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Expido la presente certificación, a instancia del interesado, en Pamplona a 15 de noviembre de 2022.

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Multiplexed network formation and Bonacich centrality*

Diego San Román[§]

Abstract

People share relationships that encompass interactions of different natures. These are called *multiplexed*. We study a non-cooperative game in which individuals participate in multiple activities and form activity-specific links in an endogenously formed multiplexed network. We explain why multiplexed relationships arise. Differently from previous literature, outcome of the game and Bonacich centrality of agents are determined simultaneously. We generalize Bonacich centrality to weighted and multiplexed networks, and show that it directly depends on the multiplexity of the network. We also provide a new condition which ensures that the Bonacich centrality takes a finite value in weighted and multiplexed networks.

1 Introduction

People share relationships that encompass interactions of different natures. They interact in multiple areas of life, such as lending money or exchanging information. This multifaceted aspect of relationships has been empirically documented and

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Figure 1a: Simplex relationship.

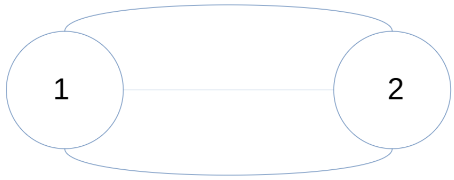

Figure 1b: Multiplexed relationship.

shown to be an important feature in network formation (Banerjee et al., 2021). Among the many models that aim to explain network formation, most consider only one type of relationship between individuals. Relationships in which individuals interact in multiple areas, as the ones documented in Banerjee et al. (2021), have been been mostly studied in other fields such as sociology (Corominas-Murtra et al., 2013), physics (Domenico et al., 2013) or computer science (Cai et al., 2005). The literature in these fields has defined the relationships in which individuals interact in only one area as *simplex* and the ones in which individuals interact in more than one area as *multiplexed*. A graphical representation of a simplex and a multiplexed relationship is shown in Figures 1a and 1b, respectively.

In the current paper, we aim to give a rationale to multiplexed networks. In order to do so, we study a game where individuals allocate a limited amount of time between multiple types of activities and form a weighted link in each of these activities. Weights of links are determined by the equilibrium values of the game, so that the network is formed endogenously. Individuals have heterogeneous preferences over activities, and activities are costly. Their motivation to participate in activities is both explained by their intrinsic preference for the activities and for the enjoyment brought by meeting and spending time with other individuals.

We base our work on the models of Chen et al. (2018) and Belhaj and Deroïan (2014), who also study games in which individuals can participate in different activities. Differently from previous literature, we study endogenous network formation and agents allocate exactly one unit of time between multiple activities. Chen et al. (2018) present a model in which individuals can participate in multiple activities, but the time agents can allocate to activities is not bounded. In reality, time is a finite resource and there are opportunity costs. Belhaj and Deroïan (2014) present a model in which individuals have one unit of time to allocate, but in which individuals can only participate in two activities. In our framework, agents can participate in more than two activities and have one unit of time to allocate between activities.

We define multiplexity of a relationship as the number of areas the two individuals interact in, and show that the enjoyment brought by meeting with other individuals plays an important role in the formation of multiplexed networks. Because we study a weighted network, we define a second measure of multiplexity which takes into account the intensity of relationships, and provide conditions which give rise to highly multiplexed network, as defined by this second measure. We furthermore find that pairs of agents spend more time practicing the same activities as costs of performing activities increases, due to convexity in costs.

Another difference with respect to Chen et al. (2018) and Belhaj and Deroïan (2014) is that these papers study an exogenous network in which equilibrium values are shaped by the structure of the network. In the model of this work, equilibrium values and structure of the network are determined simultaneously. In finite population non-cooperative games with linear-quadratic utilities where the network is exogenous, as the models of Chen et al. (2018) and Belhaj and Deroïan (2014), it has been shown by Ballester et al. (2006) that actions taken by individuals in equilibrium are defined by their Bonacich centrality. Because we study a framework in which equilibrium values and structure of the network are determined simultaneously, equilibrium values of agents are not defined by their Bonacich centrality. We find that Bonacich centrality itself is determined by the multiplexity of the network. Opsahl et al. (2010) generalize degree, closeness and betweenness centralities to weighted graphs, but do not do so for eigenvector centralities. Domenico et al. (2013) propose an eigenvector centrality for multiplexed weighted networks, in which the same importance is given to direct neighbors than to more distant neighbors, which is against the spirit of Bonacich centrality which discounts the importance of connections as the distance of connections increases. We generalize Bonacich centrality to weighted and multiplexed networks. We also find that the assumption required for the Bonacich centrality to take a finite value in weighted networks corresponds with the one used in the literature only when all links of the network take the same value. We provide a new condition which ensures that the

Bonacich centrality takes a finite value in weighted networks.

Our paper fits more broadly in the literature of friendship formation (Currarini et al., 2009, 2010, 2016), and relates to more recent works which study network multiplexity (Kobayashi and Onaga, 2022; Cheng et al., 2021).

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 presents the results on the formation of multiplexed networks. Section 4 presents the results on Bonacich centrality in weighted and multiplexed networks. Section 5 concludes.

2 The model

We consider a set $\mathcal{N} = \{1, ..., N\}$ of agents who have access to a set $\mathcal{K} = \{1, ..., K\}$ of activities, where $N, K \geq 2$. Each agent $i \in \mathcal{N}$ spends an amount of time $t_{i,k}$ performing activity $k \in \mathcal{K}$, and has an exogenously given preference parameter $a_{i,k} \geq 0$ for activity k. The preference parameter measures how much agent i enjoys performing activity k. Large values of $a_{i,k}$ indicate a high enjoyment of agent i for activity k, whereas low values of $a_{i,k}$ indicate a low enjoyment. The preference parameter only captures the enjoyment of an agent for an activity, say sport, and does not capture the enjoyment brought by spending time with other individuals who also perform the activity.

The utility brought to agent $i \in \mathcal{N}$ by spending time with individual $j \in \mathcal{N}$ in group k, where $j \neq i$, is captured by the product $t_{i,k}t_{j,k}$, where $t_{i,k}, t_{j,k} \geq 0$ are endogenously determined. Agents i and j will largely benefit from their interaction in group k if they both spend a large amount of time performing activity k, whereas they will have a low benefit from their interaction if at least either i or j spends a small amount of time performing activity k. Agents gain utility by interacting with all the other agents of set $\mathcal{N}$, across each of the activities of set $\mathcal{K}$.

While participating in activities brings utility, it also requires effort. These costs of effort are measured by parameter $c > 0$. We represent the loss of utility brought to agent i by costs of performing activity k as convex in the time agent i spends in performing activity k. With such convex costs, agents have a decreasing incentive in spending an additional amount of time in performing activity k as the amount of time they perform activity k increases.

To summarize, there are three channels which determine the utility of agents in set $\mathcal{N}$. First, agents derive utility by performing activities they enjoy. Second, they derive utility by spending time with other individuals who are performing the same activities. Third, they lose utility by performing effort-demanding activities. These forces are represented in the utility of agent $i \in \mathcal{N}$, which is written in (1):

$$U_i(t_{i,1}, ..., t_{i,k}, ..., t_{i,K}) = \sum_{k=1}^{K} a_{i,k} t_{i,k} + \sum_{k=1}^{K} \sum_{j \neq i} t_{i,k} t_{j,k} - c \sum_{k=1}^{K} t_{i,k}^2 \tag{1}$$

where $U_i : [0,1]^K \Rightarrow \mathbb{R}$ is a continuous and differentiable function in $\mathbb{R}$. We only consider agents who are interested in participating in at least one activity, i.e., $\sum_{k=1}^{K} a_{i,k} > 0$ for all $i \in \mathcal{N}$. We denote by $\{t_{i,1}^*, ..., t_{i,k}^*, ..., t_{i,K}^*\}$ the set of equilibrium values, which is the set of values $\{t_{i,1}, ..., t_{i,k}, ..., t_{i,K}\}$ which maximizes the utility of all agents $i \in \mathcal{N}$. The set $\bigcup_{i=1}^{N} \{t_{i,1}^*, ..., t_{i,k}^*, ..., t_{i,K}^*\}$ is the Nash equilibrium of the game, state in which no agent i has a profitable unilateral deviation from her equilibrium value. If we let $t'_{i,k}$ be any value of $t_{i,k}$, and $\boldsymbol{t^*_{-i,k}}$ be the set of equilibrium values of agents other than i in activity k, then the Nash equilibrium is such that, for all agents i in $\mathcal{N}$ and all activities k in $\mathcal{K}$, $U(t^*_{i,k}, \boldsymbol{t^*_{-i,k}}) \geq U(t'_{i,k}, \boldsymbol{t^*_{-i,k}})$ for all $t'_{i,k} \in [0,1]$.

The equilibrium values give rise to a weighted, undirected network in which agents are linked to each other in a maximum of K activities, in which the weight of the link between agents i and j in activity k, denoted by w_{ij}^k, takes value defined as:

$$w_{ij}^k = t_{i,k}^* t_{j,k}^* \tag{2}$$

We consider that a link in activity k is formed between agents m and n when $w_{mn}^k > 0$. A walk from node i to node j is a sequence of players $\{i, i+1, ..., j-1, j\}$ and links $\{w_{i,i+1,}^k, ..., w_{j-1,j}^k\}$ such that $w_{mn}^k > 0$ for all $m \in \{i, i+1, ..., j-1\}$ and $n = m+1$, where $k \in \mathcal{K}$. It is worth noting that walks can be a sequence in which not all nodes are distinct, and can be a sequence of nodes connected through links in different activities. The length of a walk equals the number of nodes in the sequence minus 1. A component of a network is a set $\mathcal{C}$ of nodes such that there exists a walk from any node $i \in \mathcal{C}$ to any node $j \in \mathcal{C}$. For any walk $\{i, i+1, ..., j-1, j\}$ from node i to node j, we define the last link of the walk as the link connecting nodes $j-1$ and j, which we denote by $w_{j-1,j}$. We define $\mathcal{W}_{ij}$ as the set of links between nodes i and j such that $w_{ij}^k > 0$, and $W_{ij} =| \mathcal{W}_{ij} |$ as its cardinality. We define $w_{min|ij} = \min\{\mathcal{W}_{ij}\}$ as the weakest relationship between agents i and j, and $w_{max|ij} = \max\{\mathcal{W}_{ij}\}$ as the strongest relationship between agents i and j. We define $\mathcal{W}$ as the set of links in the network, and $w_{min} = \min\{\mathcal{W}\}$ as the weakest relationship of the network. We define the *strength* s_i of node i as $s_i = \sum_{j \neq i} w_{ij}$ which is a generalization of the notion of degree in weighted networks introduced by Barrat et al. (2004). We define $\mathcal{S}$ as the set of strengths of the network, and $s_{max} = \max\{\mathcal{S}\}$ as the strength of the network with the largest value.

We give an example of a network composed of 3 agents (denoted by 1, 2 and 3) and 3 types of relationships (denoted by R, S and T). The parameters of preferences are defined as:

Preference parameters			
	Agent 1	Agent 2	Agent 3
Group R	$a_{1,R} = 0.8$	$a_{2,R} = 0.5$	$a_{3,R} = 0.4$
Group S	$a_{1,S} = 0.1$	$a_{2,S} = 0.4$	$a_{3,S} = 0.3$
Group T	$a_{1,T} = 0.1$	$a_{2,T} = 0.1$	$a_{3,T} = 0.3$

Table 1: Preference parameters

We set the costs of performing activities to $c_R = c_S = c_T = \frac{3}{2}$.
The resulting equilibrium values are given in Table 2.

Equilibrium values			
	Agent 1	Agent 2	Agent 3
Group R	$t^*_{1,R} = 0.625$	$t^*_{2,R} = 0.55$	$t^*_{3,R} = 0.525$
Group S	$t^*_{1,S} = 0.225$	$t^*_{2,S} = 0.3$	$t^*_{3,S} = 0.275$
Group T	$t^*_{1,T} = 0.15$	$t^*_{2,T} = 0.15$	$t^*_{3,T} = 0.2$

Table 2: Equilibrium values

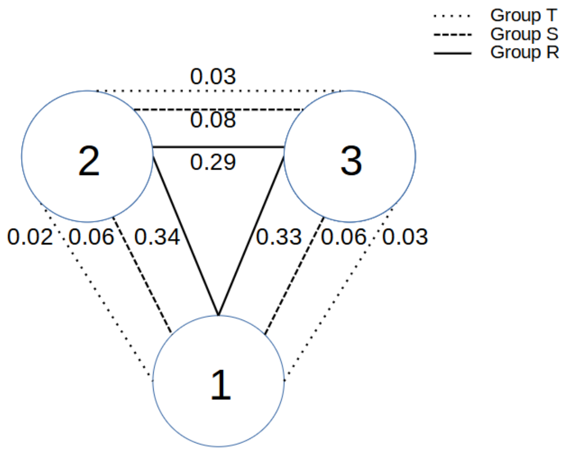

Figure 2: Weighted multiplexed network.

Given the parameter values, the subsequent equilibrium values, and the link formation process given in (2), links' weights take the values presented in Table 3 and the network represented in Figure 2 arises.

Weights of links			
	w_{12}	w_{13}	w_{23}
Group R	$w_{12}^R = 0.34$	$w_{13}^R = 0.33$	$w_{23}^R = 0.29$
Group S	$w_{12}^S = 0.06$	$w_{13}^S = 0.06$	$w_{23}^S = 0.08$
Group T	$w_{12}^T = 0.02$	$w_{13}^T = 0.03$	$w_{23}^T = 0.03$

Table 3: Weights of links

3 Multiplexed network formation

Each agent $i \in \mathcal{N}$ solves the maximization problem $\mathcal{P}$.

$$\mathcal{P}\left\{ \max \quad U_i(t_{i,1},...,t_{i,k},...,t_{i,K}) = \sum_{k=1}^{K} a_{i,k}t_{i,k} + \sum_{k=1}^{K}\sum_{j\neq i} t_{i,k}t_{j,k} - c\sum_{k=1}^{K} t_{i,k}^2. \right.$$

Whether individuals share one or multiple links depends on their equilibrium values, as indicated in (2). Their equilibrium values are given by the system of best-response functions defined below in (3). The optimal amount of time agent i spends in group k is increasing in her preference for group k and the amount of time agents other than her spend in group k, and decreasing in the cost of spending time in groups.

$$\begin{cases} t_{1,k} = \frac{a_{1,k}+\sum_{j\neq 1} t_{j,k}}{2c}, \\ \dots \\ t_{i,k} = \frac{a_{i,k}+\sum_{j\neq i} t_{j,k}}{2c}, \\ \dots \\ t_{N,k} = \frac{a_{N,k}+\sum_{j\neq N} t_{j,k}}{2c}. \end{cases} \tag{3}$$

In order to find the Nash equilibrium, both Belhaj and Deroïan (2014) and Chen et al. (2018) use utility functions similar to (1). In Belhaj and Deroïan (2014), the time agents spend in activities sums up to one unit of time and agents can participate in two activities. The amount of time an agent spends in an activity is computed, and then the time they spend in the other activity is deduced as the amount of time they have not spent in the first activity. Equilibrium values of time spent in one activity are re-scaled to 1 if their value exceeds 1 and to 0 if their value is negative. In Chen et al. (2018), agents can participate in more than two activities and have an unbounded amount of time to allocate between activities. Conditions on the heterogeneity of preference parameters and on the degree of interdependence between activities are provided for the equilibrium values to take values in the interval $[0, +\infty[$.

In the framework we present, agents have one unit of time to allocate between activities, as in Belhaj and Deroïan (2014), can participate in multiple activities, as in Chen et al. (2018), and equilibrium values lie in the interval $[0, 1]$. This framework allows us to work with the —non-negative— endogenous variable time as a finite resource, while keeping the flexibility of studying participation of agents in multiple groups.

Theorem 1. *The game admits a set of equilibrium values* $\{t^*_{1,k}, ..., t^*_{N,k}\}$ *such that* $t^*_{i,k} \in [0,1]$ *and* $\sum_{k=1}^{K} t^*_{i,k} = 1$ *for all* $i \in \mathcal{N}$ *and all* $k \in \mathcal{K}$ *if and only if* $\varphi_i(\sum_{k=1}^{K} a_{i,k}) = \eta(2c - (N-1))$ *for all* $i \in \mathcal{N}$*, for any functions* $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $\eta : \mathbb{R}^+ \to \mathbb{R}^+$*.*

We can take any values of $\sum_{k=1}^{K} a_{i,k}$ and $2c-(N-1)$ and transform them through functions φ_i and η so that agents have one unit of time to allocate between activities and spend non-negative amounts of time in activities. There is no restriction in how each parameter is transformed.[1]

We say that individuals share a multiplexed relationship if they form a link in two or more activities. We define two types of multiplexity, which are *extensive* and *intensive* multiplexity. Extensive multiplexity is an intuitive definition of multiplexity since it simply counts the different types of connections between two individuals. Formally, the extensive multiplexity of a relationship is defined as:

Definition 1 (Extensive multiplexity)**.** *The extensive multiplexity* MX^E_{ij} *of a relationship between agents* i *and* j *is defined as* $MX^E_{ij} = W_{ij}$*.*

We consider that a network fulfills extensive multiplexity if there exists two agents i and j such that $MX^E_{ij} \geq 2$. A multiplexed network of this kind arises if and only if there exists two activities that are enjoyed by at least one agent each. Because agents enjoy meeting other agents, it suffices that an agent enjoys performing an activity for her to participate in that activity, leading all other agents to also spend time in that activity, even though some may not enjoy the activity.

[1]Suppose that $K = 2$, where $a_{i,1} = 1$ and $a_{i,2} = 1$, so that $\sum_{k=1}^{K} a_{i,k} = 2$. Also suppose that $\eta(2c - (N-1)) = 4$. For agent i to have one unit of time to allocate between activities and spend non-negative amounts of time in activities, we need $\varphi_i(\sum_{k=1}^{K} a_{i,k}) = 4$. This is possible if preference parameters now take values $a_{i,1} = 2$ and $a_{i,2} = 2$, so that agent i is indifferent between activities 1 and 2, as she was before the transformation through φ_i. Even though it makes sense to have $a_{i,1} = 2$ and $a_{i,2} = 2$ in order to keep agent i's indifference between activities 1 and 2, the allocation $a_{i,1} = 4$ and $a_{i,2} = 0$, or any allocation such that $a_{i,1} + a_{i,2} = 4$ with $a_{i,1}, a_{i,2} \geq 0$, also allows her to have one unit of time to allocate between activities and spend non-negative amounts of time in activities.

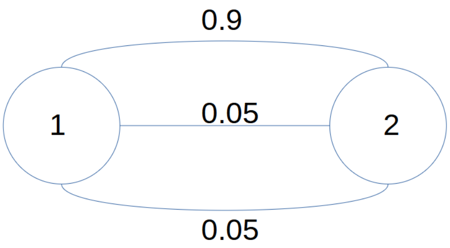

Figure 3a: 2 weak links.

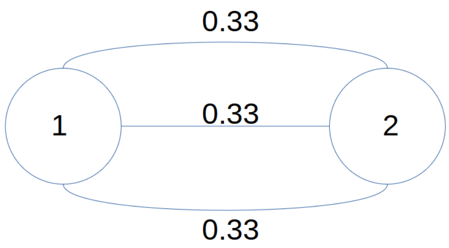

Figure 3b: 3 equally strong links.

Proposition 1. *Suppose that $\varphi_i(\sum_{k=1}^{K} a_{i,k}) = \eta(2c - (N-1))$ for all $i \in \mathcal{N}$ so that $t^*_{i,k} \in [0,1]$ and $\sum_{k=1}^{K} t^*_{i,k} = 1$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$. The network fulfills extensive multiplexity if and only if there exist two activities k and l, with $k \neq l$, such that $a_{i,k} > 0$ and $a_{j,l} > 0$ for any $i, j \in \mathcal{N}$.*

Even though extensive multiplexity is a simple and intuitive measure of multiplexity, it can be misleading in weighted networks. To see why, consider the examples given in Figures 3a and 3b.

The extensive multiplexity in both relationships is the same. However, one of the three links in Figure 3a is very strong compared to the other ones. If the two weak links took value 0, then the extensive multiplexity would decrease, even though the change in the value of weights would be very small. Therefore, we provide a measure of multiplexity that gives a higher importance to relationships in which links are equally weighted, such as the one presented in Figure 3b. We call this measure *intensive multiplexity.*[2]

Definition 2 (Intensive multiplexity)**.** *The intensive multiplexity MX^I_{ij} of a relationship between agents i and j is defined as $MX^I_{ij} = \frac{w_{min|ij}}{w_{max|ij}}$*

We consider that a network attains its maximum level of intensive multiplexity when, for each pair of nodes, all weights of links have the same value. All agents having the same preferences for activities leads them to spend the same amount of

[2]This measure only takes into account the strongest link and the weakest link of a relationship. While links of a relationship which are not the strongest nor the weakest are not considered, defining intensive multiplexity as the variance or entropy of weights in a relationship leads to the same results.

time in activities, and hence to form intensively multiplexed relationships. In the special case of two agents participating in two activities, it is sufficient that one agent enjoys participating in an activity to the same amount than the other agent enjoys participating in the other activity.

Proposition 2. *Suppose that $\varphi_i(\sum_{k=1}^{K} a_{i,k}) = \eta(2c-(N-1))$ for all $i \in \mathcal{N}$ so that $t^*_{i,k} \in [0,1]$ and $\sum_{k=1}^{K} t^*_{i,k} = 1$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$. The level of intensive multiplexity of the network attains its maximum value if either condition **1A**, **1B** or **1C** is satisfied.*
*Condition **1A**: $\mathcal{N} = \{1,2\}$ and for both agents, $a_{i,k} > 0$ for $k = 1$ or $k = 2$, and $a_{1,k} = a_{2,l}$ and $a_{1,l} = a_{2,k}$ with $a_{1,k} > 0$ and $a_{1,l} > 0$ for some activities k and l.*
*Condition **1B**: $N \geq 3$ and preferences are such that $a_{i,k} = \frac{2c-(N-1)}{K}$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$.*
*Condition **1C**: There exists at least an agent i such that $a_{i,k} > 0$ for any $k \in [3, \infty[$ and preferences are such that $a_{i,k} = \frac{2c-(N-1)}{K}$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$.*

In most cases, intensive multiplexity attains its maximum value when preferences are homogeneous (conditions **1B** and **1C**). An example of a world composed of agents with homogenous preferences is given in Table 4.

Preference parameters				
	Agent 1	Agent 2	Agent 3	Agent 4
Group R	$a_{1,R} = 0.5$	$a_{2,R} = 0.5$	$a_{3,R} = 0.5$	$a_{4,R} = 0.5$
Group S	$a_{1,S} = 0.5$	$a_{2,S} = 0.5$	$a_{3,S} = 0.5$	$a_{4,S} = 0.5$

Table 4: Homogeneous preferences

The utility of every agent is ($c = 2$):

$$U_i = 0.5 \cdot 0.5 \cdot 2 + 6 \cdot 0.25 - 2 \cdot 2 \cdot 0.5^2$$

$$U_i = 1$$

An example of a world composed of agents with heterogeneous preferences is given in Table 5.

Preference parameters				
	Agent 1	Agent 2	Agent 3	Agent 4
Group R	$a_{1,R} = 1$	$a_{2,R} = 1$	$a_{3,R} = 0$	$a_{4,R} = 0$
Group S	$a_{1,S} = 0$	$a_{2,S} = 0$	$a_{3,S} = 1$	$a_{4,S} = 1$

Table 5: Heterogeneous preferences

The utility of every agent is ($c = 2$):

$$U_i = 1 \cdot 0.6 + 0.6 \cdot (0.6 + 0.4 + 0.4) + 0.4 \cdot (0.4 + 0.6 + 0.6) - 2(0.6^2 + 0.4^2)$$

$$U_i = 1.04$$

Heterogeneity of preferences for activities leads agents to have a higher utility. Intensive multiplexity, which is in most cases attained through homogeneous preferences, undermines welfare.

Pairs of agents can not only be heterogeneous in their preferences for activities and in the weights of the links they share, but also in the time they spend together in groups. There are two reasons for which two agents i and j spend similar amounts of time in a group k. The first reason is the similarity in preference parameters for activity k, $a_{i,k}$ and $a_{j,k}$. Individuals with similar tastes for activities will more likely spend time together in the same activities. The second reason is a high value of the costs parameter c. Because costs are convex, agents are incentivized to allocate their time more equally between activities as costs increase, and disregard their preferences for activities.

Proposition 3. *Suppose that* $\varphi_i(\sum_{k=1}^{K} a_{i,k}) = \eta(2c - (N-1))$ *for all* $i \in \mathcal{N}$ *so that* $t^*_{i,k} \in [0,1]$ *and* $\sum_{k=1}^{K} t^*_{i,k} = 1$ *for all* $i \in \mathcal{N}$ *and all* $k \in \mathcal{K}$. *For any two agents* $i, j \in \mathcal{N}$ *such that* $a_{i,k} \neq a_{j,k}$, *the absolute value of the difference* $|t^*_{i,k} - t^*_{j,k}|$ *decreases as costs* c *increase.*

4 Bonacich centrality

As previously shown by Ballester et al. (2006), in finite population noncooperative games with linear-quadratic utilities such as the ones of Chen et al. (2018) and Belhaj and Deroïan (2014), equilibrium values of agents directly depend on their Bonacich centrality. Said differently, the outcome of these games directly depends on the—exogenous—structure of the network. In our framework, the network arises endogenously such that outcome of the game and structure of the network are determined simultaneously. Bonacich centrality is a centrality measure in which a node's importance is determined by the importance of its neighbors, with the particularity that the importance of neighbors is discounted as the distance increases. Centrality measures such as Bonacich centrality have been shown to play a role in many kinds of networks such as co-authorship, criminal, transportation, educational and drug abuse networks (Das et al., 2018). As Chen et al. (2018) and Belhaj and Deroïan (2014) do, the literature has focused on Bonacich centrality in unweighted and simplex networks. In order to study the role of Bonacich centrality in weighted and multiplexed networks, such as the one we study, we provide a more general expression of Bonacich centrality. To the best of our knowledge, Bonacich centrality has not been defined neither for weighted nor for multiplexed networks. Opsahl et al. (2010) generalize degree, closeness and betweenness centralities to weighted graphs, but do not do so for eigenvector centralities. Domenico et al. (2013) propose an eigenvector centrality for multiplexed weighted networks, in which the same importance is given to direct neighbors than to more distant neighbors, which is against the spirit of Bonacich centrality which discounts the importance of connections as they become more distant.

Bonacich centrality of agent i in simplex and unweighted networks, as considered by the literature, is defined in (4):

$$C_i^B = a(\sum_{x=1}^{\infty} b^{(x-1)} g_{i:x}), \tag{4}$$

where $g_{i:x}$ sums the values of last links of walks of length x emanating from i, and a and b are scalars such that $a > 0$ and $b \in [0, 1]$. As defined in Section 2, the last link of a walk from node i to node j is the link which connects node $j-1$ and j. In unweighted networks, the last links of walks take value 1. Therefore, computing $g_{i:x}$ in unweighted networks is equivalent to counting the number of walks of length x emanating from i. For any given length x, the larger the number of walks emanating from node i, the larger her Bonacich centrality. The importance of walks in Bonacich centrality is discounted as the length x increases through the term $b^{(x-1)}$.

For illustrative purposes on the computation of Bonacich centrality, let us consider Figure 4a. The value of $g_{1:1}$, which is the degree of node 1, takes value 1. There is one walk of length 1 emanating from node 1—from node 1 to node 2—, and the last link of this walk takes value 1. The value of $g_{1:1}$ is therefore $g_{1:1} = 1$. The value of $g_{1:2}$ sums the values of last links of length 2 emanating from node 1, and thus $g_{1:2} = 1$. In unweighted networks, computing $g_{i:x}$ is equivalent to counting the number of walks of length x emanating from node i.

In weighted networks, we use the same expression, but we allow links to take any values in the interval $(0, 1]$. The Bonacich centrality measure for weighted networks is given in (5):

$$C_i^{WB} = a(\sum_{x=1}^{\infty} b^{(x-1)} w_{i:x}), \tag{5}$$

where $w_{i:x}$ sums the values of last links of walks of length x, and a and b are scalars such that $a > 0$ and $b \in (0, 1)$. For illustrative purposes, let us consider Figure 4b composed of two agents linked by three links $L1$, $L2$ and $L3$ which all take value $\frac{1}{3}$. The value of $w_{1:1}$ sums the values of last links of walks of length 1 emanating from node 1. In unweighted networks, $w_{1:1}$ is the degree of node 1. In weighted networks, this is commonly defined as *strength*, which is a generalization of the notion of degree in weighted networks (Barrat et al., 2004). There are 3 walks of length 1 emanating from node 1 —Node $N1$ - Link $L1$ - Node $N2$, Node $N1$ - Link $L2$ - Node $N2$ and Node $N1$ - Link $L3$ - Node $N2$—, and the last links of all these walks take value $\frac{1}{3}$. The value of $w_{1:1}$ is therefore $w_{1:1} = 3 \cdot \frac{1}{3} = 1$.

The value of $w_{1:2}$ sums the values of last links of length 2 emanating from node 1. There are 9 walks of length 2 emanating from node 1, and the last links of all

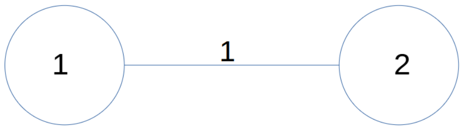

Figure 4a: Simplex network.

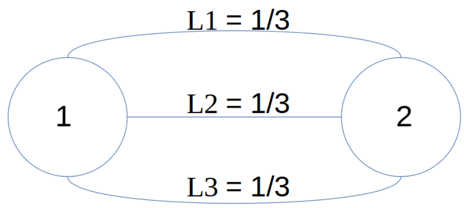

Figure 4b: Multiplexed network.

these walks take value $\frac{1}{3}$.[3] The value of $w_{1:2}$ is therefore $w_{1:2} = 9 \cdot \frac{1}{3} = 3$.
Let us assume $a = \frac{1}{2}$ and $b = \frac{1}{4}$. The weighted Bonacich centrality for the network of Figure 4a is equal to:

$$\mathbf{C}^{WB}(w, a, b) = (\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot \frac{1}{2} \cdot 1 + \frac{1}{16} \cdot \frac{1}{2} \cdot 1 ...)\mathbf{1},$$

$$\mathbf{C}^{WB}(w, a, b) \approx \begin{bmatrix} 0.66 \\ 0.66 \end{bmatrix},$$

where $\mathbf{1}$ is a vector of one's.

The weighted Bonacich centrality for the network of Figure 4b is equal to:

$$\mathbf{C}^{WB}(w, a, b) = [\frac{1}{2} \cdot (\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) + \frac{1}{4} \cdot \frac{1}{2} \cdot (\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3})...]\mathbf{1},$$

$$\mathbf{C}^{WB}(w, a, b) \approx \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Surprisingly, nodes 1 and 2 have larger Bonacich centralities in Figure 4b than in Figure 4a, even though their strength in both figures is equal. This is due to the fact that the extensive multiplexity of agents in Figure 4b is larger than the one of

[3] These 9 walks are $N1 - L1 - N2 - L1 - N1, N1 - L1 - N2 - L2 - N1, N1 - L1 - N2 - L3 - N1, N1 - L2 - N2 - L1 - N1, N1 - L2 - N2 - L2 - N1, N1 - L2 - N2 - L3 - N1, N1 - L3 - N2 - L1 - N1, N1 - L3 - N2 - L2 - N1$ and $N1 - L3 - N2 - L3 - N1$.

agents in Figure 4a. If the strength of agents 1 and 2 in Figure 4b were larger than the one in Figure 4a, we could infer that the difference in Bonacich centralities could be due to the difference in strengths. Similarly, we could also infer that the difference in Bonacich centralities could come from a difference in the number of nodes between Figures 4a and 4b, and hence, in the number of links. However, the strength and the number of agents are equal across networks. We define two networks $\boldsymbol{G}$ and $\boldsymbol{G'}$, and $f : \boldsymbol{G} \to \boldsymbol{G'}$ a bijective function which maps nodes of network $\boldsymbol{G}$ to nodes of network $\boldsymbol{G'}$ which have the same strength.

When f is bijective, there is no difference in the number nor the strength of agents across networks, and hence a difference in Bonacich centralities across networks can only be due to a difference in extensive multiplexity across networks.

It is worth noting that increasing the extensive multiplexity of a relationship between two agents has a positive impact on the Bonacich centrality of agents in the component to which these two agents belong, and a null impact on the Bonacich centrality of agents in other components.

We define a component $\boldsymbol{C} \in \boldsymbol{G}$ and its corresponding component $\boldsymbol{C'} \in \boldsymbol{G'}$, which is such that $f : \boldsymbol{C} \to \boldsymbol{C'}$ is bijective.

Theorem 2. *Let a component $\boldsymbol{C} \in \boldsymbol{G}$ have a larger extensive multiplexity than its corresponding component $\boldsymbol{C'} \in \boldsymbol{G'}$. Then, all nodes of $\boldsymbol{C}$ are more Bonacich central than the nodes of $\boldsymbol{C'}$ they are mapped to.*

If a link is removed from a simplex network, the Bonacich centrality of all agents in the component will decrease. If a link is removed from a multiplexed network, the Bonacich centrality of all agents in the component will decrease, but the reduction will be lower because agents will still be able to connect through other means. For instance, if agents can communicate through phone and e-mail, removing their phones will still allow them to communicate through e-mail. In a world where only phones exist, removing phones entails not being able to communicate.

Because there is an infinite amount of walks starting from a node, the Bonacich centrality can take infinity as a value. A solution often used in the literature is to give the scalar b a low enough value (Jackson, 2008), because it gives negligible values to benefits from very distant nodes. More specifically, b needs to be lower

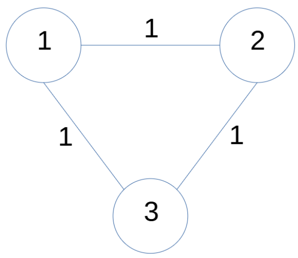

Figure 5a

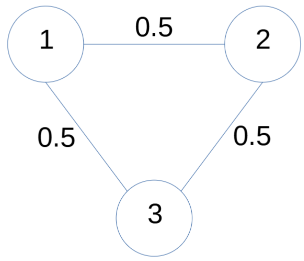

Figure 5b

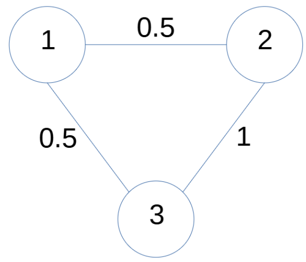

Figure 5c

than $\frac{1}{|\lambda_G|}$ where $|\lambda_G|$ is the norm of the largest eigenvalue of the adjacency matrix. For the latter to be true, it is sufficient that b be smaller than $\frac{1}{d_{max}}$ where d_{max} is the maximum degree of any agent. The inequality $b < \frac{1}{d_{max}}$ is a sufficient condition for the Bonacich centrality to be finite in unweighted networks because the maximum value $g_{i:(x+1)}$ can take is $g_{i:x} \cdot d_{max}$. Therefore, $b < \frac{1}{d_{max}}$ implies that $b^x g_{i:(x+1)} < b^{(x-1)} g_{i:x}$ for all values of x, and hence that $\lim_{x\to\infty} b^{(x-1)} g_{i:x} = 0$, so that the Bonacich centrality takes a finite value.

For illustrative purposes, let us consider Figures 5a, 5b and 5c. Consider that scalars a and b take values $a = 1$ and $b = \frac{1}{2}$. The scalar b takes value $\frac{1}{d_{max}}$ in the networks represented in Figures 5a, 5b and 5c.

In Figure 5a, the weighted Bonacich centrality of agent 1 is:

$$C_1^{WB} = 2 + \frac{1}{2} \cdot 4 + \frac{1}{4} \cdot 8 + \frac{1}{8} \cdot 16... = 2 + 2 + 2 + 2... \tag{6}$$

Because $b = \frac{1}{d_{max}}$, the weighted Bonacich centrality of agent 1 is infinite. Were b strictly lower than $\frac{1}{d_{max}}$, the largest term in the sum of (6) would be 2, and all other terms would take a value lower than 2, with an infinity of them taking a positive value close to 0, making the weighted Bonacich centrality finite. In Figure 5b, the weighted Bonacich centrality of agent 1 is:

$$C_1^{WB} = 1 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8... = 1 + 1 + 1 + 1... \tag{7}$$

Again, the weighted Bonacich centrality of agent 1 is infinite. This is because $b = \frac{1}{d_{max}}$. Were b strictly lower than $\frac{1}{d_{max}}$, the weighted Bonacich centrality would be finite.

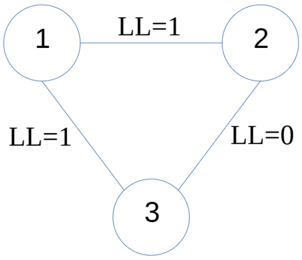

Figure 6a: Length 1

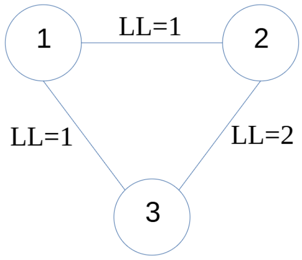

Figure 6b: Length 2

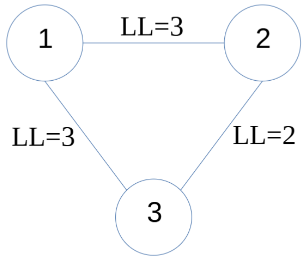

Figure 6c: Length 3

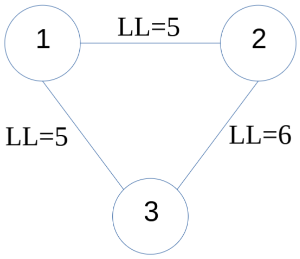

Figure 6d: Length 4

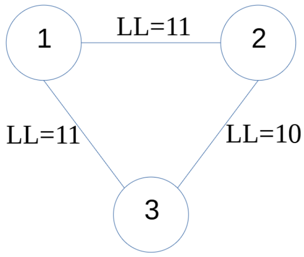

Figure 6e: Length 5

In figure 5c, the Bonacich centrality of agent 1 is infinite. To see why this is the case, consider Figures 6a, 6b, 6c, 6d and 6e which count, respectively the number of last links of walks of length 1, 2, 3, 4 and 5 starting from node 1. For example, there are two walks of length 1 emanating from node 1, and hence two corresponding last links : one connects nodes 1 and 2, and the other one connects nodes 1 and 3. As another example, there are also four walks of length 2 emanating from node 1 : one last link connects nodes 1 and 2, another last link connect nodes 1 and 3, and the other two last links connect nodes 2 and 3.

For walks of odd (even) length, the links connecting nodes 1 and 2, and nodes 1 and 3, are last links of exactly one more (less) walk than the link connecting nodes 2 and 3. Due to this pattern, the share of last links being w_{23} over the total number of last links is larger (lower) for any odd (even) length $(x+2)$ than for any length x. It follows that, if the value of w_{23} is larger (lower) than the mean between w_{12} and w_{13}, we have $w_{1:(x+2)} > w_{1:x} \cdot d_{max}^2$ for any positive odd (even) value of x. Therefore, $b < \frac{1}{d_{max}}$ is not sufficient for the weighted Bonacich centrality of agent 1 to be finite. Similarly, we have that $w_{1:(x+2)} > w_{1:x} \cdot d_{max}^2$ for any positive even value of x when the value of w_{23} is lower than the mean between w_{12} and w_{13},

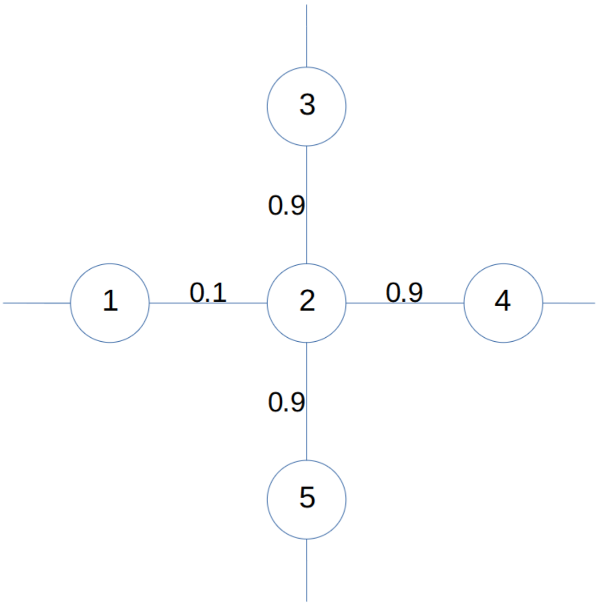

Figure 7: Subgraph of a weighted network

which also renders the condition $b < \frac{1}{d_{max}}$ insufficient for the weighted Bonacich centrality to be finite. On a more general note, for any network in which all pairs of agents share the same number of links, the inequality $w_{i:(x+2)} > w_{i:x} \cdot d_{max}^2$ is fulfilled for all odd (even) values of length x when the mean of weights w_{jk} in set $\{w_{jk} \mid j \neq i \text{ and } k \neq i\}$ is larger (lower) than the mean of weights w_{jk} in set $\{w_{jk} \mid \text{either } j = i \text{ or } k = i\}$.

The condition $b < \frac{1}{d_{max}}$ may not be sufficient for the Bonacich centrality to take a finite value when weights of links differ in value. Because the condition $b < \frac{1}{d_{max}}$ does not guarantee the finiteness of the Bonacich centrality in weighted graphs, we provide a condition which does.

The maximum value that $w_{i:(x+1)}$ can take for any $x \geq 1$ is $w_{i:x} \cdot \frac{s_{max}}{w_{min}}$, where s_{max} is the maximum strength of any node, and $w_{min} \neq 0$ is the value of the link which takes the minimum value. To see why it is the case, consider the subgraph of Figure 7.

The value of $w_{1:1}$ is 0.1. The value of $w_{1:2}$ is $w_{1:2} = 0.1 + 0.9 + 0.9 + 0.9 = 2.8$, which is the strength of node 2. If all links took value 0.1, then we would have that $w_{1:2} = 0.4 = d_{max} \cdot w_{1:1}$. However, since links which are at distance 2 from node 1 take a larger value than those which are at distance 1, we have that $w_{1:2} > d_{max} \cdot w_{1:1}$. In the subgraph of Figure 7, and more generally in any weighted and multiplexed network, the maximum value that $w_{1:2}$ can take is $\frac{s_{max}}{w_{min}} \cdot w_{1:1}$. Hence, $b < \frac{1}{\frac{s_{max}}{w_{min}}}$, or equivalently, $b < \frac{w_{min}}{s_{max}}$ is sufficient for the Bonacich centrality to be finite. It is worth noting that, in unweighted networks or any setting in which all links take the same value, $\frac{w_{min}}{s_{max}} = \frac{1}{d_{max}}$.

Proposition 4. *The condition $b < \frac{w_{min}}{s_{max}}$ is sufficient for the weighted Bonacich centrality to take a finite value.*

Contrary to the condition on b considered in unweighted networks, the condition on b in weighted networks not only depends on the agent that is the most connected, but also on the weakest link of the network.

5 Conclusion

People interact in multiple areas of life. Relationships where individuals interact in two or more areas have mainly been studied in other fields, and are called *multiplexed*. We study a game in which individuals allocate one unit of time between multiple activities, and form relationships in each of these activities. We define two measures of multiplexity and provide conditions which give rise to these two types of multiplexed networks. We furthermore find that agents spend more time together as costs of performing activities increase. We provide an expression of Bonacich centrality for weighted and multiplexed networks, and find that its value depends on the multiplexity of the network. We also study the condition for the Bonacich centrality of agent i to take a finite value in such networks, and find that the condition is consistent with the one given in the literature only when all links take the same value. We provide a new condition which ensures that the Bonacich centrality takes a finite value in any weighted network.

Appendix

Proof of Theorem 1

We prove that the game admits a set of equilibrium values $\{t^*_{1,k}, ..., t^*_{N,k}\}$ such that $t^*_{i,k} \in [0,1]$ and $\sum_{k=1}^{K} t^*_{i,k} = 1$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$ if and only if $\sum_{k=1}^{K} a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$. The proof extends to the introduction of functions φ_i and η.

Step 1: Suppose that the utility function is defined on a closed, bounded and convex set.
Brouwer's fixed-point theorem states that an equilibrium exists if the utility function $U_i(t_{i,k})$ is continuous in $t_{i,k}$, is defined on a closed, bounded and convex set, and is an endomorphism.
Suppose that the utility function is defined on a closed, bounded and convex set. Because the utility function is an ordinal concept, we can transform it and keep the order of preferences preserved as long as the transformation is positive monotonic. We define the positive monotonic function $g(U)$ such that the range of $g(U)$ is the same as its domain. The function $g(U)$ is continuous in $t_{i,k}$ and is defined on a closed, bounded and convex set. It is also an endomorphism since its domain and range are the same.
We are assuming that an equilibrium exists through $t_{i,k}$ being defined on a closed, bounded and convex set, and prove in the next steps that if an equilibrium indeed exists, the game admits a set of equilibrium values $\{t^*_{1,k}, ..., t^*_{N,k}\}$ such that $t^*_{i,k} \in [0,1]$ and $\sum_{k=1}^{K} t^*_{i,k} = 1$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$ if and only if $\sum_{k=1}^{K} a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$. Because $t^*_{i,k} \in [0,1]$ entails that the utility function is defined on a closed, bounded and convex set, an equilibrium necessarily exists.

Step 2: Prove that $t^*_{i,k} \in [0,1]$ and $\sum_{k=1}^{K} t^*_{i,k} = 1$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$ implies $\sum_{k=1}^{K} a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$.

The equalities $t^*_{i,k} \in [0,1]$ and $\sum_{k=1}^K t^*_{i,k} = 1$ for all $i \in \mathcal{N}$ imply:

$$\sum_{k=1}^K t^*_{i,k} = 1,$$

$$\frac{\sum_{k=1}^K a_{i,k} + \sum_{k=1} \sum_{j \neq i} t_{j,k}}{2c} = 1,$$

$$2c = \sum_{k=1}^K a_{i,k} + (N-1),$$

$$\sum_{k=1}^K a_{i,k} = 2c - (N-1).$$

Step 3: Prove that $\sum_{k=1}^K a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$ implies $t^*_{i,k} \in [0,1]$ and $\sum_{k=1}^K t^*_{i,k} = 1$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$.

Step 3.1: Prove that $\sum_{k=1}^K a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$ implies $\sum_{k=1}^K t^*_{i,k} = 1$ for all $i \in \mathcal{N}$.

The difference $\sum_{k=1}^K t_{i,k} - \sum_{k=1}^K t_{j,k}$ is:

$$\sum_{k=1}^K t_{i,k} - \sum_{k=1}^K t_{j,k} = \frac{\sum_{k=1}^K a_{i,k} - \sum_{k=1}^K a_{j,k} + \sum_{k=1}^K \sum_{j \neq i} t_{j,k} - \sum_{k=1}^K \sum_{i \neq j} t_{i,k}}{2c}.$$

Since $\sum_{k=1}^K a_{i,k}$ and $\sum_{k=1}^K a_{j,k}$ are both equal to $2c - (N-1)$, the difference $\sum_{k=1}^K a_{i,k} - \sum_{k=1}^K a_{j,k}$ equals 0.

$$\sum_{k=1}^K t_{i,k} - \sum_{k=1}^K t_{j,k} = \frac{\sum_{k=1}^K t_{j,k} - \sum_{k=1}^K t_{i,k}}{2c},$$

$$\sum_{k=1}^K t_{i,k} - \sum_{k=1}^K t_{j,k} = \frac{\sum_{k=1}^K t_{j,k} - \sum_{k=1}^K t_{i,k}}{\sum_{k=1}^K a_{i,k} + (N-1)}.$$

The previous equality holds if $\sum_{k=1}^K a_{i,k} + (N-1) = -1$, or if $\sum_{k=1}^K t_{i,k} = \sum_{k=1}^K t_{j,k}$. It is impossible that $\sum_{k=1}^K a_{i,k} + (N-1) = -1$ since $N \geq 2$ and $\sum_{k=1}^K a_{i,k} > 0$.

It therefore follows that $\sum_{k=1}^{K} t_{i,k} = \sum_{k=1}^{K} t_{j,k}$. Since i and j are chosen without generality, this equality holds for any pair of agents in $\mathcal{N}$.

The expression of $\sum_{k=1}^{K} t_{i,k}$ is:

$$\sum_{k=1}^{K} t_{i,k} = \frac{\sum_{k=1}^{K} a_{i,k} + \sum_{k=1}^{K} \sum_{j \neq i} t_{j,k}}{2c}.$$

Since $\sum_{k=1}^{K} t_{i,k} = \sum_{k=1}^{K} t_{j,k}$ for all $j \in \mathcal{N}$, we have that:

$$\sum_{k=1}^{K} t_{i,k} = \frac{\sum_{k=1}^{K} a_{i,k} + (N-1) \sum_{k=1}^{K} t_{i,k}}{2c},$$

$$\sum_{k=1}^{K} t_{i,k}(2c - (N-1)) = \sum_{k=1}^{K} a_{i,k},$$

$$\sum_{k=1}^{K} t_{i,k} = \frac{\sum_{k=1}^{K} a_{i,k}}{2c - (N-1)}.$$

We replace $\sum_{k=1}^{K} a_{i,k}$ by $2c - (N-1)$.

$$\sum_{k=1}^{K} t_{i,k} = \frac{2c - (N-1)}{2c - (N-1)},$$

$$\sum_{k=1}^{K} t_{i,k} = 1.$$

Step 3.2: Prove that $\sum_{k=1}^{K} a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$ implies $t_{i,k}^{*} \in [0, 1]$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$.

Step 3.2.1: Prove that $\sum_{k=1}^{K} a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$ implies $t_{i,k}^{*} \geq 0$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$.

The notation $\mathbf{x} \geq 0$ means that all components of the vector $\mathbf{x}$ are nonnegative. Farkas' lemma states that, for any $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^{m}$, exactly one of the following two assertions is true:

1. There exists an $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax}=\mathbf{b}$ and $\mathbf{x} \geq 0$.

2. There exists a $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{A}^T\mathbf{y} \geq 0$ and $\mathbf{b}^T\mathbf{y} < 0$.

The system of best-response functions can be re-written in matrix notation $\mathbf{Ax}=\mathbf{b}$, where:

$$\mathbf{A} = \begin{pmatrix} 2c & -1 & -1 & ... & ... & -1 \\ -1 & 2c & -1 & ... & ... & -1 \\ -1 & -1 & 2c & ... & ... & -1 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ -1 & -1 & -1 & ... & ... & 2c \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} t_{1,k} \\ t_{2,k} \\ t_{3,k} \\ ... \\ ... \\ t_{N,k} \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ a_{3,k} \\ ... \\ ... \\ a_{N,k} \end{pmatrix}$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$, $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{b} \in \mathbb{R}^N$. We denote by x_i the entry i of vector $\mathbf{x}$ and by y_i the entry i of vector $\mathbf{y}$. Let us suppose, *ad absurdum*, that there exists an entry i of $\mathbf{x}$, x_i, such that $x_i < 0$. It follows, by Farkas' lemma, that there exists a $\mathbf{y} \in \mathbb{R}^N$ such that $\mathbf{A}^T\mathbf{y} \geq 0$ and $\mathbf{b}^T\mathbf{y} < 0$. The vector $\mathbf{A}^T\mathbf{y}$ is:

$$\mathbf{A}^T\mathbf{y} = \begin{pmatrix} 2cy_1 - \sum_{j\neq 1} y_j \\ 2cy_2 - \sum_{j\neq 2} y_j \\ 2cy_3 - \sum_{j\neq 3} y_j \\ ... \\ ... \\ 2cy_N - \sum_{j\neq N} y_j \end{pmatrix}.$$

The value of $\mathbf{b}^T\mathbf{y}$ is:

$$\mathbf{b}^T\mathbf{y} = \sum_{k=1}^{K} a_{i,k}y_k,$$

where y_k is the entry k of vector $\mathbf{y}$.

The value of $\mathbf{b}^T\mathbf{y}$ is strictly negative if and only if there exists some y_1, chosen without loss of generality, such that $y_1 < 0$. Because $\mathbf{A}^T\mathbf{y} \geq 0$, we have that:

$$2cy_1 - \sum_{j \neq 1} y_j \geq 0,$$

which simplifies to:

$$y_1 \geq \frac{\sum_{j \neq 1} y_j}{2c}.$$

Because $y_1 < 0$, it follows that there exists some y_2, chosen without loss of generality, such that $y_2 < 0$.
Because $\mathbf{A}^T\mathbf{y} \geq 0$, we have that:

$$2cy_1 - \sum_{j \neq 1} y_j + 2cy_2 - \sum_{j \neq 2} y_j \geq 0,$$

$$2c(y_1 + y_2) - y_2 - y_1 - 2 \sum_{j \neq 1 \neq 2} y_j \geq 0,$$

$$(2c - 1)(y_1 + y_2) - 2 \sum_{j \neq 1 \neq 2} y_j \geq 0.$$

Because we have assumed $2c = \sum_{k=1}^{K} a_{i,k} + (N-1)$, the previous equation simplifies to:

$$(\sum_{k=1}^{K} a_{i,k} + N - 2)(y_1 + y_2) - 2 \sum_{j \neq 1 \neq 2} y_j \geq 0.$$

If $N = 2$, then there are two entries of $\mathbf{y}$ since $\mathbf{y} \in \mathbb{R}^N$, which implies that $\sum_{j \neq 1 \neq 2} y_j = 0$. A contradiction arises since we have that $\sum_{k=1}^{K} a_{i,k} > 0$, $y_1 < 0$ and $y_2 < 0$.
If $N \geq 3$, then $\sum_{k=1}^{K} a_{i,k} > 0$, $y_1 < 0$ and $y_2 < 0$ imply that there exists some y_3, chosen without loss of generality, such that $y_3 < 0$.
The three following inequalities are true. The first is:

$$(2c - 1)(y_1 + y_2) - 2 \sum_{j \neq 1 \neq 2} y_j \geq 0.$$

The second is:

$$(2c-1)(y_1+y_3) - 2\sum_{j\neq 1\neq 3} y_j \geq 0.$$

The third is:

$$(2c-1)(y_2+y_3) - 2\sum_{j\neq 2\neq 3} y_j \geq 0.$$

It follows that

$$(2c-1)(y_1+y_2)-2\sum_{j\neq 1\neq 2} y_j+(2c-1)(y_1+y_3)-2\sum_{j\neq 1\neq 3} y_j+(2c-1)(y_2+y_3)-2\sum_{j\neq 2\neq 3} y_j \geq 0,$$

$$(2c-1)(y_1+y_2+y_1+y_3+y_2+y_3) - 2(\sum_{j\neq 1\neq 2} y_j + \sum_{j\neq 1\neq 3} y_j + \sum_{j\neq 2\neq 3} y_j) \geq 0,$$

$$(2c-1)(2y_1+2y_2+2y_3) - 2y_3 - 2y_2 - 2y_1 - 6\sum_{j\neq 1\neq 2\neq 3} y_j \geq 0,$$

$$(2c-2)(2y_1+2y_2+2y_3) - 6\sum_{j\neq 1\neq 2\neq 3} y_j \geq 0,$$

$$(\sum_{k=1}^{K} a_{i,k} + N - 3)(2y_1+2y_2+2y_3) - 6\sum_{j\neq 1\neq 2\neq 3} y_j \geq 0.$$

If $N = 3$, then there are three entries of $\mathbf{y}$ since $\mathbf{y} \in \mathbb{R}^N$, which implies that $6\sum_{j\neq 1\neq 2\neq 3} y_j = 0$. A contradiction arises since we have that $\sum_{k=1}^{K} a_{i,k} > 0$, $y_1 < 0$, $y_2 < 0$ and $y_3 < 0$.
If $N \geq 4$, then $\sum_{k=1}^{K} a_{i,k} > 0$, $y_1 < 0$, $y_2 < 0$ and $y_3 < 0$ imply that there exists some y_4, chosen without loss of generality, such that $y_4 < 0$.
By induction of the argument, a contradiction arises for any $N \geq 4$.

Step 3.2.2: Prove that $\sum_{k=1}^{K} a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$ implies $t^*_{i,k} \leq 1$ for all $i \in \mathcal{N}$ and all $k \in \mathcal{K}$.

Let us suppose, *ad absurdum*, that there exists an agent i such that $t^*_{i,k} > 1$. Step 3.1 of the proof of Theorem 1 proves that $\sum_{k=1}^{K} a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$ implies $\sum_{k=1}^{K} t^*_{i,k} = 1$ for all $i \in \mathcal{N}$. It follows that there exists a group $l \neq k$ such that $t^*_{i,l} < 0$, which contradicts what has been proved in Step 3.2.1 of the proof of Theorem 1. $\square$

Proof of Proposition 1

Step 1: We first prove that if there exists two activities k and l, with $k \neq l$, such that $a_{i,k} > 0$ and $a_{j,l} > 0$ for any $i, j \in \mathcal{N}$, then the network is multiplexed.

Let us suppose, *ad absurdum*, that there exists two activities k and l, with $k \neq l$, such that $a_{i,k} > 0$ and $a_{j,l} > 0$ for any $i, j \in \mathcal{N}$, and that the network is simplex. The system of best-response functions of activity k is:

$$\begin{cases} t_{1,k} = \frac{a_{1,k} + \sum_{j \neq 1} t_{j,k}}{2c}, \\ \ldots \\ t_{i,k} = \frac{a_{i,k} + \sum_{j \neq i} t_{j,k}}{2c}, \\ \ldots \\ t_{N,k} = \frac{a_{N,k} + \sum_{j \neq N} t_{j,k}}{2c}. \end{cases}$$

Since $a_{i,k} > 0$, we have that $t_{i,k} > 0$. It follows that $t_{j,k} > 0$ for all $j \in \mathcal{N}$. Given the process of link formation, we have that $w^k_{ij} > 0$ for all $i, j \in \mathcal{N}$.

The system of best-response functions of activity l is:

$$\begin{cases} t_{1,l} = \frac{a_{1,l}+\sum_{j\neq 1} t_{j,l}}{2c}, \\ \dots \\ t_{i,l} = \frac{a_{i,l}+\sum_{j\neq i} t_{j,l}}{2c}, \\ \dots \\ t_{N,l} = \frac{a_{N,l}+\sum_{j\neq N} t_{j,l}}{2c}. \end{cases}$$

Because $a_{j,l} > 0$, we also have that $w_{ij}^l > 0$ for all $i, j \in \mathcal{N}$, and a contradiction arises since links are formed in activities k and l, and we assumed that the network is simplex.

Step 2: We next prove that if the network is multiplexed, then there exists two activities k and l, with $k \neq l$, such that $a_{i,k} > 0$ and $a_{j,l} > 0$ for any $i, j \in \mathcal{N}$.

Let us suppose, *ad absurdum*, that the network is multiplexed, and that there does not exist two activities k and l, with $k \neq l$, such that $a_{i,k} > 0$ and $a_{j,l} > 0$ for any $i, j \in \mathcal{N}$. Because the network is multiplexed, there exists two activities k and l, with $k \neq l$, such that $t_{i,k}^* > 0$, $t_{i,l}^* > 0$, $t_{j,k}^* > 0$ and $t_{j,l}^* > 0$. Because there does not exist two activities k and l, with $k \neq l$, such that $a_{i,k} > 0$ and $a_{j,l} > 0$ for any $i, j \in \mathcal{N}$, there exists one activity $k \in \mathcal{K}$ such that $a_{i,k} = 2c - (N-1)$ for all $i \in \mathcal{N}$, and all other activities $l \neq k$ are such that $a_{i,l} = 0$ for all $i \in \mathcal{N}$.
The difference $t_{i,k} - t_{j,k}$ for any two agents $i, j \in \mathcal{N}$ is:

$$t_{i,k} - t_{j,k} = \frac{a_{i,k} - a_{j,k} + \sum_{j\neq i} t_{j,k} - \sum_{i\neq j} t_{i,k}}{2c}.$$

Because $a_{i,k} = a_{j,k} = 2c - (N-1)$ for activity k, we have that:

$$t_{i,k} - t_{j,k} = \frac{t_{j,k} - t_{i,k}}{2c},$$

$$t_{i,k} - t_{j,k} = \frac{t_{j,k} - t_{i,k}}{\sum_{k=1}^{K} a_{i,k} + (N-1)}.$$

The previous equality holds if $\sum_{k=1}^{K} a_{i,k} + (N-1) = -1$, or if $t_{i,k} = t_{j,k}$. It is impossible that $\sum_{k=1}^{K} a_{i,k} + (N-1) = -1$ since $N \geq 2$ and $\sum_{k=1}^{K} a_{i,k} > 0$. It therefore follows that $t_{i,k} = t_{j,k}$. Since i and j are chosen without generality, this equality holds for any pair of agents in $\mathcal{N}$.
The expression of $t_{i,k}$ is:

$$t_{i,k} = \frac{a_{i,k} + \sum_{j \neq i} t_{j,k}}{2c}.$$

Because $t_{i,k} = t_{j,k}$ for all $j \in \mathcal{N}$, we have that:

$$t_{i,k} = \frac{a_{i,k} + (N-1)t_{i,k}}{2c},$$

$$t_{i,k}(2c - (N-1)) = a_{i,k},$$

$$t_{i,k} = \frac{a_{i,k}}{2c - (N-1)}.$$

Since $a_{i,k} = 2c - (N-1)$, we have that:

$$t^*_{i,k} = 1.$$

Since i is chosen without loss of generality, we have that $t^*_{i,k} = 1$ for all $i \in \mathcal{N}$.
We have just shown that in activity k, for which $a_{i,k} = 2c - (N-1)$ for all agents $i \in \mathcal{N}$, all agents are such that $t^*_{i,k} = 1$. By Theorem 1, it follows that $t^*_{i,l} = 0$ for all activities $l \neq k$ and for all agents $i \in \mathcal{N}$. A contradiction arises since we assumed that there exists two activities k and l, with $k \neq l$, such that $t^*_{i,k} > 0$, $t^*_{i,l} > 0$, $t^*_{j,k} > 0$ and $t^*_{j,l} > 0$. $\square$

Proof of Proposition 2

The intensive multiplexity takes its maximum value for all agents when w^k_{ij} takes the same value for all $i, j \in \mathcal{N}$ and all $k \in \mathcal{K}$, which happens when, for all pairs of agents i and $j \in \mathcal{N}$ and all pairs of activities k and $l \in \mathcal{K}$ such that $k \neq l$

and $i \neq j$, either *(i)* $t^*_{i,l} = t^*_{i,k}$ and $t^*_{j,l} = t^*_{j,k}$, or *(ii)* $t^*_{i,l} = t^*_{j,k}$ and $t^*_{j,l} = t^*_{i,k}$. The difference $t_{1,k} - t_{2,k}$ can be expressed as:

$$t_{1,k} - t_{2,k} = \frac{a_{1,k} - a_{2,k} + \sum_{j \neq 1} t_{j,k} - \sum_{j \neq 2} t_{j,k}}{\sum_{k=1}^{K} a_{i,k} + (N-1)},$$

$$t_{1,k} - t_{2,k} = \frac{a_{1,k} - a_{2,k} + t_{2,k} - t_{1,k}}{\sum_{k=1}^{K} a_{i,k} + (N-1)},$$

$$t_{1,k} - t_{2,k} = \frac{a_{1,k} - a_{2,k}}{\sum_{k=1}^{K} a_{i,k} + N}.$$

Let us suppose, *ad absurdum*, that $N \geq 3$, that the intensive multiplexity of the network takes its maximum value and that that $a_{1,k} > a_{2,k}$, so that not all preference parameters take the same value $a_{i,k} = \frac{2c-(N-1)}{K}$. Therefore, $t^*_{1,k} > t^*_{2,k}$. Because $\sum_{k=1}^{K} t^*_{i,k} = 1$ and $t^*_{i,k} \in [0,1]$ for all i and all k, there exists at least another group l such that $t^*_{1,l} < t^*_{2,l}$. If case *(i)* happens, such that $t^*_{1,l} = t^*_{1,k}$, then $t^*_{2,l} = t^*_{2,k}$ is impossible because we found that $t^*_{1,k} > t^*_{2,k}$ and $t^*_{1,l} < t^*_{2,l}$. If case *(ii)* happens, such that $t^*_{1,l} = t^*_{2,k}$ and $t^*_{2,l} = t^*_{1,k}$, then there is at least a third agent 3 for whom $t^*_{3,k} = t^*_{1,l}$, $t^*_{3,l} = t^*_{1,k}$, $t^*_{3,k} = t^*_{2,l}$ and $t^*_{3,l} = t^*_{2,k}$. It follows that $t^*_{1,l} = t^*_{2,l}$, and a contradiction arises since we proved $t^*_{1,l} < t^*_{2,l}$.
Let us now suppose, *ad absurdum*, that $a_{1,k} > a_{2,k}$, that the intensive multiplexity of the network takes its maximum value and that There exists at least an agent i such that $a_{i,k} > 0$ for any $k \in [3, \infty[$. Case *(i)* is impossible for the reason mentioned above: $a_{1,k} > a_{2,k}$ implies that $t^*_{1,k} > t^*_{2,k}$ and that there exists another activity l such that $t^*_{1,l} < t^*_{2,l}$. If case *(ii)* happens, such that $t^*_{1,l} = t^*_{2,k}$ and $t^*_{2,l} = t^*_{1,k}$, then there is at least a third activity m such that $t^*_{1,k} = t^*_{2,m}$ and $t^*_{2,k} = t^*_{1,m}$, and $t^*_{1,l} = t^*_{2,m}$ and $t^*_{2,l} = t^*_{1,m}$. It follows that $t^*_{1,k} = t^*_{1,l}$, and a contradiction arises since $t^*_{1,k} > t^*_{2,k}$ and $t^*_{1,l} = t^*_{2,k}$ imply $t^*_{1,k} > t^*_{1,l}$.
Let us suppose, *ad absurdum*, that $\mathcal{N} = \{1,2\}$, that both agents are such that $a_{i,k} > 0$ for $k = 1$ or $k = 2$, that the intensive multiplexity of the network takes its maximum value and that $a_{1,k} > a_{2,l}$ ($a_{1,k} < a_{2,l}$) and $a_{1,l} < a_{2,k}$ ($a_{1,l} > a_{2,k}$) with $a_{1,k} > 0$, $a_{2,l} > 0$, $a_{1,l} > 0$ and $a_{2,l} > 0$ for some activities k and l. It follows that $t^*_{1,k} > t^*_{2,l}$ ($t^*_{1,k} < t^*_{2,l}$) and $t^*_{1,l} < t^*_{2,k}$ ($t^*_{1,l} > t^*_{2,k}$), which renders case *(ii)* impossible. Case *(i)* is also impossible since $t^*_{1,k} = t^*_{1,l}$ implies $t^*_{2,k} > t^*_{2,l}$ ($t^*_{2,k} < t^*_{2,l}$).

The cases $a_{1,k} < a_{2,l}$ and $a_{1,l} = a_{2,k}$, $a_{1,k} > a_{2,l}$ and $a_{1,l} = a_{2,k}$, $a_{1,k} = a_{2,l}$ and $a_{1,l} > a_{2,k}$, $a_{1,k} = a_{2,l}$ and $a_{1,l} < a_{2,k}$, $a_{1,k} > a_{2,l}$ and $a_{1,l} > a_{2,k}$, and $a_{1,k} < a_{2,l}$ and $a_{1,l} < a_{2,k}$ are not studied since they would break with the assumption that the sum of preference parameters over activities $\sum_{k=1}^{K} a_{i,k} = 2c - (N-1)$ is constant across all agents.

Since agents 1, 2 and 3 and activities k, l and m are chosen without loss of generality, contradictions arise for any agents in $\mathcal{N}$ and any activities in $\mathcal{K}$. $\square$

Proof of Proposition 3

The difference $t_{1,k} - t_{2,k}$ is:

$$t_{1,k} - t_{2,k} = \frac{a_{1,k} - a_{2,k} + t_{2,k} - t_{1,k}}{2c},$$

which simplifies to

$$t_{1,k} - t_{2,k} = \frac{a_{1,k} - a_{2,k}}{2c+1}.$$

Increasing costs c reduces the difference $\mid t_{1,k} - t_{2,k} \mid$.

Since agents 1 and 2 are chosen without loss of generality, increasing c reduces the difference $\mid t_{i,k} - t_{j,k} \mid$ for any agents $i, j \in \mathcal{N}$. $\square$

Proof of Theorem 2

Nodes of a component of $\boldsymbol{G}$ can be more Bonacich central than the nodes of $\boldsymbol{G'}$ they are mapped to if either (i) values of weights are larger in the component of $\boldsymbol{G}$ than in the component of $\boldsymbol{G'}$, or (ii) there are more links in the component of $\boldsymbol{G}$ than in the component of $\boldsymbol{G'}$. A difference in Bonacich centralities between nodes of $\boldsymbol{G}$ and $\boldsymbol{G'}$ cannot come from a difference in values of weights, since nodes of $\boldsymbol{G}$ are mapped to nodes of $\boldsymbol{G'}$ which have the same strength. It can therefore only come from a difference in the number of links across components of networks $\boldsymbol{G}$ and $\boldsymbol{G'}$. Components of networks $\boldsymbol{G}$ and $\boldsymbol{G'}$ have the same number of nodes since f is bijective. Therefore, if the extensive multiplexity of a component of network

$\boldsymbol{G}$ is larger than the one of the component of $\boldsymbol{G'}$ to which nodes of $\boldsymbol{G}$ are mapped, then there exist values of $x \in \mathbb{N}$ such that the number of walks of length x emanating from a node in $\boldsymbol{G}$ is larger than the number of walks of length x emanating from the node in $\boldsymbol{G'}$ it is mapped to. $\square$

Proof of Proposition 4

For any x and any i, the maximum value that $w_{i:(x+1)}$ can take is $w_{i:x} \cdot \frac{s_{max}}{w_{min}}$. Therefore, $b < \frac{w_{min}}{s_{max}}$ is a sufficient condition for the weighted Bonacich centrality to be finite. $\square$

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