline{\theta}) < \overline{\theta}$, then given $\overline{\theta} < p^m(\overline{\theta})$, there is a unique $\theta^* \in (\underline{\theta}, \overline{\theta})$ such that $p^m(\theta^*) = \overline{\theta}$. For $\theta_c \to \overline{\theta}$, the price function is almost flat on $(\theta^*, \theta_c]$, but not on $(\underline{\theta}, \theta^*]$, and hence $\underline{v}$ persists and approaches

$$v^* \equiv \int_{\underline{\theta}}^{\theta^*} [u(p^m(\theta)) - u(p^m(\theta^*))] dF(\theta). \tag{28}$$

This limiting cut-off value v^* is always below $\underline{v}$ since $p^m(\theta^*) = \overline{\theta}$, $\theta^* < \overline{\theta}$, and $p^m(\theta) > p(\theta)$ for all $\theta \in (\underline{\theta}, \theta^*)$. In sum, if $p^m(\underline{\theta}) \geq \overline{\theta}$, then $\underline{v}$ approaches zero, and if $p^m(\underline{\theta}) < \overline{\theta}$, then $\underline{v}$ converges to v^*.

Proposition 4. *If $p^m(\underline{\theta}) \geq \overline{\theta}$, then for any $s > 0$, there is a sufficiently large N for which search is inactive, and if $p^m(\underline{\theta}) < \overline{\theta}$, then for any $s > v^*$, there is a sufficiently large N for which search is inactive.*

Proposition 4 shows that a sufficiently large N has the search-inactivating effect: prices are distributed in favor of inactive search in the market with a sufficiently large N. In SE, an increase in N has the price-decreasing effect on the higher-price segment since the segment approaches marginal costs when N rises. This effect generates the price dispersion above $p(\theta_c)$. However, a more critical factor for active search is the price dispersion below $p(\theta_c)$: if prices are not sufficiently distant from $p(\theta_c)$ in terms of consumer's expected utility, then the market fails to be segmented since firms at the lower-price segment can increase prices above $p(\theta_c)$ while deterring search. We find that a sufficiently large N has the price-increasing effect on the lower-price segment, and this effect discourages active search by causing the segment's prices to be more concentrated near $p(\theta_c)$. We also find that a sufficiently large N does not allow a switch from inactive to active search: in any RE, prices remain sufficiently close to $p(\overline{\theta})$ in terms of consumer's expected utility, when the market has a sufficiently large N.

5.3 Pricing with Decreasing Search Cost

In this subsection, we analyze pricing in the environment where the search cost declines, for instance, with the development of online search.

There is no impact on pricing if the decreased search cost is still higher than $\overline{v}$. A decrease in s within the range of $(\underline{v}, \overline{v})$ causes non-uniform price reductions: the price function shifts downward with greater reductions in higher prices. Firms truthfully select prices on the new price function since they get less tempted to mimic lower types when prices go down, and consumer search remains inactive since prices are less distant from the new boundary price $p(\overline{\theta})$ in terms of consumer's expected utility. On the other hand, a decrease in s within the range of $(0, \underline{v})$ has two opposing effects on prices: the higher-price segment expands as θ_c decreases, and the lower-price segment includes a price-decreasing interval below $p(\theta_c)$ as search becomes more active. This lower-price segment may also include a price-increasing interval since the expected market share for uninformed consumers, $\mu(\theta_c)$, becomes larger as θ_c decreases. We find that if the density $f(\theta_c)$ is sufficiently small, then all firms remaining at the lower-price segment decrease prices. This restriction on $f(\theta_c)$ limits the extent to which $\mu(\theta_c)$ increases with lower θ_c, $\frac{\partial \mu(\theta_c)}{\partial \theta_c} < 0$, as can be shown by (12).

Proposition 5. *(i) A decrease in s within the range of $(\underline{v}, \overline{v})$ lowers all prices with greater reductions in higher prices. (ii) Suppose s decreases within the range of $(0, \underline{v})$. The higher-price segment expands and the lower-price segment includes a price-decreasing interval. If $f(\theta_c)$ is sufficiently small, then all firms remaining at the lower-price segment decrease prices.*

Proposition 5 analyzes pricing in the environment where the search cost declines. If search is inactive, then greater price reductions arise in higher prices. If search is active, then price reductions arise only from firms remaining at the lower-price segment. The difference in price reductions may lead to opposite impacts on the price range, $p(\overline{\theta}) - p(\underline{\theta})$: if search is inactive, the range falls, but if search is active and $f(\theta_c)$ is sufficiently small, the range increases.

We conclude this section by claiming that our model captures some features of existing models as its limiting cases. First, the heterogeneity of production costs is crucial for our model to have active search and the competition-induced search-inactivating effect and price-decreasing effect on the higher-price segment. If $\overline{\theta} - \underline{\theta}$ approaches zero, then firms deter consumer search and an increase in N has a dominant price-increasing effect, as in

Rosenthal (1980) and Stahl (1989). Second, the existence of informed consumers is critical for uninformed consumers to actively engage in search. A sufficiently large U causes the search-inactivating effect as a sufficiently large N does: for any fixed $N \geq 2$, if $U \to 1$, then firms focus on uninformed consumers and select prices to maximize profits from them. Assuming $p^m(\underline{\theta}) \geq \overline{\theta}$, we can show that if $U \to 1$, then search is inactive and $p(\theta)$ approaches the equilibrium in Reinganum (1979) where all consumers are uninformed, $U = 1$. Third, our model captures the Spulber (1995) model as an extreme case of active search. If $s \to 0$, then the lower-price segment vanishes and $p(\theta)$ approaches the equilibrium in Spulber (1995) where all consumers are informed, $I = 1$.

6 Conclusion

In this paper, we propose a price-competition model in which prices are dispersed and a subset of consumers decide whether to make immediate purchases with no active price search or search sequentially. We adopt an incomplete-information setting with production heterogeneity and information friction: firms' production cost types are drawn from an interval and privately observed. We find that any symmetric pure-strategy equilibrium is classified into either Random Equilibrium in which search is inactive, or Search Equilibrium in which search is active. We then analyze how firms and consumers interact in determining pricing and active-or-inactive search when competition becomes more intense as the number of firms, N, increases. We show that whether search is active or inactive makes a difference in pricing response to competition, and the difference is pronounced in higher prices. We then show that a sufficiently large N has the search-inactivating effect: prices are distributed in favor of inactive search in the market with a sufficiently large N. We also analyze pricing in the environment where the search cost declines. We find that if search is inactive, then greater price reductions arise in higher prices, and if search is active, then price reductions arise only from firms remaining at the lower-price segment. The difference in price reductions implies that if search is inactive, the possible price range may fall, but if search is active, it may increase.

Several possible extensions for future work may be considered. One can assume that prices are observable and product quality consists of the firm-specific component that is private information and iid and the common component that is publicly observed. This would allow us to analyze pricing and consumer search in the presence of vertical product differentiation. In addition, our model suggests an empirical analysis of the following relationships: (i) the degree of active search and transaction price dispersion, (ii) competition and transaction price dispersion, (iii) competition and the degree of inactive search, and (iv) search cost and price distribution. Given the difficulty of obtaining offline sales data and information on consumer search, we may alternatively use clickstream data from online retailers to capture empirical patterns.

A Appendices

A.1 Proofs

Proof of Lemma 2. We begin by showing that the boundary value $p(\overline{\theta}) = \min\{\rho, p^m(\overline{\theta})\}$. Suppose first $\rho \geq p^m(\overline{\theta})$. A firm with $\overline{\theta}$ has no chance of winning informed consumers, $[1 - F(\overline{\theta})]^{N-1} = 0$, and if $p(\overline{\theta}) \neq p^m(\overline{\theta})$, the firm can increase its interim profit from uninformed consumers by deviating to $p^m(\overline{\theta})$. Hence, if $\rho \geq p^m(\overline{\theta})$, then $p(\overline{\theta}) = p^m(\overline{\theta})$. Suppose next $\rho \in (\overline{\theta}, p^m(\overline{\theta}))$. If $p(\overline{\theta}) > \rho$, then a firm with $\overline{\theta}$ makes no interim profit with no expected market share, but given $\rho > \overline{\theta}$, this firm can earn a positive interim profit from uninformed consumers by deviating to ρ. If $p(\overline{\theta}) < \rho$, then since $\rho < p^m(\overline{\theta})$, a firm with $\overline{\theta}$ can earn more from uninformed consumers by deviating to ρ. Hence, if $\rho \in (\overline{\theta}, p^m(\overline{\theta}))$, then $p(\overline{\theta}) = \rho$. Suppose lastly $\rho = \overline{\theta}$. Lemma 1 (ii) shows that $p(\overline{\theta}) \geq \rho$ is necessary, and for the same reason, $p_-(\overline{\theta}) \geq \rho$ is necessary, where $p_-(\overline{\theta})$ is the limit from the left. We also recall $p(\theta)$ is nondecreasing on $[\underline{\theta}, \overline{\theta}]$ as shown in Lemma 1 (ii). We obtain two findings. First, $p_-(\overline{\theta}) = \rho$: if $p_-(\overline{\theta}) > \rho$, there is $\widehat{\theta} < \overline{\theta}$ such that $p(\widehat{\theta}) > \overline{\theta}$ and $M(\widehat{\theta}) > 0$, and a firm with $\overline{\theta}$ can mimic $\widehat{\theta}$ and earn $\pi(p(\widehat{\theta}), \overline{\theta})M(\widehat{\theta}) > 0$. Second, $p(\theta)$ has no jump at $\overline{\theta}$, $p(\overline{\theta}) = p_-(\overline{\theta}) = \rho$. If $p(\theta)$ has a jump at $\overline{\theta}$, $p(\overline{\theta}) > p_-(\overline{\theta}) = \rho$, then there is a firm that can increase its price to ρ while keeping its original expected market share. Hence, if $\rho = \overline{\theta}$, then $p(\overline{\theta}) = \rho$.

Now, given the boundary value, we have $M(\theta) = \frac{U}{N} + [1 - F(\theta)]^{N-1}I$ in hand. We establish three findings. First, $p(\theta)$ is continuous on $[\underline{\theta}, \overline{\theta}]$. If $p(\theta)$ includes a jump on $(\underline{\theta}, \overline{\theta})$, then a firm can increase its price without affecting the probability of winning informed consumers. To show that there is no jump at $\{\underline{\theta}, \overline{\theta}\}$, we recall $p(\theta)$ is nondecreasing on $[\underline{\theta}, \overline{\theta}]$. If $p(\theta)$ has a jump at $\overline{\theta}$, $p_-(\overline{\theta}) < p(\overline{\theta})$, then a firm whose type is near $\overline{\theta}$ can increase its interim profit from uninformed consumers by deviating to $p(\overline{\theta})$. If $p(\theta)$ has a jump at $\underline{\theta}$, $p_+(\underline{\theta}) > p(\underline{\theta})$, where $p_+(\underline{\theta})$ is the limit from the right, then a firm with $\underline{\theta}$ can increase its interim profit by deviating to $p_+(\underline{\theta})$ while capturing the entire informed consumers.

Second, the first-order condition, $\Pi_1(\widehat{\theta}, \theta; p) = 0$ for $\widehat{\theta} = \theta$ on $(\underline{\theta}, \overline{\theta})$, is necessary, and given the boundary value and no jump at $\{\underline{\theta}, \overline{\theta}\}$, this first-order condition is sufficient to ensure that On-IC holds. The proof for this part is provided by Lemma 1 in Bagwell and Lee (2014). Third, if $p(\theta)$ satisfies On-IC, it also satisfies Off-IC. If a firm with $\theta < \overline{\theta}$ does not mimic $\overline{\theta}$, it will not select a price above $p(\overline{\theta})$ since $p(\overline{\theta}) = p^m(\overline{\theta}) > p^m(\theta)$, and if a firm with $\theta > \underline{\theta}$ does not mimic $\underline{\theta}$, it will not select a price below $p(\underline{\theta})$ since $p(\underline{\theta}) < p^m(\underline{\theta}) < p^m(\theta)$. Likewise, a firm with $\overline{\theta}$ will not select a price above $p(\overline{\theta})$, and the firm with $\underline{\theta}$ that already captures all informed consumers with $p(\underline{\theta})$ and will not select a price below $p(\underline{\theta})$. Overall, the results can be summarized as follows: the boundary value, the continuity on $[\underline{\theta}, \overline{\theta}]$, and the first-order condition on $(\underline{\theta}, \overline{\theta})$ are necessary for incentive compatibility of $p(\theta)$, and these necessary conditions are sufficient for On-IC and Off-IC. ■

Proof of Lemma 3. We present two results for the uniqueness result. First, we rewrite the differential equation in (2) as

$$\frac{dy}{d\theta} = \phi(y, \theta) \text{ where } \phi(y, \theta) \equiv -\frac{\pi(y, \theta)[\partial M(\theta)/\partial\theta]}{\pi_y(y, \theta)M(\theta)},$$

and find that two functions, $\phi(y, \theta)$ and $\frac{\partial\phi(y,\theta)}{\partial y}$, are continuous everywhere in the relevant region of (θ, y) where $\underline{\theta} < \theta < \overline{\theta}$ and $\theta < y < p^m(\theta)$. The continuity of $\frac{\partial\phi(y,\theta)}{\partial y}$ holds since demand function is assumed to be twice continuously differentiable. Given the continuity of these two functions, we can apply Picard-Lindelöf Theorem and conclude that for any point (θ_0, y_0) in the relevant region of (θ, y), the equation has a unique solution defined on some interval around θ_0. Second, we verify that the set in Lemma 2 contains no alternative function that satisfies On-IC and Off-IC. For $s \geq \overline{v}$, consider an alternative function with a boundary price $p(\overline{\theta}) < p^m(\overline{\theta})$ such that

$$s > \int_{\underline{\theta}}^{\overline{\theta}} [u(p(\theta)) - u(p(\overline{\theta}))] dF(\theta). \tag{29}$$

This function violates Off-IC since a firm with $\overline{\theta}$ can gain by deviating to a price above $p(\overline{\theta})$ while deterring search. Next, for $s \in (\underline{v}, \overline{v})$, consider an alternative function with a boundary price $p(\overline{\theta})$ such that

$$s < \int_{\underline{\theta}}^{\overline{\theta}} [u(p(\theta)) - u(p(\overline{\theta}))] dF(\theta). \tag{30}$$

A firm with $\overline{\theta}$ then has no expected market share, $M(\overline{\theta}) = 0$. If $M(\overline{\theta}) = 0$, Lemma 1 (iii) requires $p(\theta) \leq \min\{\overline{\theta}, p^m(\theta)\}$ for all $\theta < \overline{\theta}$, which means that the right-hand side of (30) cannot be above $\underline{v}$ among the functions in Lemma 2. Thus, the inequality (30) contradicts $s \in (\underline{v}, \overline{v})$. For $s \in (\underline{v}, \overline{v})$, there is another alternative function that causes the same Off-IC violation as in (29). For $s = \underline{v}$, an alternative function with $p(\overline{\theta}) > \overline{\theta}$ causes the same contradiction as in (30). ■

Proof of Lemma 4. We establish five results: (i) $p(\overline{\theta}) = \overline{\theta}$, (ii) $p(\theta)$ has no jump at $\{\underline{\theta}, \overline{\theta}\}$, (iii) the cut-off type is an interior point, $\theta_c \in (\underline{\theta}, \overline{\theta})$, (iv) $p(\theta_c) = p_-(\theta_c) = \rho$, and (v) $p(\theta)$ is continuous on $[\underline{\theta}, \theta_c]$ and on $(\theta_c, \overline{\theta}]$. For (i), recall that $p(\overline{\theta}) \geq \overline{\theta}$ from Lemma 1 (ii) and $\rho < p(\overline{\theta})$ from the text, and note that a firm with $\overline{\theta}$ makes no expected profit since $\rho < p(\overline{\theta})$ and $[1 - F(\overline{\theta})]^{N-1} = 0$. We first show that $p_-(\overline{\theta}) = \overline{\theta}$. If $p_-(\overline{\theta}) < \overline{\theta}$, there is $\widehat{\theta} < \overline{\theta}$ such that $p(\widehat{\theta}) < \widehat{\theta}$, which contradicts Lemma 1 (i). If $p_-(\overline{\theta}) > \overline{\theta}$, there is $\widehat{\theta} < \overline{\theta}$ such that $p(\widehat{\theta}) > \overline{\theta}$ and $M(\widehat{\theta}) > 0$ and a firm with $\overline{\theta}$ would mimic $\widehat{\theta}$ and earn $\pi(p(\widehat{\theta}), \overline{\theta}) M(\widehat{\theta}) > 0$. Now, given $p_-(\overline{\theta}) = \overline{\theta}$, if $p(\overline{\theta}) > \overline{\theta}$, then $p(\theta)$ has a jump at $\overline{\theta}$, $p(\overline{\theta}) > p_-(\overline{\theta}) = \overline{\theta}$, and a firm near $\overline{\theta}$ can then increase its price to $\overline{\theta}$ while keeping the original expected market share. Hence, $p_-(\overline{\theta}) = p(\overline{\theta}) = \overline{\theta}$. For (ii), suppose $p(\theta)$ has a jump at $\underline{\theta}$, $p(\underline{\theta}) < p_+(\underline{\theta})$. The firm with $\underline{\theta}$ can then increase its price to $\widehat{p} \in (p(\underline{\theta}), p_+(\underline{\theta}))$ while remaining as the lowest-price firm and keeping uninformed consumers with any $s > 0$. Hence, $p(\theta)$ has no jump at $\underline{\theta}$. It also has no jump at $\overline{\theta}$ as shown by (i).

Given no jump at $\{\underline{\theta}, \overline{\theta}\}$, we prove (iii). We use $s > 0$ and $\rho < p(\overline{\theta})$ to rule out the possibility that $\theta_c = \underline{\theta}$ or $\theta_c = \overline{\theta}$. For any $s > 0$, there is $\widehat{\theta} > \underline{\theta}$ such that $p(\widehat{\theta})$ is sufficiently close to $p(\underline{\theta})$ and deters search. Given $\rho < p(\overline{\theta})$ from the text, there is $\widehat{\theta} < \overline{\theta}$ such that $p(\widehat{\theta})$ is sufficiently close to $p(\overline{\theta})$ and induces search. Hence, $\theta_c \in (\underline{\theta}, \overline{\theta})$. For (iv), we find that if $p_-(\theta_c) > \rho$, then θ_c cannot be the cut-off type since there is $\widehat{\theta} < \theta_c$ such that $p(\widehat{\theta}) > \rho$. If $p_-(\theta_c) < \rho$, then a firm near θ_c can make a discrete price increase to ρ while keeping its expected market share approximately at the original level. Hence, $p_-(\theta_c) = \rho$. If $p(\theta)$ has a jump at θ_c, $p(\theta_c) > p_-(\theta_c) = \rho$, then a firm near θ_c can increase its price to ρ while keeping the original expected market share. Therefore, $p(\theta_c) = p_-(\theta_c) = \rho$. For (v), we note that $p(\theta)$ has no jump at $\{\underline{\theta}, \theta_c, \overline{\theta}\}$ and $p(\theta_c) = \rho$. It is also evident that $p(\theta)$ is continuous on $(\underline{\theta}, \theta_c)$ and $(\theta_c, \overline{\theta})$ since any jump at $\widehat{\theta} \in (\underline{\theta}, \theta_c)$ or at $\widehat{\theta} \in (\theta_c, \overline{\theta})$ implies that a firm can increase its price while keeping the original expected market share. ■

Proof of Lemma 5. (i) We simplify $\mu(\theta_c)$ by rearranging it as follows:

$$\begin{aligned}
\mu(\theta_c) &\equiv \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{1}{j+1} [F(\theta_c)]^j [1-F(\theta_c)]^{N-j-1} \\
&= \frac{1}{N \cdot F(\theta_c)} \sum_{j=0}^{N-1} \binom{N-1}{j} \frac{N}{j+1} [F(\theta_c)]^{j+1} [1-F(\theta_c)]^{N-j-1} \\
&= \frac{1}{N \cdot F(\theta_c)} \sum_{j=0}^{N-1} \binom{N}{j+1} [F(\theta_c)]^{j+1} [1-F(\theta_c)]^{N-j-1} \\
&= \frac{1}{N \cdot F(\theta_c)} \sum_{k=1}^{N} \binom{N}{k} [F(\theta_c)]^{k} [1-F(\theta_c)]^{N-k}.
\end{aligned}$$

For the first equality, we multiply $N \cdot F(\theta_c)$ and divide it. The second equality is given by

$$\binom{N-1}{j} \frac{N}{j+1} = \binom{N}{j+1}.$$

Letting $k = j+1$, we derive the third equality. Next, we use the probability that at least one firm has the cost type lower than θ_c,

$$\sum_{k=1}^{N} \binom{N}{k} [F(\theta_c)]^{k} [1-F(\theta_c)]^{N-k} = 1 - [1-F(\theta_c)]^N.$$

Hence, $\mu(\theta_c)$ equals the last term in (11).

(ii) We focus on $\theta_c > \underline{\theta}$ and observe that $\mu(\theta_c) > [1-F(\theta_c)]^{N-1}$ is equivalent to

$$\frac{1}{[1-F(\theta_c)]^{N-1}} > 1 + (N-1)F(\theta_c).$$

This inequality holds, since both sides are equal to 1 at $\theta_c = \underline{\theta}$ and are strictly increasing in θ_c, and at the same time, the left-hand side has a steeper slope for all $\theta_c > \underline{\theta}$. For all $\theta_c > \underline{\theta}$, $\frac{\partial \mu(\theta_c)}{\partial \theta_c} < 0$ is immediate from (12). ■

In the following Lemma 9 and 10, we report the preliminary results to simplify the proof of Lemma 6.

Lemma 9. *For any given p_1 and p_2 below r, if $p_2 > p_1$, then $\frac{\pi(p_2,\theta)}{\pi(p_1,\theta)}$ is strictly increasing in θ.*

Proof. The result follows from

$$\frac{\partial}{\partial\theta}\frac{\pi(p_2,\theta)}{\pi(p_1,\theta)}=\frac{\pi_\theta(p_2,\theta)\pi(p_1,\theta)-\pi(p_2,\theta)\pi_\theta(p_1,\theta)}{[\pi(p_1,\theta)]^2}=\frac{D(p_1)D(p_2)[p_2-p_1]}{[\pi(p_1,\theta)]^2}. \blacksquare$$

Lemma 10. *If $p(\theta)\in\mathcal{P}^D$ for $\theta\in\{(\underline{\theta},\theta_c),(\theta_c,\overline{\theta})\}$, then $\Pi(\theta,\theta;p)>\Pi(\widehat{\theta},\theta;p)$ for all θ and $\widehat{\theta}\neq\theta$ on $(\theta_c,\overline{\theta})$ and for all θ and $\widehat{\theta}\neq\theta$ on $(\underline{\theta},\theta_c)$.*

Proof. We first consider θ and $\widehat{\theta}$ on the interval $(\theta_c,\overline{\theta})$. If $\widehat{\theta}>\theta$, we find that

$$\begin{aligned}\Pi(\theta,\theta;p)-\Pi(\widehat{\theta},\theta;p) &= -\int_\theta^{\widehat{\theta}}\Pi_1(x,\theta;p)dx\\ &= \int_\theta^{\widehat{\theta}}\left[\Pi_1(x,x;p)-\Pi_1(x,\theta;p)\right]dx\\ &= \int_\theta^{\widehat{\theta}}\int_\theta^x\Pi_{12}(x,y;p)dydx>0,\end{aligned}$$

where the second equality holds given the necessary condition $\Pi_1(x,x;p)=0$, while the inequality holds given $\widehat{\theta}>\theta$ and $x\in(\theta,\widehat{\theta})$ and given the single-crossing property,

$$\Pi_{12}(x,y;p)=-\frac{\partial}{\partial x}D(p(x))M(x)>0.$$

Thus, for $\widehat{\theta}>\theta$, we have $\Pi(\theta,\theta;p)>\Pi(\widehat{\theta},\theta;p)$. We also find that given $\widehat{\theta}$, if θ is more distant from $\widehat{\theta}$, the gap between the two interim profits is greater. Next, if $\widehat{\theta}<\theta$, we also find that

$$\begin{aligned}\Pi(\theta,\theta;p)-\Pi(\widehat{\theta},\theta;p) &= \int_{\widehat{\theta}}^{\theta}\Pi_1(x,\theta;p)dx\\ &= \int_{\widehat{\theta}}^{\theta}\left[\Pi_1(x,\theta;p)-\Pi_1(x,x;p)\right]dx\\ &= \int_{\widehat{\theta}}^{\theta}\int_x^\theta\Pi_{12}(x,y;p)dydx>0,\end{aligned}$$

where the inequality holds given $\widehat{\theta}<\theta$ and $x\in(\widehat{\theta},\theta)$. The proof for θ and $\widehat{\theta}$ on the other interval $(\underline{\theta},\theta_c)$ is analogous. $\blacksquare$

Proof of Lemma 6. We first show that the jump at θ_c must satisfy the equation (13). If the jump at θ_c is made such that

$$\pi(p(\theta_c),\theta_c)M(\theta_c)>\pi(p_+(\theta_c),\theta_c)M_+(\theta_c),$$

then a firm with $x \in (\theta_c, \overline{\theta}]$ can increase its interim profit by mimicking θ_c and selecting $p(\theta_c)$: if $x \in (\theta_c, \overline{\theta}]$ is sufficiently close to θ_c, then $\pi(p(\theta_c), x)M(\theta_c) \to \pi(p(\theta_c), \theta_c)M(\theta_c)$ and $\pi(p(x), x)M(x) \to \pi(p_+(\theta_c), \theta_c)M_+(\theta_c)$.

In the remaining proof, we show that On-IC and Off-IC hold given the necessary conditions: the boundary values, $p(\theta_c) = \rho$ and $p(\overline{\theta}) = \overline{\theta}$, the jump at θ_c that satisfies (13), and the first-order condition for $\theta \in \{(\underline{\theta}, \theta_c), (\theta_c, \overline{\theta})\}$. Our proof has three parts: (a) (On- and Off-) IC for θ_c, (b) IC for $\theta \in (\theta_c, \overline{\theta})$, and (c) IC for $\theta \in [\underline{\theta}, \theta_c)$. To verify Off-IC in (a)-(c), we focus on deviations to $\widehat{p} \in (p(\theta_c), p_+(\theta_c))$, since we can easily show that other off-schedule deviations have no gains as follows. As for an off-schedule deviation to $\widehat{p} < p(\underline{\theta})$, if a firm with $\theta > \underline{\theta}$ does not mimic $\underline{\theta}$, it will not select $\widehat{p} < p(\underline{\theta})$ since $p(\underline{\theta}) < p^m(\underline{\theta}) < p^m(\theta)$. The firm with $\underline{\theta}$ that captures all informed consumers will not select a price below $p(\underline{\theta})$ since $p(\underline{\theta}) < p^m(\underline{\theta})$. As for an off-schedule deviation to $\widehat{p} > p(\overline{\theta})$, a firm with $\theta < \overline{\theta}$ will not undertake the deviation that makes zero expected profits, and a firm with $\overline{\theta}$ gains nothing from this deviation. Next, to verify On-IC in (a)-(c), we focus on On-IC for $\theta < \overline{\theta}$ and on-schedule deviations $p(\widehat{\theta})$ for $\widehat{\theta} < \overline{\theta}$, because it is obvious that a firm with $\theta < \overline{\theta}$ will not mimic $\overline{\theta}$ since $M(\overline{\theta}) = 0$ and that a firm with $\overline{\theta}$ will not mimic $\theta < \overline{\theta}$ since $p(\theta) < p(\overline{\theta}) = \overline{\theta}$ for all $\theta < \overline{\theta}$.

(a) IC for θ_c: Recall that a firm with θ_c is indifferent between $p(\theta_c)$ and $p_+(\theta_c)$,

$$\pi(p(\theta_c), \theta_c)M(\theta_c) = \pi(p_+(\theta_c), \theta_c)[1 - F(\theta_c)]^{N-1}. \tag{31}$$

A firm with θ_c has the following preferences. First, $p(\theta_c)$ is preferred to $\widehat{p} \in (p(\theta_c), p_+(\theta_c))$. Since $p_+(\theta_c) > \widehat{p}$, the equality (31) leads to the result:

$$\pi(p(\theta_c), \theta_c)M(\theta_c) > \pi(\widehat{p}, \theta_c)[1 - F(\theta_c)]^{N-1}. \tag{32}$$

Second, $p(\theta_c)$ is preferred to $p(\widehat{\theta})$ whether $\widehat{\theta} \in (\theta_c, \overline{\theta})$ or $\widehat{\theta} \in [\underline{\theta}, \theta_c)$. We first consider $\widehat{\theta} \in (\theta_c, \overline{\theta})$ and show that a firm with θ_c prefers $p_+(\theta_c)$ to $p(\widehat{\theta})$ since the firm is indifferent between $p(\theta_c)$ and $p_+(\theta_c)$. As shown in the proof of Lemma 10, for any $\widehat{\theta} \in (\theta_c, \overline{\theta})$, there is $x \in (\theta_c, \widehat{\theta})$ such that $\Pi(x, x; p) > \Pi(\widehat{\theta}, x; p)$, and given $\widehat{\theta}$, if x is more distant from $\widehat{\theta}$, the gap between the two interim profits is greater. The inequality $\Pi(x, x; p) > \Pi(\widehat{\theta}, x; p)$ therefore holds in the limit where $x \to \theta_c$, $\Pi(x, x; p) \to \pi(p_+(\theta_c), \theta_c)[1 - F(\theta_c)]^{N-1}$ and $\Pi(\widehat{\theta}, x; p) \to \pi(p(\widehat{\theta}), \theta_c)[1 - F(\widehat{\theta})]^{N-1}$. We next consider $\widehat{\theta} \in [\underline{\theta}, \theta_c)$. The proof of Lemma 10 indicates that for any $\widehat{\theta} \in [\underline{\theta}, \theta_c)$, there is $x \in (\widehat{\theta}, \theta_c)$ such that $\Pi(x, x; p) > \Pi(\widehat{\theta}, x; p)$

and that this inequality holds in the limit where $x \to \theta_c$, $\Pi(x, x; p) \to \pi(p(\theta_c), \theta)M(\theta_c)$ and $\Pi(\widehat{\theta}, x; p) \to \pi(p(\widehat{\theta}), \theta_c)M(\widehat{\theta})$. In sum, a firm with θ_c has no gain from any deviation.

(b) IC for $\theta \in (\theta_c, \overline{\theta})$: A firm with $\theta \in (\theta_c, \overline{\theta})$ has the following preferences. First, $p_+(\theta_c)$ is preferred to $p(\theta_c)$. Since $p_+(\theta_c) > p(\theta_c)$ and $\theta > \theta_c$, we can use Lemma 9 and find that

$$\frac{\pi(p_+(\theta_c), \theta)}{\pi(p(\theta_c), \theta)} > \frac{\pi(p_+(\theta_c), \theta_c)}{\pi(p(\theta_c), \theta_c)} = \frac{M(\theta_c)}{[1 - F(\theta_c)]^{N-1}},$$

where the equality is from (31). We then obtain the result:

$$\pi(p_+(\theta_c), \theta)[1 - F(\theta_c)]^{N-1} > \pi(p(\theta_c), \theta)M(\theta_c) \text{ for } \theta \in (\theta_c, \overline{\theta}).$$

Second, $p_+(\theta_c)$ is preferred to $\widehat{p} \in (p(\theta_c), p_+(\theta_c))$. Since $p_+(\theta_c) > \widehat{p}$, we obtain the result:

$$\pi(p_+(\theta_c), \theta)[1 - F(\theta_c)]^{N-1} > \pi(\widehat{p}, \theta)[1 - F(\theta_c)]^{N-1} \text{ for } \theta \in (\theta_c, \overline{\theta}).$$

Third, $p(\theta_c)$ is preferred to $p(\widehat{\theta})$ for $\widehat{\theta} \in [\underline{\theta}, \theta_c)$. Since $p(\theta_c) > p(\widehat{\theta})$ and $\theta > \theta_c$, Lemma 9 shows that

$$\frac{\pi(p(\theta_c), \theta)}{\pi(p(\widehat{\theta}), \theta)} > \frac{\pi(p(\theta_c), \theta_c)}{\pi(p(\widehat{\theta}), \theta_c)} \geq \frac{M(\widehat{\theta})}{M(\theta_c)},$$

where the second inequality is from the last result in (a). These inequalities obtain the result:

$$\pi(p(\theta_c), \theta)M(\theta_c) > \pi(p(\widehat{\theta}), \theta)M(\widehat{\theta}) \text{ for } \widehat{\theta} \in [\underline{\theta}, \theta_c).$$

In sum, the three results show that a firm with $\theta \in (\theta_c, \overline{\theta})$ prefers $p_+(\theta_c)$ to $p(\theta_c)$ and $\widehat{p} \in (p(\theta_c), p_+(\theta_c))$ and prefers $p(\theta_c)$ to $p(\widehat{\theta})$ for $\widehat{\theta} \in [\underline{\theta}, \theta_c)$. It thus suffices to show that a firm with $\theta \in (\theta_c, \overline{\theta})$ prefers $p(\theta)$ to $p_+(\theta_c)$. As shown in the proof of Lemma 10, for any $\theta \in (\theta_c, \overline{\theta})$, there is $x \in (\theta_c, \theta)$ such that $\Pi(\theta, \theta; p) > \Pi(x, \theta; p)$, and this inequality holds in the limit where $x \to \theta_c$ and $\Pi(x, \theta; p) \to \pi(p_+(\theta_c), \theta)[1 - F(\theta_c)]^{N-1}$. Therefore, we conclude that a firm with $\theta \in (\theta_c, \overline{\theta})$ has no gain from any deviation.

(c) IC for $\theta \in [\underline{\theta}, \theta_c)$: A firm with $\theta \in [\underline{\theta}, \theta_c)$ has the following preferences. First, $p(\theta_c)$ is preferred to $p_+(\theta_c)$. Since $p_+(\theta_c) > p(\theta_c)$ and $\theta_c > \theta$, Lemma 9 shows that

$$\frac{\pi(p_+(\theta_c), \theta)}{\pi(p(\theta_c), \theta)} < \frac{\pi(p_+(\theta_c), \theta_c)}{\pi(p(\theta_c), \theta_c)} = \frac{M(\theta_c)}{[1 - F(\theta_c)]^{N-1}},$$

where the equality is from (31). We thus obtain the result:

$$\pi(p(\theta_c),\theta)M(\theta_c) > \pi(p_+(\theta_c),\theta)[1-F(\theta_c)]^{N-1} \text{ for } \theta \in [\underline{\theta},\theta_c).$$

Second, $p(\theta_c)$ is preferred to $\widehat{p} \in (p(\theta_c), p_+(\theta_c))$. Given $\widehat{p} > p(\theta_c)$ and $\theta_c > \theta$, we use Lemma 9 and establish that

$$\frac{\pi(\widehat{p},\theta)}{\pi(p(\theta_c),\theta)} < \frac{\pi(\widehat{p},\theta_c)}{\pi(p(\theta_c),\theta_c)} < \frac{M(\theta_c)}{[1-F(\theta_c)]^{N-1}},$$

where the second inequality is from (32). We thus obtain the result:

$$\pi(p(\theta_c),\theta)M(\theta_c) > \pi(\widehat{p},\theta)[1-F(\theta_c)]^{N-1} \text{ for } \theta \in [\underline{\theta},\theta_c).$$

Third, $p(\theta_c)$ is preferred to $p(\widehat{\theta})$ for $\widehat{\theta} \in (\theta_c, \overline{\theta})$. Given $p(\widehat{\theta}) > p(\theta_c)$ and $\theta_c > \theta$, it follows from Lemma 9 that

$$\frac{\pi(p(\widehat{\theta}),\theta)}{\pi(p(\theta_c),\theta)} < \frac{\pi(p(\widehat{\theta}),\theta_c)}{\pi(p(\theta_c),\theta_c)} \leq \frac{M(\theta_c)}{[1-F(\widehat{\theta})]^{N-1}},$$

where the second inequality is from the second result in (a). We thus obtain the result:

$$\pi(p(\theta_c),\theta)M(\theta_c) > \pi(p(\widehat{\theta}),\theta)[1-F(\widehat{\theta})]^{N-1} \text{ for } \widehat{\theta} \in (\theta_c,\overline{\theta}).$$

Given the three results, it suffices to show that a firm with $\theta \in [\underline{\theta},\theta_c)$ prefers $p(\theta)$ to $p(\theta_c)$. For any $\theta \in [\underline{\theta},\theta_c)$, there is $x \in (\theta,\theta_c)$ such that $\Pi(\theta,\theta;p) > \Pi(x,\theta;p)$. This inequality holds in the limit where $x \to \theta_c$ and $\Pi(x,\theta;p) \to \pi(p(\theta_c),\theta)M(\theta_c)$. Therefore, we conclude that a firm with $\theta \in [\underline{\theta},\theta_c)$ has no gain from any deviation. ■

Proof of Lemma 7. (i) We use the interim profit for $\theta \leq \theta_c$ and define an implicit function:

$$\eta(\theta) \equiv \pi(p(\theta),\theta)M(\theta) - \int_\theta^{\theta_c} D(p(x))M(x)dx - \int_{\theta_c}^{\overline{\theta}} D(p(x))[1-F(x)]^{N-1}dx = 0.$$

If $\theta = \theta_c$, then the implicit function $\eta(\theta_c)$ has no second term. We find

$$\frac{\partial p(\theta)}{\partial \theta_c} = -\frac{\partial \eta(\theta)/\partial \theta_c}{\pi_p(p(\theta),\theta)M(\theta)}. \tag{33}$$

The denominator of (33) is positive and the numerator is

$$\frac{\partial \eta(\theta)}{\partial \theta_c} = \pi(p(\theta),\theta)\frac{\partial \mu(\theta_c)}{\partial \theta_c}U - \int_{\theta}^{\theta_c} D'(p(x))\frac{\partial p(x)}{\partial \theta_c}M(x)dx \tag{34}$$
$$- \int_{\theta}^{\theta_c} D(p(x))\frac{\partial \mu(\theta_c)}{\partial \theta_c}Udx - [D(p(\theta_c))M(\theta_c) - D(p_+(\theta_c))[1-F(\theta_c)]^{N-1}].$$

From (12) and Lemma 5 (ii), we know that

$$\frac{\partial \mu(\theta_c)}{\partial \theta_c} < 0 \text{ and } D(p(\theta_c))M(\theta_c) > D(p_+(\theta_c))[1-F(\theta_c)]^{N-1}.$$

We differentiate (34) with respect to θ,

$$\frac{\partial^2 \eta(\theta)}{\partial \theta \partial \theta_c} = \pi_p(p(\theta),\theta)p'(\theta)\frac{\partial \mu(\theta_c)}{\partial \theta_c}U - \frac{D'(p(\theta))}{\pi_p(p(\theta),\theta)}\frac{\partial \eta(\theta)}{\partial \theta_c}. \tag{35}$$

Since $\frac{\partial \eta(\theta_c)}{\partial \theta_c} < 0$ in (34), we find that $\frac{\partial p(\theta_c)}{\partial \theta_c} > 0$. We report further results in (a)-(c) for later use:

(a) *If* $\frac{\partial \eta(\widehat{\theta})}{\partial \theta_c} = 0$ *or* $\frac{\partial \eta(\widehat{\theta})}{\partial \theta_c} < 0$ *for any* $\widehat{\theta} < \theta_c$, *then* $\frac{\partial \eta(\theta)}{\partial \theta_c} < 0$ *for all* $\theta \in (\widehat{\theta}, \theta_c]$. This is immediate from (35): if $\frac{\partial \eta(\widehat{\theta})}{\partial \theta_c} = 0$ or $\frac{\partial \eta(\widehat{\theta})}{\partial \theta_c} < 0$, then $\frac{\partial \eta(\theta)}{\partial \theta_c}$ is strictly decreasing in θ at $\widehat{\theta}$, $\frac{\partial^2 \eta(\widehat{\theta})}{\partial \theta \partial \theta_c} < 0$.

(b) *There exists an interval* $(\widehat{\theta}, \theta_c]$ *on which* $\frac{\partial p(\theta)}{\partial \theta_c} > 0$. If $\widehat{\theta}$ is sufficiently close to θ_c, then $\frac{\partial \eta(\widehat{\theta})}{\partial \theta_c} < 0$, since the second and third terms in (34) approach zero while the remaining terms are negative. From (a), we know that $\frac{\partial \eta(\widehat{\theta})}{\partial \theta_c} < 0$ implies $\frac{\partial \eta(\theta)}{\partial \theta_c} < 0$ for all $\theta \in (\widehat{\theta}, \theta_c]$.

(c) *For an increase in* θ_c*, if the new price function crosses the original price function, it does so only once.* If there is any crossing point x^* such that $\frac{\partial \eta(x^*)}{\partial \theta_c} = 0$ (i.e., $\frac{\partial p(x^*)}{\partial \theta_c} = 0$), then $\frac{\partial \eta(\theta)}{\partial \theta_c} < 0$ (i.e., $\frac{\partial p(\theta)}{\partial \theta_c} > 0$) for all $\theta \in (x^*, \theta_c]$.

(ii) For the result $\frac{\partial \theta_c}{\partial s} > 0$, we show that

$$\frac{\partial}{\partial \theta_c}\int_{\underline{\theta}}^{\theta_c} \phi(\theta)dF(\theta) > 0 \text{ where } \phi(\theta) \equiv u(p(\theta)) - u(p(\theta_c)). \tag{36}$$

The results in (a)-(c) in (i) show that an increase in θ_c has two scenarios: the new function crosses the original function once, or there is only the price-increasing interval. We first assume that for an increase in θ_c, there is a unique crossing point $x^* \in (\underline{\theta}, \theta_c)$ such that $\frac{\partial p(x^*)}{\partial \theta_c} = 0$. Given x^*, we decompose (36) into two terms:

$$\frac{\partial}{\partial \theta_c} \int_{\underline{\theta}}^{x^*} \phi(\theta) dF(\theta) + \frac{\partial}{\partial \theta_c} \int_{x^*}^{\theta_c} \phi(\theta) dF(\theta). \tag{37}$$

The first term in (37) is positive on the price-decreasing interval $(\underline{\theta}, x^*)$,

$$\frac{\partial}{\partial \theta_c} \int_{\underline{\theta}}^{x^*} \phi(\theta) dF(\theta) = \int_{\underline{\theta}}^{x^*} \left(D(p(\theta_c)) \frac{\partial p(\theta_c)}{\partial \theta_c} - D(p(\theta)) \frac{\partial p(\theta)}{\partial \theta_c} \right) dF(\theta) > 0,$$

where $D(p(\theta_c)) \frac{\partial p(\theta_c)}{\partial \theta_c} > 0$ is implied by (i) and $D(p(\theta)) \frac{\partial p(\theta)}{\partial \theta_c} < 0$ is assumed for $\theta \in (\underline{\theta}, x^*)$. We next show the second term in (37) is also positive on the price-increasing interval (x^*, θ_c). We rewrite the term as

$$\int_{x^*}^{\theta_c} \phi(\theta) dF(\theta) = -[u(p(x^*)) - u(p(\theta_c))]F(x^*) + \int_{x^*}^{\theta_c} F(\theta) D(p(\theta)) p'(\theta) d\theta. \tag{38}$$

We then differentiate the last term in (38) with respect to θ_c,

$$\begin{aligned} \frac{\partial}{\partial \theta_c} \int_{x^*}^{\theta_c} F(\theta) D(p(\theta)) p'(\theta) d\theta = {} & F(\theta_c) D(p(\theta_c)) p'(\theta_c) \\ & + \int_{x^*}^{\theta_c} \frac{\partial}{\partial \theta_c} F(\theta) D(p(\theta)) p'(\theta) d\theta. \end{aligned} \tag{39}$$

Combining (38) and (39), we rewrite the second term in (37) as

$$\begin{aligned} \frac{\partial}{\partial \theta_c} \int_{x^*}^{\theta_c} \phi(\theta) dF(\theta) = {} & D(p(\theta_c)) p'(\theta_c) [F(\theta_c) - F(x^*)] \\ & + \int_{x^*}^{\theta_c} \frac{\partial}{\partial \theta_c} F(\theta) D(p(\theta)) p'(\theta) d\theta. \end{aligned} \tag{40}$$

The first term on the right-hand side is positive since $F(\theta_c) > F(x^*)$. The second term is also positive since Assumption 1 ensures that the term $D(p(\theta))p'(\theta) = \frac{\pi(p(\theta),\theta) D(p(\theta))}{\pi_p(p(\theta),\theta)}$ becomes greater for each θ on the price-increasing interval (x^*, θ_c).

Supposing next that there is only the price-increasing interval, we find that

$$\frac{\partial}{\partial \theta_c} \int_{\underline{\theta}}^{\theta_c} \phi(\theta) dF(\theta) = \frac{\partial}{\partial \theta_c} \int_{\underline{\theta}}^{\theta_c} F(\theta) D(p(\theta)) p'(\theta) d\theta > 0, \tag{41}$$

where the inequality is directly implied by Assumption 1. Note that (40) approaches (41) if $x^* \to \underline{\theta}$ and thus $F(x^*) \to 0$. In sum, the inequality in (36) holds.

(iii) Since $u(p(\theta)) > u(p(\theta_c))$ for all $\theta \in [\underline{\theta}, \theta_c)$, we have

$$\int_{\underline{\theta}}^{\theta_c} [u(p(\theta)) - u(p(\theta_c))] dF(\theta) > 0. \tag{42}$$

Thus, if $s \to 0$, then $\theta_c \to \underline{\theta}$ is necessary to satisfy $\Delta(\theta_c) = 0$. Indeed, if $\theta_c \to \underline{\theta}$, the value in (42) approaches zero given

$$\int_{\underline{\theta}}^{\theta_c} [u(p(\theta)) - u(p(\theta_c))] dF(\theta) < [u(p(\underline{\theta})) - u(p(\theta_c))] F(\theta_c).$$

Given the inequality (36), if $s \to \underline{v}$, then $\theta_c \to \overline{\theta}$ is necessary to satisfy $\Delta(\theta_c) = 0$. Indeed, if $\theta_c \to \overline{\theta}$, the value in (42) approaches $\underline{v}$, because $p(\theta)$ for $\theta \leq \theta_c$ approaches the price function with the boundary value $p(\overline{\theta}) = \overline{\theta}$ in RE. This approximation of $p(\theta)$ holds since if $\theta_c \to \overline{\theta}$, then $M(\theta)$ in (10) approaches $\frac{U}{N} + [1 - F(\theta)]^{N-1} I$ and $p(\theta_c)$ in (13) converges to $p(\overline{\theta}) = \overline{\theta}$. ■

Proof of Proposition 2. For any given $s \in (0, \underline{v})$, we consider the unique function that satisfies $\Delta(\theta_c) = 0$. At the observed price $p(\widehat{\theta})$, uninformed consumers have the net gain from search:

$$\Delta(\widehat{\theta}) = \int_{\underline{\theta}}^{\widehat{\theta}} [u(p(\theta)) - u(p(\widehat{\theta}))] dF(\theta) - s = \int_{\underline{\theta}}^{\widehat{\theta}} u(p(\theta)) dF(\theta) - F(\widehat{\theta}) u(p(\widehat{\theta})) - s.$$

Using integration by parts, we rewrite it as

$$\Delta(\widehat{\theta}) = -\int_{\underline{\theta}}^{\widehat{\theta}} F(\theta) u'(p(\theta)) p'(\theta) d\theta - s,$$

and find that

$$\frac{\partial \Delta(\widehat{\theta})}{\partial \widehat{\theta}} = -F(\widehat{\theta}) u'(p(\widehat{\theta})) p'(\widehat{\theta}) > 0.$$

Since $\Delta(\theta_c) = 0$ and $\frac{\partial \Delta(\widehat{\theta})}{\partial \widehat{\theta}} > 0$, it follows that $\Delta(\widehat{\theta}) < 0$ for $\widehat{\theta} < \theta_c$. Next, focusing on $\widehat{\theta} > \theta_c$, we use $\Delta(\theta_c) = 0$ and rewrite $\Delta(\widehat{\theta})$ as

$$\Delta(\widehat{\theta}) = \int_{\theta_c}^{\widehat{\theta}} u(p(\theta)) dF(\theta) - F(\widehat{\theta}) u(p(\widehat{\theta})) + F(\theta_c) u(p(\theta_c)).$$

Given the jump at θ_c, we find that

$$\lim_{\widehat{\theta}\to\theta_c} \Delta(\widehat{\theta}) = F(\theta_c)[u(p(\theta_c)) - u(p_+(\theta_c))] > 0.$$

Since $\frac{\partial \Delta(\widehat{\theta})}{\partial \widehat{\theta}} > 0$, it follows that $\Delta(\widehat{\theta}) > 0$ for $\widehat{\theta} > \theta_c$. Hence, $\Delta(\widehat{\theta}) \leq 0$ for $\widehat{\theta} \leq \theta_c$ and $\Delta(\widehat{\theta}) > 0$ for $\widehat{\theta} > \theta_c$. This result verifies optimality of sequential search with $p(\theta_c)$: with the search cost that satisfies $\Delta(\theta_c) = 0$, uninformed consumers purchase at the observed price $p(\widehat{\theta})$ if $p(\widehat{\theta}) \leq p(\theta_c)$ and search again if $p(\widehat{\theta}) > p(\theta_c)$. The remaining proofs are immediate from the text. ■

Proof of Proposition 3. (i) We use the implicit function (19) and find that

$$\frac{\partial \rho}{\partial N} = -\frac{\partial \Delta(\overline{\theta})/\partial N}{\partial \Delta(\overline{\theta})/\partial \rho} = \frac{\int_{\underline{\theta}}^{\overline{\theta}} D(p(\theta)) \frac{\partial p(\theta)}{\partial N}\Big|_{d\rho=0} dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \left(D(\rho) - D(p(\theta))\frac{\partial p(\theta)}{\partial \rho}\right) dF(\theta)}. \tag{43}$$

We fix $\rho > \overline{\theta}$ (i.e., $d\rho = 0$) and rearrange the interim profit in (18) by dividing $M(\theta)$ on both sides. By taking integral of the equation and differentiating it with respect to N, we find that

$$\begin{aligned} &\int_{\underline{\theta}}^{\overline{\theta}} D(p(\theta)) \frac{\partial p(\theta)}{\partial N}\Big|_{d\rho=0} \left(1 + \varepsilon_{D(p(\theta))} - \theta \frac{D'(p(\theta))}{D(p(\theta))}\right) dF(\theta) \\ &= \pi(\rho, \overline{\theta}) \frac{\partial}{\partial N} \int_{\underline{\theta}}^{\overline{\theta}} \frac{U/N}{M(\theta)} dF(\theta) + \frac{\partial}{\partial N} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\theta}^{\overline{\theta}} D(p(x)) \frac{M(x)}{M(\theta)} dx dF(\theta), \end{aligned} \tag{44}$$

where $\varepsilon_{D(p(\theta))}$ is the price elasticity of demand at $p(\theta)$. If demand is inelastic, then we obtain $\frac{\partial p(\theta)}{\partial \rho} = \frac{U/N}{M(\theta)}$ from (18) and in turn use (43) to derive (21). In addition, if $\overline{\theta} - \underline{\theta}$ is sufficiently small, then $\int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial p(\theta)}{\partial N}\Big|_{d\rho=0} dF(\theta) > 0$ follows from Bagwell and Lee (2014) since their Proposition 6 shows that

$$\frac{\partial}{\partial N} \int_{\underline{\theta}}^{\overline{\theta}} \frac{U/N}{M(\theta)} dF(\theta) = \frac{\partial}{\partial N} \int_0^1 \frac{1}{1 + \frac{I}{1-I} N x^{N-1}} dx > 0$$

and that $\frac{\partial}{\partial N} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\theta}^{\overline{\theta}} \frac{M(x)}{M(\theta)} dx dF(\theta)$ approaches zero for $\overline{\theta} - \underline{\theta}$ sufficiently small. Next, suppose downward-sloping demand becomes sufficiently inelastic. The left-hand side of (44) approximates the numerator of (43). In addition, if $\overline{\theta} - \underline{\theta}$ is sufficiently small, then only the first term on the right-hand side of (44) persists. The denominator of (43) is positive under Assumption 1 since an increase in the boundary price raises $p(\theta)$ for all θ. This is because the uniqueness result in Lemma 3 shows that different price functions do not cross at any point in the relevant region of $(\theta, p(\theta))$ where $\underline{\theta} < \theta < \overline{\theta}$ and $\theta < p(\theta) < p^m(\theta)$. If two functions cross only at $\underline{\theta}$ and have the same $p(\underline{\theta})$, they have the same first and second derivatives at $\widehat{\theta}$ in the limit where $\widehat{\theta} \to \underline{\theta}$, which can be verified by $\frac{dy}{d\theta}$ and $\frac{d^2y}{d\theta^2}$ in the proof of Lemma 3. This contradicts the supposition that two functions cross only at $\underline{\theta}$.

(ii) We use the interim profit for $\theta \in (\theta_c, \overline{\theta})$,

$$\varphi(\theta) = \pi(p(\theta), \theta)[1 - F(\theta)]^{N-1} - \int_{\theta}^{\overline{\theta}} D(p(x))[1 - F(x)]^{N-1} = 0,$$

and find

$$\frac{\partial p(\theta)}{\partial N} = -\frac{\varphi_N(\theta)}{\pi_p(p(\theta), \theta)[1 - F(\theta)]^{N-1}},$$

where $\varphi_N(\theta) \equiv \frac{\partial \varphi(\theta)}{\partial N}$. The sign of $\frac{\partial p(\theta)}{\partial N}$ is opposite to the sign of the numerator:

$$\begin{aligned}
\varphi_N(\theta) = {} & \pi(p(\theta), \theta)[1 - F(\theta)]^{N-1} \ln[1 - F(\theta)] \\
& - \int_{\theta}^{\overline{\theta}} D(p(x))[1 - F(x)]^{N-1} \ln[1 - F(x)] dx \\
& - \int_{\theta}^{\overline{\theta}} D'(p(x)) \frac{\partial p(x)}{\partial N} [1 - F(x)]^{N-1} dx.
\end{aligned}$$

We differentiate the numerator with respect to θ,

$$\begin{aligned}
\frac{\partial \varphi_N(\theta)}{\partial \theta} = {} & \pi_p(p(\theta), \theta) p'(\theta)[1 - F(\theta)]^{N-1} \ln[1 - F(\theta)] \\
& - \pi(p(\theta), \theta)(N - 1)[1 - F(\theta)]^{N-2} f(\theta) \left(\ln[1 - F(\theta)] + \frac{1}{N - 1} \right) \\
& - \frac{D'(p(\theta))}{\pi_p(p(\theta), \theta)} \varphi_N(\theta).
\end{aligned}$$

We then use the first-order condition,

$$\pi_p(p(\theta), \theta) p'(\theta)[1 - F(\theta)]^{N-1} = \pi(p(\theta), \theta)(N - 1)[1 - F(\theta)]^{N-2} f(\theta),$$

and simplify the earlier differentiation to

$$\frac{\partial \varphi_N(\theta)}{\partial \theta} = -\pi(p(\theta), \theta)[1 - F(\theta)]^{N-2} f(\theta) - \frac{D'(p(\theta))}{\pi_p(p(\theta), \theta)} \varphi_N(\theta). \tag{45}$$

We finally show that $\varphi_N(\theta) > 0$ (i.e., $\frac{\partial p(\theta)}{\partial N} < 0$) for all $\theta \in (\theta_c, \overline{\theta})$. Suppose there is $\widehat{\theta} \in (\theta_c, \overline{\theta})$ such that $\varphi_N(\widehat{\theta}) \leq 0$ (i.e., $\frac{\partial p(\widehat{\theta})}{\partial N} \geq 0$). From (45), we obtain $\frac{\partial \varphi_N(\widehat{\theta})}{\partial \theta} < 0$. Now, given $\varphi_N(\widehat{\theta}) \leq 0$ and $\frac{\partial \varphi_N(\widehat{\theta})}{\partial \theta} < 0$, it follows that $\varphi_N(\theta) < 0$ for all $\theta \in (\widehat{\theta}, \overline{\theta})$. From the interim profit for $\widehat{\theta}$, we obtain the profit-if-win for $\widehat{\theta}$,

$$\pi(p(\widehat{\theta}), \widehat{\theta}) = \int_{\widehat{\theta}}^{\overline{\theta}} D(p(x)) \left(\frac{1 - F(x)}{1 - F(\widehat{\theta})} \right)^{N-1} dx. \tag{46}$$

The right-hand side of (46) is strictly decreasing in N, since for $x \in (\widehat{\theta}, \overline{\theta})$, we have $\frac{1-F(x)}{1-F(\widehat{\theta})} < 1$ and

$$\frac{\partial D(p(x))}{\partial N} = -D'(p(x)) \frac{\varphi_N(x)}{\pi_p(p(x), x)[1 - F(x)]^{N-1}} < 0.$$

Hence, the left-hand side of (46) must strictly decrease in N, $\frac{\partial p(\widehat{\theta})}{\partial N} < 0$, which contradicts the supposition, $\varphi_N(\widehat{\theta}) \leq 0$. ■

Proof of Lemma 8. (i) If $\rho \geq p^m(\overline{\theta})$, then $p(\overline{\theta}) = p^m(\overline{\theta})$ for any N, and the interim profit implies

$$\pi(p(\theta), \theta) = \pi(p^m(\overline{\theta}), \overline{\theta}) \frac{U/N}{M(\theta)} + \int_{\theta}^{\overline{\theta}} D(p(x)) \frac{M(x)}{M(\theta)} dx.$$

This equation shows that $p(\underline{\theta})$ approaches $\underline{\theta}$ in the limit since the right-hand side at $\underline{\theta}$ approaches zero in the limit given

$$\frac{U/N}{M(\underline{\theta})} = \frac{U}{U + NI} \text{ and } \frac{M(x)}{M(\underline{\theta})} = \frac{U + N[1 - F(x)]^{N-1} I}{U + NI}.$$

For $\theta \in (\underline{\theta}, \overline{\theta})$, the price function in the limit, $\widetilde{p}(\theta)$, satisfies $\widetilde{p}(\theta) \leq \min\{p^m(\overline{\theta}), p^m(\theta)\}$ since $p(\theta) < p(\overline{\theta}) = p^m(\overline{\theta})$ and $p(\theta) < p^m(\theta)$ for any N. We show further that $\widetilde{p}(\theta) = \min\{p^m(\overline{\theta}), p^m(\theta)\}$. For $\theta \in (\underline{\theta}, \overline{\theta})$, if $N \to \infty$, then On-IC holds only if $p(\theta)$ approaches either the boundary value $p^m(\overline{\theta})$ or the monopoly price $p^m(\theta)$, since On-IC implies

$$\pi(p(\theta), \theta)[U + N[1 - F(\theta)]^{N-1} I] \geq \pi(p(\widehat{\theta}), \theta)[U + N[1 - F(\widehat{\theta})]^{N-1} I] \text{ for all } \widehat{\theta} > \underline{\theta},$$

where $N[1 - F(\theta)]^{N-1} I$ and $N[1 - F(\widehat{\theta})]^{N-1} I$ approach zero in the limit. Note also that a firm with $\theta \in (\underline{\theta}, \overline{\theta})$ will not mimic type $\underline{\theta}$ since $p(\underline{\theta}) \to \underline{\theta}$. Thus, $\widetilde{p}(\theta) = \min\{p^m(\overline{\theta}), p^m(\theta)\} = p^m(\theta)$ for $\theta \in (\underline{\theta}, \overline{\theta})$. Therefore, we conclude that if $N \to \infty$, then $p(\theta)$ converges to $\widetilde{p}(\theta)$ in (24).

(ii) If $\rho \in [\overline{\theta}, p^m(\overline{\theta}))$, then $p(\overline{\theta}) = \rho$ for any N. The same procedure used in (i) shows that if $N \to \infty$, then $p(\theta)$ converges to $\widetilde{p}(\theta)$ in (25).

(iii) If $\rho < \overline{\theta}$, then $p(\overline{\theta}) = \overline{\theta}$ for any N. We can rewrite the interim profit for $\theta > \theta_c$ as we did in (46). It is then clear that if $N \to \infty$, then $p(\theta) \to \theta$ for $\theta > \theta_c$. The jump at θ_c dissipates, $p(\theta_c) \to \theta_c$, in the limit since the interim profit for θ_c implies

$$\pi(p(\theta_c), \theta_c) N M(\theta_c) = \int_{\theta_c}^{\overline{\theta}} D(p(x)) N[1 - F(x)]^{N-1} dx,$$

where $NM(\theta_c)$ persists and $N[1 - F(x)]^{N-1}$ approaches zero in the limit. The interim profit for $\theta < \theta_c$ becomes

$$\pi(p(\theta), \theta) = \pi(p(\theta_c), \theta_c) \frac{M(\theta_c)}{M(\theta)} + \int_{\theta}^{\theta_c} D(p(x)) \frac{M(x)}{M(\theta)} dx.$$

If $N \to \infty$, then $p(\underline{\theta}) \to \underline{\theta}$ since the right-hand side at $\underline{\theta}$ approaches zero given that $M(\underline{\theta}) = I + \mu(\theta_c) U$ and $\mu(\theta_c) \to 0$ in the limit. For $\theta \in (\underline{\theta}, \theta_c)$, we first observe that $p(\theta) < p^m(\theta)$ and $p(\theta) < p(\theta_c)$ for any N. Recall that $p(\theta_c) \to \theta_c$ in the limit. We next verify that $\widetilde{p}(\theta) = \min\{\theta_c, p^m(\theta)\}$. For $\theta \in (\underline{\theta}, \theta_c)$, if $N \to \infty$, then On-IC holds only if $p(\theta)$ approaches either θ_c or $p^m(\theta)$, since On-IC implies that for $\widehat{\theta} \in (\underline{\theta}, \theta_c)$,

$$\pi(p(\theta), \theta)[N[1 - F(\theta)]^{N-1} I + N\mu(\theta_c) U] \geq \pi(p(\widehat{\theta}), \theta)[N[1 - F(\widehat{\theta})]^{N-1} I + N\mu(\theta_c) U],$$

where $N[1 - F(\theta)]^{N-1} I$ and $N[1 - F(\widehat{\theta})]^{N-1} I$ approach zero and $N\mu(\theta_c)$ persists in the limit. Hence, we conclude that if $N \to \infty$, then $p(\theta)$ converges to $\widetilde{p}(\theta)$ in (26). ■

Proof of Proposition 5. (i) The proof follows from two results. First, price functions in RE do not cross, as shown in the proof of Lemma 3. Second, it follows from (2) that the price function with a lower boundary value has a smaller slope $p'(\theta)$ for any $\theta \in (\underline{\theta}, \overline{\theta})$, since it conveys a smaller value of $\frac{\pi(p(\theta),\theta)}{\pi_p(p(\theta),\theta)}$ for each $\theta \in (\underline{\theta}, \overline{\theta})$ while all price functions in RE have the same $M(\theta)$.

(ii) The first part of (ii) is provided by the proof of Lemma 7: $\frac{\partial \theta_c}{\partial s} > 0$ and $\frac{\partial p(\theta)}{\partial \theta_c} > 0$ on $(\widehat{\theta}, \theta_c]$. For the second part of (ii), suppose $f(\theta_c)$ is sufficiently small and an increase in θ_c shifts the price function such that $\frac{\partial p(\theta)}{\partial \theta_c} < 0$ on $(\underline{\theta}, x^*)$ and $\frac{\partial p(\theta)}{\partial \theta_c} > 0$ on (x^*, θ_c). This supposition implies $\frac{\partial \eta(\theta)}{\partial \theta_c} > 0$ on $(\underline{\theta}, x^*)$ in (33), which in turn implies $\frac{\partial^2 \eta(\theta)}{\partial \theta \partial \theta_c} > 0$ on $(\underline{\theta}, x^*)$ in (35). This is because the first term in (35) vanishes since it includes $\frac{\partial \mu(\theta_c)}{\partial \theta_c}$ that vanishes for $f(\theta_c)$ sufficiently small, regardless of how $p'(\theta)$ changes with $f(\theta_c)$, whereas the second term persists for any $f(\theta_c)$. Now, given $\frac{\partial \eta(\theta)}{\partial \theta_c} > 0$ and $\frac{\partial^2 \eta(\theta)}{\partial \theta \partial \theta_c} > 0$ on $(\underline{\theta}, x^*)$, it is impossible to satisfy $\frac{\partial \eta(\theta)}{\partial \theta_c} < 0$ on (x^*, θ_c) and hence the supposition is contradicted. ■

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