Cournot Competition and Green Innovation in a Dynamic Oligopoly*

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Abstract

In this paper we analyze a dynamic Cournot oligopoly to study the relationship between competition and green innovation. Firms face a tax on emissions and react to this tax investing in an abatement technology. The tax is given by the feedback Stackelberg equilibrium of a dynamic policy game between a regulator and a polluting oligopoly where environmental damages depend on the pollution stock. For constant marginal damages, we find that firms' R&D investment increases monotonically with the number of firms in the industry because competition increases the tax. This effect is explained by the fact that the tax can be decomposed in two terms, one negative that reflects the divergence between the price and the marginal revenue because of the market power of firms, and another positive that reflects the divergence between the social valuation of the pollution stock and the private valuation. When the number of firms in the industry increases, the absolute value of the first term decreases and the tax increases leading to more investment. Moreover, as in this case firms increase their stock of abatement capital, net emissions decrease and this causes a reduction of the pollution stock.

Keywords: oligopoly, homogeneous good, Cournot competition, green R&D, end-of-the-pipe abatement technology, emission tax, differential games

JEL Classification System: H23, L12, L51, Q52, Q55

1 Introduction

The effects of competition on firms' innovation is a classical issue in the literature on industrial organization. It goes back to the indirect debate between Schumpeter (1942) and Arrow (1962) focusing on the so-called Schumpeterian hypothesis: one should expect to observe an inverse relationship between innovation and the intensity of competition, because monopoly rents would vanish as competition becomes stronger. An hypothesis discussed by Arrow (1962) who claims that a competitive firm has a larger incentive to innovate than a monopolist who could be interested in postponing the R&D investment (a review of this literature can be found in Tirole (1988)). This debate received a new impulse with the publication of a paper by Aghion et al. (2005) that based on a neo-Schumpeterian endogenous growth model provides evidence of an inverted-U relationship between aggregate R&D and the intensity of market competition using UK panel data.[1] A more recent paper by Aghion et al. (2023) investigates the effects of consumers' environmental concerns and market competition on firms' decision to innovate in clean technologies. They find a significant positive effect of environmental concerns on the probability for a firm to innovate in the clean direction, a positive effect that is larger the higher the competition is.

Despite the abundant literature on this issue, only a few scholars and very recently have been interested in the relationship between green innovation and the competitive pressure. The list of papers addressing this issue consists of Feichtinger et al. (2016), Lambertini et al. (2017), Menezes and Pereira (2017), and Dragone et al. (2022). Our aim is to contribute to the literature with new insights analyzing this issue in the framework of a dynamic Cournot oligopoly that produces an homogeneous good where firms react to an emission tax investing in green R&D that generates some spillovers.[2] The tax is given by

[1] Hashmi (2013) revisits the inverted-U relationship by using US data finding a mildly negative relationship between competition and innovation. An assessment of the lessons learnt from Schumpeterian growth theory can be found in Aghion et al. (2013).

[2] An excellent review of the effects of competition on innovation in the framework of the oligopoly theory can be found in Vives (2008). Recently, Yanese and Long (2023) has revisited this issue using a dynamic model of an industry consisting of a few large firms and a fringe of small firms that pro-

the feedback Stackelberg equilibrium of a dynamic policy game between a regulator and a polluting oligopoly where environmental damages depend on the stock of pollution. The regulator playing as the leader chooses an emission tax to maximize net social welfare, and the firms acting as followers selects their R&D investment in an abatement technology and output to maximize profits. We compute the tax for a linear-state policy game. Our findings show that firms' R&D investment increases with the number of firms in the industry. This effect operates through the *positive* influence that the tax has on R&D investment.[3] The tax leads firms to invest in the abatement technology, but we find that the tax increases with the number of firms in the industry, the result is that more competition translates into more R&D investment. This effect is explained by the fact that the optimal tax is the addition of a subsidy that corrects the divergence between the price and the marginal revenue because of the market power of firms, and a tax that closes the divergence between the social valuation of the pollution stock and the private valuation because of the negative externality. If the environmental damages are high enough, the second component dominates the first one and the optimal policy consists of taxing emissions.[4] Thus, if the number of firms in the industry increases, the market power of firms decreases and the absolute value of the first component of the tax also decreases causing an increase in the tax and the corresponding increases in investment. In our model, this effect is independent of the degree of spillovers because the tax does not

duce horizontally differentiated products, where each firm's marginal cost depends of a common pool of knowledge that accumulates over time due to large firms' investment. The authors find that for the open-loop Nash equilibrium, the relationship between the number of large firms and the steady-state stock of knowledge capital has an inverted-U shape. However, this relationship does not necessarily appear for the Markov-perfect Nash equilibria.

[3]Dijkstra and Gil-Moltó (2018) find for the case of a static Cournot oligopoly that the effect of the strictness of the environmental policy on green innovation is non-monotonic. Our results do not support this conclusion, but it should be taken into account for assessing this divergence that we are considering different types of innovation.

[4]Obviously, if damages are low the tax could become a subsidy and firms would not invest in an abatement technology. In this case, the more severe problem with the market allocation would be the lack of competence and not the environmental problem. In this paper, we are interested in the cases where emission taxation is justified.

depend on spillovers. In the model, spillovers are associated to the abatement capital so that they do not affect the decisions of firms on output and R&D investment, but only to the dynamics of the pollution stock. Moreover, as firms increase their stock of abatement capital, net emissions decrease and this causes a reduction of the pollution stock. Thus, our findings go in the line of those obtained by Aghion et al. (2023) that supports the idea that market competition promotes the adoption of cleaner technologies.

We also find that the steady state is a global asymptotically stable point so that the regulated market converges asymptotically to the steady-state abatement capacity and pollution stock from any initial conditions. In the paper, we focus on some particular initial conditions that we consider the more interesting case. We assume that the initial value of the abatement capacity is zero and that the initial pollution stock is higher than the steady-state pollution stock. Assuming that the initial value of abatement capacity is zero is consistent with the idea that firms will only invest in R&D if a tax is set up by the regulator. Thus, if the initial conditions reflect the state of the market before regulation it seems reasonable to assume that the initial abatement capacity is zero. On the other hand, if the initial value of pollution stock were lower than the steady-state pollution stock, regulations would lead to an increase in the pollution stock. We are more interesting in the case where taxation reduces the pollution stock.[5] For these initial conditions, the abatement capacity increase monotonically. However, the pollution stock could increase provided that the initial stock of pollution is not too large, but only during an initial period of time. In the long run, the pollution stock will decrease.

Our research contributes to the literature of competition and innovation that has been commented at the beginning of this section, and also to the literature on environmental regulation of firms with market power in a dynamic setting. The seminal paper of this literature is Benchekroun and Long (1998). In this paper a subgame-perfect tax rule that implements the efficient outcome for a Cournot oligopoly is designed when environmental damage is caused by a stock pollutant.[6] Later contributions to this literature are Yanase

[5] Nevertheless, in Appendix C we completely characterize the dynamics of the model considering all possible initial conditions with respect to the steady-state values.

[6] Benchekroun and Long (2002) focused on the case of a polluting monopoly. For this case, they show

(2009), Benchekroun and Chaudhuri (2011), Feichtinger et al. (2016), Martín-Herrán and Rubio (2018a, 2023) and Dragone et al. (2022).[7] Benchekroun and Chaudhuri (2011) show that the imposition of a tax that depends on the pollution stock can induce stable cartelization in a polluting oligopoly as the one analyzed by Benchekroun and Long (1998). Yanase (2009) was the first paper introducing abatement activities by firms. Abatement activities reduce emissions in each period of time, but firms do not invest in R&D. He examines a dynamic policy game between national governments that fix taxes or standards in a model of international pollution control for duopolists that compete myopically in quantities in a third country with product differentiation.[8] The same approach is adopted by Martín-Herrán and Rubio (2018a, 2023) in their analysis of the optimal environmental policy for the case of a polluting monopoly developed in Martín-Herrán and Rubio (2018a), and for the case of an oligopoly addressed by Martín-Herrán and Rubio (2023). In Feichtinger et al. (2016) firms invest in productive capacity and abate emissions with some spillovers in each period of time, but as in Yanase (2009), they do not invest in R&D. In their model, the environmental regulator charges an emission tax rate on the *accumulated emissions* of each firm, and also fixes the price of the output eliminating in this way the interdependence between firms through their influence on price. Finally, they assume that the optimal tax is the tax that maximizes the steady-state level of social welfare and that the regulated price depends on the number of firms in the industry. Their results show that there exists a constellation of parameter values

that tax rules are not unique. Lambertini (2018) reviews the literature on dynamic polluting oligopolists.

[7] We could also include in this list the papers by Stimming (1999), Feenstra et al. (2001) and more recently Walter (2018), but these papers consider a flow pollutant and focus on the effect of the environmental policy on the accumulation of capital. Stimming (1999) and Feenstra et al. (2001) investigate the effects of taxes and standards on the accumulation of productive capital for the case of a duopoly, whereas Walter (2018) studies the effect of a tax on emissions over the investment in R&D also in a duopolistic market. Xepapadeas (1992) and Kort (1996) also address these issues, but in their papers the market structure where the polluting firm operates is not clearly recognized.

[8] More recently, Yanase and Kamei (2022) study a two-country differential game model of transboundary pollution with international polluting oligopolies. The authors assume that governments use permits to regulate pollution. They compare autarky and bilateral free trade and conclude that free trade is better for the environment than autarky.

wherein the aggregate abatement of the industry at steady state is non-monotonic in the number of firms, presenting in some case an inverted-U relationship. They claim that this result is a consequence of some form of regulation, in their paper the regulation of the price, that modifies the aggregate behavior of the industry. Our analysis does not detect this kind of relationship, but our model diverges from theirs in several aspects. We study an oligopoly model where firms invest in R&D, the price is endogenous and the tax is charged on *current emissions* as in the seminal paper by Benchekroun and Long (1998). Dragone et al. (2022) model presents the essential elements of Feichtinger et al. (2016) model, but they do not include investment in productive capacity or price regulation, although as in Feichtinger et al. (2016) tax is charged on accumulated emissions. Their analysis also yields an inverted-U relationship between the aggregate abatement and the intensity of competition, but they highlight the role of spillovers in abatement activities to explain this result instead of the role of regulation as Feichtinger et al. (2016).

Finally, we would like to comment the papers by Menezes and Pereira (2017) and Martín-Herrán and Rubio (2018b) where firms invest in R&D. Menezes and Pereira (2017) study the dynamic competition of a duopoly in supply schedules. The focus of the paper is on the characterization of the optimal policy mix consisting of a tax on emissions and a subsidy on investment costs. They find that an increase in the intensity of the competition increases the tax and reduces the subsidy, but they do not address the effect on investment. Martín-Herrán and Rubio (2018b) analyze the second-best emission tax for a polluting monopoly so that the issue studied in this paper is outside the scope of their analysis.

The remainder of the paper is organized as follows. Section 2 presents the model of a polluting oligopoly. Section 3 characterizes the feedback Stackelberg equilibrium. In Section 4 the relationship between competition and green innovation is studied in the framework of a linear-state dynamic game. Section 5 offers some concluding remarks and points out lines for future research.

2 The Model

We consider a Cournot oligopoly that faces a market demand represented by the decreasing inverse demand function $P(Q(t))$ where $Q(t) = \sum_{i=1}^{n} q_i(t)$ is the output of the industry at time t and $n \geq 2$ is the number of firms. Firms produce a homogeneous good using the same productive technology described by the cost function $PC(q_i(t)) = cq_i(t)$. The production process generates pollution emissions, but after an appropriate choice of measurement units we can say that each unit of output generates one unit of pollution. Emissions are subject to a per unit tax $\tau(t)$. As a response to the tax, firms can either decrease their output or invest in R&D to reduce the emission per unit of output. We assume that the firm adopts an end-of-the-pipe abatement technology such that net emissions are $e_i(t) = q_i(t) - a_i(Y_i(t))$, where $a_i(Y_i(t))$ is the abatement function and $Y_i(t)$ is the *effective stock of R&D capital.*[9] Function $a_i(Y_i(t))$ satisfies the following properties: $a_i(0) = 0$, $a_i(Y_i(t)) \leq q_i$, $a'(Y_i(t)) > 0$ and $a_i''(Y_i(t)) < 0$ for all $Y_i(t) \geq 0$. With positive spillovers, if the firms R&D capital stocks are $y_1(t), ..., y_n(t)$, respectively, the firm i's effective stock of R&D capital is $Y_i(t) = y_i(t) + \beta \sum_{j \neq i}^{n} y_j(t)$, where $\beta \in [0, 1]$ measures the degree of spillovers. However, in this paper we adhere to the approach proposed by D'Aspremont and Jacquemin (1988) for modelling innovation in a Cournot duopoly assuming that $a_i(Y_i(t))$ is a linear function and that the decreasing returns of the abatement function are captured by a strictly convex R&D cost function $IC(w_i(t))$ where $w_i(t)$ stands for the R&D investment. In fact, we redefine $Y_i(t)$ in terms of abated emissions so that this variable can be also interpreted as the abatement capacity of the firm and net emissions can be written as follows $e_i(t) = q_i(t) - y_i(t) - \beta \sum_{j \neq i}^{n} y_j(t)$.[10] Thus, the dynamics of the abatement capacity for each firm is defined by the differential equation

$$\dot{y}_i(t) = w_i(t) - \delta_y y_i(t), \quad y_i(0) = y_0 \geq 0, \quad i = 1, \ldots, n, \tag{1}$$

[9] Notice that this type of abatement does not reduce the coefficient gross emissions/output, but the coefficient net emissions/output, since $e_i(t)/q_i(t) = 1 - (a_i(Y_i(t))/q_i(t))$. Thus, for a given value of the output, the higher the abatement, the lower the ratio $e_i(t)/q_i(t)$.

[10] This approach was first adopted by Poyago-Theotoky (2007) in a static model, and more recently has been used by Menezes and Pereira (2017) and Martín-Herrán and Rubio (2018b) in a dynamic context.

where δ_y stands for the depreciation rate of the abatement capacity. The focus of the paper is on a stock pollutant that evolves according to the following differential equation

$$\dot{x}(t) = \sum_{i=1}^{n} \left(q_i(t) - y_i(t) - \beta \sum_{j \neq i}^{n} y_j(t) \right) - \delta_x x(t), \quad x(0) = x_0 \geq 0, \tag{2}$$

where $x(t)$ stands for the pollution stock and $\delta_x > 0$ for the decay rate of pollution stock. Environmental damages are given by $D(x(t))$ with $D_x > 0$ and $D_{xx} \geq 0$.

The differential equation describing the dynamics of the pollution stock can be equivalently rewritten as follows

$$\dot{x}(t) = \sum_{i=1}^{n} q_i(t) - (1 + \beta(n-1)) \sum_{i=1}^{n} y_i(t) - \delta_x x(t), \quad x(0) = x_0 \geq 0. \tag{3}$$

The objective of firms is to choose output and R&D investment in order to maximize the discount present value of net profits given by the following expression

$$\max_{q_i(t), w_i(t)} \int_0^{\infty} e^{-rt} \left\{ P(Q(t))q_i(t) - cq_i(t) - IC(w_i(t)) \right.$$
$$\left. - \tau(t) \left(q_i(t) - y_i(t) - \beta \sum_{j \neq i}^{n} y_j(t) \right) \right\} dt, \tag{4}$$

subject to differential Eqs. (1) and (3), initial conditions and the usual non-negativity constraints where r is the time discount rate.

On the other hand, the regulator chooses the emission tax with the aim of maximizing net social welfare given by the sum of consumer surplus and firms' net profits plus tax revenues minus environmental damages. As firms' tax expenses and regulator tax revenues cancel out, the dynamic optimization problem for the regulator can be written as follows:

$$\max_{\tau(t)} \int_0^{\infty} e^{-rt} \left\{ \int_0^{Q(t)} P(z(t))dz(t) - cQ(t) - \sum_{i=1}^{n} IC(w_i(t)) - D(x(t)) \right\} dt. \tag{5}$$

The regulator also solves this problem subject to differential Eqs. (1) and (3), initial conditions and the usual non-negativity constraints.

Thus, the optimal tax rate is defined by the solution to a dynamic policy game given by (4) and (5) and differential Eqs. (1) and (3).

3 The Feedback Stackelberg Equilibrium

In this paper, we are interested in characterizing a time-consistent tax. For this reason, we assume that the regulator cannot commit in the *long run* to future taxes and compute a subgame perfect Stackelberg equilibrium in Markov strategies using dynamic programming. However, it is straightforward from expression (5) that if the regulator cannot commit in the *short run*, its capacity for affecting the firms' decisions vanishes. For this reason, we assume that in each period a three-stage game is played where firstly the regulator chooses the tax, in the second stage the firms select the level of R&D investment and finally decide the output to put in the market. Nevertheless, we consider that firms are forward looking players that are aware of the dynamic strategic interdependence with the regulator, since they realize that their current decisions will influence the regulator's decisions because their current decisions on output and investment affect the dynamics of the pollution stock. In this case, the output selection must satisfy the following Hamilton-Jacobi-Bellman (HJB) equation:[11]

$$rV_i(x,\bar{y}) = \max_{q_i} \left\{ P(Q)q_i - cq_i - IC(w_i) - \tau \left(q_i - y_i - \beta \sum_{j \neq i}^{n} y_j \right) \right.$$
$$\left. + \frac{\partial V_i}{\partial x} \left(\sum_{i=1}^{n} q_i - (1+\beta(n-1)) \sum_{i=1}^{n} y_i - \delta_x x \right) \right\},$$

where $\bar{y} = (y_1, ..., y_n)$ with $i = 1, 2, ..., n$ and $V_i(x, \bar{y})$ stands for the maximum discounted present value of net profits for the current values of the pollution stock and abatement capacities.

From the first-order condition (FOC) for the maximization of the right-hand side (RHS) of the HJB equation, we obtain that

$$P'(Q)q_i + P(Q) = c + \tau - \frac{\partial V_i}{\partial x}, \tag{6}$$

where the left-hand side (LHS) represents the marginal revenue of the firm and the RHS the marginal costs. These costs are formed by the marginal cost of production, the tax

[11]Time argument will be eliminated when no confusion arises. As the dynamics of the abatement capacity does not depend on output, we omit them from the HJB equation at this stage.

and the firm's shadow price of the pollution stock. The latter is given by the reduction in the discounted present value of the firm's net profits because of the increase in the pollution stock produced by the increase in production. Adding condition (6) for the number of firms we obtain an expression that implicitly defines the dependence of total output with respect to the tax and state variables.

$$P'(Q)\sum_{i=1}^{n} q_i + nP(Q) = n(c+\tau) - \sum_{i=1}^{n}\frac{\partial V_i}{\partial x},$$

$$P'(Q)Q + nP(Q) = n(c+\tau) - \sum_{i=1}^{n}\frac{\partial V_i}{\partial x}. \tag{7}$$

From this expression we have that

$$\frac{\partial Q}{\partial \tau} = \frac{n}{P''(Q)Q + (n+1)P'(Q)}, \tag{8}$$

so that $P'' \leq 0$ is a sufficient condition to obtain that an increase in the tax reduces the output of the industry for given values of the state variables. Condition (7) implicitly defines $Q(\tau, x, \bar{y})$ so that using (6) we can write the firms' output as a function of the tax and state variables.

$$q_i(\tau, x, \bar{y}) = \frac{1}{P'(Q(\tau, x, \bar{y}))}\left(c + \tau - \frac{\partial V_i}{\partial x} - P(Q(\tau, x, \bar{y}))\right). \tag{9}$$

Once we recognize the dependence of the output with respect to the tax, we can calculate the optimal tax as the solution to the following differential game:

$$\max_{w_i} \int_0^{\infty} e^{-rt}\left\{P(Q(\tau, x, \bar{y}))q_i(\tau, x, \bar{y}) - cq_i(\tau, x, \bar{y}) - IC(w_i)\right.$$
$$\left. -\tau\left(q_i(\tau, x, \bar{y}) - y_i - \beta\sum_{j\neq i}^{n} y_j\right)\right\} dt, \tag{10}$$

$$\max_{\tau} \int_0^{\infty} e^{-rt}\left\{\int_0^{Q(\tau,x,\bar{y})} P(z)dz - cQ(\tau, x, \bar{y}) - \sum_{i=1}^{n} IC(w_i) - D(x)\right\} dt, \tag{11}$$

subject to differential equations:

$$\dot{x} = Q(\tau, x, \bar{y}) - (1 + \beta(n-1))\sum_{i=1}^{n} y_i - \delta_x x, \; x(0) = x_0 \geq 0,$$
$$\dot{y}_i = w_i - \delta_y y_i, \quad y_i(0) = y_0 \geq 0, \quad i = 1, \ldots, n.$$

Now, expression (11) clearly shows the dependence of net social welfare on the tax because the regulator can influence the market equilibrium by imposing a tax on emissions.

Next, we derive the Markov perfect Nash equilibrium (MPNE) of the differential game defined by (10) and (11). Denoting by $W(x, \bar{y})$ the regulator's value function, the following HJB equation has to be satisfied:

$$rW(x,\bar{y}) = \max_{\tau} \left\{ \int_0^{Q(\tau,x,\bar{y})} P(z)dz - cQ(\tau,x,\bar{y}) - \sum_{i=1}^{n} IC(w_i) - D(x) \right.$$

$$\left. + \frac{\partial W}{\partial x} \left(Q(\tau,x,\bar{y}) - (1+\beta(n-1)) \sum_{i=1}^{n} y_i - \delta_x x \right) + \sum_{k=1}^{n} \frac{\partial W}{\partial y_k}(w_k - \delta_y y_k) \right\}. \tag{12}$$

The FOC yields

$$\left(P(Q(\tau,x,\bar{y}) - c + \frac{\partial W}{\partial x} \right) \frac{\partial Q}{\partial \tau} = 0.$$

As $\partial Q/\partial \tau \neq 0$, this condition requires that

$$P(Q(\tau,x,\bar{y}) = c - \frac{\partial W}{\partial x}. \tag{13}$$

Thus, the price must be equal to marginal costs that now include the marginal cost of production and the *social* shadow price of the pollution stock.

On the other hand, the optimal R&D investment must satisfy the following HJB equation:

$$rV_i(x,\bar{y}) = \max_{w_i} \{ P(Q(\tau,x,\bar{y}))q_i(\tau,x,\bar{y}) - cq_i(\tau,x,\bar{y}) - IC(w_i)$$

$$-\tau \left(q_i(\tau,x,\bar{y}) - y_i - \beta \sum_{j \neq i} y_j \right)$$

$$\left. + \frac{\partial V_i}{\partial x} \left(Q(\tau,x,\bar{y}) - (1+\beta(n-1)) \sum_{i=1}^{n} y_i - \delta_x x \right) + \sum_{k=1}^{n} \frac{\partial V_i}{\partial y_k}(w_k - \delta_y y_k) \right\}. \tag{14}$$

As the output does not depend on the R&D investment, we obtain the following condition for the maximization of the RHS of the HJB equation

$$IC'(w_i) = \frac{\partial V_i}{\partial y_i}, \quad i = 1, 2, \ldots, n. \tag{15}$$

The LHS of the condition stands for marginal investment cost. The RHS stands for the marginal benefit that is defined by the increase in the discounted present value of net profits coming from the increase in the abatement capacity. These two conditions (13) and (15) implicitly defines the optimal feedback strategies for the tax and investment that characterize the MPNE. At this point, we should point out that according to conditions (13) and (15), the optimal tax does not depend on firm's investment and the optimal investment does not depend either on the tax. This observation allows us to claim that

Proposition 1 *The feedback Stackelberg equilibrium (FSE) of the differential game defined by (10) and (11) coincides with the MPNE.*

The FSE allows the leader to enjoy of a stagewise first-mover advantage that can be interpreted as the ability to commit in the short run. In this case, backward induction is applied to derive the equilibrium, substituting the follower's reaction function in the leader's HJB equation, and computing the optimal strategy for the leader maximizing the RHS of the equation. However, the reaction functions defined by conditions (13) and (15) are orthogonal and, therefore, directly give the optimal strategies for the tax and R&D investment. Thus, the backward induction procedure does not provide any difference with respect to the optimal strategies that are given by the MPNE regardless of whether the regulator moves first or after firms decide on investment. [12]

From the previous proposition it is straightforward to conclude that

Corollary 1 *The FSE when the regulator is the leader of the game is time consistent in the short run and in the long run.*

The short-run consistency occurs just because of the orthogonality of the reaction functions defined by (13) and (15). Suppose the timing of the game is changed and the regulator selects the tax after firms have taken their decision on the R&D investment level. But as the tax does not depend on the R&D investment, firms cannot influence the decision of the regulator even when they decide first and the result is that we would obtain the same tax as when the regulator moves first. In the long run, the tax is also time consistent because we compute a subgame perfect equilibrium in Markov strategies.

[12] In Rubio (2006) the conditions that explain this coincidence in economic applications of differential games are studied.

Observe that when the regulator is the leader of the differential game (10)-(11) is as well the leader of the complete differential game presented in the previous section and consequently Corollary 1 also applies in this case.

Finally, we characterize the tax of the FSE. Using (7) and (13) we can derive the following expression

$$\tau = \frac{1}{n}\frac{P}{\xi} - \left(\frac{\partial W}{\partial x} - \frac{1}{n}\sum_{i=1}^{n}\frac{\partial V_i}{\partial x}\right), \tag{16}$$

where ξ is the price elasticity of the demand curve. This tax corrects two market distortions. One caused by the market power of firms and the other by a negative externality. The first term of the RHS of (16) corrects the first distortion and consequently is a subsidy, the term is negative. As is well known we find that the lower the elasticity the higher the subsidy. The second term, that is equal to the difference between the *social* shadow price of the pollution stock and the addition of the *private* shadow price of the pollution stock, corrects the negative externality and is expected to be positive. Unfortunately, at this point we cannot advance in the analysis of the tax terms and their signs without giving more structure to our model because at this level of generality the shadow prices of the pollution stock are given by unknown value function derivatives. For this reason, in the next section, we investigate this issue addressing a linear-state policy game where environmental damages are linear.

4 The Linear-State (LS) Policy Game

The LS differential game we study in this section considers an oligopoly that faces a linear (inverse) demand function given by $P = a - Q$ with $a > c$, and operates with a quadratic investment cost function $IC(w_i) = \gamma w_i^2/2$ with $\gamma > 0$. Moreover, the environmental damages are given by the linear function $D(x) = dx$ with $d > 0$.

For this specification of the policy game, the FOC (6) reads

$$a - Q_{-i} - 2q_i = c + \tau - \frac{\partial V_i}{\partial x}, \quad i = 1, 2, ..., n,$$

where $Q_{-i} = \sum_{j \neq i}^{n} q_i$ and the LHS is the marginal revenue of the firm for a linear demand function. Using this condition we can write the aggregate and individual outputs as a function of the tax

$$Q(\tau, x, \bar{y}) = \frac{1}{n+1}\left(n(s-\tau)+\sum_{i=1}^{n}\frac{\partial V_i}{\partial x}\right), \quad \frac{\partial Q}{\partial \tau} = -\frac{n}{n+1}, \tag{17}$$

$$q_i(\tau, x, \bar{y}) = \frac{1}{n+1}\left(s-\tau-\sum_{k=1}^{n}\frac{\partial V_k}{\partial x}\right)+\frac{\partial V_i}{\partial x}, \tag{18}$$

where $s = a - c$.

On the other hand, FOC (13) yields

$$Q(\tau, x, \bar{y}) = s + \frac{\partial W}{\partial x}, \tag{19}$$

so that (17) and (19) allow us to derive the feedback strategy for the tax

$$\tau(x, \bar{y}) = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{\partial V_i}{\partial x} - (n+1)\frac{\partial W}{\partial x} - s\right), \tag{20}$$

and by substitution in (18) the feedback strategy for the firms' output

$$q_i(x, \bar{y}) = \frac{1}{n}\left(s + \frac{\partial W}{\partial x} - \sum_{k=1}^{n}\frac{\partial V_k}{\partial x}\right) + \frac{\partial V_i}{\partial x}. \tag{21}$$

Finally, FOC (15) gives us directly the feedback strategy for the R&D investment

$$w_i(x, \bar{y}) = \frac{1}{\gamma}\frac{\partial V_i}{\partial y_i}. \tag{22}$$

Now, for calculating the value functions we have to substitute the feedback strategies in the HJB equations and solve them. Substituting the aggregate output and the investment in the regulator's HJB equation and rearranging terms we obtain the following partial differential equation:

$$rW(x, \bar{y}) = \frac{1}{2}\left(s + \frac{\partial W}{\partial x}\right)^2 - \frac{1}{2\gamma}\sum_{i=1}^{n}\left(\frac{\partial V_i}{\partial y_i}\right)^2 - dx$$

$$-\frac{\partial W}{\partial x}\left((1+\beta(n-1))\sum_{i=1}^{n} y_i + \delta_x x\right) + \sum_{k=1}^{n}\frac{\partial W}{\partial y_k}\left(\frac{1}{\gamma}\frac{\partial V_k}{\partial y_k} - \delta_y y_k\right). \tag{23}$$

Now, substituting the aggregate and individual outputs, the tax and the investment in the firms' HJB equation yields the following differential equation:

$$rV_i(x, \bar{y}) = -\frac{\partial W}{\partial x}\left(\frac{1}{n}\left(s + \frac{\partial W}{\partial x} - \sum_{i=1}^{n}\frac{\partial V_i}{\partial x}\right) + \frac{\partial V_i}{\partial x}\right) - \frac{1}{2\gamma}\left(\frac{\partial V_i}{\partial y_i}\right)^2$$

$$-\frac{1}{n}\left(\sum_{i=1}^{n}\frac{\partial V_i}{\partial x}-(n+1)\frac{\partial W}{\partial x}-s\right)\left(\frac{1}{n}\left(s+\frac{\partial W}{\partial x}-\sum_{i=1}^{n}\frac{\partial V_i}{\partial x}\right)+\frac{\partial V_i}{\partial x}-y_i-\beta\sum_{j\neq i}^{n}y_j\right)$$

$$+\frac{\partial V_i}{\partial x}\left(s+\frac{\partial W}{\partial x}-(1+\beta(n-1))\sum_{i=1}^{n}y_i-\delta_x x\right)+\sum_{k=1}^{n}\frac{\partial V_i}{\partial y_k}\left(\frac{1}{\gamma}\frac{\partial V_k}{\partial y_k}-\delta_y y_k\right). \tag{24}$$

For solving this pair of equations, we conjecture linear representations of the value functions[13]

$$W(x,\bar{y})=A_w x+\sum_{i=1}^{n}B_w^i y_i+C_w,\quad V_i(x,\bar{y})=A_i x+B_i^i y_i+\sum_{k\neq i}^{n}B_i^k y_k+C_i, \tag{25}$$

that gives $\partial W/\partial x = A_w$, $\partial W/\partial y_i = B_w^i$, $\partial V_i/\partial x = A_i$, $\partial V_i/\partial y_i = B_i^i$, $\partial V_i/\partial y_k = B_i^k$ with A_w, B_w^i, A_i, B_i^i and B_i^k unknowns to be calculated.

Substituting W, $\partial W/\partial x$, $\partial W/\partial y_i$ and $\partial V_i/\partial y_i$ into Eq. (23) and V_i, $\partial V_i/\partial x$, $\partial W/\partial x$, $\partial V_i/\partial y_i$ and $\partial V_i/\partial y_k$ into Eq. (24) and equalizing the coefficients of variables x, y_i and y_k we obtain a unique solution for the coefficients of the values functions[14]

$$A_w=-\frac{d}{r+\delta_x},\quad B_w^i=\frac{d(1+\beta(n-1))}{(r+\delta_x)(r+\delta_y)},$$

$$A_i=0,\quad B_i^i=\frac{(n+1)d-s(r+\delta_x)}{n(r+\delta_x)(r+\delta_y)},\quad B_i^k=\frac{\beta((n+1)d-s(r+\delta_x))}{n(r+\delta_x)(r+\delta_y)}.$$

From this solution we can conclude that

Proposition 2 *The optimal strategies for the production and R&D investment are*

$$q_i^*=\frac{1}{n}\left(s-\frac{d}{r+\delta_x}\right),\quad w_i^*=\frac{d(n+1)-s(r+\delta_x)}{n\gamma(r+\delta_x)(r+\delta_y)}. \tag{26}$$

The two variables satisfy the non-negativity constraint provided that

$$d\in\left[\frac{s(r+\delta_x)}{n+1},\ s(r+\delta_x)\right]. \tag{27}$$

Observe that if d is large enough, it does not make sense to produce the good from an economic perspective because the environmental damages are extremely huge. Instead, if d is too low, it is not profitable to invest in abatement capacity because the marginal

[13] Where the subscripts w and i stand for the regulator and firms, respectively.

[14] Details for this calculation are given in Appendix A. We omit coefficients C_w and C_i because they do not affect the results obtained in this section.

benefit of abatement capacity for firms, $\partial V_i/\partial y_i = B_i^i$, is negative. Moreover, it is easy to check that an increase in marginal damages decreases production and increases R&D investment. On the other hand, we see that spillovers have no influence in the optimal strategies of production and R&D investment. Spillovers affect emissions and consequently will influence the dynamics of the pollution stock as we will see below, but they do not affect the firms' decisions on output and R&D investment. Notice that, although $\partial V_i/\partial y_k = B_k^i$ depends on the degree of spillovers so that they will affect the discount present value of firms' net profits, the marginal benefit of own abatement capacity given by $\partial V_i/\partial y_i = B_i^i$ does not depend on β and consequently spillovers do no influence the decision of firms on R&D investment. Finally, we investigate the effect of competition on production and R&D investment. The effect on output is clear, competition decrease the production of firms. However, the output of the industry is constant. This result is explained by condition (13) that requires for the maximization of net social welfare that the price be equal to the marginal cost of production plus the social shadow price of the pollution stock, but as this is constant, the price of the good must be constant and consequently the output of the industry. The effect of competition on R&D investment is not so obvious, but taking the derivative of w_i^* with respect to n we obtain the following expression

$$\frac{\partial w_i^*}{\partial n} = \frac{s(r+\delta_x) - d}{n^2\gamma(r+\delta_x)(r+\delta_y)}, \tag{28}$$

that is positive if condition (27) is satisfied. We highlight this result with the following proposition

Proposition 3 *If the firms invest in abatement capacity, the investment increases with competition.*

This result adheres to the hypothesis that competition promotes innovation, in our model, green innovation. If more firms compete in quantities in a polluting oligopoly the result is that firms end investing more in abatement capacity.

Finally, using (20) we calculate the emission tax.

Proposition 4 *The optimal emission tax is*

$$\tau^* = -\frac{1}{n}\left(s - \frac{d}{r+\delta_x}\right) + \frac{d}{r+\delta_x} = -q_i^* - \frac{\partial W}{\partial x}, \tag{29}$$

which is positive if condition (27) holds.

According to expression (16), the first term of the tax should be $P/n\xi$. It is easy to check that if the demand function is $Q = a - P$, $P/n\xi = -q_i$. The second term is the difference between the social shadow price of the pollution stock, $-\partial W/\partial x = -A_w = d/(r+\delta_x)$, that is positive, and the average of the private shadow prices $\partial V_i/\partial x = A_i$, that for constant marginal damages is zero. Thus, the second term is positive and the sign of the tax depends on the severity of environmental damages. If these damages are large enough as justify a positive R&D investment, the tax is positive and increasing with marginal damages and competition. In any case, it should be highlighted that the tax is lower than the social shadow price of the pollution stock, just because the tax also corrects the distortion caused by the firms' market power.

We would also like to point out that firms do not associate any price to the pollution stock what means that the optimal decision on output is given by the maximization of current net profits, i.e. by the static Cournot equilibrium. Thus, for constant marginal damages the Stackelberg equilibrium can be also computed assuming that firms myopically select the output level in the third stage and that they choose the investment in the second stage for a given constant tax, and that finally the regulator selects the optimal level of the tax. According to this procedure the firms' value functions are calculated in the second stage assuming a constant tax rate. In this case, at the second stage the optimal strategy of the investment depends explicitly on the tax. However, the same value functions and strategies than those derived in this section are obtained once the optimal tax is substituted in the optimal strategy of the investment calculated at the second stage. In fact, using (29) we can retrieve the optimal strategy of the investment depending on the tax substituting in (26). Notice that the tax can be rewritten as follows

$$\tau^* = \frac{(n+1)d - s(r+\delta_x)}{n(r+\delta_x)},$$

so that $(n+1)d - s(r+\delta_x) = \tau^* n(r+\delta_x)$. The substituting the numerator of w_i^* in (26) by $\tau^* n(r+\delta_x)$ we obtain the optimal strategy of the investment depending on the tax:

$$w_i^* = \frac{\tau^*}{\gamma(r+\delta_y)}. \tag{30}$$

Thus, the higher the tax, the higher the firms' R&D investment and as the tax increases with competition, we can conclude that more competition leads to more investment. This expression clarifies that competition influences investment through the effect that competition has on the optimal tax.

We end the study of the policy game analyzing the dynamics of the state variables that is given by the following system of differential equations:

$$\dot{x} = \frac{(r+\delta_x)s - d}{r+\delta_x} - n(1+\beta(n-1))y - \delta_x x, \;\; x(0) = x_0 \geq 0, \tag{31}$$

$$\dot{y} = \frac{d(n+1) - s(r+\delta_x)}{\gamma n(r+\delta_y)(r+\delta_x)} - \delta_y y, \;\; y(0) = y_0 \geq 0. \tag{32}$$

The steady state of the system is:

$$x_{ss}^* = \frac{1}{\delta_x(r+\delta_x)}\left(\frac{(\delta_y\gamma(r+\delta_y) + 1 + \beta(n-1))(r+\delta_x)s}{\delta_y\gamma(r+\delta_y)} - \frac{((1+\beta(n-1))(n+1) + \delta_y\gamma(r+\delta_y))d}{\delta_y\gamma(r+\delta_y)}\right), \tag{33}$$

$$y_{ss}^* = \frac{d(n+1) - s(r+\delta_x)}{\delta_y n\gamma(r+\delta_x)(r+\delta_y)}, \tag{34}$$

and for industry emissions

$$E_{ss}^* = Q^* - n\left(1+\beta(n-1)\right) y_{ss}^* = x_{ss}^*\delta_x, \tag{35}$$

where

$$Q^* = \frac{(r+\delta_x)s - d}{r+\delta_x}.$$

The steady-state values are non-negative for values of d in the following interval

$$d \in \left[\frac{s(r+\delta_x)}{n+1}, \; \frac{(\delta_y\gamma(r+\delta_y) + 1 + \beta(n-1))(r+\delta_x)s}{(1+\beta(n-1))(n+1) + \delta_y\gamma(r+\delta_y)}\right], \tag{36}$$

where the upper bound of this interval is lower than the upper bound of the interval in (27).

It can be seen that the steady-state abatement capacity does not depend on spillovers. However, it increases with the number of firms in the industry. Notice that $\partial y^*_{ss}/\partial n = (1/\delta_x)\partial w^*/\partial n$ and we have established the R&D investment increases with competition. It is also straightforward from (35) that steady-state emissions decrease with the spillovers and the number of firms in the industry. Notice that gross emissions, that depends on the industry output, are independent of the number of firms and spillovers and that the steady-state abatement capacity increases with competition. Thus an augmentation in the degree of spillovers increases abatement and also an increase in competition, the result is a decrease in net emissions, and consequently, a reduction in the steady-state pollution stock.

Although our LS policy game yields constant values for the control variables, the net emissions, the abatement capacity and the pollution stock evolve in time. In order to know how these variables evolve over time and in particular, the type of stability of its steady state, we evaluate the trace and determinant of the following 2 x 2 Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} -\delta_x & -n(1+\beta(n-1)) \\ 0 & -\delta_y \end{bmatrix}.$$

The trace is $\Upsilon(J) = -(\delta_x + \delta_y) < 0$ while the determinant is $\triangle(J) = \delta_x \delta_y > 0$. Therefore, the steady state equilibrium is a *global asymptotically stable point* and we can conclude that

Proposition 5 *The system of differential equations for the stock of pollution and abatement capacity has a unique positive steady state provided that the marginal damages d belongs to the interior of interval (36). The steady state is a stable node and is globally stable, i.e. the market converges asymptotically to the steady-state abatement capacity and pollution stock from any initial condition. Moreover, an increase in competition increases the steady-state abatement capacity and reduces the steady-state pollution stock.*

Finally, we solve the system of differential equation describing the dynamics of the pollution stock and the abatement capital stock.

The differential equation of the abatement capital stock can be solved independently of the equation of the pollution stock. The solution to equation (32) reads:

$$y(t) = y^*_{ss}\left(1 - e^{-\delta_y t}\right) + y_0 e^{-\delta_y t}, \tag{37}$$

where y^*_{ss} is the the steady-state value of the abatement capital stock given in (34).

The solution to equation (19) reads[15]:

- If $\delta_x \neq \delta_y$, then

$$x(t) = x^*_{ss} + (x_0 - x^*_{ss})\, e^{-\delta_x t} + \frac{n\,(1+\beta(n-1))\,(y^*_{ss} - y_0)}{\delta_x - \delta_y}(e^{-\delta_y t} - e^{-\delta_x t}). \tag{38}$$

- If $\delta_x = \delta_y = \delta$, then

$$x(t) = x^*_{ss} + (x_0 - x^*_{ss})\, e^{-\delta t} + n\,(1 + \beta(n-1))\,(y^*_{ss} - y_0) t e^{-\delta t}. \tag{39}$$

where x^*_{ss} is the steady-state value of the pollution stock given in (33).

The dynamics of the abatement capacity depends on the initial value of this stock. It increases (decreases) if the initial abatement capacity is lower (larger) than its steady state value. However, the dynamics of the pollution stock is more complex and depends not only on the initial value, but also on the relationship between the depreciation rate of abatement capacity and the natural rate of decay of the pollution stock. In Appendix C, the reader can be found a detailed analysis of the dynamics of the model. Here, we focus on a particular case that we think that is the more interesting. First, we suppose that the initial value of the abatement capacity is zero consistently with the idea that firms do not invest in R&D if no tax is applied on emissions. Second, we suppose that the initial value of the pollution stock is larger than its steady-state value. If this is not the case, the optimal policy will lead to an accumulation of emissions. A case that it does not seem very interesting. For $x_0 > x^*_{ss}$ and $y_0 = 0$, (38) for the general case $\delta_x \neq \delta_y$ simplifies to yield

$$x(t) = x^*_{ss} + (x_0 - x^*_{ss})\, e^{-\delta_x t} + \frac{n\,(1+\beta(n-1))\,y^*_{ss}}{\delta_x - \delta_y}(e^{-\delta_y t} - e^{-\delta_x t}),$$

[15] The details of the computation are presented in Appendix B.

and the first derivative with respect to time is

$$\frac{\partial x}{\partial t} = \left(-(x_0 - x_{ss}^*) + \frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\right)\delta_x e^{-\delta_x t} - \frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\delta_y e^{-\delta_y t}, \tag{40}$$

where the first term of the first parenthesis on the RHS is negative and the sign of the other terms depends on the relationship between δ_x and δ_y. For $\delta_x > \delta_y$, $x(t)$ will present an extreme if the following condition is satisfied

$$\frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\delta_x e^{-\delta_x t} = \frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\delta_y e^{-\delta_y t} + (x_0 - x_{ss}^*)\delta_x e^{-\delta_x t}.$$

This condition can be rewritten as follows

$$\frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\delta_x = \frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\delta_y e^{(\delta_x - \delta_y)t} + (x_0 - x_{ss}^*)\delta_x \tag{41}$$

where the LHS is constant with respect to time and the RHS is a strictly convex increasing function. Thus, if (41) has a solution it must be satisfied that

$$\frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\delta_x > \frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\delta_y + (x_0 - x_{ss}^*)\delta_x,$$

i.e. the LHS of (41) must be higher than the value of the RHS for $t = 0$. This condition requires that

$$x_0 < x_{ss}^* + \frac{1}{\delta_x} n\,(1+\beta(n-1))\,y_{ss}^*,$$

that using (35) yields

$$x_0 < \frac{1}{\delta_x} Q^* = \frac{(r + \delta_x)s - d}{\delta_x(r + \delta_x)}. \tag{42}$$

Thus, we can conclude that if this condition is satisfied, the pollution stock increases until it reaches a maximum for decreasing afterwards. On the contrary, if x_0 is higher than this upper bound, the pollution stock decreases for all $t \geq 0$. When $\delta_x < \delta_y$, there could be a maximum too if the following condition is satisfied for a finite value of t.

$$-\frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\delta_y e^{-\delta_y t} = \left(-(x_0 - x_{ss}^*) + \frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\right)\delta_x e^{-\delta_x t},$$

This condition can be rewritten as follows

$$-\frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\delta_y = \left(x_0 - x_{ss}^* - \frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}\right)\delta_x e^{(\delta_y - \delta_x)t}, \tag{43}$$

where the LHS is constant with respect to time and the RHS is a strictly convex increasing function. Thus, if (43) has a solution it must be satisfied that

$$-\frac{n\,(1+\beta(n-1))\,y^*_{ss}}{\delta_x-\delta_y}\delta_y > \left(x_0-x^*_{ss}-\frac{n\,(1+\beta(n-1))\,y^*_{ss}}{\delta_x-\delta_y}\right)\delta_x,$$

i.e. the LHS of (41) must be higher than the value of the RHS for $t=0$. From where, we obtain the same condition that the one derived for $\delta_x > \delta_y$. Finally, we analyze the case with $\delta_x = \delta_y$. If these two parameters are identical, the derivative of the pollution stock with respect to time is

$$\frac{\partial x}{\partial t} = (-\delta\,(x_0 - x^*_{ss}) + n\,(1+\beta(n-1))\,y^*_{ss}(1-\delta t))e^{-\delta t},$$

where the first term between parenthesis is negative and the sign of the second term depends on t. In this case, this derivative will be zero for a finite time provided that

$$x_0 < x^*_{ss} + \frac{1}{\delta}n\,(1+\beta(n-1))\,y^*_{ss},$$

that is the same condition obtained for the other two cases. All these results are summarized in the following proposition:

Proposition 6 *When $x_0 > x^*_{ss}$ and $y_0 = 0$, the abatement capacity increases and the pollution stock decreases if the initial stock of pollution is higher than the threshold value defined by (42). However, when the initial pollution stock is lower, it first increases until it reaches a maximum for decreasing afterwards.*

This analysis shows that for the initial conditions we consider the reaction of firms to the tax is to build an abatement capacity that reduces industry emissions. However, even if the initial values of the pollution stock is larger than the steady-state value the stock of pollution could increase, but only during an initial period of time. In the long run, the pollution stock will decrease.

Finally, we evaluate the effect of spillovers and competition on the optimal temporal paths of abatement capacity and pollution stock. As was established above spillovers do not affect the R&D investment and consequently they do not have any effect on the optimal path of abatement capacity. However, we have seen that competition increases the steady-state abatement capacity so that according to (37) competition increases abatement capacity at any time. The effects on the pollution stock are not so straightforward are the effects on abatement capacity are. To evaluate these effects we rewrite (38) as follows

$$x(t) = x_0 e^{-\delta_x t} + x_{ss}^*(1 - e^{-\delta_x t}) + \frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}(e^{-\delta_y t} - e^{-\delta_x t}).$$

Now, using (35) we can eliminate x_{ss}^*

$$x(t) = x_0 e^{-\delta_x t} + \frac{1}{\delta_x}(Q^* - n\,(1 + \beta(n-1))\,y_{ss}^*)(1 - e^{-\delta_x t})$$
$$+\frac{n\,(1+\beta(n-1))\,y_{ss}^*}{\delta_x - \delta_y}(e^{-\delta_y t} - e^{-\delta_x t}),$$

where Q^* does not depend on either β or n. Taking common factor, we obtain the following expression for

$$x(t) = x_0 e^{-\delta_x t} + \frac{1}{\delta_x}Q^*(1 - e^{-\delta_x t})$$
$$+n\,(1 + \beta(n-1))\,y_{ss}^*\left(\frac{e^{-\delta_y t} - e^{-\delta_x t}}{\delta_x - \delta_y} - \frac{1}{\delta_x}(1 - e^{-\delta_x t})\right). \tag{44}$$

Then, as $\partial y_{ss}^*/\partial\beta = 0$ and $\partial y_{ss}^*/\partial n > 0$, the signs of $\partial x(t)/\partial\beta$ and $\partial x(t)/\partial n$ will depend on the sign of

$$\frac{e^{-\delta_y t} - e^{-\delta_x t}}{\delta_x - \delta_y} - \frac{1}{\delta_x}(1 - e^{-\delta_x t}). \tag{45}$$

In Appendix D it is showed that this expression is negative for all $t > 0$ and we can conclude that $\partial x(t)/\partial\beta$ and $\partial x(t)/\partial n$ are negative for the general case $\delta_x \neq \delta_y$.

When $\delta_x = \delta_y = \delta$, for $x_0 > x_{ss}^*$ and $y_0 = 0$, (39) simplifies to yield

$$x(t) = x_{ss}^* + (x_0 - x_{ss}^*)\,e^{-\delta t} + n\,(1 + \beta(n-1))\,y_{ss}^* t e^{-\delta t}.$$

Proceeding as in the general case, eliminating x_{ss}^* using (35), we have:

$$x(t) = x_0 e^{-\delta t} + \frac{1}{\delta}Q^*(1 - e^{-\delta t}) - n(1 + \beta(n-1))y_{ss}^*\left(t + \frac{1}{\delta}\right)e^{-\delta t},$$

and we obtain that also in case $\delta_x = \delta_y = \delta$, at any time, $\partial x/\partial\beta$ and $\partial x/\partial n$ are negative.

These results are summarized in the last proposition of the paper:

Proposition 7 *When* $x_0 > x_{ss}^*$ *and* $y_0 = 0$*, the abatement capacity increases with competition for all* $t > 0$*. However, the stock of pollution decreases. On the other hand, an increase in the degree of spillover does not affect the abatement capacity, but reduces the pollution stock.*

This proposition says us that the effects that spillovers and competition have on the steady-state values of the abatement capacity and pollution stock also apply at any time. Thus, the model presents some general results on the effects of spillovers and competition on the state variables that are not constrained to the steady state values.

5 Conclusions

In this paper a dynamic Cournot oligopoly is used with the aim of studying the relationship between competition and green innovation. The intensity of competition is approached by the number of firms in the industry. The firms face a tax on emissions and react to this tax investing in R&D to reduce the emissions per unit of output (green innovation). R&D accumulates and determines the abatement capacity of firms. The optimal tax rate is given by the feedback Stackelberg equilibrium of a dynamic policy game between a regulator and a polluting oligopoly for a stock pollutant. We compute the tax for a linear-state policy game. Our analysis shows that firms' R&D investment increases with competition. This effect occurs because the optimal tax increases with competition. In a polluting oligopoly the tax is lower than the difference between the social shadow price of the pollution stock and its private shadow price because the tax has to correct also the market distortion caused by the market power of the firms. But as the competition increases, this distortion weakens and the tax increases to reflect the difference between the social shadow price of the pollution stock and its private shadow price. Thus, more competition implies a higher abatement capacity, lower emissions and finally a lower pollution stock. This effect does not depend on the degree of spillovers since the tax does not depend on spillovers. In our model, spillovers are associated to the abatement capital and consequently they do not affect the decisions on output and R&D investment, but only to the dynamics of the pollution stock. However, spillovers reduce net emission and we find that the higher the spillovers, the lower the pollution stock. Our model also shows that the steady state is a global asymptotically stable point with a dynamics for the state variables that depends on the initial conditions. For the abatement capacity we find that is monotonically increasing if it is assumed that its initial value is zero. An assumption that seems consistent with the idea that firms' only invest in R&D if emissions are taxed. We also find that the pollution stock could increase even if its initial value is higher than the steady-state pollution stock. Nevertheless, in the long run, the pollution stock will decrease.

Our paper assumes that firms adopt an end-of-the-pipe abatement technology that yields a net emission function that is additively separable in production (gross emissions) and abatement. This specification facilities the analytical computation of the solutions, but restricts the analysis to a particular form of green innovation. In order to extend the analysis performed in this paper, we could consider that firms invest in a cleaner technology. This type of innovations can be modelled by a reduction of the technological coefficient emissions/production.[16] Another feature of this paper is that environmental policy is based on the use of only one policy instrument, the tax on emissions, when there are several distortions that affect the market allocation. There is a negative externality due to pollution, but also a positive externality due to spillovers. Moreover, firms have market power. It could be very interesting to characterize the first-best policy based on a combination of different policy instruments to assess how the environmental policy could affect the relationship between competition and green innovation. The recent paper by Aghion et al. (2023) highlights the effects of consumers' environmental concerns on innovation, and also the paper by Langinier and Chaudhuri (2020) analyzes innovation with green consumers. In this line, we could extend our dynamic model to incorporate

[16] Two recent papers addressing this issue are Walter (2018) and Langinier and Chaudhuri (2020). In both papers, the R&D investment reduces the coefficient emissions/production. Walter (2018) studies in a dynamic setting the effects of an emission tax on the coefficient emissions/production depending on the degree of cooperation when firms invest in R&D. Langinier and Chaudhuri (2020) investigate in a static setting the effect of patent policies and emission taxes on green innovation, and on the emission level in the presence of green consumers.

the pollution stock to the utility function of consumers. In this case, the willingness to pay for the good would depend on the level of the pollution stock and firms would have an incentive to invest in R&D even if no tax is charged on emissions. Finally, we have concentrated on a model where the intensity of competition is given by the number of firms in the industry. Thus, it would be also interesting to look at an oligopoly with product differentiation to consider other types of intensity in competition as the variation in the degree of product substitutability.

Appendix

A Calculating the Coefficients of the Value Functions

Using the value functions

$$W(x,\bar{y}) = A_w x + \sum_{i=1}^{n} B_w^i y_i + C_w, \quad V_i(x,\bar{y}) = A_i x + B_i^i y_i + \sum_{k\neq i}^{n} B_i^k y_k + C_i,$$

we can rewrite the HJB equations (23) and (24) as follows

$$r\left(A_w x + \sum_{i=1}^{n} B_w^i y_i + C_w\right) = \frac{1}{2}(s+A_w)^2 - \frac{1}{2\gamma}\sum_{i=1}^{n}(B_i^i)^2 - dx$$

$$-A_w\left((1+\beta(n-1))\sum_{i=1}^{n} y_i + \delta_x x\right) + \sum_{k=1}^{n} B_w^k\left(\frac{1}{\gamma}B_k^k - \delta_y y_k\right), \quad (46)$$

$$r\left(A_i x + B_i^i y_i + \sum_{k\neq i}^{n} B_i^k y_k + C_i\right) = -A_w\left(\frac{1}{n}\left(s + A_w - \sum_{i=1}^{n} A_i\right) + A_i\right) - \frac{1}{2\gamma}\left(B_i^i\right)^2$$

$$-\frac{1}{n}\left(\sum_{i=1}^{n} A_i - (n+1)A_w - s\right)\left(\frac{1}{n}\left(s + A_w - \sum_{i=1}^{n} A_i\right) + A_i - y_i - \beta\sum_{j\neq i}^{n} y_j\right)$$

$$+A_i\left(s + A_w - (1+\beta(n-1))\sum_{i=1}^{n} y_i - \delta_x x\right) + \sum_{k=1}^{n} B_i^k\left(\frac{1}{\gamma}B_k^k - \delta_y y_k\right). \quad (47)$$

At this point, we look for a symmetric solution taking into account that the cross effects of the y variables in the value function of firms are not identical to the own effects. Notice that if we look at differential equation (3) that describes the dynamics of the pollution stock and the differential equation showing the dynamics of the abatement capacity of each firm, we see that the effect of control variables of each firm on the dynamics of the state variables is completely symmetric for all firms. However, from the expression of firm i's current net profits

$$\pi_i = (a - Q)q_i - cq_i - \frac{\gamma}{2}w_i^2 - \tau\left(q_i - y_i - \beta\sum_{j\neq i}^{n} y_j\right)$$

we realize that the effect of y_i is different from the effect of y_j with $j \neq i$.

Therefore, we cannot assume that the value functions of all firms are identical, but we can assume the following symmetric properties: A_i takes the same value for all firms, B_i^i also takes the same value for all firms, and the same occurs for B_i^k. In this case, the HJB equation (47) yields

$$\begin{aligned} r\left(A_i x + B_i^i y_i + (n-1)B_i^k y_k + C_i\right) &= -A_w\left(\frac{1}{n}\left(s + A_w - nA_i\right) + A_i\right) - \frac{1}{2\gamma}\left(B_i^i\right)^2 \\ -\frac{1}{n}\left(nA_i - (n+1)A_w - s\right)&\left(\frac{1}{n}\left(s + A_w - nA_i\right) + A_i - y_i - \beta(n-1)y_k\right) \\ &+A_i\left(s + A_w - \left(1 + \beta(n-1)\right)y_i - \left(1 + \beta(n-1)\right)y_i - \delta_x x\right) \\ &+B_i^i\left(\frac{1}{\gamma}B_i^i - \delta_y y_i\right) + (n-1)B_i^k\left(\frac{1}{\gamma}B_k^k - \delta_y y_k\right). \end{aligned} \quad (48)$$

Now grouping the coefficients of x, we obtain that $(r + \delta)A_i x = 0$ which implies that $A_i = 0$. This allows us to simplify the expression (48)

$$\begin{aligned} r\left(B_i^i y_i + (n-1)B_i^k y_k + C_i\right) &= -A_w\frac{1}{n}\left(s + A_w\right) - \frac{1}{2\gamma}\left(B_i^i\right)^2 \\ &+\frac{1}{n}\left((n+1)A_w + s\right)\left(\frac{1}{n}\left(s + A_w\right) - y_i - \beta(n-1)y_k\right) \\ &+B_i^i\left(\frac{1}{\gamma}B_i^i - \delta_y y_i\right) + (n-1)B_i^k\left(\frac{1}{\gamma}B_k^k - \delta_y y_k\right). \end{aligned} \quad (49)$$

Grouping terms in y_i gives

$$(rB_i^i + \frac{1}{n}((n+1)A_w + s) + B_i^i\delta_y)y_i = 0,$$

that allows us to calculate

$$B_i^i = -\frac{(n+1)A_w + s}{n(r + \delta_y)}. \quad (50)$$

Finally, grouping terms in y_k gives

$$(n-1)(rB_i^k + \frac{1}{n}((n+1)A_w + s)\beta + B_i^k \delta_y) y_k = 0,$$

from where we obtain that $B_i^k = \beta B_i^i$.

Now, because of the symmetric role of all y variables in the regulator's problem, and focussing on symmetric solutions we assume that B_w^i are identical for all y variables, and (46) simplifies as follows

$$r\left(A_w x + nB_w^i y_i + C_w\right) = \frac{1}{2}(s + A_w)^2 - \frac{1}{2\gamma} n(B_i^i)^2 - dx$$
$$-A_w\left((1+\beta(n-1))\, y_i + \delta_x x\right) + nB_w^i \left(\frac{1}{\gamma} B_i^i - \delta_y y_i\right). \tag{51}$$

From this expression we can derive the following condition for A_w

$$((r+\delta_x)A_w + d)x = 0,$$

so that the value of A_w is

$$A_w = -\frac{d}{r+\delta_x}. \tag{52}$$

We can also obtain a condition for B_w^i

$$n((r+\delta_y)B_w^i + A_w(1+\beta(n-1)))y_i = 0,$$

that using (52) yields

$$B_w^i = \frac{d(1+\beta(n-1))}{(r+\delta_x)(r+\delta_y)}. \tag{53}$$

Finally, substituting A_w in (50) we obtain the coefficient

$$B_i^i = \frac{(n+1)d - s(r+\delta_x)}{n(r+\delta_x)(r+\delta_y)}. \tag{54}$$

The coefficient C_i and C_w can be also calculated using (49) and (51), but as the optimal strategies do not depend on these coefficient in order to save space we will omit these calculations.

B Solution to the differential equation describing the dynamics of the pollution stock

The dynamics of the pollution stock in (31) once the solution of the abatement capital stock given in (37) is substituted reads:

$$\dot{x} + \delta_x x = \frac{(r+\delta_x)s - d}{r+\delta_x} - n(1+\beta(n-1))(y^*_{ss}\left(1-e^{-\delta_y t}\right) + y_0 e^{-\delta_y t}). \tag{55}$$

First, we solve the homogeneous first order linear equation $\dot{x} + \delta_x x = 0$. The general solution to this equation reads: $x^h(t) = Ce^{-\delta_x t}$, with C a constant of integration. Second, we postulate a particular solution of the non-homogenous equation as follows: $x^{nh}(t) = A + Be^{-\delta_y t}$ if $\delta_x \neq \delta_y$ and $x^{nh}(t) = A + Bte^{-\delta t}$ if $\delta_x = \delta_y = \delta$.

Substituting in the differential equation (55) if $\delta_x \neq \delta_y$ we get:

$$-\delta_y Be^{-\delta_y t} + \delta_x(A + Be^{-\delta_y t}) = \frac{(r+\delta_x)s - d}{r+\delta_x} - n(1+\beta(n-1))(y^*_{ss}\left(1-e^{-\delta_y t}\right) + y_0 e^{-\delta_y t}).$$

Identifying terms:

$$\begin{aligned} -\delta_y B + \delta_x B &= -n(1+\beta(n-1))(-y^*_{ss} + y_0), \\ \delta_x A &= \frac{(r+\delta_x)s - d}{r+\delta_x} - n(1+\beta(n-1))y^*_{ss}. \end{aligned}$$

Hence,

$$\begin{aligned} A &= \frac{1}{\delta_x}\left[\frac{(r+\delta_x)s - d}{r+\delta_x} - n(1+\beta(n-1))y^*_{ss}\right], \\ B &= \frac{n\left(1+\beta(n-1)\right)\left(y^*_{ss} - y_0\right)}{\delta_x - \delta_y}. \end{aligned} \tag{56}$$

Simplifying the expression of A we can easily show that A coincides with the steady-state value of the pollution stock, x^*_{ss}, given in (33). The solution to equation (55) is given by $x(t) = x^h(t) + x^{nh}(t) = Ce^{-\delta_x t} + x^*_{ss} + Be^{-\delta_y t}$, with B given in (56). From the initial condition $x(0) = x_0$, we determine the constant of integration C which is given by $C = x_0 - x^*_{ss} - B$.

Substituting in the differential equation (55) if $\delta_x = \delta_y = \delta$ we get:

$$(1-\delta t)Be^{-\delta t} + \delta(A + Bte^{-\delta t}) = \frac{(r+\delta_x)s - d}{r+\delta_x} - n(1+\beta(n-1))(y^*_{ss}\left(1-e^{-\delta_y t}\right) + y_0 e^{-\delta_y t}).$$

Identifying terms:

$$
\begin{aligned}
B &= n\left(1+\beta(n-1)\right)\left(y_{ss}^{*}-y_0\right), \\
\delta A &= -n(1+\beta(n-1))y_{ss}^{*}+\frac{(r+\delta_x)s-d}{r+\delta_x}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
A &= \frac{1}{\delta}\left[-n(1+\beta(n-1))y_{ss}^{*}+\frac{(r+\delta_x)s-d}{r+\delta_x}\right]=x_{ss}^{*}, \\
B &= n(1+\beta(n-1))(y^{*}-y_0).
\end{aligned} \tag{57}
$$

Again, the solution to equation (55) is given by $x(t)=x^{h}(t)+x^{nh}(t)=Ce^{-\delta t}+x_{ss}^{*}+Bte^{-\delta t}$, with B given in (57). From the initial condition $x(0)=x_0$, we have $C=x_0-x_{ss}^{*}$.

C The dynamics of the state variables

In order to evaluate the dynamics of the pollution stock, we calculate the first derivative of (38) that can be rearranged as follows

$$
\dot{x}(t)=(x_{ss}^{*}-x_0)\delta_x e^{-\delta_x t}-\frac{n(1+\beta(n-1))(y_0-y_{SS}^{*})}{\delta_x-\delta_y}\left(\frac{\delta_x}{e^{\delta_x t}}-\frac{\delta_y}{e^{\delta_y t}}\right). \tag{58}
$$

To evaluate the sign of this derivative and to know whether the stock is increasing or decreasing, we begin studying whether $\dot{x}(t)=0$ has a solution. For $\dot{x}(t)=0,$ expression (58) yields

$$
(x_{ss}^{*}-x_0)\delta_x=\frac{n(1+\beta(n-1))(y_0-y_{ss}^{*})}{\delta_x-\delta_y}\left(\delta_x-\delta_y e^{(\delta_x-\delta_y)t}\right). \tag{59}
$$

On the left-hand side we have a constant and on the right-hand side a function of t with the following features: initial value, $n(1+\beta(n-1))(y_0-y_{ss}^{*})$; first derivative, $-n(1+\beta(n-1))(y_0-y_{ss}^{*})\delta_y e^{(\delta_x-\delta_y)t}$; and second derivative, $-n(1+\beta(n-1))(y_0-y_{ss}^{*})\delta_y(\delta_x-\delta_y)e^{(\delta_x-\delta_y)t}$. Thus, the existence of a solution for this equation depends on the sign of the differences: $x_{ss}^{*}-x_0,\ y_0-y_{ss}^{*}$ and $\delta_x-\delta_y$.

We initiate the analysis considering $\delta_x>\delta_y$ and $y_0<y_{ss}^{*}$. In this case, the function on the right-hand side of (59) is an increasing convex function with a negative initial value, and consequently equation (59) has a unique positive solution provided that $(x_{ss}^{*}-x_0)\delta_x>$

$n(1+\beta(n-1))(y_0 - y^*_{ss})$. This condition is satisfied if $(y^*_{ss} - \delta_x(x_0 - x^*_{ss}))/(n(1+\beta(n-1))) < y_0$, i.e. if vector (x_0, y_0) is below isocline $\dot{x} = 0$. Notice that isocline $\dot{x} = 0$ is a straight line with slope $-\delta_x/(n(1+\beta(n-1)))$, so that we can write it as follows $y - y^*_{ss} = -\delta_x/(n(1+\beta(n-1)))(x - x^*_{ss})$. Thus, we can conclude that if (x_0, y_0) is below isoclines $\dot{x} = \dot{y} = 0$, the stock of pollution first increases until line $\dot{x} = 0$ is reached and decreases afterwards. However, if (x_0, y_0) is below isocline $\dot{y} = 0$ but above isocline $\dot{x} = 0$, the pollution stock is a monotone decreasing function of time. Next, we suppose that $y_0 > y^*_{SS}$. In this case, the function on the right-hand side of (59) is a decreasing concave function with a positive initial value, and then equation (59) has a unique positive solution provided that $(x^*_{SS} - x_0)\delta_x < /(n(1+\beta(n-1)))(y_0 - y^*_{ss})$, i.e. if vector (x_0, y_0) is above isocline $\dot{x} = 0$. This implies that if (x_0, y_0) is above isoclines $\dot{x} = \dot{y} = 0$, the stock of pollution first decreases until line $\dot{x} = 0$ is reached and increases afterwards. But in the case that (x_0, y_0) is below isocline $\dot{x} = 0$ and above isocline $\dot{y} = 0$, the pollution stock is a monotone increasing function of time.

We continue the analysis considering $\delta_x < \delta_y$ and $y_0 < y^*_{ss}$. In this case, the right-hand side of (59) is an increasing concave function of time with a negative initial value that converges to the following value

$$\lim_{t\to+\infty} \frac{(y_0 - y^*_{ss})(n(1+\beta(n-1)))}{\delta_x - \delta_y}\left(\delta_x - \delta_y e^{(\delta_x - \delta_y)t}\right) = \frac{y_0 - y^*_{ss}}{\delta_x - \delta_y}\delta_x n(1+\beta(n-1)) > 0.$$

In this case, equation (59) has a unique positive solution if

$$(y_0 - y^*_{ss})n(1+\beta(n-1)) < (x^*_{ss} - x_0)\delta_x < \frac{y_0 - y^*_{ss}}{\delta_x - \delta_y}n(1+\beta(n-1))\delta_x. \tag{60}$$

This condition implies, on the one hand, that $y_0 < y^*_{ss} - \delta_x(x_0 - x^*_{ss})/(n(1+\beta(n-1)))$ and, on the other hand, that $y_0 < n(1+\beta(n-1))y^*_{ss} + (x_0 - x^*_{ss})(\delta_y - \delta_x)$ that requires that (x_0, y_0) is below isocline $\dot{x} = 0$ and below the straight line that passes through the stationary point (x^*_{ss}, y^*_{ss}) and has as direction vector the eigenvector $(1, (\delta_y - \delta_x)/(n(1+\beta(n-1))))$. If this is the case, the pollution stock first increases until isocline $\dot{x} = 0$ is reached and decreases afterwards. When this condition is not satisfied because the initial point is above isocline $\dot{x} = 0$, the pollution stock is a monotone decreasing function; while if it is not satisfied because the initial point is above the line $y - y^*_{ss} = (\delta_y - \delta_x)/(n(1+\beta(n-1)))(x - x^*_{ss})$, the pollution stock is a monotone increasing function.

Next, we suppose that $y_0 > y_{ss}^*$. When the initial value of the abatement capacity is larger than its steady-state value, the right-hand side of (59) is a decreasing convex function of time with a positive initial value that converges to the negative value: $n(1+\beta(n-1))(y_0-y_{ss}^*)\delta_x/(\delta_x-\delta_y)$. Now, equation (59) has a unique positive solution provided that

$$n(1+\beta(n-1))\frac{y_0-y_{ss}^*}{\delta_x-\delta_y}\delta_x < (x_{ss}^*-x_0)\delta_x < n(1+\beta(n-1))(y_0-y_{ss}^*).$$

This condition holds now if (x_0, y_0) is above isocline $\dot{x}=0$ and line $y-y_{ss}^*=(\delta_y-\delta_x)/(n(1+\beta(n-1)))(x-x_{ss}^*)$. If this is the case, the pollution stock first decreases until isocline $\dot{x}=0$ is reached and increases afterwards. When this condition does not hold, two possibilities arise. In the first possibility the initial point is below isocline $\dot{x}=0$, and then the pollution stock is a monotone increasing function of time. In the second possibility the initial point is below the line with direction vector $(1,(\delta_y-\delta_x)/(n(1+\beta(n-1))))$, and in this case the stock of pollution is a monotone decreasing function of time.

Finally, we address the case $\delta_x=\delta_y$. When the decay rate of the pollution stock is equal to the depreciation rate of the abatement capacity the two roots of the characteristic equation (δ_x,δ_y) are identical and equal to δ. Then, there are no changes in the solution of differential equation (32) that can be written as follows

$$y(t)=(y_0-y_{ss}^*)e^{-\delta t}+y_{ss}^*.$$

However, now the solution for the differential equation of the pollution stock reads

$$x(t)=x_0e^{-\delta t}+(1-e^{-\delta t})x_{ss}^*-n(1+\beta(n-1))(y_0-y_{ss}^*)te^{-\delta t}.$$

For this solution the first derivative is

$$\dot{x}(t)=e^{-\delta t}\left((x_{ss}^*-x_0)\delta-n(1+\beta(n-1))(y_0-y_{ss}^*)(1-t\delta)\right),$$

so that equation $\dot{x}(t)=0$ has a positive solution provided that the following expression is positive

$$t=\frac{1}{\delta}+\frac{x_0-x_{ss}^*}{(y_0-y_{ss}^*)n(1+\beta(n-1))}. \tag{61}$$

It is obvious that this expression is positive if $x_0 > x_{ss}^*$ and $y_0 > y_{ss}^*$ or if $x_0 < x_{ss}^*$ and $y_0 < y_{ss}^*$. When $x_0 > x_{ss}^*$ and $y_0 < y_{ss}^*$, (61) is positive if $y_0 < y_{ss}^*-\delta(x_0-x_{ss}^*)$, i.e. if the initial point is below isocline $\dot{x} = 0$. When $x_0 < x_{ss}^*$ and $y_0 > y_{ss}^*$, (61) is positive if $y_0 > y_{ss}^* - \delta(x_0 - x_{ss}^*)/(n(1 + \beta(n-1)))$, i.e. if the initial point is above isocline $\dot{x} = 0$. Thus, if the initial point is below (above) isoclines $\dot{x} = \dot{y} = 0$, the pollution stock first increases (decreases) until isocline $\dot{x} = 0$ is reached and then begins to decrease (increase). In the other cases, the pollution stock increases (decreases) if the initial stock is below (above) its steady-state value.

D Determination of the sign of (45)

Next we analyze function

$$f(t) = \frac{e^{-\delta_y t} - e^{-\delta_x t}}{\delta_x - \delta_y} - \frac{1}{\delta_x}(1 - e^{-\delta_x t}).$$

Function $f(t)$ is exclusively zero at $t = 0$ and takes a negative value if $t > 0$.

It is clear that $f(0) = 0$ and to show that $f(t) < 0$ if $t > 0$, we prove that $f(t)$ is a strictly decreasing function for any $t > 0$.

The derivative of function $f(t)$ reads:

$$f'(t) = \frac{1}{\delta_x - \delta_y}\left(-\delta_y e^{-\delta_y t} + \delta_x e^{-\delta_x t}\right) - e^{-\delta_x t}.$$

Then,

$$f'(t) < 0 \Leftrightarrow \frac{1}{\delta_x - \delta_y}\left(-\delta_y e^{-\delta_y t} + \delta_x e^{-\delta_x t}\right) - e^{-\delta_x t} < 0.$$

Multiplying by $e^{\delta_x t}$, we have

$$f'(t) < 0 \Leftrightarrow \frac{1}{\delta_x - \delta_y}\left(-\delta_y e^{(\delta_x - \delta_y)t} + \delta_x\right) - 1 < 0 \Leftrightarrow \frac{1}{\delta_x - \delta_y}\left(-\delta_y e^{(\delta_x - \delta_y)t} + \delta_x\right) < 1.$$

Therefore, if $\delta_x > \delta_y$, then

$$f'(t) < 0 \Leftrightarrow -\delta_y e^{(\delta_x - \delta_y)t} + \delta_x < \delta_x - \delta_y \Leftrightarrow -\delta_y e^{(\delta_x - \delta_y)t} < -\delta_y \Leftrightarrow e^{(\delta_x - \delta_y)t} > 1.$$

Last inequality always applies under assumptions $\delta_x > \delta_y$ and $t > 0$.

If $\delta_x < \delta_y$, then

$$f'(t) < 0 \Leftrightarrow -\delta_y e^{(\delta_x - \delta_y)t} + \delta_x > \delta_x - \delta_y \Leftrightarrow -\delta_y e^{(\delta_x - \delta_y)t} > -\delta_y \Leftrightarrow e^{(\delta_x - \delta_y)t} < 1.$$

Last inequality always applies under assumptions $\delta_x < \delta_y$ and $t > 0$

Consequently, we have proved that $f'(t) < 0$ for any $t > 0$, regardless of how δ_x and δ_y compares, and hence, $f(t) < 0$ for any $t > 0$.

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