Multiproduct-Firm Pricing Games and Transformed Potentials*

Volker Nocke† Nicolas Schutz‡

March 6, 2024

Abstract

We adopt a potential games approach to study multiproduct-firm pricing games where products can be local complements or substitutes. We show that any such game based on an IIA demand system admits an ordinal potential, giving rise to a simple proof of equilibrium existence. We introduce the concept of transformed potential, and characterize the class of demand systems that give rise to multiproduct-firm pricing games admitting such a potential, as well as the associated transformation functions. The resulting demand systems allow for substitutability or complementarity patterns that go beyond IIA, and can resemble those induced by "one-stop shopping" behavior.

Keywords: Multiproduct firms, potential game, oligopoly pricing, IIA demand, complementary goods

Journal of Economics Literature Classification: L13, D43

1 Introduction

Multiproduct firms selling horizontally-differentiated goods are ubiquitous and many markets are dominated by a small number of firms wielding market power. This is reflected in

*We thank Francesc Dilmé, Özlem Bédré-Defolie, and seminar and conference participants at the Bergamo Workshop on Advances in IO, the Conference of the IO committee of the German Economic Association, the CRC TR 224 Fall Retreat, the MaCCI-CREST IO Workshop, the MaCCI Summer Institute, University of Southern California, and the World Congress of the Econometric Society for helpful comments and suggestions. We gratefully acknowledge financial support from the German Research Foundation (DFG) through CRC TR 224 (Project B03).

†Department of Economics and MaCCI, University of Mannheim. Also affiliated with CEPR. Email: volker.nocke@gmail.com.

‡Department of Economics and MaCCI, University of Mannheim. Also affiliated with CEPR. Email: schutz@uni-mannheim.de.

the empirical industrial organization literature, where multiproduct-firm oligopoly features prominently (e.g., Berry, Levinsohn, and Pakes, 1995; Nevo, 2001; Miller and Weinberg, 2017).

The theoretical analysis of such markets, however, is hampered by a number of technical difficulties, such as payoff functions failing to be quasi-concave (Spady, 1984; Hanson and Martin, 1996) and/or (log-)supermodular (e.g., Whinston, 2007, footnote 8). In Nocke and Schutz (2018), those difficulties are circumvented by means of an aggregative games approach, but at the cost of imposing some technical regularity conditions as well as, more substantially, the restriction to substitute products.

In this paper, we propose a different approach to the proof of equilibrium existence in multiproduct-firm oligopoly, namely one based on the theory of potential games. A normal-form game is said to admit a *potential* if there exists a function, called the potential function, such that whenever a player changes her action, the variation in her payoff is equal to the variation in the potential function (Monderer and Shapley, 1996b). Under the weaker concept of an ordinal potential, all that is required is that the variation in the deviating player's payoff has the same sign as the variation in the ordinal potential function. In such games, equilibrium existence can be established without solving a multidimensional fixed point problem (as in the best-response approach) or a nested fixed point problem (as in the aggregative games approach): An action profile that globally maximizes the (ordinal) potential is a Nash equilibrium.

In the first part of the paper (Section 2), we study multiproduct-firm pricing games based on demand systems satisfying the independence of irrelevant alternatives (IIA) property. We show that any such game admits an ordinal potential.[1] Based on this insight, we then prove existence of equilibrium under minimal assumptions on demand by showing that the ordinal potential function has a global maximizer. Importantly, the demand framework does not impose that products be substitutes, but instead allows products to be local complements or substitutes, depending on the level of prices.

Our results in the first part of the paper raise the question whether there may be other demand systems such that the induced multiproduct-firm pricing game admits an ordinal potential. Unfortunately, there is no known way of providing a complete solution to this problem: While Monderer and Shapley (1996b) provide a cross-partial derivatives test that allows to verify easily whether a given game admits a potential, no such test is known for the weaker concept of ordinal potential.[2] The starting point of our approach to this question

[1] Thus, such games are both aggregative and ordinal potential games. Connections between (variants of) aggregative games and (variants of) potential games have been explored in earlier work by Dubey, Haimanko, and Zapechelnyuk (2006) and Jensen (2010).

[2] For example, Monderer and Shapley (1996b) write, "Unlike (weighted) potential games, ordinal potential games are not easily characterized. We do not know of any useful characterization [...] for differentiable

is the observation that the multiproduct-firm pricing game studied in the first part of the paper admits a *log-potential.* That is, the multiproduct-firm pricing game with logged payoffs admits a potential. More generally, classic examples of games admitting an ordinal potential that is not a potential—such as the homogeneous-goods Cournot model with symmetric firms (Kukushkin, 1994; Monderer and Shapley, 1996b) and thus the lottery contest with symmetric players—also admit a log-potential.

We introduce the novel concept of a *transformed potential*: We say that a normal-form game admits a transformed potential if there exists a strictly monotone transformation function G such that the game that results from applying this transformation to all players' payoffs admits a potential. The advantage of this approach is that, for a given transformation function (such as the logarithm), Monderer and Shapley (1996b)'s cross-partial derivatives test can be applied.

In the second part of the paper (Section 3), we address the following two related questions. What classes of demand systems give rise to a transformed-potential multiproduct-firm pricing game? What is the associated set of transformation functions? In answering these questions, we require that the demand system induces a game admitting a transformed potential regardless of the ownership structure of products (i.e., which product is offered by which firm) and the vector of marginal costs. With a slight abuse of terminology, we will often say that such a demand system admits a transformed potential. Solving (systems of) ordinary and partial differential equations, we show that the only classes of demand systems admitting a transformed potential are of the "generalized linear" or IIA forms. In the latter case, the corresponding transformation function is of the log type, whereas it is of the linear type in the former case.

In the final part of the paper (Section 4), we relax the requirement that the demand system induces a game admitting a transformed potential *regardless of the ownership structure* by, instead, fixing the ownership structure of products. Although we continue to find that the only admissible transformations are of the linear and log types, we identify a richer class of demand systems. For a given ownership structure, the class of demand systems that corresponds to linear transformation functions continues to be of the generalized linear form, albeit in a slightly richer form which we completely characterize.

The system of partial differential equations that characterizes the class of demand systems corresponding to log transformation functions is hard to solve in general. We provide a complete solution for the case of two firms. In that case, the demand system has a nest

ordinal potential games." In recent work, Ewerhart (2017) provides derivatives-based necessary conditions for a smooth game to admit an ordinal potential. However, as those conditions are not sufficient, they do not permit a complete characterization of smooth ordinal potential games. Moreover, Ewerhart's derivatives-based test must be performed at a Nash equilibrium action profile, which further limits its applicability for the questions addressed in this paper.

structure that permits patterns of substitutability and complementarity that go beyond those implied by the IIA property. In particular, the nest structure allows products to be complements within a firm, but substitutes across firms, as would arise in models featuring "one-stop shopping" (Stahl, 1982; Bliss, 1988; Chen and Rey, 2012). Although we are not able to provide a complete solution of the general case of three or more firms, we provide two rich classes of demand systems that admit a log-potential for a given ownership structure. In the first such class, each firm owns one or more entire nests of products, so that competition takes place across nests, but not within nests. In the second such class, each firm owns products in only one nest and may face competition from rival firms in that same nest, as well as from firms in different nests.[3]

Related literature. Our paper is motivated by, and contributes to, the literature on multiproduct-firm pricing games with horizontally-differentiated products.[4] As a multiproduct firm's profit function typically fails to be quasi-concave in own price, Caplin and Nalebuff (1991)'s existence result for single-product-firm pricing games does not extend. As a result, equilibrium existence had, until recently, been shown only in special cases of demand systems satisfying some variants of the IIA property: Multinomial logit demand (Spady, 1984; Konovalov and Sándor, 2010), CES demand (Konovalov and Sándor, 2010), and nested multinomial logit demand where each firm owns a nest of products (Gallego and Wang, 2014). In recent work, Nocke and Schutz (2018) adopt an aggregative games approach to unify and extend those results to the larger class of demand systems that can be derived from (multi-stage) discrete/continuous choice, under some restrictions on the relationship between the nest and ownership structures. The present paper further generalizes these earlier equilibrium existence results, and more substantially, allows products to be not only substitutes but also (local) complements, depending on the level of prices.[5]

Our paper also contributes to the literature on potential games, pioneered by Slade (1994) and Monderer and Shapley (1996b). Potential games have been shown to have desirable properties. For example, the Nash equilibrium that maximizes the potential function satisfies the finite improvement property (Monderer and Shapley, 1996b), the fictitious play property (Monderer and Shapley, 1996a), local asymptotic stability (Slade, 1994), and is robust to

[3]For a model with competition within and across nests, under the assumption of nested CES or multinomial logit demand with substitutes, see Nocke and Schutz (2019).

[4]The focus on horizontally-differentiated products is shared by the empirical industrial organization literature (e.g., Berry, 1994; Berry, Levinsohn, and Pakes, 1995; Nevo, 2001; Berry, Levinsohn, and Pakes, 2004; Miller and Weinberg, 2017). There is a separate theoretical literature on multiproduct-firm oligopoly with pure vertical product differentiation (see Champsaur and Rochet, 1989; Johnson and Myatt, 2003, 2006).

[5]For an equilibrium existence result with complements and substitutes, see Quint (2014). His framework, however, differs from ours in two important ways: First, on the demand side, products are perfect complements within a nest and substitutes across nests; second, he restricts attention to single-product firms.

incomplete information (Ui, 2001).

Closer to our work, Slade (1994) proposes a class of inverse demand systems for differentiated products such that the induced single-product firm quantity-setting game admits a potential. She does not, however, provide a complete characterization of the demand systems satisfying that property. By contrast, we introduce the concept of a transformed potential and characterize the set of demand systems such that the induced multiproduct-firm pricing game admits a transformed potential. In unpublished work, Quint (2006) notes that the single-product-firm pricing game with logged payoffs, multinomial logit demand, and costless production admits a potential. We show that this property holds for a considerably larger class of demand systems with multiproduct firms and costly production.

Building on the seminal paper of Gentzkow (2007), there is a growing literature in empirical industrial organization focusing on the estimation of consumer demand in the presence of complementarities. Recent contributions include Thomassen, Smith, Seiler, and Schiraldi (2017), Ershov, Orr, and Laliberté (2019), and Iaria and Wang (2019). Unlike the existing literature on multiproduct-firm oligopoly, our approach to equilibrium existence can accommodate such complementarities.

Price-dependent patterns of substitutability/complementarity are at the heart of Rey and Tirole (2019)'s analysis of the effects of cooperative price caps. Price caps (or floors) can be shown to break the convex-valuedness of the best-response correspondence, resulting in serious issues for existing approaches to equilibrium existence based on the Kakutani fixed-point theorem or on aggregative games techniques. By contrast, our equilibrium existence results extend readily to competition in the presence of arbitrary price caps (or floors).

2 Multiproduct-Firm Oligopoly with IIA Demand

In this section, we use a potential games approach to study a multiproduct-firm pricing model where demand satisfies the IIA property, and products can be (local) complements or substitutes, depending on the level of prices. The potential games approach allows us to establish equilibrium existence under minimal assumptions.

2.1 The Model

Consider an industry with a finite set of differentiated products $\mathcal{N}$. The representative consumer's quasi-linear indirect utility is given by:

$$y + V(p) = y + \Psi\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right),$$

where y denotes income, p_j the price of product j, and Ψ and h_j are differentiable functions of a single variable. Roy's identity yields the demand for product i:

$$D_i(p) = -h_i'(p_i)\Psi'\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right).$$

Well-known special cases of this class of demand system include multinomial logit demand (with $\Psi(H) = \log(1+H)$ and $h_i(p_i) = \exp[(a_i - p_i)/(\lambda)]$) and CES demand ($\Psi = \log$ and $h_i(p_i) = a_i p_i^{1-\sigma}$). More generally, Nocke and Schutz (2018) provide necessary and sufficient conditions for this demand system to be derivable from multistage discrete/continuous choice. The choice process is sequential, with the consumer first observing the value of an outside option, and deciding whether to take it. If he does not take it, he observes a vector of product-specific taste shocks, and chooses the product that delivers the highest indirect utility. Finally, he decides how much of that product to consume. Under this micro-foundation, $\log h_j$ corresponds to the mean utility delivered by good j, whereas the function Ψ reflects the distribution of the value of the outside option. Nocke and Schutz (2018)'s necessary and sufficient conditions, which we assume to hold throughout, are:

(i) Each h_i is $\mathcal{C}^1$, strictly positive, strictly decreasing, and log-convex.

(ii) Ψ is $\mathcal{C}^1$ with non-negative derivative, and $H \mapsto H\Psi'(H)$ is non-decreasing.

To streamline the exposition, we strengthen condition (ii) slightly, imposing that Ψ' be everywhere strictly positive.

This demand system has the IIA property as

$$D_i(p)/D_j(p) = h_i'(p_i)/h_j'(p_j)$$

is independent of the price of any third product k. Despite the demand system being derivable from discrete/continuous choice, products can be complements. Specifically, products are (local) complements if Ψ' is locally increasing and local substitutes if Ψ' is locally decreasing. The reason why complementarities can arise is that a reduction in the price of good j reduces the probability that a consumer takes the outside option, thereby potentially increasing the *ex ante* choice probability for good $k \neq j$.

On the supply side, the set of firms, $\mathcal{F}$, is a partition of the set of products, $\mathcal{N}$. We assume that there are at least two firms. Firms produce under constant returns to scale; the vector of constant unit costs for all products is denoted $c = (c_j)_{j\in\mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}$.

Setting $h_j(\infty) \equiv \lim_{p_j\to\infty} h_j(p_j)$, and adopting the convention that the sum of an empty collection of reals is equal to zero, the profit of firm f is given by:

$$\pi^f(p) = \sum_{\substack{k\in f:\\ p_k<\infty}} (p_k - c_k)(-h_k'(p_k))\Psi'\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right),\ \forall p \in (0,\infty]^{\mathcal{N}},$$

As in Nocke and Schutz (2018), the compactification of action sets permitted by infinite prices will be useful to establish existence of equilibrium. The assumption is that an infinite price on a product results in zero profit from that product. As $(p_k - c_k)h_k'(p_k) \underset{p_k \to \infty}{\longrightarrow} 0$ by log-concavity of h_k, this assumption is consistent with what one would obtain if one were to take limits in the profit function, as long as the limiting vector of industry prices has at least one finite component.[6]

Firms compete by setting prices simultaneously. For every firm $f \in \mathcal{F}$, define

$$\mathcal{P}^f \equiv \left\{ p^f \in (0,\infty]^f : \quad \sum_{\substack{k\in f: \\ p_k<\infty}} (p_k - c_k)(-h_k'(p_k)) > 0 \right\}.$$

As price vectors outside $\mathcal{P}^f$ are strictly dominated for firm f, we redefine the action set of firm f as $\mathcal{P}^f$ in the following.

2.2 Equilibrium Existence: A Potential Games Approach

For every $p \in \prod_{g\in\mathcal{F}} \mathcal{P}^g$, define

$$W(p) \equiv \Psi' \left(\sum_{j\in\mathcal{N}} h_j(p_j) \right) \prod_{g\in\mathcal{F}} \sum_{\substack{k\in g: \\ p_k<\infty}} (p_k - c_k)\left(-h_k'(p_k)\right). \qquad (1)$$

With a slight abuse of notation, let (p^f, p^{-f}) be the vector of prices when firm f sets the price vector p^f and rivals set p^{-f}. Since, for every $f \in \mathcal{F}$,

$$W(p) = \pi^f(p) \times \underbrace{\prod_{g\neq f} \sum_{\substack{k\in g: \\ p_k<\infty}} (p_k - c_k)\left(-h_k'(p_k)\right)}_{>0,\text{ independent of } p^f},$$

we have that, for every $p = (p^f, p^{-f}) \in \prod_{g\in\mathcal{F}} \mathcal{P}^g$ and $p^{f\prime} \in \mathcal{P}^f$,

$$\pi^f(p^{f\prime}, p^{-f}) - \pi^f(p^f, p^{-f}) > 0 \Longleftrightarrow W(p^{f\prime}, p^{-f}) - W(p^f, p^{-f}) > 0.$$

The function $W(\cdot)$ is therefore an ordinal potential for the multiproduct-firm pricing game defined in the previous subsection.

As shown in Monderer and Shapley (1996b), the ordinal potential can be used to obtain a simple proof of equilibrium existence: If p^* solves the maximization problem

$$\max_{p\in\prod_{g\in\mathcal{F}} \mathcal{P}^g} W(p),$$

[6] Profit functions do not necessarily have a limit as *all* prices tend to infinity. Examples of demand systems where the limit does not exist include CES and the multinomial logit without outside options. See Section II.3 in the Online Appendix to Nocke and Schutz (2018) for a detailed discussion of infinite prices.

then for every $f \in \mathcal{F}$ and $p^f \in \mathcal{P}^f$,

$$\pi^f(p^{*f}, p^{*-f}) \geq \pi^f(p^f, p^{*-f}),$$

and so p^* is a Nash equilibrium. Equilibrium existence can thus be established by showing the existence of a global maximizer of the ordinal potential.

Applying this insight to our multiproduct-firm pricing game, we obtain equilibrium existence under minimal restrictions:

Proposition 1. *Suppose that Ψ is twice differentiable. Then, for any firm partition $\mathcal{F}$ and any marginal cost vector c, the associated ordinal potential function $W(\cdot)$ has a global maximizer, and the multiproduct-firm pricing game has a pure-strategy Nash equilibrium.*

Proof. See Appendix A. □

A substantial economic contribution relative to Nocke and Schutz (2018) consists in deriving equilibrium existence results allowing for complements. In the present framework, whether products are local substitutes or complements depends on the local behavior of Ψ', and thus on the level of prices. Such price-dependent patterns of complementarity/substitutability are at the core of Rey and Tirole (2019).

Importantly, the equilibrium existence result of Proposition 1 continues to hold even in the presence of arbitrary price caps and floors. Such price caps (and floors) may arise because of regulation or, as recently advocated by Rey and Tirole (2019), due to cooperative agreements. Suppose that for all $i \in \mathcal{N}$, there exists a price cap $\overline{p}_i \leq \infty$ and a price floor $\underline{p}_i \geq c_i$ such that p_i has to satisfy $\underline{p}_i \leq p_i \leq \overline{p}_i$. As this type of regulation breaks the convex-valuedness of best responses, standard approaches to equilibrium existence based on the Kakutani fixed point theorem or aggregative games techniques do not apply.[7] By contrast, the potential games approach still delivers equilibrium existence: As $\prod_{j \in \mathcal{N}} [\underline{c}_j, \overline{c}_j]$ is compact, the potential function continues to have a global maximizer, and so a Nash equilibrium exists.[8]

We close this section by discussing some of the more technical aspects of Proposition 1, providing first an overview of the key steps of its proof. Suppose first that products are never complements, so that price vectors that contain components at or below marginal cost are strictly dominated. We can thus restrict the domain of the potential function W to the set $\prod_{j \in \mathcal{N}} (c_j, \infty] \cap \prod_{f \in \mathcal{F}} \mathcal{P}^f$. The next step consists in showing that the function W can

[7] Recall from Spady (1984) and Hanson and Martin (1996) that multinomial logit profit functions can fail to be quasi-concave. In the presence of price caps or floors, this failure of quasi-concavity can result in the failure of uni-modality.

[8] More generally, an equilibrium exists provided action sets are closed.

be extended in a continuous way to the set $\prod_{j\in\mathcal{N}}[c_j,\infty]$, and has a global maximizer p^* on that set. This p^* must necessarily be an element of $\prod_{f\in\mathcal{F}}\mathcal{P}^f$, for otherwise, $W(p^*)$ would be equal to zero, and so W could not be maximized at p^*. A difficulty in establishing existence of p^* involves showing that the extension of $W(\cdot)$ to $\prod_{j\in\mathcal{N}}[c_j,\infty]$ is continuous even when all components of the price vector are infinite—see Lemma C in the appendix for details.

To allow products to be complements involves additional technical difficulties as a firm might want to price some of its products at zero (and thus below marginal cost) to boost the demand for its other products. This is problematic as the demand system is not defined at such prices. The assumption that Ψ is twice differentiable is a weak technical condition ensuring that such a pricing incentive does not exist.

A more technical contribution of Proposition 1 relative to Nocke and Schutz (2018) consists in deriving equilibrium existence under weaker regularity and monotonicity assumptions: In the baseline of Nocke and Schutz (2018), it was assumed that Ψ is equal to the logarithm, and each h_i is $\mathcal{C}^3$ and such that the elasticity of $-h_i'$ is non-decreasing.[9] Without such regularity and monotonicity assumptions, it is easy to construct examples of multiproduct-firm pricing games with IIA demand where best responses are neither convex-valued nor monotone. Despite such classic conditions failing to hold, Proposition 1 implies that those pricing games have a Nash equilibrium.

Finally, the potential games approach provides a new method to compute equilibria. Instead of solving a multidimensional fixed point problem (as with the best-response approach) or a nested fixed point problem (as with the aggregative games approach of Nocke and Schutz, 2018), it involves finding the global maximizer of the ordinal potential function W.

3 Transformed Potentials and Demand Systems

The multiproduct-firm pricing game analyzed in the previous section has an important feature: Despite that game not having a potential, the game resulting from taking a well-chosen monotone transformation of the payoff functions does have a potential.

Specifically, the normal form game with payoff functions $\log \pi^f$ for every $f \in \mathcal{F}$ has a potential: $U \equiv \log W$, where W is the ordinal potential defined in equation (1). That is, the demand system of Section 2 has the following property: There exists a transformation function G (here, $G \equiv \log$) such that—regardless of the vector of marginal costs c and of the firm partition $\mathcal{F}$—the normal form game with payoff function $G \circ \pi^f$ for every firm f has a

[9] In their online appendix, Nocke and Schutz (2018), allow Ψ to differ from the logarithm. However, their existence proof requires numerous additional technical assumptions. (See Assumption (iii) in their online appendix.)

potential. In such a case, we say that D admits a *transformed potential* or, more specifically, a *G-potential.* In this section, we fully characterize, first, the set of demand systems that admit a transformed potential and, second, the associated transformation functions.

3.1 Demand Systems Admitting Transformed Potentials: A Complete Characterization

Let the demand system D be a continuous mapping from $\mathbb{R}^{\mathcal{N}}_{++}$ to $\mathbb{R}^{\mathcal{N}}_{+}$. Let $\mathcal{Q} \equiv \{p \in \mathbb{R}^{\mathcal{N}}_{++} : D(p) \in \mathbb{R}^{\mathcal{N}}_{++}\}$ be the vector of prices at which the demand for all products is strictly positive. By continuity, $\mathcal{Q}$ is open.

We impose the following technical restrictions on the demand system D. The set $\mathcal{Q}$ is non-empty and convex. Moreover, D is $\mathcal{C}^2$ on $\mathcal{Q}$ and satisfies Slutsky symmetry and strict monotonicity: For all $p \in \mathcal{Q}$ and all $i, j \in \mathcal{N}$, $\partial_i D_j(p) = \partial_j D_i(p)$ and $\partial_i D_i(p) < 0$.[10] It also satisfies non-zero substitution almost everywhere: For all $i, j \in \mathcal{N}$ and almost every $p \in \mathcal{Q}$, $\partial_j D_i(p) \neq 0$. We also assume that for every product $i \in \mathcal{N}$ there exists a price vector p such that $\partial_i[p_i D_i(p)] < 0$; that is, the revenue on product i is not everywhere increasing in the price of that product. Slutsky symmetry and the convexity of $\mathcal{Q}$ imply the existence of a function V such that $\partial_i V(p) = -D_i(p)$ for every $p \in \mathcal{Q}$ and $i \in \mathcal{N}$. We assume that the level sets of V are connected surfaces, in the sense that any two points on the same level set can be connected by a continuously differentiable path.[11]

In the following, we seek to characterize potential functions on the set $\mathcal{Q}$.[12] We restrict attention to transformation functions G that have the following two properties: First, those functions are defined on an interval of strictly positive reals that include all attainable, strictly positive profit levels; second, those functions G are $\mathcal{C}^3$ with $G' > 0$. Given the classes of demand systems identified in the theorem below, the fact that we are not attempting to define transformation functions over non-positive reals turns out to be irrelevant.

Theorem 1. *Let D be a demand system and G a transformation function satisfying the assumptions made above. Then, the following assertions are equivalent:*

(a) *D admits a G-potential.*

[10] Notation: $\partial_i \kappa$ denotes the partial derivative of the function κ with respect to its ith argument; $\partial^2_{ij}\kappa$ denotes the cross-partial derivative with respect to the ith and jth arguments.

[11] This assumption will later allow us to invoke results by Goldman and Uzawa (1964) and Anderson, Erkal, and Piccinin (2020) to integrate systems of partial differential equations. If $\mathcal{Q} = \mathbb{R}^{\mathcal{N}}_{++}$, then the assumption is automatically satisfied if V is convex, i.e., if the demand system D can be derived from quasi-linear utility maximization.

[12] The differential techniques we are using in this paper do not allow us to deal with kinks in demand systems. Such kinks typically occur at price vectors at which the demand for one product vanishes.

(b) *One of the following assertions holds true:*

(i) For any attainable, strictly positive profit level π*,* $G(\pi) = A+B\log\pi$*. The demand system* D *takes the IIA form*

$$D_i(p) = -h_i'(p_i)\Psi'\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right). \tag{2}$$

(ii) For any attainable, strictly positive profit level π*,* $G(\pi) = A + C\pi$*. The demand system* D *takes the generalized linear form*

$$D_i(p) = -h_i'(p_i) + \sum_{j\neq i} \alpha_{ij} p_j, \tag{3}$$

with $\alpha_{ij} = \alpha_{ji}$ *for every* i, j*.*

(iii) For any attainable, strictly positive profit level π*,* $G(\pi) = A + B\log\pi + C\pi$*. The demand system* D *takes the perfect complements form*

$$D_i(p) = \gamma_i\left(\beta - \sum_{j\in\mathcal{N}} \gamma_j p_j\right), \tag{4}$$

with $\gamma_i > 0$ *for every* i *and* $\beta > 0$*.*

The theorem thus shows that D admits a G-potential if and only if one of the three following conditions holds. First, D takes the IIA form analyzed in Section 2. In this case, the function G is necessarily an affine transformation of the logarithm. Second, D takes the generalized linear form, a special case of which is the linear demand system of Bowley (1924) and Shubik and Levitan (1980). In this case, the function G is necessarily affine. Third, D takes a form that is consistent with both the IIA and generalized linear forms, thus implying that products are perfect complements and demand is linear in prices. In this case, the function G is a combination of the logarithmic form of the IIA demand and the affine form of the generalized linear demand.

We close this subsection by providing expressions for the associated potential functions. The potential function for part (b)-(i) of the theorem can be found by taking the logarithm of the ordinal potential function in equation (1):

$$U(p) = \log\Psi'\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right) + \sum_{f\in\mathcal{F}} \log\left(\sum_{j\in f}(p_j - c_j)(-h_j'(p_j))\right). \tag{5}$$

A potential function for part (b)-(ii) can be obtained by integrating the payoff gradient:

$$U(p) = \sum_{k\in\mathcal{N}}(p_k - c_k)(-h_k'(p_k)) + \frac{1}{2}\sum_{\substack{j,k\in\mathcal{N}\\ j\neq k}} \alpha_{jk} p_j p_k + \frac{1}{2}\sum_{f\in\mathcal{F}}\sum_{\substack{j,k\in\mathcal{N}\\ j\neq k}} \alpha_{jk}(p_k - c_k)^2. \tag{6}$$

Finally, potential functions for part (b)-(iii) can be obtained by linearly combining the two functions $U(p)$ defined above, *mutatis mutandis*.

3.2 Proof of the Theorem

The fact that (b) implies (a) follows immediately by noting that the gradient of the potential function defined in equation (5) (respectively, equation (6)) is equal to the payoff gradient (see Monderer and Shapley, 1996b).

In the remainder of this subsection, we show that (a) implies (b). Suppose that the demand system D admits a G-potential. We begin by introducing new notation. For every $i \in \mathcal{N}$, let $\overline{\pi}_i \equiv \sup_{p \in \mathcal{Q}} p_i D_i(p)$ be the supremum of the revenue from product i, and let $\overline{\pi} \equiv \max_{i \in \mathcal{N}} \overline{\pi}_i$. Define

$$\pi_i : \ (p, c_i) \in \{(p, c_i) \in \mathcal{Q} \times \mathbb{R}_{++} : \ p_i > c_i\} \mapsto (p_i - c_i) D_i(p).$$

The range of π_i is the open interval $(0, \overline{\pi}_i)$.

For every $\pi \in (0, \overline{\pi}_i)$, let

$$Q_i(\pi) = \{p \in \mathcal{Q} : \ p_i D_i(p) > \pi\}.$$

For every π, $Q_i(\pi)$ is non-empty and open, and the set function $Q_i(\cdot)$ is non-increasing: $Q_i(\pi) \subseteq Q_i(\pi')$ whenever $\pi \geq \pi'$. Moreover, $p \in Q_i(\pi)$ if and only if there exists $c_i < p_i$ such that $\pi_i(p, c_i) = \pi$.

Let $\overline{\Pi} \equiv \max_{i \in \mathcal{N}} \sup_{p \in \mathcal{Q}} p_{-i} \cdot D_{-i}(p)$ be the supremum of the profits of a (multiproduct) firm in the industry. (Since, by assumption, there are always at least two firms in the industry, a given firm earns profits on at most $|\mathcal{N}| - 1$ products.) The transformation function G is defined over the domain $\left(0, \overline{\Pi}\right)$. Its curvature is denoted

$$\epsilon(\pi) \equiv -\pi \frac{G''(\pi)}{G'(\pi)}.$$

Applying Theorem 4.5 in Monderer and Shapley (1996a), we show that ϵ solves a certain parameterized ordinary differential equation:

Lemma 1. *For every $i, j \in \mathcal{N}$ with $i \neq j$, for every $\pi \in (0, \overline{\pi}_i)$, and $p \in Q_i(\pi)$,*

$$\partial_j D_i \left(D_i + \pi \frac{\partial_i D_i}{D_i} \right) \left(\epsilon'(\pi) + \frac{\epsilon(\pi)(1 - \epsilon(\pi))}{\pi} \right) = \partial^2_{ij} D_i \left(1 - \epsilon(\pi)\right), \tag{7}$$

where the function D_i and its partial derivatives are all evaluated at p.

Proof. See Appendix B. □

Exploiting Lemma 1, we characterize the admissible transformation functions:

Lemma 2. *There exist constants A, B, and C such that $B + C\pi > 0$ and*

$$G(\pi) = A + B \log \pi + C\pi \tag{8}$$

for every $\pi \in (0, \overline{\pi})$.

Proof. See Appendix B. $\square$

Using again Theorem 4.5 in Monderer and Shapley (1996a) and the above transformation functions, we show that the demand system must satisfy certain partial differential equations:

Lemma 3. *If $B \neq 0$ in equation (8), then for every $p \in \mathcal{Q}$,*

$$\forall (i,j,k) \in \mathcal{N}^3 \text{ with } k \neq i,j, \quad \partial_k \frac{D_i(p)}{D_j(p)} = 0,$$

$$\forall (i,j) \in \mathcal{N}^2,\ \forall (k,l) \in (\mathcal{N} \setminus \{i,j\})^2 \quad \partial^2_{ik} \log \frac{D_j(p)}{D_l(p)} = 0.$$

If $C \neq 0$ in equation (8), then for every $i,j,k \in \mathcal{N}$ with $k \neq i,j$ and every $p \in \mathcal{Q}$, $\partial^2_{ik} D_j(p) = 0$.

Proof. See Appendix B. $\square$

Integrating the system of partial differential equations from the second part of the previous lemma (which is straightforward) as well as from the first part (which relies on earlier results by Goldman and Uzawa (1964) and Anderson, Erkal, and Piccinin (2020)) yields:

Lemma 4. *If $B \neq 0$ in equation (8), then the demand system D takes the IIA form of equation (2) on the domain $\mathcal{Q}$.*
If $C \neq 0$ in equation (8), then the demand system D takes the generalized linear form of equation (3) on the domain $\mathcal{Q}$.

Proof. See Appendix B. $\square$

The special case of part (iii) of the theorem arises when $B, C \neq 0$ in equation (8). As shown below, the resulting demand system is linear and features perfect complements:

Lemma 5. *If $B, C \neq 0$ in equation (8), then the demand system D takes the perfect complements form of equation (4) on the domain $\mathcal{Q}$.*

Proof. See Appendix B. $\square$

Finally, we show that the transformation functions identified in Lemma 2 extend to the entire domain $\left(0, \overline{\Pi}\right)$:

Lemma 6. *In the statement of Lemma 2, $\overline{\pi}$ can be replaced by $\overline{\Pi}$.*

Proof. See Appendix B. $\square$

4 Nested Demand Systems: An Exploration

In the previous section, we fully characterized the demand systems and transformation functions that give rise to a potential game *regardless of the firm partition*. In this section, we analyze whether, for a fixed firm partition $\mathcal{F}$, there are richer demand systems that induce a multiproduct-firm pricing game admitting a transformed potential.

We say that $(D, \mathcal{F})$ admits a G-potential if, for every marginal cost vector c, the multiproduct-firm pricing game with payoff function $G \circ \pi^f$ for any $f \in \mathcal{F}$ has a potential.

Proposition 2. *Let D be a demand system and G a transformation function satisfying the assumptions in Section 3.1. Let $\mathcal{F}$ be a firm partition. Then, the following assertions are equivalent:*

(a) *$(D, \mathcal{F})$ admits a G-potential.*

(b) *One of the following assertions holds true:*

(i) *For any attainable, strictly positive profit level π, $G(\pi) = A + B \log \pi$. The demand system D satisfies the following properties: For every $f, g \in \mathcal{F}$ with $f \neq g$, $i, j \in f$, and $k, l \in g$, $\partial_k D_i / D_j = 0$ and $\partial^2_{ik} \log D_j / D_l = 0$.*

(ii) *For any attainable, strictly positive profit level π, $G(\pi) = A + C\pi$. The demand system D takes the generalized linear form: For any $i \in f \in \mathcal{F}$*

$$D_i(p) = -\partial_i \psi^f(p^f) + \sum_{j \notin f} \alpha_{ij} p_j,$$

where p^f is the vector of prices set by firm f.

(iii) *For any attainable, strictly positive profit level π, $G(\pi) = A + B \log \pi + C\pi$. The demand system D takes the perfect complements form of equation (4).*

Proof. See Appendix C. □

The admissible transformation functions are the same as in Theorem 1, as is the class of demand systems identified in part (b)-(iii) of the proposition. The class of demand systems in part (b)-(ii) of the proposition is similar to before, except that the (potentially) non-linear part of D_i now depends on the vector of prices of the firm owning product i. The associated potential function is:

$$U(p) = \sum_{f \in \mathcal{F}} \left[\sum_{j \in f} (p_j - c_j) \left(-\partial_j \psi^f(p^f) \right) + \frac{1}{2} \sum_{\substack{j \in f \\ k \in \mathcal{N} \setminus f}} \alpha_{jk} p_j p_k \right].$$

In contrast to part (b)-(i) of Theorem 1, part (b)-(i) of the proposition does not fully characterize the resulting demand system, but instead provides a system of partial differential equations that the demand system must solve. Unfortunately, we have not been able to solve this system in the general case with three or more firms.

We now fully characterize part (b)-(i) of Proposition 2 for the case of two firms:

Proposition 3. *Suppose $|\mathcal{F}| = 2$. Then, $(D, \mathcal{F})$ admits a log-potential if and only if the demand system D takes the following form: For every $i \in f \in \mathcal{F}$ and $p \in \mathcal{Q}$,*

$$D_i(p) = -\partial_i \psi^f(p^f) \Psi' \left(\sum_{g \in \mathcal{F}} \psi^g(p^g) \right),$$

where p^g is the vector of prices set by firm g.

Proof. See Appendix D. □

The associated potential function is:

$$U(p) = \log \Psi' \left(\sum_{f \in \mathcal{F}} \psi^f(p^f) \right) + \sum_{f \in \mathcal{F}} \log \left(\sum_{j \in f} (p_j - c_j) \left(-\partial_j \psi^f(p^f) \right) \right). \tag{9}$$

The class of demand systems characterized in Proposition 3 permits more flexibility than that in part (b)-(i) of Theorem 1 in that the $\psi^g(\cdot)$ functions do not need to be additively separable in the components of the price vector p^g. A possible micro-foundation for this new demand system is a three-stage discrete/continuous choice process: First, the consumer decides whether to take up the outside option (as a function of the realization of the taste shock for that option); if not, second, the consumer chooses from which firm to purchase (as a function of the realizations of the taste shocks for the two firms); finally, the consumer chooses which products to purchase (and how much) from the selected firm (as a function of the product-level taste shocks). With this micro-foundation, $\log \psi^f$ can be interpreted as the consumer's mean utility of choosing firm f, whereas Ψ reflects the distribution of the taste shock for the outside option. A special case of the class of demand systems characterized in Proposition 3 is a nested multinomial logit (or nested CES) demand system where each firm owns one or several nests of products, and each nest can consist of several sub-nests, etc.

The fact that ψ^g need not be additively separable in p^g permits some substitution patterns that go beyond those implied by the IIA property. Specifically, the ratio of demands for goods i and j can depend on the price of a third product k provided that product k is owned by a firm that also owns at least one of the two products i and j.

The class of demand systems of Section 2 had the property that, at any given vector of prices, all products were either local substitutes or local complements to one another. The new class of demand systems of Proposition 3 permits more flexibility in this regard: For example, product 1 could be a complement to product 2 and a substitute to product 3, with all three products owned by the same firm, and at the same time a substitute to all products owned by the rival firm. Such demand patterns frequently arise through "one-stop shopping," where products offered by different stores are substitutes, but products offered by the same store can be complements.

Before discussing the case of three or more firms, we provide a brief description of the key steps in our proof of Proposition 3. First, we fix two products i and j that are owned by different firms, and integrate the partial differential equation $\partial^2_{ij} \log D_i/D_j = 0$ to obtain the functional equation

$$\rho_i(p_{-i})D_i(p) = \rho_j(p_{-j})D_j(p)$$

for some functions $\rho_i(\cdot)$ and $\rho_j(\cdot)$. Using Roy's identity, this functional equation can be rewritten as

$$\rho_i(p_{-i})\partial_i V(p) = \rho_j(p_{-j})\partial_j V(p),$$

where $V(\cdot)$ is the representative consumer's indirect utility function. Integrating this partial differential equation with the method of characteristics yields:

$$V(p) = \Psi\left(\psi_i(p_{-i}) + \psi_j(p_{-j}), p_{-\{i,j\}}\right),$$

for some functions $\Psi(\cdot)$, $\psi_i(\cdot)$, and $\psi_j(\cdot)$. Exploiting the other partial differential equations in part (b)-(i) of Proposition 2, we show that the arguments $p_{-\{i,j\}}$ can be dropped from the function $\Psi(\cdot)$, and that $\psi_i(p_{-i}) + \psi_j(p_{-j})$ is additively separable in the two firms' price vectors. It is that final step of the proof that is hard to generalize to the case of three or more firms.

We close this section by discussing the case of three or more firms, providing examples of rich classes of demand systems that admit a log-potential for a given firm partition f. The first example is the class of demand systems identified in Proposition 3, but with an arbitrary number of firms:

Proposition 4. *Let $\mathcal{F}$ be a firm partition. Then, $(D, \mathcal{F})$ admits a log-potential if the demand system D takes the following form: For every $i \in f \in \mathcal{F}$ and $p \in \mathcal{Q}$,*

$$D_i(p) = -\partial_i \psi^f(p^f)\Psi'\left(\sum_{g\in\mathcal{F}} \psi^g(p^g)\right).$$

Proof. The result follows immediately by noticing that the gradient of the potential function in equation (9) coincides with the payoff gradient. □

That first example has the feature that a product of firm f is an equally good substitute (or complement) to a product of firm f' as to one of firm f''. The second example relaxes that feature:

Proposition 5. *Let $\mathcal{F}$ be a firm partition and $\mathcal{E}$ a partition of $\mathcal{F}$. Then, $(D, \mathcal{F})$ admits a log-potential if the demand system D takes the following form: For every $i \in f \in e \in \mathcal{E}$ and $p \in \mathcal{Q}$,*

$$D_i(p) = -\partial_i \psi^f(p^f) \Psi^{e\prime} \left(\sum_{g \in e} \psi^g(p^g) \right) \Psi' \left[\sum_{\epsilon \in \mathcal{E}} \Psi^\epsilon \left(\sum_{g \in \epsilon} \psi^g(p^g) \right) \right].$$

Proof. The result follows immediately by defining the potential

$$U(p) = \log \Psi' \left[\sum_{\epsilon \in \mathcal{E}} \Psi^\epsilon \left(\sum_{g \in \epsilon} \psi^g(p^g) \right) \right] + \sum_{\epsilon \in \mathcal{E}} \log \Psi^{\epsilon\prime} \left(\sum_{g \in \epsilon} \psi^g(p^g) \right) + \sum_{\epsilon \in \mathcal{E}} \sum_{g \in \epsilon} \log \left(\sum_{j \in g} (p_j - c_j) \left(-\partial_j \psi^g(p^g) \right) \right)$$

and noticing that its gradient coincides with the payoff gradient. □

One interpretation for this class of demand systems is that consumers make multi-stage discrete/continuous choices, with different subsets of firms having their products in different nests. Under this interpretation, $\log \Psi^e$ is the mean utility delivered by nest e.

5 Conclusion

In this paper, we have made several contributions. First, we have proven existence of equilibrium for multiproduct-firm pricing games with IIA demand under minimal restrictions on demand. Importantly, the demand framework has the property that, depending on the level of prices, products can be local substitutes or complements. Moreover, the equilibrium existence result holds even in the presence of price caps and floors—instances in which other approaches to equilibrium existence face serious difficulties.

Second, we have introduced the novel concept of a transformed potential. The advantage of this new concept is that, in contrast to an ordinal potential, a cross-partial derivatives test is available for transformed potentials. We have fully characterized the class of demand systems admitting such a transformed potential regardless of the ownership structure, along with the associated transformation functions. Those demand systems are of the generalized linear or IIA types.

Third, for a given ownership structure, we have shown that the only adissible transformation functions are either of the linear or the logarithmic type. We have completely characterized the class of demand systems admitting a potential with a linear transformation function, as well as partially characterized the demand systems admitting a potential with a logarithmic transformation function. The latter demand systems can have nest structures, permitting patterns of substitutability and complementarity that go beyond those implied by the IIA property. For instance, products can be complements within a firm but substitutes across firms, as in models of one-stop shopping.

The system of partial differential equations that characterizes the demand systems admitting a log-potential for a given ownership structure is hard to solve in the general case of three or more firms. We conjecture that the solutions are demand systems with a (multi-level) nest structure, and only those demand systems. A formal proof of this conjecture is left for future research.

A Proof of Proposition 1

Preliminaries. We begin by introducing new notation. The function Ψ is defined over the interval $(\underline{H}, \overline{H})$, where

$$\underline{H} \equiv \sum_{j \in \mathcal{N}} h_j(\infty) \quad \text{and} \quad \overline{H} \equiv \sum_{j \in \mathcal{N}} h_j(0).$$

For every $f \in \mathcal{F}$, $p^f \in (0, \infty]^f$, and $p \in (0, \infty]^{\mathcal{N}}$, we let

$$u^f(p^f) = \sum_{\substack{k \in f \\ p_k < \infty}} (p_k - c_k)(-h_k'(p_k)),$$
$$H(p) = \sum_{j \in \mathcal{N}} h_j(p_j),$$

This allows us to rewrite firm f's profit and the ordinal potential function as follows:

$$\pi^f(p) = \Psi'(H(p)) u^f(p^f),$$
$$W(p) = \Psi'(H(p)) \prod_{g \in \mathcal{F}} u^g(p^g).$$

The following lemma will be useful to determine the limits of u^f and W as some (or all) of the prices tend to infinity:

Lemma A. *For every $j \in \mathcal{N}$ and $\alpha \in [0, 1)$,*

$$\lim_{p_j \to \infty} p_j \frac{h_j'(p_j)}{h_j(p_j)^\alpha} = 0.$$

Proof. We drop the product subscript to ease notation. Let $\alpha \in [0, 1)$ and $\phi(p) \equiv h(p)^{1-\alpha}$ for every $p > 0$. Since $(1 - \alpha)\phi'(p) = h'(p)/h(p)^\alpha$, all we need to do is show that $p\phi'(p) \underset{p \to \infty}{\longrightarrow} 0$.

As h is 0-convex (i.e., log-convex), it is ρ-convex for every $\rho \geq 0$. It follows in particular that ϕ is convex. Moreover, since h is positive and decreasing, so is ϕ. This implies that $\phi(\infty) \equiv \lim_{p \to \infty} \phi(p)$ exists and is finite.

By the fundamental theorem of calculus, we have:

$$\phi(p) - \phi\left(\frac{p}{2}\right) = \int_{p/2}^{p} \phi'(t) dt \leq \int_{p/2}^{p} \phi'(p) dt = \frac{1}{2} p \phi'(p) \leq 0,$$

where the first inequality follows by the convexity of ϕ. Since $\phi(\infty)$ is finite, we have that $\phi(p) - \phi(p/2) \underset{p \to \infty}{\longrightarrow} 0$, which implies that $p\phi'(p) \underset{p \to \infty}{\longrightarrow} 0$ by the sandwich theorem. □

For what follows, it is useful to extend the domains of the functions $u^f(\cdot)$, $H(\cdot)$, and $W(\cdot)$ to price vectors for which some of the prices are equal to zero and/or some of the firms make strictly negative profits. For every firm f, let

$$\mathcal{P}_0^f \equiv \left\{ p^f \in [0,\infty]^f : \; p_j^f > 0 \text{ for every } j \text{ such that } h_j'(0) = -\infty \right\}.$$

We begin by extending the domains of u^f and H:

Lemma B. *For every $f \in \mathcal{F}$, u^f has a continuous and real-valued extension to $\mathcal{P}_0^f$. Moreover, H has a continuous and real-valued extension to $\mathcal{P}_0 \equiv \prod_{g \in \mathcal{F}} \mathcal{P}_0^g$.*

Proof. Since u^f is additively separable in p^f, all we need to do is show that, for every $j \in f$, (i) $\lim_{p_j \to \infty}(p_j - c_j)h_j'(p_j) = 0$, and (ii) if $h_j'(p_j) > -\infty$, then $\lim_{p_j \to 0}(p_j - c_j)h_j'(p_j)$ is finite. The former follows by Lemma A, whereas the latter holds trivially.

Next, we turn our attention to H. For every j such that $h_j'(0) > -\infty$, we have that $h_j(0) < \infty$. Hence, any such h_j can be extended in a continuous and real-valued way to $[0,\infty]$. Since H is additively separable in p, the result follows. $\square$

Let us now extend the domain of W to $(0,\infty]^{\mathcal{N}}$ by defining for every p in that set

$$W(p) = \begin{cases} \Psi'(H(p)) \prod_{g \in \mathcal{F}} u^g(p^g) & \text{if } p_j \neq \infty \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, W is continuous on $(0,\infty)^{\mathcal{N}}$. The following lemma states that W is also continuous at price vectors containing infinite components, and extends its domain to $\mathcal{P}_0^* \equiv \mathcal{P}_0 \setminus \{0\}$:

Lemma C. *The function W has a continuous and real-valued extension to $\mathcal{P}_0^*$.*

Proof. Let $\hat{p} \in \mathcal{P}_0^*$. Suppose that $\hat{p}_k = 0$ for some k. By Lemma B, H is continuous at $\hat{p}$ and u^g is continuous at $\hat{p}^g$ for every firm g. We now show that $\lim_{H \to H(\hat{p})} \Psi'(H)$ exists and is finite. Since $\hat{p} \in \mathcal{P}_0^*$, we have that $\hat{p}_i > 0$ for some i. Moreover, since $\hat{p}_k = 0$, it follows that $H(\hat{p}) \in (\underline{H}, \overline{H})$. Therefore, $\lim_{H \to H(\hat{p})} \Psi'(H) = \Psi'(H(\hat{p}))$, which is indeed finite. Hence,

$$\lim_{p \to \hat{p}} W(p) = \Psi'(H(\hat{p})) \prod_{g \in \mathcal{F}} h^g(\hat{p}^g) \equiv W(\hat{p}).$$

Likewise, for every $\tilde{p} \in \mathcal{P}_0^*$ containing at least one finite component,

$$\lim_{p \to \tilde{p}} W(p) = \Psi'(H(\tilde{p})) \prod_{g \in \mathcal{F}} h^g(\tilde{p}^g) = W(\tilde{p}).$$

Hence, W is continuous on $\mathcal{P}_0^* \setminus \{(\infty,\infty,\ldots,\infty)\}$.

Finally, we show that W is continuous at $(\infty,\infty,\ldots,\infty)$. Let $(p(n))_{n \geq 0}$ be a sequence such that $p(n) \neq (\infty,\infty,\ldots,\infty)$ for every n and $\lim_{n \to \infty} p(n) = (\infty,\infty,\ldots,\infty)$. Then,

$H(p(n)) \underset{n\to\infty}{\longrightarrow} \underline{H}$. Since $H\Psi'(H)$ is positive and non-decreasing, it has a finite limit as H tends to $\underline{H}$. We have:

$$
\begin{aligned}
|W(p(n))| &= |H(p(n))\Psi'(H(p(n)))| \times \prod_{g\in\mathcal{F}} \left| \sum_{\substack{k\in g\\ p_k(n)<\infty}} \frac{(p_k(n)-c_k)(-h_k'(p_k(n)))}{H(p(n))^{\frac{1}{|\mathcal{F}|}}} \right| \\
&\leq H(p(n))\Psi'(H(p(n))) \times \prod_{g\in\mathcal{F}} \sum_{\substack{k\in g\\ p_k(n)<\infty}} \frac{|p_k(n)-c_k|\,(-h_k'(p_k(n)))}{H(p(n))^{\frac{1}{|\mathcal{F}|}}} \\
&\leq H(p(n))\Psi'(H(p(n))) \times \prod_{g\in\mathcal{F}} \sum_{\substack{k\in g\\ p_k(n)<\infty}} \frac{-p_k(n)h_k'(p_k(n))}{h_k(p_k(n))^{\frac{1}{|\mathcal{F}|}}} \\
&\underset{n\to\infty}{\longrightarrow} 0 = W(\infty,\dots,\infty),
\end{aligned}
$$

where we have used Lemma A and the fact that $\lim_{H\to\underline{H}} H\Psi'(H) \in [0,\infty)$. □

Proof of the proposition.

Proof. Let $(p(n))_{n\geq 0}$ be a sequence over $\mathcal{P}$ such that

$$\lim_{n\to\infty} W(p(n)) = \sup_{p\in\mathcal{P}} W(p).$$

For every $i\in\mathcal{N}$, the sequence $(p_i(n))_{n\geq 0}$ is either bounded or unbounded. In the former case, we can extract a subsequence that converges to some $p_i^*\in[0,\infty)$. In the latter case, we can extract a subsequence that converges to $p_i^*=\infty$. Doing so (sequentially) for every $i\in\mathcal{N}$, we obtain a subsequence $(p'(n))_{n\geq 0}$ that tends to some limiting price vector $p^*\in[0,\infty]^{\mathcal{N}}$ as n tends to infinity. To ease notation, we relabel $(p'(n))_{n\geq 0}$ as $(p(n))_{n\geq 0}$. Our goal is to show that $p^*\in\mathcal{P}$. The result will then follow by the continuity of W on $\mathcal{P}$ (see Lemma C).

We begin by showing that $p^*\in\mathcal{P}_0^*$. Clearly, $p^*\neq 0$. To see this, note that if p^* had all of its components equal to zero, then we would have $p_j(n)<c_j$ for every j for n sufficiently high, and so $p(n)$ could not belong to $\mathcal{P}$.

Assume for a contradiction that $p_i^*=0$ for a product i for which $h_i'(0)=-\infty$, and let f be the firm that owns product i. Then, $(p_i(n)-c_i)(-h_i'(p_i(n))) \underset{n\to\infty}{\longrightarrow} -\infty$. By Lemma A, $\lim_{p_j\to\infty} p_j h_j'(p_j)=0$ for every j. Hence, there exists $P_j>c_j$ such that $(p_j-c_j)(-h_j'(p_j))\leq 1$ for every $p_j\geq P_j$. Moreover, since $p_j\mapsto(p_j-c_j)(-h_j'(p_j))$ is continuous on the compact set $[c_j,P_j]$, it is bounded above by some real K_j. As $(p_j-c_j)(-h_j'(p_j))<0$ for $p_j<c_j$, this implies that $(p_j-c_j)(-h_j'(p_j))\leq\max(1,K_j)\equiv K_j'$ for every $p_j\in\mathbb{R}_{++}$. Hence, for every $n\geq 0$,

$$u^f(p^f(n)) \leq \sum_{\substack{j\in f\\ j\neq i}} K_j' + (p_i(n)-c_i)\left(-h_i'(p_i(n))\right) \underset{n\to\infty}{\longrightarrow} -\infty.$$

Thus, $u^f(p^f(n)) < 0$ for n sufficiently high, and so $p^f(n) \notin \mathcal{P}^f$, a contradiction.

Summing up, $p^* \neq 0$ and $p_j^* > 0$ for every j such that $h_j'(0) = -\infty$. It follows that $p^* \in \mathcal{P}_0^*$. Since W is continuous on $\mathcal{P}_0^*$ by Lemma C, it follows that $\sup_{p\in\mathcal{P}} W(p) = W(p^*)$.

Next, we show that $\Psi'(H(p^*))$ and $u^g(p^{g*})$ are finite and strictly positive for every firm g. Clearly, $W(p) > 0$ for every $p \in \prod_{j\in\mathcal{N}}(c_j, \infty)$, and so $W(p^*) > 0$. This implies that p^* has at least one finite component, so that $H(p^*) > \underline{H}$. Moreover,

$$H(p^*) = \sum_{\substack{j\in\mathcal{N}\\ p_j^*=0}} h_j(0) + \sum_{\substack{j\in\mathcal{N}\\ p_j^*>0}} h_j(p_j^*) < \sum_{j\in\mathcal{N}} h_j(0) = \overline{H}$$

since $p_i^* > 0$ for at least one product i and $h_j(0) < \infty$ for every product j such that $p_j^* = 0$. Hence, $H(p^*) \in (\underline{H}, \overline{H})$, and so $\Psi'(H(p^*))$ is finite and strictly positive. We therefore have:

$$W(p^*) = \underbrace{\Psi'(H(p^*))}_{\text{finite, }>0} \times \prod_{g\in\mathcal{F}} u^g(p^{g*}) > 0. \tag{10}$$

Since $p^* \in \mathcal{P}_0^*$, $u^g(p^{g*})$ is finite for every firm g. Moreover, since $p^g(n) \in \mathcal{P}^g$ for every n, p^{g*} belongs to the closure of $\mathcal{P}^g$, and so $u^g(p^{g*}) \geq 0$. Combining this with inequality (10), we conclude that $u^g(p^{g*})$ is finite and strictly positive for every g.

Finally, we show that $p_j^* > 0$ for every j. Assume for a contradiction that $p_i^* = 0$ for some product i, and let f be the firm owning that product. Since $u^f(p^{f*}) > 0$, there exists another product k owned by firm f such that $p_k^* \in (c_k, \infty)$. Since u^f is continuous, there exists $\underline{p}_k \in (c_k, p_k^*)$ such that for every $p_k \in (\underline{p}_k, p_k^*]$, $u^f(p_k, p_{-k}^{f*}) > 0$, where (p_k, p_{-k}^{f*}) denotes the vector obtained by replacing the kth element of p^{f*} by p_k. We now show that

$$W(p_k, p_{-k}^*) \leq W(p^*) \quad \forall p_k \in (\underline{p}_k, p_k^*). \tag{11}$$

To see this, define the sequence $\tilde{p}(n)$ as follows: For every $n \geq 0$ and $j \in \mathcal{N}$,

$$\tilde{p}_j(n) = \begin{cases} p_k & \text{if } j = k, \\ p_j(n) & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{p}(n) \underset{n\to\infty}{\longrightarrow} (p_k, p_{-k}^*)$. Moreover, by continuity of u^f, we have that $u^f(\tilde{p}^f(n)) > 0$ for n high enough. Since $\tilde{p}_j(n) > 0$ for every j and n, this implies that $\tilde{p}(n) \in \mathcal{P}$ for n high enough. Hence, for sufficiently high n, we have $W(\tilde{p}(n)) \leq W(p^*)$. Taking limits and using the continuity of W, we obtain condition (11).

Since $u^g(p^{g*}) > 0$ for every g, condition (11) can be rewritten as

$$\frac{\Psi'\left[H(p^*)\right] u^f(p^{f*}) - \Psi'\left[H(p_k, p_{-k}^*)\right] u^f\left(p_k, p_{-k}^{f*}\right)}{p_k^* - p_k} \geq 0.$$

Let $(p_k^n)_{n\geq 0}$ be a sequence over $(\underline{p}_k, p_k^*)$ such that $p_k^n \underset{n\to\infty}{\longrightarrow} p_k^*$. Then, for every $n \geq 0$,

$$\Psi'\left[H(p^*)\right] \frac{u^f(p^{f*}) - u^f\left(p_k^n, p_{-k}^{f*}\right)}{p_k^* - p_k^n} + u^f\left(p_k^n, p_{-k}^{f*}\right) \frac{\Psi'\left[H(p^*)\right] - \Psi'\left[H(p_k^n, p_{-k}^*)\right]}{p_k^* - p_k^n} \geq 0. \quad (12)$$

As n tends to infinity, the second term on the left-hand side tends to

$$u^f(p^{f*}) h_k'(p_k^*) \Psi''(H(p^*)),$$

where we have used the fact that h_k and Ψ' are differentiable and u^f is continuous.

As for the first term on the left-hand side, note that

$$\begin{aligned}\frac{u^f(p^{f*}) - u^f\left(p_k^n, p_{-k}^{f*}\right)}{p_k^* - p_k^n} &= \frac{(p_k^* - c_k)(-h_k'(p_k^*)) - (p_k^n - c_k)(-h_k'(p_k^n))}{p_k^* - p_k^n} \\ &= -h_k'(p_k^*) + (p_k^n - c_k) \underbrace{\frac{h_k'(p_k^n) - h_k'(p_k^*)}{p_k^* - p_k^n}}_{\equiv \delta_k^n}.\end{aligned}$$

Since h_k is convex and $p_k^n < p_k^*$, we have that $\delta_k^n \leq 0$ for every n. If $(\delta_k^n)_{n\geq 0}$ were unbounded, then we could extract a subsequence that diverges to $-\infty$. Along that subsequence, the left-hand side of condition (12) would then diverge to $-\infty$, which cannot be. It follows that $(\delta_k^n)_{n\geq 0}$ is bounded, and we can extract a subsequence that converges to some $\delta_k \in (-\infty, 0]$.

Assume for a contradiction that $\delta_k = 0$. By log-convexity of h_k, we have that, for every n,

$$\begin{aligned}0 &\leq \frac{1}{p_k^* - p_k^n}\left(\frac{h_k'(p_k^*)}{h_k(p_k^*)} - \frac{h_k'(p_k^n)}{h_k(p_k^n)}\right) \\ &= \frac{h_k'(p_k^*) - h_k'(p_k^n)}{p_k^* - p_k^n} \frac{1}{h_k(p_k^*)} + h_k'(p_k^n) \frac{1}{p_k^* - p_k^n}\left(\frac{1}{h_k(p_k^*)} - \frac{1}{h_k(p_k^n)}\right), \\ &\underset{n\to\infty}{\longrightarrow} -\frac{h_k'(p_k^*)^2}{h_k(p_k^*)^2} < 0,\end{aligned}$$

where we have taken the limit along the aforementioned subsequence and used the fact that $\delta_k = 0$. We have thus obtained a contradiction, which implies that $\delta_k < 0$.

Taking limits along the convergent subsequence in condition (12), we obtain:

$$(p_k^* - c_k) \frac{-\delta_k}{-h_k'(p_k^*)} \leq 1 - \frac{\Psi''(H(p^*))}{\Psi'(H(p^*))} u^f(p^{f*}),$$

which, since $p_k^* > c_k$ and $\delta_k < 0$, implies that

$$1 - \frac{\Psi''(H(p^*))}{\Psi'(H(p^*))} u^f(p^{f*}) > 0.$$

We now perform the same exercise for product i (for which $p_i^* = 0$). The argument used above implies that

$$\frac{\Psi'\left[H(p_i, p_{-i}^*)\right] u^f\left(p_i, p_{-i}^{f*}\right) - \Psi'\left[H(p^*)\right] u^f(p^{f*})}{p_i} \leq 0$$

for every $p_i > 0$. Let $(p_i^n)_{n\geq 0}$ be a strictly positive sequence of prices converging to zero. Using the above inequality, we obtain that, for every n,

$$\Psi'\left[H(p^*)\right] \frac{u^f\left(p_i^n, p_{-i}^{f*}\right) - u^f(p^{f*})}{p_i^n} + u^f\left(p_i^n, p_{-i}^{f*}\right) \frac{\Psi'\left[H(p_i^n, p_{-i}^*)\right] - \Psi'\left[H(p^*)\right]}{p_i^n} \leq 0. \quad (13)$$

As before, the second term on the left-hand side tends to

$$u^f(p^*)h_i'(0)\Psi''(H(p^*))$$

as n tends to infinity. Moreover,

$$\frac{u^f\left(p_i^n, p_{-i}^{f*}\right) - u^f(p^{f*})}{p_i^n} = -h_i'(0) + (p_i^n - c_i) \underbrace{\frac{h_i'(0) - h_i'(p_i^n)}{p_i^n}}_{\equiv \delta_i^n}.$$

The sequence $(\delta_i^n)_{n\geq 0}$ is non-positive. If it were unbounded, we could extract a subsequence that diverges to $-\infty$. Since $p_i^n - c_i \underset{n\to\infty}{\longrightarrow} -c_i < 0$, the left-hand side of condition (13) would then diverge to $+\infty$, which cannot be. Hence, $(\delta_i^n)_{n\geq 0}$ is bounded and we can extract a subsequence that converges to some $\delta_i \in (-\infty, 0]$.

Taking limits along the convergent subsequence in condition (13), we obtain:

$$c_i \frac{\delta_i}{-h_i'(0)} \geq 1 - \frac{\Psi''(H(p^*))}{\Psi'(H(p^*))} u^f(p^{f*}).$$

This is a contradiction since the left-hand side of the above inequality is non-positive, whereas the right-hand side is strictly positive, as shown above. Therefore, $p_j^* > 0$ for every j, and, since $u^f(p^{f*}) > 0$ for every f, we have that $p^* \in \mathcal{P}$. This concludes the proof. □

B Proof of Theorem 1

B.1 Proof of Lemma 1

Proof. Let $i, j \in \mathcal{N}$ with $i \neq j$, $\pi \in (0, \overline{\pi}_i)$, and $p \in Q_i(\pi)$. Choose $c_i > 0$ such that $\pi_i(p, c_i) = \pi$, and fix some arbitrary c_k in $(0, p_k)$ for every $k \neq i$. By Theorem 4.5 in

Monderer and Shapley (1996b) applied to the pricing game in which all firms are single-product firms and the marginal cost vector is $(c_k)_{k\in\mathcal{N}}$, we have that

$$\frac{\partial^2}{\partial p_i \partial p_j} G(\pi_i(p, c_i)) = \frac{\partial^2}{\partial p_i \partial p_j} G(\pi_j(p, c_j)).$$

Since this condition must hold for every $c_i < p_i$ and the right-hand side does not depend on c_i, we obtain

$$\frac{\partial^3}{\partial c_i \partial p_i \partial p_j} G(\pi_i(p, c_i)) = 0.$$

We have:

$$\begin{aligned}
\frac{\partial G(\pi_i(p, c_i))}{\partial c_i} &= -D_i G'(\pi_i), \\
\frac{\partial G(\pi_i(p, c_i))}{\partial c_i \partial p_j} &= -\partial_j D_i G'(\pi_i) - D_i (p_i - c_i) \partial_j D_i G''(\pi_i), \\
&= -\partial_j D_i G'(\pi_i) \left(1 - \epsilon(\pi_i)\right), \\
\frac{\partial G(\pi_i(p, c_i))}{\partial c_i \partial p_j \partial p_i} &= -\partial^2_{ij} D_i G'(\pi_i) \left(1 - \epsilon(\pi_i)\right) \\
&\quad - \partial_j D_i \left[D_i + (p_i - c_i)\partial_i D_i\right] \left[G''(\pi_i)(1 - \epsilon(\pi_i)) - G'(\pi_i)\epsilon'(\pi_i)\right], \\
&= -G' \left(\partial^2_{ij} D_i (1 - \epsilon) - \partial_j D_i \left[D_i + (p_i - c_i)\partial_i D_i\right] \left[\frac{\epsilon(1-\epsilon)}{\pi_i} + \epsilon'\right] \right), \\
&= -G' \left(\partial^2_{ij} D_i (1 - \epsilon) - \partial_j D_i \left[D_i + \pi \frac{\partial_i D_i}{D_i}\right] \left[\frac{\epsilon(1-\epsilon)}{\pi} + \epsilon'\right] \right),
\end{aligned}$$

which proves the lemma. □

B.2 Proof of Lemma 2

To prove Lemma 2, we split it into a series of technical lemmas. The case where $\epsilon(\pi) = 1$ for every $\pi \in (0, \overline{\pi})$ is trivial: There exists constants of integration A and B such that $G(\pi) = A + B \log \pi$ for every $\pi \in (0, \overline{\pi})$. Moreover, B must be strictly positive since $G' > 0$.

Next, we turn to the more-involved case where ϵ is not always equal to 1:

Lemma D. *Suppose that $\epsilon(\hat{\pi}) \neq 1$ and $\partial^2_{ij} D_i(\hat{p}) \neq 0$ for some $i, j \in \mathcal{N}$ ($i \neq j$), $\hat{\pi} \in (0, \overline{\pi}_i)$, and $\hat{p} \in Q_i(\hat{\pi})$. Then, for almost every $p \in Q_i(\hat{\pi})$, $\partial^2_{ij} D_i(p) \neq 0$. Moreover, there exists $\lambda < 0$ such that, for almost every $p \in Q_i(\hat{\pi})$,*

$$\frac{D_i(p)\partial_j D_i(p)}{\partial^2_{ij} D_i(p)} = \lambda \text{ and } \frac{\partial_i D_i(p) \partial_j D_i(p)}{D_i(p) \partial^2_{ij} D_i(p)} = \frac{1}{2},$$

and for every $\pi \in (0, \overline{\pi}_i)$,

$$\left(\lambda + \frac{1}{2}\pi\right) \left[\epsilon'(\pi) + \frac{\epsilon(\pi)(1 - \epsilon(\pi))}{\pi}\right] = 1 - \epsilon(\pi). \tag{14}$$

Proof. Since ϵ is continuous, there exists $\eta > 0$ such that $\epsilon(\pi) \neq 1$ for every $\pi \in [\hat{\pi}-2\eta, \hat{\pi}-\eta]$. Using equation (7) with price vector $\hat{p}$ and profit level π in $[\hat{\pi}-2\eta, \hat{\pi}-\eta]$, this implies that

$$\frac{\epsilon(\pi)(1-\epsilon(\pi))}{\pi} + \epsilon'(\pi) \neq 0 \tag{15}$$

for every π in that interval.

Let $p \in Q_i(\hat{\pi}-\eta)$ such that $\partial_j D_i(p) \neq 0$ (which, by assumption, holds almost everywhere) and $\pi \in [\hat{\pi}-2\eta, \hat{\pi}-\eta] \setminus \{-D_i(p)^2/\partial_i D_i(p)\}$. Combining inequality (15) with equation (7) at price vector p and profit level π, we obtain that $\partial^2_{ij} D_i(p) \neq 0$.

Using the above and rearranging terms in equation (7), we obtain that, for every $\pi \in (0, \overline{\pi}_i)$ and almost every $p \in Q_i\left(\max(\hat{\pi}-\eta, \pi)\right)$,

$$\left(\frac{D_i(p)\partial_j D_i(p)}{\partial^2_{ij} D_i(p)} + \pi \frac{\partial_i D_i(p)\partial_j D_i(p)}{D_i(p)\partial^2_{ij} D_i(p)}\right)\left(\epsilon'(\pi) + \frac{\epsilon(\pi)(1-\epsilon(\pi))}{\pi}\right) = 1 - \epsilon(\pi). \tag{16}$$

Combining equation (16) with inequality (15), this implies that

$$\frac{D_i(p)\partial_j D_i(p)}{\partial^2_{ij} D_i(p)} + \pi \frac{\partial_i D_i(p)\partial_j D_i(p)}{D_i(p)\partial^2_{ij} D_i(p)} = \frac{1-\epsilon(\pi)}{\epsilon'(\pi) + \frac{\epsilon(\pi)(1-\epsilon(\pi))}{\pi}},$$

for every $\pi \in [\hat{\pi}-2\eta, \hat{\pi}-\eta]$ and almost every $p \in Q_i\left(\hat{\pi}-\eta\right)$. Since the right-hand side does not depend on p, this implies the existence of two constants, $\lambda \neq 0$ and $\mu \neq 0$, such that

$$\frac{D_i(p)\partial_j D_i(p)}{\partial^2_{ij} D_i(p)} = \lambda \text{ and } \frac{\partial_i D_i(p)\partial_j D_i(p)}{D_i(p)\partial^2_{ij} D_i(p)} = \mu$$

for almost every $p \in Q_i\left(\hat{\pi}-\eta\right)$.

Dividing the second equality by the first and using the fact that D_i is $\mathcal{C}_1$, we obtain that $\partial_i D_i(p)/D_i^2(p) = \mu/\lambda$ for every p in $Q_i\left(\hat{\pi}-\eta\right)$. Rewriting, this means that

$$\frac{\partial}{\partial p_i}\left(\frac{1}{D_i(p)} + \frac{\mu}{\lambda} p_i\right) = 0 \tag{17}$$

for every p in $Q_i\left(\hat{\pi}-\eta\right)$.

Let $\check{p}$ in $Q_i(\hat{\pi}-\eta)$. Since that set is open, there exists $\varepsilon > 0$ such that $\mathcal{B}$, the open ball of radius ε centered in $\check{p}$, is contained in $Q_i(\hat{\pi}-\eta)$. Since equation (17) holds everywhere in $\mathcal{B}$ and $\mathcal{B}$ is convex, we obtain the existence of a $\mathcal{C}^2$ function $\phi(\cdot)$ such that

$$\frac{1}{D_i(p)} = -\frac{\mu}{\lambda} p_i + \phi(p_{-i})$$

for every $p \in \mathcal{B}$.

Using the above expression for D_i, we obtain the partial derivatives

$$\partial_j D_i = -D_i^2 \partial_j \phi \quad \text{and} \quad \partial^2_{ij} D_i = -2 D_i (\partial_i D_i)(\partial_j \phi)$$

everywhere on $\mathcal{B}$. Plugging those partial derivatives into the definition of μ yields $\mu = 1/2$. Since $\partial_i D_i < 0$, this implies that $\lambda < 0$. Plugging λ and μ into equation (16) then proves the lemma. □

Lemma E. *Suppose that, for some $i, j \in \mathcal{N}$ ($i \neq j$) and $\hat{p} \in \mathcal{Q}$, $\partial^2_{ij} D_i(\hat{p}) \neq 0$. Then, there exists $\pi \in (0, \overline{\pi}_i)$ such that $\epsilon(\pi) = 1$.*

Proof. Assume for a contradiction that $\epsilon(\pi) \neq 1$ for every $\pi \in (0, \overline{\pi}_i)$. Applying Lemma D to price vector $\hat{p}$ and every profit level $\hat{\pi} \in (0, \hat{p}_i D_i(\hat{p}))$, we obtain that $\partial^2_{ij} D_i(p) \neq 0$ for almost every p in $\mathcal{Q}$. Let $\tilde{p} \in \mathcal{Q}$ such that $\partial_i[\tilde{p}_i D_i(\tilde{p})] < 0$ (which exists by assumption). Since the partial derivative is continuous and $\partial^2_{ij} D_i(p)$ is different from zero almost everywhere, we can choose $\tilde{p}$ such that $\partial^2_{ij} D_i(\tilde{p}) \neq 0$.

Let $c_i \equiv \partial_i[\tilde{p}_i D_i(\tilde{p})]/\partial_i D_i(\tilde{p})$ and $\pi \equiv \pi_i(\tilde{p}, c_i)$, and note that $c_i \in (0, \tilde{p}_i)$ so that $\pi > 0$ and $\tilde{p} \in Q_i(\pi)$. By definition, we have that

$$D_i(\tilde{p}) + \pi \frac{\partial_i D_i(\tilde{p})}{D_i(\tilde{p})} = 0.$$

Hence, at price vector $\tilde{p}$ and profit level π, the left-hand side of equation (7) is equal to zero, whereas the right-hand side is $\partial^2_{ij} D_i(\tilde{p})(1 - \epsilon(\pi)) \neq 0$, a contradiction. □

Lemma F. *Suppose that $\epsilon(\hat{\pi}) \neq 1$ for some $i \in \mathcal{N}$ and $\hat{\pi} \in (0, \overline{\pi}_i)$. Then, $\partial^2_{ij} D_i(p) = 0$ for every $j \in \mathcal{N} \setminus \{i\}$ and $p \in Q_i(\hat{\pi})$.*

Proof. Assume for a contradiction that, for some $\hat{p} \in Q_i(\hat{\pi})$ and $j \neq i$, $\partial^2_{ij} D_i(\hat{p}) \neq 0$. By Lemma E, there exists $\tilde{\pi} \in (0, \overline{\pi}_i)$ such that $\epsilon(\tilde{\pi}) = 1$. Moreover, by Lemma D, we have that, for every $\pi \in (0, \overline{\pi}_i) \setminus \{-2\lambda\}$, $\epsilon'(\pi) = F(\pi, \epsilon(\pi))$, where

$$F(p, \eta) \equiv (1 - \eta) \left(\frac{1}{\lambda + \frac{1}{2}\pi} - \frac{\eta}{\pi} \right) \quad \forall (\pi, \eta) \in ((0, \overline{\pi}_i) \setminus \{-2\lambda\}) \times \mathbb{R}$$

and λ is a strictly negative constant.

Assume for a contradiction that $\pi_0 \equiv -2\lambda \notin (0, \overline{\pi}_i)$. Then, F is $\mathcal{C}^1$ on $(0, \overline{\pi}_i) \times \mathbb{R}$. By the Picard-Lindelöf theorem, the initial value problem $\eta' = F(\pi, \eta)$ with initial condition $\eta(\tilde{\pi}) = 1$ therefore has a unique maximal solution, and this solution is $\epsilon(\cdot)$. Note however that the constant function $\pi \in (0, \overline{\pi}_i) \mapsto 1$ is trivially a maximal solution to that initial value problem. It follows that $\epsilon(\pi) = 1$ for every π, which is a contradiction.

Hence, $\pi_0 \in (0, \overline{\pi}_i)$. Moreover, $\epsilon(\pi_0) = 1$ (see equation (14) in Lemma D). Since $\epsilon(\pi) \underset{\pi \to \pi_0}{\longrightarrow} \epsilon(\pi_0) = 1$ and $\lambda + \frac{1}{2}\pi \underset{\pi \to \pi_0}{\longrightarrow} 0$, there exists $\theta > 0$ such that $\frac{1}{\lambda + \frac{1}{2}\pi} - \frac{\epsilon(\pi)}{\pi} < 0$ for all $\pi \in (\pi_0 - \theta, \pi_0)$ and $\frac{1}{\lambda + \frac{1}{2}\pi} - \frac{\epsilon(\pi)}{\pi} > 0$ for all $\pi \in (\pi_0, \pi_0 + \theta)$.

Assume for a contradiction that $\epsilon(\pi) \neq 1$ for every $\pi \in (\pi_0 - \theta, \pi_0)$. We distinguish two cases. Assume first that $\epsilon(\pi) > 1$ for every π in $(\pi_0 - \theta, \pi_0)$. Then, $\epsilon'(\pi)$ is strictly positive for every π in that interval. Therefore, $\epsilon(\pi_0) = \lim_{\pi \uparrow \pi_0} > 1$, which is a contradiction. Assume instead that $\epsilon(\pi) < 1$ for every π in $(\pi_0 - \theta, \pi_0)$. Then, $\epsilon'(\pi) < 0$ for every such π, and so $\epsilon(\pi_0) < 1$, which is again a contradiction. Therefore, there exists $\pi_1 \in (\pi_0 - \theta, \pi_0)$ such that $\epsilon(\pi_1) = 1$.

Next, assume for a contradiction that $\epsilon(\pi) \neq 1$ for every $\pi \in (\pi_0, \pi_0 + \theta)$. If $\epsilon(\pi) > 1$ for every $\pi \in (\pi_0, \pi_0 + \theta)$, then $\epsilon'(\pi) < 0$ for every such π, and so $1 = \epsilon(\pi_0) > 1$! If instead $\epsilon(\pi) < 1$ for every $\pi \in (\pi_0, \pi_0 + \theta)$, then $\epsilon'(\pi) > 0$ for every such π, and so $1 = \epsilon(\pi_0) < 1$! Therefore, there exists $\pi_2 \in (\pi_0, \pi_0 + \theta)$ such that $\epsilon(\pi_2) = 1$.

The restriction of ϵ to $(0, \pi_0)$ solves the initial value problem $\eta' = F(\pi, \eta)$ with initial condition $\eta(\pi_1) = 1$ on the interval $(0, \pi_0)$. Since F is $\mathcal{C}^1$ on $(0, \pi_0) \times \mathbb{R}$, that initial value problem has a unique maximal solution by the Picard-Lindelöf theorem. As the constant function $\pi \in (0, \pi_0) \mapsto 1$ is trivially a maximal solution, it follows that $\epsilon(\pi) = 1$ for every $\pi \in (0, \pi_0)$. The same reasoning implies that $\epsilon(\pi) = 1$ for every $\pi \in (\pi_0, \overline{\pi}_i)$. By continuity, it follows that $\epsilon(\pi) = 1$ for every $\pi \in (0, \overline{\pi}_i)$, a contradiction. □

Lemma G. *Suppose that $\epsilon(\hat{\pi}) \neq 1$ for some $i \in \mathcal{N}$ and $\hat{\pi} \in (0, \overline{\pi}_i)$. Then, for every $\pi \in (0, \overline{\pi})$,*

$$\epsilon'(\pi) = -\frac{\epsilon(\pi)(1 - \epsilon(\pi))}{\pi}. \tag{18}$$

Proof. Let $j \in \mathcal{N} \setminus \{i\}$. By Lemma F, $\partial^2_{ij} D_i(p) = 0$ for every $p \in Q_i(\hat{\pi})$. By Lemma 1, this implies that

$$\partial_j D_i(p) \left[D_i(p) + \pi \frac{\partial_i D_i(p)}{D_i(p)} \right] \left[\frac{\epsilon(\pi)(1 - \epsilon(\pi))}{\pi} + \epsilon'(\pi) \right] = 0$$

for every $\pi \in (0, \overline{\pi}_i)$ and $p \in Q_i\left(\max(\hat{\pi}, \pi)\right)$. As $\partial_j D_i(p) \neq 0$ almost everywhere and D_i is $\mathcal{C}^1$, it follows that

$$\left[D_i(p) + \pi \frac{\partial_i D_i(p)}{D_i(p)} \right] \left[\frac{\epsilon(\pi)(1 - \epsilon(\pi))}{\pi} + \epsilon'(\pi) \right] = 0$$

for every $\pi \in (0, \overline{\pi}_i)$ and $p \in Q_i\left(\max(\hat{\pi}, \pi)\right)$. As ϵ is $\mathcal{C}^1$ and, for fixed p, the first term of the product on the left-hand side is different from zero whenever $\pi \neq -D_i(p)^2 / \partial_i D_i(p)$, the above equation reduces to $\epsilon'(\pi) = H(\pi, \epsilon(\pi))$ for every $\pi \in (0, \overline{\pi}_i)$, where

$$H: \ (\pi, \eta) \in (0, \overline{\pi}) \times \mathbb{R} \mapsto -\frac{\eta(1 - \eta)}{\pi}.$$

Assume for a contradiction that $\epsilon(\pi_0) = 1$ for some $\pi_0 \in (0, \overline{\pi}_i)$. Then, ϵ is a solution on $(0, \overline{\pi}_i)$ of the initial value problem $\eta' = H(\pi, \eta)$ with initial condition $\epsilon(\pi_0) = 1$. As H is $\mathcal{C}^1$, that problem has a unique solution on $(0, \overline{\pi}_i)$ by the Picard-Lindelöf theorem. Since the constant function $\pi \in (0, \overline{\pi}_i) \mapsto 1$ is trivially a solution, it follows that $\epsilon(\pi) = 1$ for every $\pi \in (0, \overline{\pi}_i)$, a contradiction. Therefore, $\epsilon(\pi) \neq 1$ for every $\pi \in (0, \overline{\pi}_i)$. The above argument can then be repeated for every $k \in \mathcal{N}$ to show that $\epsilon(\pi) \neq 1$ and $\epsilon'(\pi) = H(\pi, \epsilon(\pi))$ for every $\pi \in (0, \overline{\pi}_k)$. □

Lemma H. *Suppose that, for some $\hat{\pi} \in (0, \overline{\pi})$, $\epsilon(\hat{\pi}) \neq 1$. Then, there exist $A, B, C \in \mathbb{R}$ such that $B \geq 0$, $C \neq 0$, and, for every $\pi \in (0, \overline{\pi})$, $B + C\pi > 0$ and*

$$G(\pi) = A + B \log \pi + C\pi.$$

Proof. By Lemma G, ϵ solves the ordinary differential equation (18) on $(0, \overline{\pi})$. Let us re-express this differential equation in terms of G and its derivatives. As

$$\epsilon'(\pi) = \frac{\pi G''^2(\pi) - G'(\pi) \left(\pi G'''(\pi) + G''(\pi)\right)}{G'^2(\pi)},$$

we have that

$$\begin{aligned}
0 &= \epsilon'(\pi) + \frac{\epsilon(\pi)(1 - \epsilon(\pi))}{\pi} \\
&= \frac{\pi G''^2(\pi) - G'(\pi) \left(\pi G'''(\pi) + G''(\pi)\right)}{G'^2(\pi)} - \frac{G''(\pi)}{G'(\pi)} \left(1 + \pi \frac{G''(\pi)}{G'(\pi)}\right) \\
&= -\frac{1}{G'(\pi)} \left(\pi G'''(\pi) + 2G''(\pi)\right) \\
&= -\frac{1}{G'(\pi)} \frac{d^2}{d\pi^2} \left[\pi G'(\pi)\right].
\end{aligned}$$

Hence, there exist $B, C \in \mathbb{R}$ such that $\pi G'(\pi) = B + C\pi$ for every $\pi \in (0, \overline{\pi})$. Dividing by π and integrating once more yields $G(\pi) = A + B \log \pi + C\pi$ for some constant of integration A. The inequalities $B \geq 0$, $C \neq 0$, and $B + C\pi \geq 0$ follow immediately as G must be strictly increasing and $\epsilon(\hat{\pi}) \neq 1$. $\square$

B.3 Proof of Lemma 3

Proof. Let $f = \{i, j\}$ and $g = \{k, l\}$, and consider the firm partition $\mathcal{F} \equiv \{f, g, \mathcal{N} \setminus (f \cup g)\}$. For every $i' \in \mathcal{N} \setminus (f \cup g)$, fix some $c_{i'} \in (0, p_{i'})$. Let $c_j \in (0, p_j)$ and, if $i \neq j$, $c_i = p_i$. Similarly, let $c_l \in (0, p_l)$ and, if $k \neq l$, $c_k = p_l$. We have thus defined a multiproduct-firm pricing game. Note that, by construction, $\pi^f(p) \in (0, \overline{\pi}_j)$ and $\pi^g(p) \in (0, \overline{\pi}_l)$.

By Theorem 4.5 in Monderer and Shapley (1996b),[13] we have that

$$\frac{\partial^2}{\partial p_i \partial p_k} G\left[\pi^f(p)\right] = \frac{\partial^2}{\partial p_i \partial p_k} G\left[\pi^g(p)\right].$$

[13] Although Monderer and Shapley stated their theorem for uni-dimensional action sets, it is straightforward to extend it to multi-dimensional action sets.

If $i \neq j$, then

$$
\begin{aligned}
\partial^2_{ik} G(\pi^f) &= \partial^2_{ik} \left[B \log \pi^f + C \pi^f \right] \\
&= B \partial_k \frac{D_i + (p_j - c_j)\partial_i D_j}{(p_j - c_j) D_j} + C \left[\partial_k D_i + (p_j - c_j)\partial^2_{ik} D_j \right] + \\
&= B \left[\frac{1}{p_j - c_j} \partial_k \frac{D_i}{D_j} + \partial^2_{ik} \log D_j \right] + C \left[\partial_k D_i + (p_j - c_j)\partial^2_{ik} D_j \right].
\end{aligned}
$$

If instead $i = j$, then we obtain the same expression:

$$
\begin{aligned}
\partial^2_{ik} G(\pi^f) &= B \partial^2_{ik} \log D_j + C \left[\partial_k D_j + (p_j - c_j)\partial^2_{ik} D_j \right] \\
&= B \left[\frac{1}{p_j - c_j} \partial_k \frac{D_i}{D_j} + \partial^2_{ik} \log D_j \right] + C \left[\partial_k D_i + (p_j - c_j)\partial^2_{ik} D_j \right].
\end{aligned}
$$

Similarly, we obtain

$$
\partial^2_{ik} G(\pi^g) = B \left[\frac{1}{p_l - c_l} \partial_i \frac{D_k}{D_l} + \partial^2_{ik} \log D_l \right] + C \left[\partial_i D_k + (p_l - c_l)\partial^2_{ik} D_l \right].
$$

Plugging those expressions into the above condition on cross-partial derivatives and using the fact that $\partial_k D_i = \partial_i D_k$ yields:

$$
\begin{aligned}
B \left[\frac{1}{p_j - c_j} \partial_k \frac{D_i}{D_j} + \partial^2_{ik} \log D_j - \frac{1}{p_l - c_l} \partial_i \frac{D_k}{D_l} - \partial^2_{ik} \log D_l \right] & \\
+ C \left[(p_j - c_j)\partial^2_{ik} D_j - (p_l - c_l)\partial^2_{ik} D_l \right] = 0. & \quad (19)
\end{aligned}
$$

Suppose that $C \neq 0$. As Condition (19) must hold on an open set of costs c_j and c_l, we immediately obtain that $\partial^2_{ik} D_j = 0$ and $\partial^2_{ik} D_l = 0$ (regardless of whether $B \neq 0$).

The above implies that, regardless of whether $C \neq 0$, Condition (19) reduces to

$$
B \left[\frac{1}{p_j - c_j} \partial_k \frac{D_i}{D_j} + \partial^2_{ik} \log D_j - \frac{1}{p_l - c_l} \partial_i \frac{D_k}{D_l} - \partial^2_{ik} \log D_l \right] = 0.
$$

Suppose that $B \neq 0$. As the above condition must hold for an open set of costs c_j and c_l, we obtain that $\partial_k (D_i / D_j) = 0$, $\partial_i (D_k / D_l) = 0$, and $\partial^2_{ik} \log(D_j / D_l) = 0$. $\square$

B.4 Proof of Lemma 4

To prove Lemma 4, we split it into two technical lemmas. We begin by integrating the system of partial differential equations in the second part of Lemma 3:

Lemma I. *Suppose that $\partial^2_{ik} D_j(p) = 0$ for every $i, j, k \in \mathcal{N}$ with $k \neq i, j$ and every $p \in \mathcal{Q}$. Then, the demand system D takes the generalized linear form of equation* (3).

Proof. Fix some j in $\mathcal{N}$. Since $\partial_k(\partial_j D_j) = 0$ for every $k \neq j$, we have that $\partial_j D_j$ is independent of p_{-j}. Therefore, there exist functions ϕ_j and ψ_j such that $D_j(p) = \phi_j(p_j) + \psi_j(p_{-j})$ for every $p \in \mathcal{Q}$. Next, let $i \neq j$. Then, for every $k \neq i, j$, $\partial^2_{ik}\psi_j = \partial^2_{ik} D_j = 0$, implying that $\partial_i \psi_j$ does not depend on $p_{-i,j}$. Therefore, there exist functions ψ^i_j and χ such that $\psi_j(p_{-j}) = \psi^i_j(p_i) + \chi(p_{-i,j})$ for every p. Repeating this argument for every i, we can rewrite D_j as

$$D_j(p) = \phi_j(p_j) + \sum_{i \neq j} \psi^i_j(p_i)$$

for every $p \in \mathcal{Q}$.

Since, for every i and j with $i \neq j$, $\partial_i D_j = \partial_j D_i$, we have that $\psi'^i_j(p_i) = \psi'^j_i(p_j)$ for every $p \in \mathcal{Q}$. Hence, there exist scalars $\alpha_{ij} = \alpha_{ji}$ such that $\psi'^i_j(p_i) = \alpha_{ji}$ and $\psi'^j_i(p_j) = \alpha_{ij}$ for every p_i and p_j. It follows that, for some constants of integration $(\beta_{ji})_{i \neq j}$, D_j is given by

$$D_j(p) = \phi_j(p_j) + \sum_{j \neq i} \left(\alpha_{ji} p_i + \beta_{ji}\right).$$

Setting

$$h'_j(p_j) \equiv -\phi_j(p_j) - \sum_{j \neq i} \beta_{ji}$$

concludes the proof. □

Next, we turn to the system of partial differential equations in the first part of Lemma 3:

Lemma J. *Suppose that for every $p \in \mathcal{Q}$,*

$$\forall (i,j,k) \in \mathcal{N}^3 \text{ with } k \neq i,j, \quad \partial_k \frac{D_i(p)}{D_j(p)} = 0,$$

$$\forall (i,j) \in \mathcal{N}^2,\ \forall (k,l) \in (\mathcal{N} \setminus \{i,j\})^2 \quad \partial^2_{ik} \log \frac{D_j(p)}{D_l(p)} = 0.$$

Then, the demand system D takes the IIA form of equation (2).

Proof. Suppose first that $|\mathcal{N}| \geq 3$. Then, the result follows from Proposition 1 in Anderson, Erkal, and Piccinin (2020), the proof of which we replicate here. We have that, for every $p \in \mathcal{Q}$ and every $i, j, k \in \mathcal{N}$ such that $k \neq i, j$, $\partial_k(\partial_i V(p)/\partial_j V(p)) = 0$. Thus, using terminology introduced by Goldman and Uzawa (1964), the function $-V$ is strongly separable with respect to the partition $\{\{n\}\}_{n \in \mathcal{N}}$. Moreover, that function is $\mathcal{C}^3$ on $\mathcal{Q}$, its level sets are connected surfaces, and its partial derivatives are strictly positive everywhere on $\mathcal{Q}$. Theorem 1 in Goldman and Uzawa (1964) then implies that $-V$ takes the form[14]

$$-V(p) = -\Psi\left(\sum_{j \in \mathcal{N}} h_j(p_j)\right).$$

[14] Although Goldman and Uzawa stated their results for utility functions defined on the entire non-negative orthant, it is straightforward to see that their proofs continue to go through for utility functions defined over a convex subset of that orthant.

Suppose instead that $|\mathcal{N}| = 2$, and write $\mathcal{N} = \{1, 2\}$. As $\partial^2_{12} \log(D_1/D_2) = 0$, there exist functions ϕ_1 and ϕ_2 such that

$$\log \frac{D_1(p)}{D_2(p)} = \phi_1(p_1) - \phi_2(p_2)$$

for every $p \in \mathcal{Q}$. Taking exponentials, this implies that

$$\frac{D_1(p)}{D_2(p)} = \frac{\mathrm{e}^{\phi_1(p_1)}}{\mathrm{e}^{\phi_2(p_2)}}.$$

For $i = 1, 2$, let h_i be an anti-derivative of e^{ϕ_i}, so that

$$\frac{\partial_1 V(p)}{\partial_2 V(p)} = \frac{h_1'(p_1)}{h_2'(p_2)},$$

which means that there exists a function λ such that

$$\frac{\partial_1 V(p)}{h_1'(p_1)} = \lambda(p) = \frac{\partial_2 V(p)}{h_2'(p_2)}.$$

By Lemma 1 in Goldman and Uzawa (1964), there thus exists a function Ψ such that

$$V(p) = \Psi\left(h_1(p_1) + h_2(p_2)\right). \qquad \square$$

B.5 Proof of Lemma 5

Proof. Suppose first that $|\mathcal{N}| \geq 3$. Since $B \neq 0$, Lemma 4 implies the existence of functions Ψ and h_k such that D takes the IIA form of equation (2). Moreover, since $C \neq 0$, Lemma 3 implies that, for every triple of pairwise-distinct products (i, j, k) and every $p \in \mathcal{Q}$,

$$0 = \partial^2_{ik} D_j(p) = -\underbrace{h_j'(p_j) h_i'(p_i) h_k'(p_k)}_{\neq 0 \text{ since } p \in \mathcal{Q}} \Psi'''\left(\sum_{l \in \mathcal{N}} h_l(p_l)\right).$$

Hence, $\Psi'''(H) = 0$ for every H in the domain of Ψ. Since that domain is an interval, there exist constants a and b such that $\Psi'(H) = a + bH$ for every H. Moreover, $b \neq 0$ since substitution effects are non-zero almost everywhere.

By Lemma 3, we also have that, for every pair of distinct products (i, j) and $p \in \mathcal{Q}$,

$$0 = \partial^2_{ij} D_j(p) = -h_j''(p_j) h_i'(p_i) b.$$

Hence, $h_j''(p_j) = 0$ for every p_j in the domain of h_j, which is again an interval. It follows that, for some constants a_j and $b_j \neq 0$, $h_j(p_j) = a_j - b_j p_j$ for every p_j.

Hence, for every $j \in \mathcal{N}$ and $p \in \mathcal{N}$,

$$D_j(p) = b_j \left(a + b \sum_{k \in \mathcal{N}} (a_k - b_k p_k)\right) = b_j \left(\tilde{a} - b \sum_{k \in \mathcal{N}} b_k p_k\right),$$

where $\tilde{a} \equiv a + b\sum_{k\in\mathcal{N}} a_k$. Since $0 > \partial_j D_j = -bb_j^2$, the parameter b must be strictly positive. Setting $\beta \equiv \tilde{a}/\sqrt{b}$ and $\gamma_i \equiv b_i\sqrt{b}$ then yields equation (4). Since $0 < D_i/D_j = \gamma_i/\gamma_j$, all the γ-parameters must have the same sign, and we adopt the convention that they are all strictly positive. This implies that the parameter β must also be strictly positive. $\square$

B.6 Proof of Lemma 6

We assume throughout this subsection that $|\mathcal{N}| \geq 3$. (If $|\mathcal{N}| = 2$, there is nothing to prove.) We introduce new notation. For every $j \in \mathcal{N}$, let $\overline{\Pi}_{-j} \equiv \sup_{p\in\mathcal{Q}} p_{-j} \cdot D_{-j}(p)$, and note that $\overline{\Pi} = \max_{k\in\mathcal{N}} \Pi_{-k}$. For every $\pi \in (0, \overline{\Pi}_{-j})$, define the open set

$$Q_{-j}(\pi) \equiv \{p \in \mathcal{Q} : \ p_{-j} \cdot D_{-j}(p) > \pi\}.$$

Finally, for every $p \in \mathcal{Q}$ and $c \in \prod_{k\in\mathcal{N}}(0, p_k)$, let $\pi_{-j}(p, c) \equiv (p_{-j} - c_{-j}) \cdot D_{-j}(p)$ denote the profit of a hypothetical multiproduct firm owning all the products but product j. Observe that, for every $\pi \in (0, \overline{\Pi}_{-j})$ and $p \in Q_{-j}(\pi)$, there exists $c \in \prod_{k\in\mathcal{N}}(0, p_k)$ such that $\pi_{-j}(p, c) = \pi$.

We begin by proving two technical lemmas, which will be useful to prove Lemma 6:

Lemma K. *Let $j \in \mathcal{N}$ and $\hat{\pi} \in (0, \overline{\Pi}_{-j})$. Suppose that $\partial\pi_{-j}(p, c)/\partial p_i = 0$ for every $i \in \mathcal{N} \setminus \{j\}$, and every $p \in Q_{-j}(\hat{\pi})$ and $c \in \prod_{k\in\mathcal{N}}(0, p_k)$ such that $\pi_{-j}(p, c) = \hat{\pi}$. Then, there exist an open and convex set $O \subseteq Q_{-j}(\hat{\pi})$, a function $h_j(\cdot)$, and scalars $\gamma_i > 0$ ($i \neq j$) and β such that for every $p \in O$ and $i \in \mathcal{N} \setminus \{j\}$,*

$$D_i(p) = \frac{\gamma_i}{\beta + \frac{1}{\hat{\pi}}\left[\sum_{i\neq j}\gamma_i p_i - h_j(p_j)\right]}. \tag{20}$$

Moreover, $B \neq 0$ and $C = 0$ in equation (8).

Proof. Let $\hat{p} \in Q_{-j}(\hat{\pi})$. Choose $\varepsilon > 0$ sufficiently small so that O, the open ball centered in $\hat{p}$ and of radius ε, is contained in $Q_{-j}(\hat{\pi})$.

As a first step, we show that $\partial_i(D_i(p)/D_k(p)) = 0$ for every $i, k \neq j$ and $p \in O$. Let $p \in O$ and $c \in \prod_{l\in\mathcal{N}}(0, p_l)$ such that $\pi_{-j}(p, c) = \hat{\pi}$. Fix some $i, k \in \mathcal{N} \setminus \{j\}$ with $i \neq k$. For every $\delta > 0$, define the marginal cost vector $c(\delta)$ as follows: For every $l \in \mathcal{N}$,

$$c_l(\delta) \equiv \begin{cases} c_l + \delta & \text{if } l = i, \\ c_l - \delta\frac{D_i(p)}{D_k(p)} & \text{if } l = k, \\ c_l & \text{otherwise.} \end{cases}$$

Clearly, for δ sufficiently small, $c(\delta) \in \prod_{l\in\mathcal{N}}(0, p_l)$ and $\pi_{-j}(p, c(\delta)) = \hat{\pi}$. It follows that, for every such δ,

$$\begin{aligned}
0 = \frac{\partial \pi_{-j}(p, c(\delta))}{\partial p_i} &= D_i(p) + \sum_{l\neq j}(p_l - c_l(\delta))\partial_i D_l(p) \\
&= \underbrace{D_i(p) + \sum_{l\neq j}(p_l - c_l)\partial_i D_l(p)}_{=0} + \left[-\partial_i D_i(p) + \frac{D_i(p)}{D_k(p)}\partial_i D_k(p)\right]\delta \\
&= -\delta D_k(p)\partial_i \frac{D_i(p)}{D_k(p)}.
\end{aligned}$$

Hence, $\partial_i(D_i(p)/D_k(p)) = 0$.

Suppose that $B \neq 0$ in equation (8), so that, by Lemma 4, there exist functions Ψ and $(h_i)_{i\in\mathcal{N}}$ such that D takes the IIA form of equation (2). As, for every $p \in O$ and every $i, k \in \mathcal{N} \setminus \{j\}$ such that $i \neq k$,

$$0 = \partial_i \frac{D_i(p)}{D_k(p)} = \frac{h_i''(p_i)}{h_k'(p_k)},$$

the function $h_i'(\cdot)$ is constant on the projection of O onto the i-th dimension. Put $\gamma_i \equiv -h_i'(p_i) \neq 0$. We then have that, for some constant of integration α_i, $h_i(p_i) = \alpha_i - \gamma_i p_i$ for every $p \in O$. Let $\tilde{h}_j(p_j) \equiv \sum_{i\neq j}\alpha_i + h_j(p_j)$.

We have thus shown that, for every $p \in O$ and $c \in \prod_{l\in\mathcal{N}}(0, p_l)$,

$$\pi_{-j}(p, c) = \left(\underbrace{\sum_{i\neq j} p_i\gamma_i}_{\equiv P(p)} - \underbrace{\sum_{i\neq j} c_i\gamma_i}_{\equiv C(c)}\right)\Psi'\left(\tilde{h}_j(p_j) - \sum_{i\neq j}\gamma_i p_i\right).$$

Suppose that c is such that $\pi_{-j}(p, c) = \hat{\pi}$, or, equivalently,

$$(P(p) - C(c))\,\Psi'\left(\tilde{h}_j(p_j) - P(p)\right) = \hat{\pi}.$$

Then, for $i \neq j$

$$0 = \frac{\partial \pi_{-j}(p, c)}{\partial p_i} = \gamma_i \Psi'\left(\tilde{h}_j(p_j) - P(p)\right) - (P(p) - C(c))\,\gamma_i \Psi''\left(\tilde{h}_j(p_j) - P(p)\right).$$

It follows that

$$\frac{\Psi''\left(\tilde{h}_j(p_j) - P(p)\right)}{\Psi'\left(\tilde{h}_j(p_j) - P(p)\right)^2} = \frac{1}{\hat{\pi}}.$$

This means that, for every H in the non-empty interval

$$(\underline{H}, \overline{H}) \equiv \left(\inf_{p \in O} h_j(p_j) - P(p), \sup_{p \in O} h_j(p_j) - P(p) \right),$$

Ψ satisfies $\Psi''(H)/\Psi'(H)^2 = 1/\hat{\pi}$. Integrating this differential equation, we obtain that

$$\Psi'(H) = \frac{1}{\beta - \frac{H}{\hat{\pi}}}$$

for some constant of integration β. Hence, for every $i \neq j$, D_i takes the form announced in the statement of the lemma. Moreover, since $\gamma_i/\gamma_k = D_i/D_k > 0$, all the γ-coefficients must have the same sign. We adopt the convention that they are strictly positive.

Next, assume for a contradiction that $C \neq 0$ in equation (8), so that, by Lemma 4, there exist functions $(h_i)_{i \in \mathcal{N}}$ and scalars $(\alpha_{ik})_{\substack{i,k \in \mathcal{N} \\ i \neq k}}$ such that $\alpha_{ik} = \alpha_{ki}$ for every i and k, and D takes the generalized linear form of equation (3). For every $p \in O$ and every $i, k \in \mathcal{N} \setminus \{j\}$ such that $i \neq k$, we have:

$$0 = \partial_i \frac{D_i(p)}{D_k(p)} = \frac{-h_i''(p_i) D_k(p) - \alpha_{ik} D_i(p)}{D_k(p)^2}.$$

Thus,

$$\frac{D_i(p)}{D_k(p)} = -\frac{h_i''(p_i)}{\alpha_{ik}},$$

and, since the left-hand side does not depend on p_i, h_i'' must be constant on the projection of O onto the i-th dimension. Thus, for some constants γ and $\tilde{\beta}$, $h_i'(p_i) = -\tilde{\beta} - \gamma p_i$.

For every $k \in \mathcal{N}$, let $\gamma_k = \alpha_{ik}$ if $k \neq i$, and $\gamma_k = \gamma$ if $k = i$. The above analysis implies that, for every $k \in \mathcal{N} \setminus \{j\}$,

$$D_k(p) = \frac{\gamma_k}{\gamma_i} D_i(p) = \frac{\gamma_k}{\gamma} \left(\tilde{\beta} + \sum_{l \in \mathcal{N}} \gamma_l p_l \right) = \gamma_k \frac{\hat{\pi}}{\gamma} \left(\beta + \frac{1}{\hat{\pi}} \sum_{l \in \mathcal{N}} \gamma_l p_l \right),$$

with $\beta \equiv \tilde{\beta}/\hat{\pi}$. Thus, D_k takes the form

$$D_k(p) = \gamma_k \Psi' \left(\tilde{h}_j(p_j) - \sum_{l \neq j} \gamma_l p_l \right),$$

where $\tilde{h}_j(p_j) \equiv -\gamma_j p_j$ and $\Psi'(H) = \frac{\hat{\pi}}{\gamma} \left(\beta - \frac{H}{\hat{\pi}} \right)$. Yet, the analysis for the case where $B \neq 0$ in equation (8) implies that Ψ must then satisfy $\Psi''/(\Psi')^2 = 1/\hat{\pi}$ on an open set, which it clearly does not. We have thus obtained a contradiction. $\square$

Lemma L. *Let $j \in \mathcal{N}$ and $\pi \in (0, \overline{\Pi}_{-j})$. Suppose that, for some $i \in \mathcal{N} \setminus \{j\}$, $\hat{p} \in Q_{-j}(\hat{\pi})$, and $\hat{c} \in \prod_{k \in \mathcal{N}} (0, \hat{p}_k)$, we have that $\pi_{-j}(\hat{p}, \hat{c}) = \hat{\pi}$ and $\partial \pi_{-j}(\hat{p}, \hat{c})/\partial p_i \neq 0$. Then, there exist $\tilde{p} \in Q_{-j}(\hat{\pi})$ and $\tilde{c} \in \prod_{k \in \mathcal{N}} (0, \tilde{p}_k)$ such that $\pi_{-j}(\tilde{p}, \tilde{c}) = \hat{\pi}$, $\partial \pi_{-j}(\tilde{p}, \tilde{c})/\partial p_i \neq 0$, and $\partial \pi_{-j}(\tilde{p}, \tilde{c})/\partial p_j \neq 0$.*

Proof. We begin by establishing the existence of an open set $O \subseteq Q_{-j}(\pi)$ such that for every $p \in O$, there exists $c(p) \in \prod_{k \in \mathcal{N}} (0, p_k)$ such that $\pi_{-j}(p, c(p)) = \hat{\pi}$ and $\partial \pi_{-j}(p, c(p))/\partial p_i \neq 0$. For every $p \in Q_{-j}(\hat{\pi})$, define the marginal cost vector $c(p)$ as follows: For every $k \in \mathcal{N}$, $c_k(p)$ is equal to $\hat{c}_k$ if $k \neq i$, and to

$$p_i + \frac{1}{D_i(p)} \left(\sum_{l \neq i,j} (p_l - c_l) D_l(p) - \hat{\pi} \right)$$

if $k = i$. Clearly, $c(\cdot)$ is continuous and $c(p) \in \prod_{k \in \mathcal{N}} (0, p_k)$ for every p in some open neighborhood O' of $\hat{p}$. By construction, $\pi_{-j}(p, c(p)) = \hat{\pi}$ for every $p \in O'$. Moreover, since the partial derivative $\partial_i \pi_{-j}$ is continuous, there exists an open neighborhood $O \subseteq O'$ of $\hat{p}$ such that $\partial_i \pi_{-j}(p, c(p)) \neq 0$ for every $p \in O$.

Suppose that $B \neq 0$ in equation (8), so that, by Lemma 4, the demand system takes the IIA form of equation (2). Since substitution effects are non-zero almost everywhere, there exists a $p \in O$ such that $\partial_j D_i(p) \neq 0$. For every $k \in \mathcal{N} \setminus \{j\}$, we have:

$$\partial_j D_k(p) = -h'_j(p_j) h'_k(p_k) \Psi'' \left(\sum_{l \in \mathcal{N}} h_l(p_l) \right) = \frac{h'_k(p_k)}{h'_i(p_i)} \partial_j D_i(p) = \frac{D_k(p)}{D_i(p)} \partial_j D_i(p).$$

It follows that the partial derivatives $\partial_j D_k(p)$ $(k \neq j)$ all have the same sign. Since $p_k - c_k(p) > 0$ for every k, this implies that

$$\partial_j \pi_{-j}(p, c(p)) = \sum_{k \neq j} (p_k - c_k(p)) \partial_j D_k(p)$$

is different from zero.

Next, suppose that $C \neq 0$ in equation (8), so that, by Lemma 4, the demand system takes the generalized linear form of equation (3). Assume for a contradiction that for every $p \in O$ and $c \in \prod_{k \in \mathcal{N}} (0, p_k)$, $\partial_j \pi_{-j}(p, c) = 0$ whenever $\pi_{-j}(p, c) = \hat{\pi}$ and $\partial_i \pi_{-j}(p, c) \neq 0$. Let $p \in O$ and $k \neq i$ in $\mathcal{N} \setminus \{j\}$. For every $\delta > 0$, consider the marginal cost vector $\tilde{c}(\delta)$, obtained by adding δ to the i-th component of vector $c(p)$ and subtracting $\delta D_i(p)/D_k(p)$ from the k-th component. Clearly, $\pi_{-j}(p, \tilde{c}(\delta)) = \hat{\pi}$ and, by continuity of $\partial_i \pi_{-j}$, $\partial_i \pi_{-j}(p, \tilde{c}(\delta)) \neq 0$ for δ sufficiently small. Hence,

$$0 = \partial_j \pi_{-j}(p, \tilde{c}(\delta)) = \underbrace{\partial_j \pi_{-j}(p, c(p))}_{=0} + \delta \left(-\partial_j D_i(p) + \frac{D_i(p)}{D_k(p)} \partial_j D_k(p) \right).$$

It follows that, for every $p \in O$ and $k \in \mathcal{N} \setminus \{i, j\}$,

$$0 < \frac{D_k(p)}{D_i(p)} = \frac{\partial_j D_k(p)}{\partial_j D_i(p)} = \frac{\alpha_{jk}}{\alpha_{ji}},$$

so that all the partial derivatives $\partial_j D_k(p) = \alpha_{jk}$ $(k \neq j)$ have the same sign. The reasoning in the previous paragraph then implies that $\partial_j \pi_{-j}(p, c(p)) \neq 0$, which is a contradiction. $\square$

We can now prove Lemma 6:

Proof. By Lemma 2, there exist A, B, and C such that $B+C\pi > 0$ and $G(\pi) = A+B\log\pi + C\pi$ for every $\pi \in (0, \overline{\pi})$. Let $\tilde{G}(\pi) \equiv A + B\log\pi + C$ for every $\pi \in (0, \overline{\Pi})$. Our goal is to show that $G(\pi) = \tilde{G}(\pi)$ and $B + c\pi > 0$ for every $\pi \in (0, \overline{\Pi})$. Note that, once the former will have been proven, the latter will follow immediately as G must be increasing. Assume that $\overline{\Pi} > \underline{\pi}$ (otherwise, there is nothing to prove).

Assume for a contradiction that $G'(\pi') \neq \tilde{G}'(\pi')$ for some $\pi' \in (\overline{\pi}, \overline{\Pi})$, and define

$$\hat{\pi} \equiv \inf\{\pi > \overline{\pi} : \ G'(\pi) \neq \tilde{G}'(\pi)\}.$$

By continuity, $G'(\hat{\pi}) = G'(\hat{\pi})$. Let $j \in \mathcal{N}$ such that $\overline{\Pi}_{-j} > \hat{\pi}$.

Assume first that the assumptions of Lemma L are satisfied for profit level $\hat{\pi}$. Then, there exist $i \in \mathcal{N} \setminus \{j\}$, $\tilde{p} \in Q_{-j}(\hat{\pi})$, and $\tilde{c} \in \prod_{k\in\mathcal{N}}(0, \tilde{p}_k)$ such that $\pi_{-j}(\tilde{p}, \tilde{c}) = \hat{\pi}$, $\partial_i\pi_{-j}(\tilde{p}, \tilde{c}) \neq 0$, and $\partial_j\pi_{-j}(\tilde{p}, \tilde{c}) \neq 0$. For every $\pi \in (0, \overline{\Pi})$, define the marginal cost vector $c(\pi)$ as follows: $c_k(\pi)$ is equal to $\tilde{c}_k$ if $k \neq i$, and to

$$\tilde{p}_i + \frac{1}{D_i(\tilde{p})}\left(\sum_{l\neq i,j}(\tilde{p}_l - c_l)D_l(\tilde{p}) - \pi\right)$$

if $k = i$. By construction, $\pi_{-j}(\tilde{p}, c(\pi)) = \pi$ and $c(\pi) \in \prod_{k\in\mathcal{N}}(0, \tilde{p}_k)$ for π close enough to $\hat{\pi}$, say, for $\pi \in (\hat{\pi} - \eta', \hat{\pi} + \eta')$ with $\eta' > 0$. Moreover, by continuity of the partial derivatives $\partial_i\pi_{-j}$ and $\partial_j\pi_{-j}$, there exists $\eta \in (0, \eta')$ such that $\partial_i\pi_{-j}(\tilde{p}, c(\pi))$ and $\partial_j\pi_{-j}(\tilde{p}, c(\pi))$ are both different from zero whenever $\pi \in (\hat{\pi} - \eta, \hat{\pi} + \eta)$.

For every $\pi \in (\hat{\pi} - \eta, \hat{\pi} + \eta)$, consider the pricing game with marginal cost vector $c(\pi)$ and firm partition $\mathcal{F} = \{f, g\}$, where $f = \mathcal{N} \setminus \{j\}$ and $g = \{j\}$. By Theorem 4.5 in Monderer and Shapley (1996b), we have that

$$\frac{\partial^2}{\partial p_i \partial p_j} G\left[\pi_{-j}(\tilde{p}, c(\pi))\right] = \frac{\partial^2}{\partial p_i \partial p_j} G\left[\pi_j(\tilde{p}, c_j(\pi))\right] = \frac{\partial^2}{\partial p_i \partial p_j} \tilde{G}\left[\pi_j(\tilde{p}, c_j(\pi))\right],$$

where the second equality follows as $\pi_j(\tilde{p}, c_j(\pi)) < \overline{\pi}$. As

$$\frac{\partial^2}{\partial p_i \partial p_j} G\left[\pi_{-j}(\tilde{p}, c(\pi))\right] = \partial_i\pi_{-j}(\tilde{p}, c(\pi))\partial_j\pi_{-j}(\tilde{p}, c(\pi))G''(\pi) + \partial^2_{ij}\pi_{-j}(\tilde{p}, c(\pi))G'(\pi),$$

this implies that G' solves the initial value problem

$$G''(\pi) = \frac{1}{\partial_i\pi_{-j}(\tilde{p}, c(\pi))\partial_j\pi_{-j}(\tilde{p}, c(\pi))G''(\pi)}\left[\partial^2_{ij}\tilde{G}\left[\pi_j(\tilde{p}, c_j(\pi))\right] - \partial^2_{ij}\pi_{-j}(\tilde{p}, c(\pi))G'(\pi)\right]$$

on the interval $(\hat{\pi} - \eta, \hat{\pi} + \eta)$ with initial condition $G(\hat{\pi}) = \tilde{G}(\pi)$. As the right-hand side of the differential equation is $\mathcal{C}^1$ in (π, G), the Picard-Lindelöf theorem implies that that initial value problem has a unique solution. Since $\tilde{G}$ clearly solves the initial value problem (see the potential functions defined at the end of Section 3.1), it follows that G and $\tilde{G}$ coincide on $(\hat{\pi} - \eta, \hat{\pi} + \eta)$, contradicting the definition of $\hat{\pi}$.

Assume instead that the assumptions of Lemma L are not satisfied for profit level $\hat{\pi}$, so that, by Lemma K, $B \neq 0$ and $C = 0$, and the demand system is as in equation (20) on an open set $O \subseteq Q_{-j}(\hat{\pi})$. As $C = 0$, we have that ϵ, the curvature of G, is identically equal to 1 on $(0, \overline{\pi})$. Moreover, $\epsilon(\hat{\pi}) = 1$ by continuity. Let $\tilde{p} \in O$ and $\tilde{c} \in \prod_{k \in \mathcal{N}}(0, \tilde{p}_k)$ such that $\pi_{-j}(\tilde{p}, \tilde{c}) = \hat{\pi}$ and $\partial_j D_i(\tilde{p}) \neq 0$ for some $i \neq j$ (recall that substitution effects are non-zero almost everywhere). For some small enough $\eta > 0$ and every $\pi \in (\hat{\pi} - \eta, \hat{\pi} + \eta)$, define $c(\pi)$ as we did in the first part of the proof.

Applying again Theorem 4.5 in Monderer and Shapley (1996b) to the pricing game with marginal cost vector $c(\pi)$ and firm partition $\mathcal{F} = \{f, g\}$, where $f = \mathcal{N} \setminus \{j\}$ and $g = \{j\}$, and differentiating once more with respect to c_i, we obtain:

$$\frac{\partial^2}{\partial p_i \partial p_j \partial c_i} G\left[\pi_{-j}(\tilde{p}, c(\pi))\right] = 0.$$

Before computing the above partial derivative, it is useful to obtain simplified expressions for $\partial_j \pi_{-j}(p, c)$ and $\partial_i \pi_{-j}(p, c)$. Let $\Psi'(H) = 1/(\beta - H/\hat{\pi})$, and note that $\Psi''/(\Psi')^2 = 1/\hat{\pi}$. We have:

$$\begin{aligned}
\partial_j \pi_{-j}(p, c) &= \left(\sum_{k \neq j}(p_k - c_k)\gamma_k\right) \partial_j \Psi'\left(\overbrace{h_j(p_j) - \sum_{k \neq j} \gamma_k p_k}^{\equiv H}\right) \\
&= \left(\sum_{k \neq j}(p_k - c_k)\gamma_k \Psi'(H)\right) h_j'(p_j) \frac{\Psi''(H)}{\Psi'(H)} \\
&= \pi_{-j}(p, c) \frac{h_j'(p_j)\gamma_i \Psi''(H)}{\gamma_i \Psi'(H)} \\
&= \pi_{-j}(p, c) \frac{\partial_j D_i(p)}{D_i(p)}, \\
\partial_i \pi_{-j}(p, c) &= \gamma_i \Psi'(H) - \left(\sum_{k \neq j}(p_k - c_k)\gamma_k\right) \gamma_i \Psi''(H) \\
&= D_i(p) - \left(\sum_{k \neq j}(p_k - c_k)\gamma_k\right) \Psi'(H)\gamma_i \Psi'(H) \frac{\Psi''(H)}{\Psi'(H)^2} \\
&= D_i(p) \left[1 - \frac{\pi_{-j}(p, c)}{\hat{\pi}}\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial G\left[\pi_{-j}(\tilde{p}, c(\pi))\right]}{\partial c_i} &= -D_i(\tilde{p})G'(\pi), \\
\frac{\partial^2 G\left[\pi_{-j}(\tilde{p}, c(\pi))\right]}{\partial c_i \partial p_j} &= -\partial_j D_i(\tilde{p})G'(\pi) - D_i(\tilde{p})\partial_j \pi_{-j}(\tilde{p}, c(\pi))G''(\pi) \\
&= -\partial_j D_i(\tilde{p})\left[G'(\pi) + \pi G''(\pi)\right] = -\partial_j D_i(\tilde{p})\left[1 - \epsilon(\pi)\right], \\
\frac{\partial G\left[\pi_{-j}(\tilde{p}, c(\pi))\right]}{\partial c_i \partial p_j \partial p_i} &= -\partial^2_{ij} D_i(\tilde{p})\left[1 - \epsilon(\pi)\right] + \partial_j D_i(\tilde{p})\partial_i \pi_{-j}(\tilde{p}, c(\pi))\epsilon'(\pi) \\
&= -\partial^2_{ij} D_i(\tilde{p})\left[1 - \epsilon(\pi)\right] + D_i(\tilde{p})\partial_j D_i(\tilde{p})\left[1 - \frac{\pi}{\hat{\pi}}\right]\epsilon'(\pi).
\end{aligned}$$

Hence, ϵ solves the differential equation

$$\epsilon'(\pi) = \frac{\partial^2_{ij} D_i(\tilde{p})}{D_i(\tilde{p})\partial_j D_i(\tilde{p})}\hat{\pi}\frac{1 - \epsilon(\pi)}{\hat{\pi} - \pi} \tag{21}$$

on the interval $(\hat{\pi}, \hat{\pi} + \eta)$. Moreover, ϵ must satisfy the boundary condition $\lim_{\pi \downarrow \hat{\pi}} \epsilon(\pi) = 1$. Note that

$$\frac{\partial^2_{ij} D_i(p)}{\partial_j D_i(p)} = \frac{-\gamma_i^2 h_j'(p_j)\Psi'''(H)}{\gamma_i h_j'(p_j)\Psi''(H)} = -\frac{2}{\hat{\pi}} D_i(p) < 0.$$

Assume for a contradiction that $\epsilon(\pi) \neq 1$ for every $\pi \in (\hat{\pi}, \hat{\pi} + \eta)$. If $\epsilon(\pi) > 1$ everywhere on that interval, then $\epsilon'(\pi) < 0$ for every $\pi \in (\hat{\pi}, \hat{\pi} + \eta)$, which implies that $\lim_{\pi \downarrow \hat{\pi}} \epsilon(\pi) > 1$, violating the boundary condition. If instead $\epsilon < 1$, then $\epsilon'(\pi) > 0$ for every $\pi \in (\hat{\pi}, \hat{\pi} + \eta)$, which implies that $\lim_{\pi \downarrow \hat{\pi}} \epsilon(\pi) < 1$, violating again the boundary condition.

Hence, there exists a $\pi_0 \in (\hat{\pi}, \hat{\pi} + \eta)$ such that $\epsilon(\pi_0) = 1$. This means that ϵ solves the differential equation (21) with initial condition $\epsilon(\pi_0) = 1$ on the interval $(\hat{\pi}, \hat{\pi} + \eta)$. As the right-hand side of the differential equation is $\mathcal{C}^1$ in (π, ϵ), this initial value problem has a unique solution. Since the constant function $\pi \mapsto 1$ is trivially a solution, it follows that $\epsilon(\pi) = 1$ for every $\pi \in (\hat{\pi}, \hat{\pi} + \eta)$. Hence, there exist constants A' and B' such that $G(\pi) = A' + B' \log \pi$ for every $\pi \in (\hat{\pi}, \hat{\pi} + \eta)$. Since G must be $\mathcal{C}^1$ on $(\hat{\pi} - \eta, \hat{\pi} + \eta)$ and $G(\pi) = A + B \log \pi$ for every $\pi \in (\hat{\pi} - \eta, \hat{\pi})$, it follows that $A' = A$ and $B' = B$. Hence, G and $\tilde{G}$ coincide on $(0, \hat{\pi} + \eta)$, a contradiction. $\square$

C Proof of Proposition 2

TBW

D Proof of Proposition 3

TBW

References

Anderson, S. P., N. Erkal, and D. Piccinin (2020): "Aggregative games and oligopoly theory: short-run and long-run analysis," *RAND Journal of Economics*, 51(2), 470–495.

Berry, S., J. Levinsohn, and A. Pakes (2004): "Differentiated Products Demand Systems from a Combination of Micro and Macro Data: The New Car Market," *Journal of Political Economy*, 112(1), 68–105.

Berry, S. T. (1994): "Estimating Discrete-Choice Models of Product Differentiation," *RAND Journal of Economics*, 25(2), 242–262.

Berry, S. T., J. A. Levinsohn, and A. Pakes (1995): "Automobile Prices in Market Equilibrium," *Econometrica*, 63(4), 841–90.

Bliss, C. (1988): "A Theory of Retail Pricing," *Journal of Industrial Economics*, 36(4), 375–91.

Bowley, A. L. (1924): *The Mathematical Groundwork of Economics: An Introductory Treatise*. Oxford University Press, Oxford.

Caplin, A., and B. Nalebuff (1991): "Aggregation and Imperfect Competition: On the Existence of Equilibrium," *Econometrica*, 59(1), 25–59.

Champsaur, P., and J.-C. Rochet (1989): "Multiproduct Duopolists," *Econometrica*, 57(3), 533–57.

Chen, Z., and P. Rey (2012): "Loss Leading as an Exploitative Practice," *American Economic Review*, 102(7), 3462–82.

Dubey, P., O. Haimanko, and A. Zapechelnyuk (2006): "Strategic Complements and Substitutes, and Potential Games," *Games and Economic Behavior*, 54(1), 77 – 94.

Ershov, D., S. Orr, and J.-W. Laliberté (2019): "Mergers in a Model with Complementarity," Unpublished manuscript.

Ewerhart, C. (2017): "Ordinal potentials in smooth games," ECON - Working Papers 265, Department of Economics - University of Zurich.

Gallego, G., and R. Wang (2014): "Multiproduct Price Optimization and Competition Under the Nested Logit Model with Product-Differentiated Price Sensitivities," *Operations Research*, 62(2), 450–461.

Gentzkow, M. (2007): "Valuing New Goods in a Model with Complementarity: Online Newspapers," *American Economic Review*, 97(3), 713–744.

Goldman, S. M., and H. Uzawa (1964): "A Note on Separability in Demand Analysis," *Econometrica*, 32(3), 387–398.

Hanson, W., and K. Martin (1996): "Optimizing Multinomial Logit Profit Functions," *Management Science*, 42(7), 992–1003.

Iaria, A., and A. Wang (2019): "Identification and Estimation of Demand for Bundles," Unpublished manuscript.

Jensen, M. (2010): "Aggregative Games and Best-Reply Potentials," *Economic Theory*, 43(1), 45–66.

Johnson, J. P., and D. P. Myatt (2003): "Multiproduct Quality Competition: Fighting Brands and Product Line Pruning," *American Economic Review*, 93(3), 748–774.

——— (2006): "Multiproduct Cournot Oligopoly," *RAND Journal of Economics*, 37(3), 583–601.

Konovalov, A., and Z. Sándor (2010): "On Price Equilibrium with Multi-Product Firms," *Economic Theory*, 44(2), 271–292.

Kukushkin, N. (1994): "A Condition for the Existence of a Nash Equilibrium in Games with Public and Private Objectives," *Games and Economic Behavior*, 7(2), 177–192.

Miller, N. H., and M. C. Weinberg (2017): "Understanding the Price Effects of the MillerCoors Joint Venture," *Econometrica*, 85(6), 1763–1791.

Monderer, D., and L. Shapley (1996a): "Fictitious Play Property for Games with Identical Interests," *Journal of Economic Theory*, 68(1), 258–265.

Monderer, D., and L. S. Shapley (1996b): "Potential Games," *Games and Economic Behavior*, 14(1), 124 – 143.

Nevo, A. (2001): "Measuring Market Power in the Ready-to-Eat Cereal Industry," *Econometrica*, 69(2), 307–42.

Nocke, V., and N. Schutz (2018): "Multiproduct-Firm Oligopoly: An Aggregative Games Approach," *Econometrica*, 86(2), 523–557.

——— (2019): "An Aggregative Games Approach to Merger Analysis in Multiproduct-Firm Oligopoly," CEPR Discussion Papers 12905 (v. 2), C.E.P.R. Discussion Papers.

Quint, D. (2006): "Economics of Patent Pools When Some (but not all) Patents are Essential," Discussion Papers 06-028, Stanford Institute for Economic Policy Research.

——— (2014): "Imperfect competition with complements and substitutes," *Journal of Economic Theory*, 152(C), 266–290.

Rey, P., and J. Tirole (2019): "Price Caps as Welfare-Enhancing Coopetition," *Journal of Political Economy*, 127(6), 3018–3069.

Shubik, M., and R. Levitan (1980): *Market Structure and Behavior*. Cambridge, MA: Harvard University Press.

Slade, M. E. (1994): "What Does an Oligopoly Maximize?," *The Journal of Industrial Economics*, 42(1), 45–61.

Spady, R. H. (1984): "Non-Cooperative Price-Setting by Asymmetric Multiproduct Firms," Unpublished manuscript.

Stahl, K. (1982): "Location and Spatial Pricing Theory with Nonconvex Transportation Cost Schedules," *Bell Journal of Economics*, 13(2), 575–582.

Thomassen, O., H. Smith, S. Seiler, and P. Schiraldi (2017): "Multi-category Competition and Market Power: A Model of Supermarket Pricing," *American Economic Review*, 107(8), 2308–51.

Ui, T. (2001): "Robust Equilibria of Potential Games," *Econometrica*, 69(5), 1373–1380.

Whinston, M. D. (2007): "Chapter 36: Antitrust Policy toward Horizontal Mergers," in *Handbook of Industrial Organization*, ed. by M. Armstrong, and R. Porter, vol. 3, pp. 2369 – 2440. Elsevier.