Network formation and heterogeneous risks *

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Abstract

We study a new model to study the effect of contract externalities that arise through shock transmission. We model a financial network where good firms enjoy direct and indirect benefits from linking with one another. Bad risks benefit from having a connection with a good firm, but they are a cost to both direct and indirect connections. In efficient networks the good risks should form large connected components with very few bad risks attached. The equilibrium networks, on the other hand, have many more bad risks attached, they are core-periphery structures, and components are also smaller than the efficient ones. We also study extensions with heterogenous "bad risks," with diversity in the costs to good risk firms of linking with bad risks, and with incomplete information.

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1 Introduction

The interconnected structure of the economy plays a crucial role in the transmission and the effects of social and economic shocks. Previous literature in economics has allowed us to understand this role relatively well (see e.g. Acemoglu et al. 2012, Cabrales, Gale and Gottardi 2016) and to identify what are the efficient network structures from the point of view of prevention of shocks and their contagion. But we know less regarding which types of networks are formed in equilibrium and how they relate to efficient ones.

In this paper we analyze a novel, tractable model of agents interacting in a social or business network and characterize the pattern of the relationships which agents choose to establish and hence the resulting level of activity. By comparing the equilibrium to the socially efficient pattern of linkages we can then determine the effects of the contractual externalities arising from the relationships formed among agents on the overall pattern of linkages which arises. We show that the presence of heterogeneity among agents and of informational asymmetries further amplifies the adverse consequences of these externalities. We believe that the applications of this model are broad and its simplicity makes it useful in many environments. We now briefly discuss a few of them and further elaborate in the paper.

One leading application of the model is to a financial network with *borrowers* and *lenders*. Borrowers need the support of lenders to take to fruition a risky investment opportunity. Lenders not only provide capital to borrowers, but also mutual insurance and hedging among them. As a result, lenders enjoy direct and indirect benefits from linking with one another and can earn profits from the loans granted to borrowers. Borrowers, on the other hand, benefit from having a connection with some lender, which provides the funds needed to realize their opportunity. However, the establishment of a relationship with a borrower and the provision of funds this requires from the lender also constitutes a cost for all the other lenders who are directly or indirectly connected with it, as this relationship means they will be exposed to the risk of a chain of financial shocks and defaults if the investment fails to deliver. The key assumption we will make is that contracting among agents (borrowers and lenders, or among lenders) is bilateral, so that a borrower can compensate her lender for the possible direct harm inflicted, but indirect connections do not get a compensation. This externality drives an inefficiency of equilibrium networks.

Another application is to a production system where there are some (*brown*) firms whose activity has adverse consequences for the environment. These firms benefit from the presence of a direct relationship with other, more sustainable (*green*) firms. But those same brown firms generate negative externalities for the rest of the firms with which they are directly or indirectly connected (both *brown* and *green*). Connections between green firms are instead beneficial for all the (green) firms that are directly or indirectly affected. Our applications extend beyond the business realm. We can also think of the model as describing a situation where agents benefit from social interaction but this at the same time affects the chances of disease contagion. Individuals differ for their exposure to the risk of getting infected. There is a group of people whose activity is safe and, as long as they only interact with each other, have then a very limited risk of becoming infected. Social interaction among these agents is beneficial for all the ones affected. At the same time, each member of this group may enjoy forming relationships with external partners, even though they have a higher chance of getting infected. These connections bring costs to the rest of the group, because of the increased chance of disease contagion. Because of that, they affect the level of social interaction within the group.

The main result of the paper is the characterization of efficient and equilibrium networks. We refer to the lenders/green firms as the B firms, and to the borrowers/brown firms as the C firms. The efficient networks in the context of our model have the property that the B firms should form a set of minimally connected components that are symmetric (i.e. all of the same size). In each of those components C firms may also be present. Any C firm is linked to at most one B firm, while each B firm is directly connected with few C firms, sometimes none. This is because the costs of these connections are borne by all the B firms of the component and it is efficient to limit the number of C firms in a component to limit the burden of those costs falling on the component members.

Equilibrium networks, on the other hand, are quite different. As in most of the network literature, we consider a notion of equilibrium that allows for bilateral deviations to avoid trivial coordination failures. Since the formation of a linkage between two firms, when it involves a B and a C firm, entails a cost rather than a benefit for one of the two parties (the B firm), we need to allow for the possibility of monetary transfers when such linkages are considered, to compensate the B firms. We consider therefore an equilibrium notion that combines some elements of a bilateral equilibrium (Goyal, Vega Redondo 2007) and of a pairwise equilibrium (Bloch, Jackson 2007). In this notion, the network linkages and the transfers from C to B firms have to be immune to profitable individual *and pairwise* deviations. The latter means that any pair of firms contemplating a possible deviation is able to delete any subset of their existing linkages and form a new linkage between them (possibly with a transfer).

Equilibrium and efficient network structures differ in several dimensions. First, the number of B firms present in all (but at most one) components in equilibrium is smaller than the socially efficient one. In contrast, the number of C linkages formed by any B firm in equilibrium is larger than at the social optimum. This is due to the fact that when a B and a C firm decide to establish a linkage, they do not take into account the (negative) effects this linkage has on other B and C firms in the component. Interestingly, this has additional effects on the network structure: it becomes in fact less profitable for a B firm to form linkages with other B firms, since such linkages expose the firm to an excessively large number of indirect connections to C firms. As a result, B firms establish too few connections among them.

In our model we show that both the equilibrium and the efficient networks have a *core-*

periphery structure. There is group of "centrally" located agents that are all (directly or indirectly) connected among themselves (the B firms), and a group of (typically smaller) agents (the C firms) that connect to only one of the *core* agents. This structure arises almost directly from the assumptions of our model and our focus is more on the pattern and intensity of connections at the core and the periphery. It is however still worth mentioning it since this structure does appear in reality, as we refer to it when we discuss the related literature.

The various effects of the externality associated with the formation of linkages lead to a sufficiently large departure from efficiency to warrant scope for regulatory interventions. The easiest remedy would be to impose a limit on connections between B and C firms. But this may not always be implementable, as it requires significant information to be available to the regulator (particularly in the presence of additional heterogeneity among firms, as in the situations studied later in the paper). Limiting the formation in general of linkages among firms, on the other hand, may not be beneficial, since the problem lies in the types, rather than the number, of connections that are formed in equilibrium. Another possibility is to tax the transfer payments among firms, which are needed, as we argued, for linkages between B and C firms to form. Taxing them reduces the surplus generated by such linkages so that fewer of them would be formed in equilibrium. Even if the tax is restricted to be uniform, because of the informational problems mentioned above, it may still have a different effect on the various types of firms than a regulation on the number of linkages.

We also examine the consequences of introducing additional forms of heterogeneity among firms. We study first the case where C firms are of various types, that differ in the relative magnitude of the cost suffered by B firms when a direct and an indirect relationship with them is established. This generates an additional dimension in which equilibrium networks differ from socially efficient ones and thus expands the efficiency gap. We show that in equilibrium B firms choose to connect more with those C firms for whom the costs of direct connections are lower while those of indirect connections are higher. In contrast, efficiency requires that B firms should connect primarily with different types of C firms, those featuring lower costs of indirect connections even though direct connections are more costly.

We then explore the consequences of another type of heterogeneity, this time concerning B firms, and of the possible presence of informational asymmetries. B firms now differ in the magnitude of the costs they bear for any (direct and indirect) linkage to C firms. This induces some heterogeneity in the propensity of B fims to entertain relationships with C firms: the low cost B's prefer to have more C's attached to them, while the high cost B's prefer less C's. It in turn affects the desirability of different types of B firms to become partners of a B firm. When the types of the B firms are public information, the equilibrium exhibits assortative matching among the different types of B firms. The high cost types form "closed" components, composed only of high types, so as to benefit of their lower propensity to establish C connections.

On the other hand, when the type of a B firm is only privately observed by it, the formation of linkages among B firms becomes more complex. A B firm in this case cannot condition its proposals of link formation to the type of its counterpart. As a consequence the type of the firms they are matched with becomes uncertain. In equilibrium, the expectation of B firms about it properly reflects the distribution of requests for linkages expressed by the population of B types. In this case, the low cost B firms choose to form an even larger number of connections with other B firms than when types are observable, because they anticipate that now they may end up being connected with some high cost B firms who form fewer C connections and thus carry lower externalities. The opposite happens for high cost types. Both effects contribute to worsen the average quality - as a counterpart - of B firms wanting to establish connections. This in turn affects the overall pattern of connections that are formed in the system. We show that total connectivity is reduced as a result of the presence of asymmetric information, as the low types reduce their connection by more than what the high types increase.

Our paper is organized as follows. Section 2 discusses the related literature. Section 3 describes the main model, with its microfoundations, and analyzes its efficient and equilibrium outcomes. Section 4 does the same for a model with heterogeneous C firms. Section 5 concentrates in a model with different B firms, and does so both under symmetric and asymmetric information about the types of those firms. Section 6 concludes.

2 Related literature

We organize our review of connections with the literature according to the different parts of our results.

Our findings on the efficiency properties of equilibria for the baseline model, without heterogeneities, are related with the ones obtained by Faria-e-Castro (2015) for a quantitative model of the interbank market. In his case the inefficiencies are a consequence of limited liability, and only lead to excessive risk-taking, not to lower connections between central agents in the system, as we show. Farboodi (2023) is also related. In her model lenders benefit by joining larger components in order to hedge the risk that investment opportunities do not arise at the borrowers they are directly linked to. At the same time, this insurance can only occur by forming intermediation chains. It is efficient that intermediation is operated by lenders, not by potential borrowers, as this lowers default. Because of the way in which gains from trade are assumed to be shared, efficient intermediation does not always obtain in equilibrium. In contrast, we do not have intermediation. Hedging is ensured by the formation of linkages among lenders and we show that too few of such linkages are formed because each individual lender benefits by establishing an excessively large number of relationships with borrowers.

From an empirical viewpoint, the core-periphery structure which we show characterizes equilibrium networks, is typical of real-life interbank markets (see, for example, 't Veld, and Van Lelyveld (2014) for the Netherlands, Craig and von Peter (2014) for Germany).[1] Our findings are also consistent with the findings of Silva et al. (2016), who use stochastic frontier analysis to show empirically that core–periphery structures are risk-taking inefficient, because they add higher systemic risk levels in the financial system. A more recent paper, by Coen and Coen (2023) considers a stylized model of interbank borrowing and lending where risk contagion generates a contractual externality analogous to the one in our paper. They then estimate the model structurally, and thus the can quantify the size of the welfare losses due to these externalities, and propose public policy measures to mitigate these losses.

As we also argue in this paper, the risk of contagion has important effects on the pattern of social connections that are formed even beyond financial markets. Alfaro et al. (2024) construct a quantitative model of disease contagion and endogenous social interaction, and show that the level of interaction is heavily influenced by the probability of contagion, even in the presence of altruistic agents that partially internalize the effect of their contagion externality. They do not consider a network model, but one where externalities relate to aggregate behavior. In terms of empirical findings, Eisenberg et al. (2013) show that the fear of contagion in mental health settings leads to lower social interactions among college roommates.

The effects we find are important even at the level of the evolution of species. Social interaction can be useful, for example, to exchange information. But it brings the possibility of disease contagion. Cooney et al. (2022) show, in a quantitative evolutionary model, that this tradeoff is relevant. They establish numerically that "For a wide range of assumptions about the benefits and costs of infection [...] the evolutionarily-stable sociality strategy (ESS) is distinct from the collective optimum—the level of sociality that would be best for all individuals."

An important feature of our analysis is the consideration of rich dimensions of heterogeneity among the agents present in the network. There is very little work in the literature related to this issue in the context of network formation. We should mention two papers (though rather distantly related) who also point out the effects arising from agents' heterogeneity. In a context of disease contagion, Angrisani, Samek and Serrano Padial (2024) estimate a heterogeneous-agent model (where heterogeneity concerns both infection risk and the cost of behavior adoption) and show that the existence of misspecified beliefs entail higher infection rates. The existence of heterogeneity is crucial in their results. For example, the presence of only a fraction of rational agents disincentivizes other agents' from adopting the effective behaviors in limiting contagion. Ba et al. (2020) explore theoretically

[1] The connection to our model goes even further. Craig and Ma (2020) have identified a group of banks in the German interbank market that are consistent lenders and another group that are consistent borrowers. The lenders in the German interbank system also have relationships between themselves (Craig and Ma (2020), figure 1).

and empirically a model with heterogeneous financial risks. They show that institutions with a higher probability of a very large shock dominate the bankruptcy risk of the market.

Our results on the consequences of asymmetric information for network formation are somewhat in the spirit of Akerlof: the high propensity of low types (with low cost) to establish linkages deter high types from establishing connections. This further reduces the size of components that are formed in equilibrium and at the same time lowers the quality of connections. This result resonates with the emphasis of Acemoglu et al. (2020) on the anticipation of shocks as a way to generate market freezes. Unlike in their case, in our model "freezes" arise because of heterogeneities in the ability of different agents to be weather shocks. Our view is consistent with the evidence presented by Gabrieli and Georg (2014) that market freezes in the Eurosystem did not affect all institutions equally, as some were almost unaffected, whereas others could not operate. This resonates well with our assumption that some B firms have a lower cost of interacting with others. It is also consistent with animal behavior in disease contagion contexts. Lopes, Block and König (2016) show experimentally that when manipulating the disease susceptibility of different animals, "immune-challenged animals showed reduced connectivity to their social groups" as predicted by our model.

3 Benchmark model

In this model we have two types of agents located on a network. One type of agents, of which there are N, we denote as B. The other type of firms, of which there are FN, we denote as C. We should think of the B agents as being large relative to the C ones. There are a number N of B agents and FN of C agents. The B agents are intermediaries who provide services to the C agents. We model the relationships existing between C and B firms in terms of undirected linkages in a network. Our main assumptions are that a linkage to a B agent is always beneficial for its counterpart, both when it is a C or a B agent, while a linkage to a C agent always entails some cost for its counterpart. Furthermore, not only the direct linkages to C agents but also the indirect ones entail costs. The essence is that some agents - like the C ones - create a negative externality on everyone that is directly or indirectly connected to them, while other agents - the B ones - generate a positive externality. There may then be a net benefit in establishing a linkage between a B and a C agent or between a pair of B agents, but the formation of this linkage in turn affects the profitability of the other linkages of these agents as well as of other agents' linkages.

There are different possible applications of this model. One is the interbank market. As mentioned in the introduction, Craig and Ma (2020) showed that for the German interbank market some banks are most of the time lenders and others are most of the time borrowers. The borrowers, our C firms, are banks which identify local investment opportunities and invest in them, but lack adequate local funding and need the support of external lenders, the B firms, to fund those opportunities. The latter firms are also banks but are able to operate on a larger scale and have so greater access to funds. The investments can experience negative shocks which generate financing problems not only for the C firms originating those loans but also for the B firms which contribute to their funding and may then extend to their own B partners. This in turn will affect other C firms, which rely for funding on such B firms which are only indirectly affected by the shocks.

A different interpretation is that C firms are climate-change-inducing, *brown* firms.[2] Each of them benefits from a contact with a clean *green* B firm (which, say, supplies them an input). The C agents need not be firms. They could also be individuals working in environments with high probability of direct exposure to viruses. B agents work instead in environments where virus contagion can only be indirect (i.e. through contacts with C agents).

We focus our attention on a specification of the model that embeds the general elements outlined above, but makes some simplifications for tractability of the analysis. We assume C firms never benefit from having more than one relationship with B banks. This assumption is in line with the evidence that relationship banking, which necessarily limits the number of banks with whom a firm (a borrower, or another, usually smaller, bank) is connected, are good for efficiency (Elyasiani and Goldberg 2004). It is also fairly descriptive of some banking systems, as the main bank system in Japan (Hoshi 1995) or Germany (Behr and Guettler 2007).[3] The other main simplification concerns the linearity of the costs of linkages. A detailed microfoundation for our specific model will be provided in Section 3.1.

The key, payoff relevant features of a network are then described by the following variables: for each B firm i, n_i^{BD} describes the number of B firms and n_i^{CD} the number of C firms to which firm i is directly linked, while n_i^{BI} and n_i^{CI} are the numbers of B and of C firms indirectly connected to firm i. The indirect connections arise from the linkages established by the B firms which are directly linked with i as well as by their partners. For each C firm j, n_j^{CBD} describes the number of C firms (not including j) who are all directly connected to the same B firm as j (so they are indirectly connected among them, with paths of distance 2 going through the same B).

Since a linkage between a B and a C firm always entails a cost for the first one and a possible benefit for the latter, to allow for the possibility that such link may be formed in a decentralized, network formation process, we must also allow for transfers made by C to B firms to compensate them for their costs.

The payoff of each firm depends then on the properties of the network described by the above parameters and the transfers made to or received from other firms. Denote by Γ the network formed among the firms in the economy and t the vector of transfers received or paid by the various firms. The payoffs for a generic B firm i and a generic C firm j are, respectively:

[2] Or they produce other environmental problems.

[3] Another way to think about the model, as in Huremovic et al. (2020), is a situation in which C firms are non-financial firms and B firms are the banks that supply them credit.

$$u_{B,i}(\Gamma,t) = g\left(n_i^{BD} + n_i^{BI}\right) - c\left(n_i^{CI} + n_i^{CD}\right) + t_i - \alpha l_i \tag{1}$$

$$u_{C,j}(\Gamma,t) = f\left(n_j^{BD}\right) - cn_j^{CBD} - t_j \tag{2}$$

with

$$f\left(n_j^{BD}\right) = \begin{cases} K_B \text{ if } n_j^{BD} \geq 1 \\ 0 \text{ if } n_j^{BD} = 0 \end{cases}$$

for some $K_B > 0$, and $g(.)$ being a differentiable, increasing and strictly concave function. The functions $g(.)$ and $f(.)$ capture the benefits, respectively for the B and the C firms, of their linkages to other B firms, while c captures the cost of the linkages to other C firms.

More specifically, the $g(.)$ function describes the benefits a generic B firm i obtains from its direct and indirect linkages to other B firms. All such linkages provide a benefit to i, decreasing in the total number of existing linkages. Also, for simplicity the benefit of direct and indirect linkages is assumed the same.[4] The B firms are the "normal", non-externality inducing agents.

The benefits for a C firm, described by the $f(.)$ function, are positive and equal to K_B as long as there is a direct linkage to at least one B firm (multiple linkages entail no additional benefits). Thus the C firm just benefits from one connection, as we can say what counts is gaining access to the market. The connection with a C firm in turn generates a cost not only for the B firm that is its direct counterpart but also for the other C firms directly linked with that B firm and for the other B firms with an indirect linkage to the C firm. We assume for simplicity these direct and indirect costs are identical and the unit costs are also the same and equal to c for B and C firms. More details on how such cost specification can arise are provided in Section 3.1.

The variable t_j denotes the net transfer paid by a C firm j to the B firm to whom j is directly linked, to compensate such firm for the services and costs it has to incur because of this linkage (recall the B firm receives no benefit, only direct and indirect costs from the linkage with a C firm); t_i is then the total transfer received by B firm i.

Finally, l_i denotes the number of direct linkages that the B type firm i has with other B firms and α the cost to be paid for those - undirected - links. We should think of α as a small positive number. In the environment considered, where the benefits of direct and indirect linkages are the same, this feature generates a clear preference for forming indirect rather than direct connections with other B firms, or for minimally connected components among them.

[4] We can interpret the linkages to other B firms as providing risk sharing/trading possibilities among them (hence the concavity), something quite reasonable for big intermediaries in the interbank market (as mentioned earlier, we supply a specific context where this is true in section 3.1).

3.1 Model microfoundation

In this section we provide a more explicit microfoundation for the payoffs of the model, by focusing first on our leading example where the linkages are credit relationships among financial intermediaries and with firms.

3.1.1 Banks, lending and risk sharing

Shocks, funds and priorities Suppose each B firm/bank i has one unit of funds, which can be used to meet the funding needs of C firms (they could be other, "local", banks or also firms) or of other B firms. Requests for such funds may come from the C firms directly connected to i, when hit by a shock, or from other B firms directly or indirectly connected to i which may need some funds to help them to meet the requests coming from their direct C contacts as well as to face shocks which may hit them.

There are three types of shocks that may hit the different firms.

1. The C firms can be stressed by a *small shock*: any C firm is hit by such shock with probability p and this event is i.i.d. across firms. The firm that is hit must have an amount k (< 1) of funds available, to stay in business.

2. The C firms can also receive a *large shock*: any C firm is hit by a large shock with probability q_C. To survive a large shock, the firm must have 1 unit of funds.

3. The B firms can in turn be stressed by a *big shock:* any B firm is hit by a large shock with probability q_B. To stay solvent, the B firm that receives the shock must have an amount of funds greater or equal than the value of the shock, given by a realization of a Pareto distribution with parameter γ (with $\gamma > 1$).

We assume that at most one big shock or one large shock may happen at a time. That is, either one B firm is hit by a big shock, or one C firm receives a large shock, but not both and possibly none. The intuition for this stochastic structure is that large C shocks or big B shocks are rather uncommon.[5] In contrast, small C shocks are common and easily diversifiable.

The priority for the use of the funds available in a connected component is given to the requests of funds coming from C firms hit by small shocks. Hence we have:

1. The unit of funds available to each B firm is used in the first place to meet the needs of the C firms directly connected to B who are hit by a small shock. We assume $Fk < 1$ so that the funds available are always sufficient to cover these shocks.

[5] We should think of q_C as sufficiently small so that $(1 - q_C)^{FN}$ is close to one, or that there is some negative correlation that forces at most a single large shock in the system to occur. Similarly for the big B shocks.

2.a If a C firm is also hit by a large shock, this firm can use all the remaining funds (left after the payments due to face the small shocks) of the B firm directly connected to C. If these funds are not sufficient to meet the large shock, the affected C firm has to go to the market, which entails an extra cost d per unit solicited (on account of potential adverse selection since the market does not have an ongoing relationship with that firm).

2.b If instead one of the B firms receives a big shock, the firm can only rely on the funds remaining in the other B firms lying in the same component, after meeting the requests coming from C firms to face their small shocks. The B firms are unable to access the market because they are larger and have a more complex balance sheet, so informational asymmetries are potentially much larger. They have instead an ongoing relationship, which mitigates the informational asymmetries, with the other B firms to whom they are directly or indirectly connected.

Costs as a function of the size of the components The structure of the shocks and the priorities in access to available funds described in the previous subsection determine the expected payoffs of the firms in a component, given by the benefits of staying in business minus the costs due to the payments which need to be made. Such payoffs clearly depend on the network structure.

Costs and benefits for the C firms Under the stated assumption $Fk < 1$, the funds of the B firm are always sufficient to meet the small shock hitting C firms with a direct linkage to B, hence the C firms suffer no cost when hit by a small shock (provided they are connected to a B firm). If a number f of the C firms (directly) connected to the B firm are stressed with a small shock, they need a total amount fk of funds to face the liquidity needs coming from these small shocks. This leaves the B firm a positive amount of funds equal to $1 - fk$. If one of these C firms is also hit by a large shock, it can access the funds $1 - fk$ remaining at the B firm to which it is directly connected. The C firm has so a shortfall of fk. It cannot use the funds left at other, only indirectly connected, B firms but can obtain fk in the market at unit cost d.

That means a C firm j, when linked to some B firm, always survives the shocks which may hit it. Hence the C firm always earns K_B from staying in business, but faces a cost equal to the expected payment it needs to make to get the funds it needs in the market. This cost depends on the network structure, in particular on the number n_j^{CBD} of C firms who are sharing with the C firm j a relationship with the same B firm, and is equal, in expectation, to $dkpn_j^{CBD}$. This cost arises with probability q_C, hence the expected value of the cost is linear in n_j^{CBD}. In addition, the C firm must pay some amount t_j to the B firm to compensate it for the access to its funds in case of need.

So overall, the expected payoff for C firm j (excluding the transfers payments to B

firms) is:

$$u_{C,j}\left(\Gamma,t\right)+t_j=K_B-q_Cdkpn_j^{CBD},$$

similar to (2).

Costs and benefits for the B **firms** Remember $n_i^{BD}+n_i^{BI}+1$ is the number of B firms lying in the same component where B firm i is, that is directly or indirectly connected to i, and $n_i^{CI}+n_i^{CD}$ is the number of direct and indirect C contacts of B firm i. Recall our assumption that when a big shock hits B firm i, this firm can use all the funds that are left at all the B firms in the same component, after each of them made the requested payments to the C firms directly connected to it, on account of the small shocks they received. Hence the funds available to B firm i are $n_i^{BD}+n_i^{BI}+1-p\left(n_i^{CI}+n_i^{CD}\right)k+t_i-\alpha l_i$. If this value is greater or equal than the size η of the big shock hitting the firm, B firm i will survive, otherwise it will fail.[6]

We assume that the value of a B firm is A if it survives, and 0 if it does not. Since η follows a Pareto distribution with parameter γ, the expected payoff of B firm i (again excluding the transfers received as well as the costs of link formation) is:

$$u_{B,i}\left(\Gamma,t\right)-t_i+\alpha l_i=A_1\left(1-\frac{q_B}{\left(n_i^{BD}+n_i^{BI}+1+t_i-\alpha l_i-p\left(n_i^{CI}+n_i^{CD}\right)k\right)^{\gamma}}\right)$$

This expression of $u_{B,i}\left(\Gamma,t\right)$ is increasing and concave in $n_i^{BD}+n_i^{BI}$, as the first and second derivatives with respect to it are

$$\frac{\partial u_B}{\partial\left(n_i^{BD}+n_i^{BI}\right)}=\frac{A\gamma q_B}{\left(n_i^{BD}+n_i^{BI}+1+t_i-\alpha l_i-p\left(n_i^{CI}+n_i^{CD}\right)k\right)^{\gamma+1}}>0,$$

$$\frac{\partial^2 u_B}{\partial(n_i^{BD}+n_i^{BI})^2}=\frac{-Aq_B\gamma\left(\gamma+1\right)}{\left(n_i^{BD}+n_i^{BI}+1+t_i-\alpha l_i-p\left(n_i^{CI}+n_i^{CD}\right)k\right)^{\gamma+2}}<0.$$

It is also decreasing in $n_i^{CI}+n_i^{CD}$ (as well as in l_i) as the first derivative with respect to this variable is

$$\frac{\partial u_B}{\partial\left(n_i^{CI}+n_i^{CD}\right)}=\frac{-Aq_B\gamma pk}{\left(n_i^{BD}+n_i^{BI}+1+t_i-\alpha l_i-p\left(n_i^{CI}+n_i^{CD}\right)k\right)^{\gamma+1}}>0,$$

as in the specification adopted in (1).[7]

[6] This is in contrast with C firms which never fail, because they can tap in the external market to fund any shortfall remaining after receiving the payment from the B firm to whom they are directly connected. The implicit assumption here is that a B firm on the one hand benefits from its access to the funds available at any other B directly or indirectly connected to it, on the other hand is unable to access the external market. This can be justified on the basis of informational issues, as argued above, and of the fact that this kind of firm is larger and so are its needs.

[7] The expression we derived for $u_{B,i}$ is strictly convex in $n_i^{CI}+n_i^{CD}$:

$$\frac{\partial^2 u_B}{\partial\left(n_i^{CI}+n_i^{CD}\right)^2}=\frac{Aq_B\gamma\left(\gamma+1\right)\left(pk\right)^2}{\left(n_i^{BD}+n_i^{BI}+1+t_i-\alpha l_i-p\left(n_i^{CI}+n_i^{CD}\right)k\right)^{\gamma+2}}>0$$

3.1.2 Other microfoundations: sustainability and disease transmission

As mentioned earlier, one alternative interpretation is that C firms are unsustainable "brown" firms. These firms benefit from a contact with a clean B firm which supplies them an input. The contact with a brown firm generates instead a cost for the B firm as well as for its direct or indirect B and C contacts. This could be related with the fact that C firms are riskier, because the transition towards a greener economy means a brown firm is more likely to disappear abruptly, either because people stop using GHG emitting technology (many technologies enjoy strong network complementarities, so changes can be abrupt), or because of regulation shifts. They could thereby create serious financial shocks. On the other hand, B firms benefit from being connected with one another.

In this interpretation the B firms operate in a production network. The marginal benefits coming from the size of that network are decreasing, hence the decreasing returns in $g(.)$. Each B firm can rely on supplies coming from - and going to - other B firms in the network, which are sustainable (let's say green). A B firm can also rely on supplies from other, C firms. These firms are unsustainable (let's say brown), are small and provide a specialized input, which they can produce relatively cheaply because of their unsustainability. By virtue of their size and specialization they can only work for one B firm. Unsustainable firms are more susceptible to climate change risk (which could be transition risk or physical risk). In this story n_i^{CD} is the number of brown firms a B firm i chooses to have as suppliers. The cost from brown C connections arises because their risk exposure means they are less reliable in their supply activities, and this unreliability affects not only their direct B partners, but other B firms in the network.

Another possibility mentioned earlier is that the C agents are individuals working in environments with high probability of direct exposure to viruses. B agents work instead in environments where virus contagion can only be indirect (i.e. through contacts with C agents). That is, B agents live in a community with relatively low and rather safe contacts.

A linear term in $n_i^{CI}+n_i^{CD}$ thus obtains with a first order approximation. There is some asymmetry in the fact that we linearize in one dimension, $n_i^{CI}+n_i^{CD}$, but not the other, $n_i^{BD}+n_i^{BI}$. This is done for analytical simplicity, but an alternative formulation based on a slightly different assumption regarding the priority of funds would allow to achieve the result without the need of a linearization. Suppose B had priority in the use of its funds to meet a big shock hitting the firm: thus if a B firm receives a big shock it has at its disposal the full $n_i^{BD}+n_i^{BI}+1$ units available to B firms in its component. This means in such case the B firm survives whenever the size of the shock is smaller or at most equal to $n_i^{BD}+n_i^{BI}+1$. In that event each C firm hit by a small shock must then go to the market to get the funds needed (k) to face that shock. The B firm hit by the big shock must then compensate all the C firms in the component for the additional, unit cost d they must bear getting funds in the market. The expected number of affected firms in a component is $p\left(n_i^{CI}+n_i^{CD}\right)$ and thus the expected cost of the compensation B must pay to C firms when it is hit by a big shock will be $kdp\left(n_i^{CI}+n_i^{CD}\right)$. We obtain so the following expression for the payoff to B firm i:

$$u_{B,i}(h,t)-t_i+\alpha l_i = A\left(1-q_B\frac{1}{(n_i^{BD}+n_i^{BI}+1)^{\gamma}}\right)-q_B kdp\left(n_i^{CI}+n_i^{CD}\right),$$

again similar to (1).

The B agents provide some service to the C agents each of whom commutes (at varying and endogenous frequencies) to other communities where the virus is more present (one could think of C agents as health professionals and B agents as individuals outside the profession but who may have reasons to interact with them). In the 1990s, the spread of AIDS in Africa was very related to the activities of long distance truckers, or people in a village who went to markets (Orubuloye, Caldwell and Caldwell 1995 or Gysels, Pool and Bwanika 2001).

More precisely, the B agents thus generate benefits for the whole group, described by the $g(.)$ function. The C linkages are connections outside this group, with people that are somehow more exposed to viruses, say because their jobs require lots of outside interactions. The cost of C connections comes then from the fact that a larger group of outside contacts increases the likelihood of contracting the disease and that spreads to the whole B group: c is the cost of getting the disease.

3.2 Efficient networks

We now characterize the optimal networks in this economy, where we take a utilitarian approach and simply consider the total payoff earned by all firms in the economy. Note that since preferences are quasi-linear, the transfers t cancel out when we take the sum of all payoffs and we can then concentrate on the surplus generated by the structure of the linkages among B and with C firms.

We assume the number of C firms is sufficiently large, relative to the number of B firms:

Assumption 1. *$F > (K_B + c)/2c$.*

We similarly assume that the bilateral surplus generated by a linkage between a B and a C firm is positive and sufficiently so. In many of the applications it is in fact natural to think of B firms as being large relative to C firms, and hence at c as being small (relative to K_B).

Assumption 2. *$K_B > 3c$.*

The following properties of an optimal network are a fairly immediate implication of the assumptions made on firms' payoffs. Let n_B denote the number of B firms who are directly or indirectly connected among them, that is lying in the same connected component: note that $n_B \geq 1$ and $= 1$ when there is only one B firm. We then have the following:

Proposition 1. *In the optimal network structure each connected component has a* core periphery structure *where:*

1. *every C firm is linked to only one B firm;*

2. *B firms are minimally connected among them;*

3. *each B firm lying in a component with n_B firms of type B is linked to the same number C_B^* of C firms, with C_B^* being the only integer in the interval*

$$\left[\max\left\{\frac{K_B - n_B c}{2c}, 0\right\}, \max\left\{\frac{K_B - c(n_B - 2)}{2c}, 0\right\}\right]$$

Proof. *See Appendix.*

At the optimum, as shown in Proposition 1, the number of C linkages is the same for all the B firms present in a given component. The total surplus generated by the linkages among the firms lying in a connected component is then: [8]:

$$W^{comp}(n_B, C_B) = n_B[g(n_B - 1) - cC_B n_B] + n_B C_B (K_B - c(C_B - 1)).$$

It depends so on the number n_B of B firms present in the component and the number of linkages C_B that each of them has with C firms.

The optimal number of C linkages, C_B^*, is such that total surplus does not increase neither adding or deleting a C linkage. Note that under assumption 1 the value of C_B^* stated in Proposition 1 is less or equal than F, for all $n_B = 1, ..N$. Hence the optimal number of C linkages for each B firm in a component of size n_B, as described by C_B^*, is always feasible. Also, the expression of C_B^* shows there is a tradeoff between linkages to C and to B firms. Under Assumption 2, it is always beneficial for a B firm with no connection to other B firms to link with some C firms. However, as the number of linkages to other B firms increases, the profitability of C linkages decreases and hence the optimal number of C linkages becomes smaller.

If we then substitute for C_B in $W^{comp}(n_B, C_B)$ the value of C_B^* we found in the previous proposition, we obtain an expression that is only a function of the number n_B of B firms present in the component. In so doing we abstract for simplicity from the integer constraint regarding the number of C linkages and approximate C_B^* with the lower bound of the interval in Proposition 1.[9] We have so[10]:

$$\begin{aligned} \phi(n_B) &\equiv W^{comp}(n_B, \max\left\{\frac{K_B - n_B c}{2c}, 0\right\}) \qquad (3) \\ &= n_B g(n_B - 1) + \frac{n_B}{4c}(\max\{K_B - n_B c, 0\})^2 + cn_B \max\left\{\frac{K_B - n_B c}{2c}, 0\right\} \end{aligned}$$

To characterize the optimal network structure it remains then to determine the optimal allocation of B firms across components. To this end, the first key step is finding the

[8]The expression is obtained summing the payoffs of the n_B type B firms present in the component and of the C_B type C firms connected to each of these B firms, ignoring the linkage costs among B firms, assumed negligible. Note that the transfers between the two types of firms cancel out.

[9]As we said above, in the applications it is natural to think of c as small, so the approximation is close to exact.

[10]The details of the derivation are in the Appendix.

optimal component size n_B^*, that is the value that maximizes the payoff of each firm[11] in the component, formally:

$$
\begin{aligned}
n_B^* \in \arg\max_{n_B \in [1,N]} \frac{\phi(n_B)}{n_B} &= g(n_B - 1) \\
&+ \frac{1}{4c}\left(\max\{K_B - n_B c, 0\}\right)^2 + c \max\left\{\frac{K_B - n_B c}{2c}, 0\right\}.
\end{aligned} \tag{4}
$$

We allow n_B^* to be any real number greater or equal than one. We ignore so the integer constraint also on this variable. This has no significant economic effect and we thus refrain from considering it.

Observing the above expression we see that a key tradeoff emerges in the determination of n_B^* (which reflects the contrasting forces which we noticed above shape the optimal value of C_B): the first term of $\phi(n_B)/n_B$ is in fact increasing in n_B while the second one is decreasing in n_B. In terms of economic forces, the trade-off is between the benefits of risk sharing among B firms, which are always increasing in the size of a component, and the benefits of the services provided to C firms, which are always decreasing in n_B because of the higher indirect costs generated by the linkages to C firms in larger components. In general terms, a higher value of K_B (higher value of the services provided by B to C firms) should lead to a smaller value of the optimal component size n_B^* (since it increases the optimal number C_B^* of C firms linked to any B firm), while an upward shift of $g(n_B - 1)$ for every n_B (higher benefits of risk sharing among B firms) leads to bigger n_B^*. Also, a larger c, increasing the costs of - direct and indirect - connections to C firms, reduces not only the value C_B^* of optimal C linkages of B firms but also that of linkages among B firms, n_B^*, as it increases the cost of indirect linkages to C firms and hence the costs of linking to other B firms.

We can argue such a tradeoff is also present for the other interpretations of the model. In the example of sustainable firms, the tradeoff is between the resort to smaller and not so sustainable C suppliers, which are cheaper but also risky for the group, and that to more sustainable and less risky B suppliers, which are better for the group but more expensive. Similarly, there is a tradeoff in the case of disease transmission between restricting connections to own group members or having also external connections. Limiting interactions to members of the same B group is safer and benefits the other members of the group, but it means forgoing the private benefits of interacting also with agents outside the group.

In order to get a clearer sense of the tradeoff we mention in the previous paragraph, it is useful to consider the case where the gain from risk sharing takes a constant elasticity form:

$$
g(n_B - 1) = A\left({n_B}^{1-\gamma} - 1\right)/(1-\gamma), \tag{5}
$$

where we assume $\gamma > 0$. The following proposition then makes more precise the tradeoff between the benefits of risk sharing (parametrized by the constant A in the function $g(.)$ in

[11] Strictly speaking, it maximizes the payoff of each B firm and its C connections.

(5)) and the surplus generated by the relationship with C firms (parametrized by K_B).

PROPOSITION 2. *Suppose $g(n_B)$ is as in (5). The optimal component size is either minimal (1), with a single B firm in each component, or maximal (N), except when we have: (i) $A > \frac{K_B}{2}$,*

$$\text{(ii) } A^{\frac{1}{1+\gamma}} c^{\frac{\gamma}{1+\gamma}} 2^{-\frac{\gamma}{1+\gamma}} \left(\gamma^{\frac{-\gamma}{1+\gamma}} + \gamma^{\frac{1}{1+\gamma}}\right) - \frac{K_B + c}{2} < 0, \tag{6}$$

and (iii) N is neither too large nor too small[12]*. Under these conditions (i)-(iii), the optimal component size is interior.*

Proof. *See Appendix.*

Note that inequality (6) can be equivalently written as

$$\left(\frac{c}{2\gamma}\right)^{\frac{\gamma}{1+\gamma}} A^{\frac{1}{1+\gamma}} (1+\gamma) - \frac{c}{2} < \frac{K_B}{2}$$

which is satisfied for K_B large and/or c sufficiently small, that is, when the benefits of connections between B and C firms are sufficiently large relative to the costs of linkages to C firms, in line with Assumption 2. The benefit of linkages among B firms, captured by A, should on the other hand be not too large, while satisfying the condition $A > \frac{K_B}{2}$.

A network configuration where all the disjoint components in the system are of the optimal size n_B^* is generally not feasible when the optimum is interior: feasibility only holds when N/n_B^* is exactly an integer. The integer constraint concerning the optimal number of components constitutes a more serious issue than the one concerning the optimal component size n_B^* (or the optimal number of C linkages) and that is why do not ignore it. In the first case, having one member more or less in a business group, as argued above, has quite a limited impact on the network structure. But components are business groups. They could be quite large, and a fraction of a group would be a non trivial number of firms. The next step is then to determine how the gap induced by feasibility, which prevents all components from being of the optimal size, should be distributed, in particular whether it should be symmetrically or asymmetrically allocated across the different components. This depends on the properties of the function $\phi(n_B)$.

PROPOSITION 3. *Suppose $g'(.)$ is convex and $\phi(n_B)$ is uniformly strictly concave or uniformly strictly convex for all $n_B \leq N$. Then, generically, the optimal network structures, that maximize total surplus in the economy, are symmetric, with connected components all of the same size $\hat{n}_B$,*[13] *given by the closest feasible point to n_B^* either above or below it.*

Proof. *See Appendix.*

[12] A more precise statement of the condition on N is in the proof in the Appendix.

[13] There may be an optimal network with two component sizes, n and 1, in the (nongeneric) case where $\phi'(n) = \phi(n)/n = \phi'(1)$ as we show in the proof of Proposition 3 in the Appendix.

In order to get a clearer sense of when the uniform concavity or convexity properties hold,[14] Proposition A1 in the Appendix gives precise conditions for their validity when the $g(.)$ function is as in (5).[15]

3.3 Equilibrium networks

In this section we analyze the network structures which result from the decisions of each individual firm regarding which linkages to form. To this end we need to describe first the network formation game. A strategy for a B firm is given by undirected link proposals to other B firms as well as to C firms, in the latter case accompanied by transfer requests, for a C firm it is given by link proposals plus proposed transfers to B firms.

One key assumption we make is that the compensation from a C firm j to a B firm i only depends on the establishment of a linkage among them. That is, the compensation and the agreement to form a linkage cannot be conditional on the number and type of other connections of these two firms (in particular of the B firm, who is likely to have other linkages). This means the terms of the agreement reached (or the contract signed) among the two parties cannot internalize the externalities that the $i-j$ connection imposes on other relationships.[16] Similar considerations apply to the link proposals among B firms. Of course, firms in the game anticipate others' action and will make decisions anticipating them, even if they cannot be controlled contractually.

More formally, a strategy of B firm i is given by $x_i = (s_i, t_i)$. The first term,[17] $s_i = [s_{ik}]_{k \in B \setminus i}$, describes the link proposals to other B firms, with $s_{ik} \in \{0, 1\}$, where $s_{ik} = 1$ is a proposal by i to form a link with B firm k. The second term, $t_i = [t^i_{ij}]_{j \in C}$, is given by the transfer requests (and hence also link proposals) to C firms, with $t^i_{ij} \in \mathbb{R}$ constituting the (minimal) transfer requested by i from C firm j for i to agree to establish a link with firm j.[18] Analogously, for a C firm j a strategy is given by $x_j = t_j = [t^j_{ji}]_{i \in B}$, where $t^j_{ji} \in \mathbb{R}$ is the transfer offered by j to B firm i to form a link with j. Let X_B and X_C denote the set of admissible strategies, respectively, for a B and a C firm.

Given the strategies of all firms, described by $x = \left\{ (x_i)_{i \in B}, (x_j)_{j \in C} \right\}$, a link is created between two B firms i and k if $s_{ik} = s_{ki} = 1$ and it is not created otherwise, while a link

[14] Under the assumptions stated so far, neither the first term, $n_B g(n_B - 1)$, nor the second one, $\frac{n_B}{4c} (\max\{K_B - n_B c, 0\})^2$, are ensured to be uniformly concave or convex. Hence some additional conditions are needed.

[15] To get some understanding for the conditions stated in that proposition, note that under (5) the concavity of the first term in $\phi(n_B)$, $n_B g(n_B - 1)$, depends on whether γ is bigger or smaller than 2. If A is large with respect to K_B the concavity or convexity of the first term of $\phi(n_B)$ dominates the second term (which is concave for small n_B and convex for larger n_B, hence the need for conditions beyond the one on γ), while the third term is linear.

[16] We can think of a situation where contracting between B and C firms is bilateral. See Duffie and Wang (2017) for an alternative view when contracts are complete and can take into account all the externalities in the network. Predictably, the outcome is efficient in that case.

[17] With some abuse of notation we let B denote also the set of firms of type B; similarly for C.

[18] This is effectively a reservation price at which i is willing to link to j.

between a B firm i and a C firm j is created if the transfer request is compatible with the transfer offered: $t_{ij}^{i}+t_{ji}^{j}\geq 0$. A strategy profile x induces so a network $h(x)$ and a set of transfers $t(x)$ and hence the values of the payoffs of any B firm i, $\pi_i(x)\equiv u_{B,i}(h(x),t(x))$, and any C firm j, $\pi_i(x)\equiv u_{C,j}(h(x),t(x))$, where the functions $u(.)$ are as specified in section 3.

We will use the notion of *bilateral pairwise equilibrium* (which combines some features of the bilateral equilibrium of Goyal and Vega-Redondo (2007) with others of the pairwise equilibrium of Bloch and Jackson (2007)). For any strategy profile x, let x_{-i-j} be the profile of strategies for all players other than i and j. We then have:

Definition 1. *A profile of strategies x^* is a bilateral pairwise equilibrium if*

1. *For any B firm i, $\pi_i\left(x_i^*,x_{-i}^*\right)\geq\pi_i\left(x_i,x_{-i}^*\right)$ for all $x_i\in X_B$; similarly for any C firm j, $\pi_j\left(x_j^*,x_{-j}^*\right)\geq\pi_j\left(x_j,x_{-j}^*\right)$ for all $x_j\in X_C$;*

2. *For any pair of B firms i,k and any pair of alternative strategies of these two firms with regard to the proposals of linkages to other B firms $(s_i.t_i^*),(s_k.t_k^*)\in X_B$,*

$$\pi_i\left((s_i.t_i^*),(s_k.t_k^*),x_{-i-k}^*\right)>\pi_i\left(x_i^*,x_k^*,x_{-i-k}^*\right)$$
$$\Longrightarrow\pi_k\left((s_i.t_i^*),(s_k.t_k^*),x_{-i-k}^*\right)<\pi_k\left(x_i^*,x_k^*,x_{-i-k}^*\right);$$

3. *For any pair of a B firm i and C firm j and any pair of alternative strategies of these two firms with regard to the transfer requested by i from j, $\left(s_i^*.t_{ij}^i,t_{i-j}^{i^*}\right)\in X_B$, and of the transfer proposed $\left(t_{ji}^j,t_{j-i}^{j^*}\right)\in X_C$ from j to i :*

$$\pi_i\left(\left(s_i^*.t_{ij}^i,t_{i-j}^{i^*}\right),\left(t_{ji}^j,t_{j-i}^{j^*}\right),x_{-i-j}^*\right)>\pi_i\left(x_i^*,x_j^*,x_{-i-j}^*\right)$$
$$\Longrightarrow\pi_j\left(\left(s_i^*.t_{ij}^i,t_{i-j}^{i^*}\right),\left(t_{ji}^j,t_{j-i}^{j^*}\right),x_{-i-j}^*\right)<\pi_j\left(x_i^*,x_j^*,x_{-i-j}^*\right)$$

Condition 1. requires that no single firm has a profitable deviation by revising its strategy (in the sense of rejecting proposals that are accepted in equilibrium - i.e. deleting linkages - and of accepting proposals that are rejected in equilibrium). Conditions 2. and 3. then deal with bilateral deviations, where any pair of firms can agree to form a new link among them. In so doing they may at the same time revise their individual strategies with respect to the other firms, that is delete links with some of them, as we explain more precisely in what follows.

In particular, condition 2. considers the possibility of a deviation by any pair of B firms, who jointly modify the proposal they make to each other and revise the proposal each of them makes to all other B firms. For example, they could agree to form a link between themselves while deleting some of the links that each of them has in equilibrium with other B firms. Their connections to C firms are instead kept unmodified. In contrast, condition 3. examines the possible deviations by a pair given by a B and a C firm, who may create a new connection among them, with appropriate transfers, if profitable for both. In this case the rest of their strategy profile, with respect to all other firms, is kept unchanged.

In other words, a bilateral pairwise equilibrium is a bilateral equilibrium as far as the linkages among the B firms are concerned, and a pairwise equilibrium with regard to the linkages between B and C firms. We need to combine the two notions, because the notion of bilateral equilibrium does not contemplate transfers between players. These transfers are instead present in our environment, associated to linkages between B and C firms, and conditions 1. and 3. allow their terms to be modified. The notion of pairwise equilibrium considers transfers and the possibility of changing them, but not the simultaneous link creation and destruction, as condition 2. does with regard to the linkages among B firms.

Notice there is some asymmetry in the fact that two B firms, when they consider adding a link among them, can coordinate suitable changes in their strategy with respect to all other B firms (e.g., deleting some linkages). In contrast, when a B and a C firm add a connection among them, they are not able to modify any component of their strategy towards other firms. A motivation for this feature and the asymmetry in treatment of the relationships among B firms and between B and C firms comes from our microfoundation. B firms are well established, large intermediaries. There are relatively few of them and we can argue the connections among them are not too hard to observe. C firms are, on the other hand, smaller customers and the credit lines from large firms to such customers are more difficult to monitor (this could also be illegal for competition policy reasons).

PROPOSITION 4. *In a bilateral pairwise equilibrium, a number $\overline{C}_B$ of C firms given by the only integer in the interval $\left[\min\left\{F, \frac{K_B-c}{c}\right\}, \min\left\{F, \frac{K_B}{c}\right\}\right]$ is linked to each B firm. In addition, all network components are minimally connected trees and all of them, except at most one, have a number $\overline{n}_B$ of B firms, with the remaining component, if it exists, having a number of B firms strictly smaller than $\overline{n}_B$.*

To understand how $\overline{C}_B$ is determined, notice that the bilateral surplus from adding a connection between a C firm and a B firm i, when such B firm is already connected to a number $\widehat{C}_{B,i}$ of C firms, is:[19]

$$-c + K_B - c\hat{C}_{B,i} \tag{7}$$

So as long as $\widehat{C}_{B,i}$ is such that

$$\widehat{C}_{B,i} \geq \frac{K_B - c}{c}$$

it is not (bilaterally) profitable to create a connection between a B firm and a (yet unmatched) C firm. Similarly, deleting one of the existing C linkages generates the following

[19]Letting $\hat{C}_B$ be the number of C linkages of the other $n_B - 1$ type B firms in the component and $\hat{C}_{B,i}$ the linkages of B firm i, i's payoff is $g(n_B - 1) - cC_B(n_B - 1) - c\hat{C}_{B,i} + t_i - \alpha l_i$. We then see that, in this situation, if a new linkage were created between firm i and another C firm j, i's payoff, ignoring the additional transfer received, varies by $-c$ while the change in j's payoff is $K_B - c\hat{C}_{B,i}$ (ignoring the transfers between the two).

variation in bilateral surplus for the B and C firms involved:

$$c - K_B + c(\hat{C}_{B,i} - 1)$$

Thus if

$$\widehat{C}_{B,i} \leq \frac{K_B}{c},$$

it is also not profitable to delete an existing link.

Then if the number of C firms in the system is so large that we also have[20] $F > K_B/c$, C connections will be created in equilibrium until all bilateral surplus is exhausted. In that case a subset of the C firms will remain unmatched in equilibrium and the surplus generated by the C connections is equal to zero.

On the other hand, if $F \leq K_B/c$,

$$\overline{C}_B = F,$$

and so all the C firms will be matched in equilibrium. In this second case, because the C firms are on the "short" side of the market, the transfer t_i will be such that all the surplus that is generated, positive in this case, is appropriated by them. This establishes the first part of the claim in the proposition.

Note that the equilibrium value of $\overline{C}_B$ stated in the proposition is always positive under Assumption 2. Hence in equilibrium all B firms always establish some linkages with C firms (no matter how profitable linkages with B firms are). The rest of the proof of the proposition is then in the Appendix.

It is then interesting to compare the value we just found for the number of linkages between C and B firms that are formed in an equilibrium network with the corresponding value in the efficient network, stated in Proposition 1. The two are clearly different. This is due to the fact that, when a B firm, say i, and a C firm decide to establish a connection, they only consider their own benefits and ignore the effects this has on the payoff of the other firms. Each of the $\hat{C}_{B,i}$ other type C firms who are already connected to B firm i sees its payoff reduced by this additional connection (by an amount equal to c). A second, also negative, external effect is on the other B firms linked with i, who experience a loss, in total equal to $(n_B - 1)c$, when i establish an additional linkage with an additional C firm. As we saw in condition (7) above, the effect of an additional connection for the two parties involved is $-c + K_B - c\hat{C}_{B,i}$. Once we subtract from this expression the cost of the negative effect on the other C firms connected with B firm i, $-c\widehat{C}_{B,i}$, and the one on the B firms linked with i, $-(n_B - 1)c$, we obtain the effect on social surplus, derived in the proof of Proposition 1.

Since these external welfare effects of a linkage between a B and a C firms are ignored

[20] Recall that Assumption 1 states that $F > (K_B + c)/2c$ but we could still have $K_B/c > F > (K_B + c)/2c$ and hence feasibility binding in equilibrium.

and, as argued above, they are both negative, it follows that, as it is immediate to verify,[21] $C_B^* < \overline{C}_B$. Thus in equilibrium any B firm forms an inefficiently high number of linkages with C firms. The extra terms in (the expressions of the lower and upper bounds for) C_B^* relative to $\overline{C}_B$ reflect the negative effect of a C linkage on other firms in the system.

We turn now to the analysis of the number of connections that are formed among B firms in equilibrium. As claimed in Proposition 4, this number is the same for all but at most one component. The number is the one that maximizes the joint surplus of the B firms lying in a component, taking as given the number $\overline{C}_B$ of the connections each of them forms with C firms.

$$\overline{n}_B \in \arg\max_{n_B \geq 1} \left\{ g\left(n_B - 1\right) - cn_B\overline{C}_B \right\}, \tag{8}$$

where we again approximate $\bar{C}_B$ with the lower bound of the interval in Proposition 4, analogously to what we did for the derivation of n_B^*.

If we compare the function that is maximized in (8) with $W^{comp}(n_B, C_B^*)/n_B$, whose maximization in (4) yielded the socially optimal size n_B^* of a component, we notice two important differences. One is that the number of C linkages is the equilibrium value $\overline{C}_B$ and not the socially optimal value C_B^*. The second difference is that in (8) only the benefits of the B firms appear, while in (4) we also have the benefits of the C firms linked with them. We then see that the marginal benefit of an additional connection among B firms is equal to $g'\left(n_B - 1\right)$ and is then the same as in the case of the social optimum. But the marginal cost of such connection is different as it is now $c\bar{C}_B$, increasing in the number of C firms that are connected to any B firm. In the expression of the marginal cost for the social optimum derived in the proof in the Appendix, in addition to the analogous term cC_B^*, which describes the cost of adding another B firm in a component, due to the presence of other C firms directly connected to this firm, there is an additional term, $c/2$, which reflects the effect that increasing n_B has on the level of C_B^*. As we show in the proof, we always have $(C_B^* + 1/2) < \bar{C}_B$. Hence the component size is lower in equilibrium, $\overline{n}_B < n_B^*$, than in the efficient solution. Formally:

PROPOSITION 5. *The number n_B of B firms in all (but at most one) components in equilibrium is smaller than the socially optimal component size: $\overline{n}_B < n_B^*$. Furthermore, the number of C firms connected to every B firm in equilibrium is larger than at the social optimum: $\overline{C}_B > C_B^*$*

Proof. *See Appendix.*

The economic significance of this result is that in equilibrium there is an excessive level of intermediation $\overline{C}_B > C_B^*$ as each B firm does not internalize the consequences that

[21] It is immediate to verify that the lower bound of the interval where $\bar{C}_B$ lies is strictly greater, by an amount greater than 1, than the one where C_B^* is: $\frac{K_B - c}{c} - \frac{K_B - n_B c}{2c} = \frac{K_B - c}{2c} + \frac{n_B - 1}{2} > 1$ by Assumption 2 and the fact that $n_B \geq 1$. The same is true for the upper bound: $\frac{K_B}{c} - \frac{K_B - c(n_B - 2)}{2c}$, which takes the same value. Hence the integer values in the two intervals cannot be the same.

forming linkages with C firms has, in terms of higher risk exposure, for all its B contacts. We then see that B firms respond to this by reducing the linkages among them. We have so an inefficiently low level of risk sharing among B firms, who anticipate the formation by each of them of a large number of harmful C contacts. Interestingly the equilibrium, unlike the social optimum, exhibits asymmetry in the components that are formed: the size of each component except one is in fact equal to the individually optimal value $\bar{n}_B$. The remaining component is smaller. The B firms in this smaller component achieve a lower surplus than firms in components of size $\bar{n}_B$. Since monetary transfers among B firms are precluded in our construction, such firms are however unable to induce other B firms to join their component.

For the other interpretations of the model the implications are also interesting. In the production network case, the equilibrium features each sustainable B firm having too many connections with cheaper and less sustainable C firms. As a consequence of the risk and instability those connections bring to the groups of B firms connected among them, these groups end up being smaller than optimality would prescribe. Similarly, the private enjoyment of C connections in the disease transmission interpretation generates an excess of C connections and too small B groups from a social welfare point of view.

Note that equilibrium and efficient configurations may be very different. This happens, for example, when $g(.)$ is as in (5) in which case, as we show in the next proposition, $\overline{n}_B$ is a constant, independent of N,when N is large enough.

Proposition 6. *If* $g(n_B - 1) = A\left(n_B^{1-\gamma} - 1\right)/(1-\gamma)$, *then* $\overline{n}_B = \min\left\{\left(\frac{A}{c\overline{C}_B}\right)^{\frac{1}{\gamma}}, N\right\}$.

Proof. *See Appendix*

In contrast, for this specification of $g(.)$ it is easy to verify that, when $\gamma < 1$, the efficient network always features a single connected component comprised of the whole population of N B-type firms (as long as N is sufficiently large).[22] Thus in such a situation component sizes in the equilibrium network are much smaller than at at the efficient network.

A natural question is whether there are policies that could reduce the disparity between equilibrium and efficiency. In the stylized framework we considered here, the introduction of a limit on the number of connections between B and C firms would clearly do the job. But, as we will see below, the optimal pattern of connections becomes more complicated in the presence of some heterogeneity among B (or C) firms, in which case the design of interventions which are improving requires more knowledge from the regulator. An alternative, perhaps more realistic, route is to tax the transfers t_{ij} between the B and C firms, so that the surplus in the pair is reduced. In that way, less connections would result in equilibrium.

[22]When $\gamma < 1$, the function $\lim_{n_B \to \infty} g(n_B - 1) = \infty$. So, $\lim_{n_B \to \infty} \frac{\phi(n_B)}{n_B} = \lim_{n_b \to \infty} g(n_B - 1) = \infty$. Thus, when N is sufficiently large, $n_B^* = N$

4 Heterogeneous types of C firms

We extend now the analysis to the case where C firms are heterogeneous: in particular, we suppose now there are two "types of C firms", denoted by C_1 and C_2. We assume the cost for a B firm of forming a direct linkage is smaller with C_1 firms than with C_2 firms, while the indirect costs are bigger in the case of connections to C_2 than to C_1 firms. Formally, we have the following expression of the payoffs of firms:

$$u_{B,i}(\Gamma,t) = g\left(n_i^{BD} + n_i^{BI}\right) - c_A\left(n_i^{C_1D} + n_i^{C_2I}\right) - c_F\left(n_i^{C_2D} + n_i^{C_1I}\right) + t_i - \alpha l_i$$

and

$$\begin{aligned} u_{C_1,j}(\Gamma,t) - t_i &= f\left(n_j^{BD}\right) - c(n_j^{C_1BD} + n_j^{C_2BD}) - t_i \\ u_{C_2,j}(\Gamma,t) - t_i &= f\left(n_j^{BD}\right) - c(n_j^{C_1BD} + n_j^{C_2BD}) - t_i \end{aligned}$$

for

$$f\left(n_j^{BD}\right) = \begin{cases} K_B \text{ if } n_j^{BD} \geq 1 \\ 0 \text{ if } n_j^{BD} = 0 \end{cases}$$

with $c_A < c_F$.

The new element here is the composition of the linkages to C firms between C_1 and C_2 types in the equilibrium and the efficient network. We assume, similarly to the homogeneous case, that the number of C firms of each type, relative to the number of B firms, is sufficiently large and that the bilateral benefit of a link to a C firm exceeds its cost, whatever its type:

Assumption 3. *1.* $F_1 \geq \frac{(K_B - c_A)}{2c}$, $F_2 \geq \frac{(K_B - c_F)}{2c}$. *2.* $K_B > \max\{c_A, c_F\}$.

We show in what follows that in equilibrium B firms choose to establish linkages only with C_1 firms, that is with firms for whom direct linkage costs are smaller but indirect costs are bigger. More precisely, each B firm forms the same number $\bar{C}_{Bh}$ of linkages with C_h firms, for $h \in \{1,2\}$:

$$\begin{aligned} \bar{C}_{B1} &= \min\left\{\frac{1}{c}(K_B - c_A), 0\right\} \\ \bar{C}_{B2} &= 0 \end{aligned}$$

In contrast, the socially optimal network structure features a completely opposite pattern of linkages to C firms, only with type 2 firms. Letting n_B^* denote as before the optimal component size of B firms, the efficient number of linkages $C_{Bh}^*(n_B^*)$ that each of these B firms should form with C firms of type $h \in \{1,2\}$, whenever $n_B^* > 2$, is:

$$\begin{aligned} C_{B1}^*(n_B^*) &= 0 \\ C_{B2}^*(n_B^*) &= \max\left\{\frac{K_B - (c_F + c_A(n_B^* - 1))}{2c}, 0\right\} \end{aligned}$$

When instead $n_B^* < 2$, the opposite property holds at the optimum: $C_{B1}^*(n_B^*) > 0 = C_{B2}^*(n_B^*)$.

We summarize the finding in the following:

Proposition 7. *Under 3, in equilibrium B firms only connect to C_1 firms: $\overline{C}_{B1} > 0 = \overline{C}_{B2}$. In contrast, at the social optimum, whenever the component size is not too small ($n_B^* > 2$), they should only connect to C_2 firms: $C_{B2}^*(n_B^*) > 0 = C_{B1}^*(n_B^*)$.*

Proof. *See Appendix.*

To gain some understanding for the above result, we need first to think about the characteristics of the two kinds of C firms, and what they represent. Coming back to the microfoundation proposed in Section 3.1, we could assume that C_2 firms are somewhat safer, as they have a lower likelihood to have a small shock than C_1 firms, $p_2 < p_1$. But at the same time the bilateral total benefits generated by a connection of a C_1 firm with a B firm are higher than for a C_2 firm. This could be due to different reasons, from a knowledge spillover leading to a productivity increase, to a bribe. These benefits more than compensate the larger costs incurred when the B firm suffers a big shock. However, while bilateral benefits are larger in the case of a connection of a B firm with a C_1 firm, the indirect costs for other B firms are also higher. There are, of course alternative interpretations of the heterogeneity of C firms, related to the other microfoundations we discussed. In the case of environmental externalities, C_1 firms could be energy firms that have lower costs of production/extraction so that a direct relationship with them is more profitable for B firms, but these firms generate more harmful emissions. In constrast, C_2 firms are less profitable for their partners but use less dirty technologies and generate so less externalities on other firms.

Now, recall that C_1 customers are privately more profitable but socially more harmful than C_2 customers, since $c_A < c_F$ (the magnitude of the externalities induced by linkages with such firms is bigger). In equilibrium each B firm chooses then to form linkages only with C_1 customers. This is the opposite to what efficiency requires whenever the optimal component size features not too few B firms in the same component: linkages should only be formed with C_2 and not with C_1 customers. When there are more than two B firms in an optimal component, the consideration of the indirect effects prevails over that of the direct effect. Hence the optimum prescribes the formation of linkages only with firms which generate less externalities, even though the direct cost of such linkages is higher. When instead the number of B firms in an optimal component is smaller than two, the consideration of the direct effects prevails and hence the pattern of C linkages at the optimum is the same as in equilibrium.

This means that in equilibrium trading relationships are only formed with the firms that adopt the least sustainable technologies, which are dirtier but cheaper, hence creating more profits within the pair, but worsening the situation for the other connected B firms; or that friendship relationships are formed only with the more attractive, but also more contagious, C_1 individuals.

Thus the presence of heterogeneity among C firms generates new dimensions of inefficiency of equilibria, which in turn tends to further reduce the size of the components which form in equilibrium. This seems relevant in our applications.

5 Heterogeneous types of B firms and asymmetric information

We now examine a different type of heterogeneity, this time concerning B firms, consisting in the presence of different costs of providing services to C firms. We will consider both the case where the type of a B firm is observable to all other firms and when instead it is unobservable, that is we have asymmetric information.

More precisely, there are now two types of B firms, H and L. The number of H type firms is N_H and that of L firms is N_L, with $N_H + N_L = N$. It is convenient here to simplify the model, by assuming that the payoff for B firm i only depends on the number of its direct connections to other B firms, n_i^{BD}, and on the number of C firms connected to itself or to any of its direct B contacts. The reason is that this allows to keep the analysis tractable in the presence of asymmetric information, since i only needs to form beliefs over the types of its direct connections, not also of the indirect ones.

The payoff of an arbitrary B firm i, of type $r = H, L$, is then now:

$$u_{B_i}(\Gamma, t; r) = g\left(n_i^{BD}\right) - c_r n_i^{BCD} + t_i - \alpha l_i$$

where we denote by n_i^{BCD} the number of C firms who are all directly connected to the B firms to which firm i is directly connected, including i; that is, all C firms at distance 2 or 1 from i in the graph. We assume that the different types of B firms have different costs of providing services to C firms, so that $c_H > c_L$.

The payoff for C firms is instead unchanged:

$$u_{C_j}(\Gamma, t) = f\left(n_j^{BD}\right) - c n_j^{CBD} - t_i$$

where

$$f\left(n_j^{BD}\right) = \begin{cases} K_B \text{ if } n_j^{BD} \geq 1 \\ 0 \text{ if } n_j^{BD} = 0 \end{cases}$$

The condition we imposed that the cost of C connections for B firms arises only from their own C connections and the ones of their direct B connections can be justified on the basis of the presence of sufficient decay in how the disruption costs generated by C firms travel through the network. When that is not the case, the effects we find in this section will likely be a lower estimate of the actual ones.

The main objective of this section is to examine the consequences of asymmetric information over firms' types for the properties of the network configurations arising in equilibrium. For that reason we will focus our attention on equilibrium oucomes, and compare the scenario where types are observable to the one where they are only privately observed.

5.1 C linkages for different types of B firms

We will assume throughout that the type of a B firm is transparent to her C contacts. Hence any possible informational asymmetry only concerns the formation of direct linkages among B firms. The assumption is not unreasonable in our context. We can think in fact of the C suppliers as having a long term relationship with B firms, and it is likely they know each other well.

Under this assumption, the number of C firms with whom a B firm establishes a direct linkage is the same both when her type is publicly observable and when it is not. To determine the C connections of a B firm of type r, notice that the bilateral surplus from a connection between a C firm and a B firm, when the latter is already connected to a number $\widehat{C}_B$ of C firms is:

$$K_B - c_r - c\widehat{C}_B$$

So if $\widehat{C}_B$ is such that

$$K_B - c_r - c\widehat{C}_B > 0$$

it is profitable create a connection between the two. Hence, if we denote by $\overline{C}_{Br}$ the number of C firms with whom an r type of B firm chooses to establish a linkage, we have that if

$$F \geq \frac{K_B - c_L}{c},$$

then

$$\overline{C}_{BH} = \frac{K_B - c_H}{c}, \ \overline{C}_{BL} = \frac{K_B - c_L}{c}. \tag{9}$$

If instead

$$F < \frac{K_B - c_L}{c}$$

the pattern of equilibrium C linkages for the two types of B firms is obtained solving the following system of equations:

$$\begin{aligned} K_B - c_H - c\overline{C}_{BH} &= K_B - c_L - c\overline{C}_{BL} \quad (10) \\ \overline{C}_{BH}N_H + \overline{C}_{BL}N_L &= FN \end{aligned}$$

and is then given by:

$$\overline{C}_{BH} = F - \left(\frac{c_H - c_L}{c}\right)\frac{N_L}{N}, \ \overline{C}_{BL} = F + \left(\frac{c_H - c_L}{c}\right)\frac{N_H}{N}. \tag{11}$$

To understand why, when the number of C firms is not large enough, we need to solve jointly for the number of C connections of H and L firms, as in (10), it helps to point out the following. Note first that the surplus generated by a given number of C connections is different when we consider an L or a H type of B firm. Since C firms are identical, in equilibrium they must be indifferent between establishing a connection with any of the two types of B firms. Because in this situation C firms are on the short side of the market, they appropriate all the surplus. Hence in equilibrium the surplus generated by a connection with a B firm must be the same whatever the type r of the B firm, as in condition (10).

Examining the equilibrium values we found for the number of C connections in 9 and 11, we see that in both cases we have:

$$\overline{C}_{BL} - \overline{C}_{BH} = \frac{c_H - c_L}{c} > 0, \tag{12}$$

that is, an L type of B firm establishes a larger number of C connections than an H type. This is intuitive since the cost of forming such connections is lower for L firms.

5.2 B linkages with symmetric information

When the type of a B firm is publicly observable, the equilibrium pattern of linkages among B firms features assortative matching. It is clear that H types of B firms only want to establish linkages with B firms of the same type H (provided the size of the component these firms choose to form in equilibrium is smaller than the number of H firms in the population). This is true because, as we saw above, each of the H types has fewer connections with C firms than the L types, hence the indirect costs of forming a linkage with an H type B firm are lower than with an L type. As a consequence, both types of B firms would like to form linkages with firms of type H. However in equilibrium a type H firm always rejects an offer made by a type L firm, since such firm brings too many negative externalities (too many C connections).[23]

Given this property, the number of direct linkages n_r^{BD} which a B firm of type r forms in equilibrium with B firms of the same type is obtained by solving:

$$\max_{n_r^{BD} \geq 0} g\left(n_r^{BD}\right) - c_r\left(n_r^{BD} + 1\right)\overline{C}_{Br}$$

The heterogeneity in the cost of C linkages and then in $\overline{C}_{Br}$ induces some heterogeneity in the preference for the number of B connections. Hence the equilibrium value of n_r^{BD} varies with r. Except for this, and the fact that now components are segregated according to the type of B firms, the general structure of the equilibrium network is otherwise similar to the one we obtained with homogeneous firms in Section 3.3.

5.3 B linkages with asymmetric information

When the type of a B firm is not observable by other B firms, the network formation game and the equilibrium notion need to be appropriately extended to deal with the presence

[23] This is somewhat reminiscent of the logic of Farrell and Scotchmer (1988). They posit a coalition formation game with equal division within coalitions. As they point out (p. 280) "When players can be ranked by ability, the equal-sharing rule makes the ablest people reluctant to admit the less able to their group: while they would always like a big group so as to exploit economies of scale, they do not want to go too far down the distribution of abilities, since they then subsidize less able members."

of asymmetric information. The matching in this case is in fact more complicated. As we saw above, B firms of different types wish to have different numbers of C connections and this implies that their desirability as partners is different. However, when types are not observable, offers can no longer be rejected on the basis of the type of the proponent. It becomes so difficult to find a decentralized procedure of bilateral proposals as in Definition 1 that guarantees that every B firm ends up with the number of B connections she desires and is consistent with the fact that types are only privately known. We consider then a centralized matching procedure which allows to meet all the requests for B linkages expressed by any B firm. We abstract so from coordination issues which may prevent the attainment of this outcome and focus instead on the implications of asymmetric information for the expected quality of B connections and hence the equilibrium level of connectivity between B firms.

More precisely, we consider here a game in which each B firm announces the number $n_r^{BD}$of direct linkages she wishes to form with other B firms. All the announcements are collected by a mediator, who then forms the matches randomly and anonymously, so as to implement the desired number of linkages announced by each type of B firm. Even though the announcement made by each B firm could reveal something about its type, the anonymous and random matching procedure ensures that none of this information is conveyed to other B firms. As we show in the Appendix, for N sufficiently large a matching process can always be found so that every firm achieves the desired number of matches, that is, no rationing occurs. In what follows we will then focus on such case.

We will consider the Bayes-Nash equilibria of the network formation game described above. Matches between any B firm and C firms occur then as described in the previous section (they are not affected by private information and do not depend on the linkages formed among B firms): hence each B firm of type r establishes linkages with the same number $\overline{C}_{Br}$ of C firms. Recall that we assumed in this section that each B firm only cares - besides its direct C connections - for the number of its direct B connections and the number of C connections formed by each of them. Since the latter number depends on their type, a B firm cares not only for the number n_r^{BD} of linkages to other B firms - which the firm chooses - but also for the composition of these linkages. That is what is the proportion of such linkages to H and to L firms, which is exogenous to the firm. A B firm does not know the types of the other B firms with whom it is matched and can only form expectations based on the equilibrium strategies of the two types. For any B connection that is formed, the expected number of additional (indirect) C connections which are induced (the key aspect to determine the indirect cost of a B connection), is:

$$\overline{C}_{BL}\frac{\pi_L n_L^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}}+\overline{C}_{BH}\frac{\pi_H n_H^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}},$$

where $\pi_r = N_r/N$ is the fraction of B firms in the population which are of type $r = H, L$.

Hence the number n_r^{BD} of B linkages which a type r firm chooses to form in equilibrium maximizes:

$$g\left(n_r^{BD}\right)-c_r\left(\overline{C}_{Br}+n_r^{BD}\left(\overline{C}_{BL}\frac{\pi_L n_L^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}}+\overline{C}_{BH}\frac{\pi_H n_H^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}}\right)\right)$$

The first order condition of this problem is:

$$g'\left(n_r^{BD}\right)=c_r\left(\overline{C}_{BL}\frac{\pi_L n_L^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}}+\overline{C}_{BH}\frac{\pi_H n_H^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}}\right)$$

Since by (12) we have $\overline{C}_{BL}>\overline{C}_{BH}$, the marginal cost for a B firm of type L of forming a linkage with another B firm under asymmetric information,

$$c_L\left(\overline{C}_{BL}\frac{\pi_L n_L^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}}+\overline{C}_{BH}\frac{\pi_H n_H^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}}\right),$$

is lower than under symmetric information, $c_L\overline{C}_{BL}$. It then follows from the above first order condition that low cost type firms establish more linkages under asymmetric than with symmetric information. By an analogous argument, high cost types form fewer connections under asymmetric information, since

$$c_H\left(\overline{C}_{BL}\frac{\pi_L n_L^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}}+\overline{C}_{BH}\frac{\pi_H n_H^{BD}}{\pi_L n_L^{BD}+\pi_H n_H^{BD}}\right)>c_H\overline{C}_{BH}$$

and so the marginal cost of forming a linkage for a type B firm of type H is higher under asymmetric information. Letting $n_r^{BD,AI}$ and $n_r^{BD,SI}$ denote respectively the number of linkages to other B firms that a r type chooses to form, respectively, with asymmetric and with symmetric information, we have:

Proposition 8. *In equilibrium, when N is sufficiently large*[24] *we have $n_L^{BD,AI}>n_L^{BD,SI}$ and $n_H^{BD,AI}<n_H^{BD,SI}$.*

Furthermore, when we compare the number of direct connections to other B firms established by L and H types, we see the relationship is rather different under asymmetric and symmetric information:

Proposition 9. *For N sufficiently large:*

1. *under asymmetric information L types establish more connections to B firms than H types: $n_L^{BD,AI}>n_H^{BD,AI}$;*

[24] The argument in the proof relies on the fact that the number of linkages formed in equilibrium equals the firms' desired number, which is obtained from their first order conditions. A sufficient condition for those values to be feasible with asymmetric information (see the algorithm description in the Appendix for details) is that $n_H^{BD,AI}\leq N$, and $n_L^{BD,AI}-n_H^{BD,AI}\leq p_L N$, both of which will be satisfied for N large.

The equilibria we consider are constructed assuming firms believe that rationing does not occur even when they contemplate possible deviations. This however only makes deviations more profitable, thus the equilibria we find are robust to the possibility of rationing following deviations.

2. under symmetric information, the opposite property holds, for some parameter values:

(a) when $F \geq \frac{K_B - c_L}{c}$, $n_L^{BD,SI} < n_H^{BD,SI}$ *iff* $c_H + c_L > K_B$,

(b) when $F < \frac{K_B - c_L}{c}$, $n_L^{BD,SI} < n_H^{BD,SI}$ *iff* $\frac{c_L}{c}\pi_H + \frac{c_H}{c}\pi_L > F$.

Proof. *See Appendix*

With asymmetric information, matching is anonymous and so the expected number of indirect C connections which is induced by the formation of a linkage with a B firm is the same for both types. Thus the only thing that matters for the number of B linkages formed by the two types in equilibrium is the fact that $c_L < c_H$, that is the fact that the cost of such indirect linkages is always lower for L types. Therefore we unambiguously have $n_L^{BD,AI} > n_H^{BD,AI}$.

The situation is rather different with symmetric information, as we have several forces at play. In this case, as we observed, B linkages are formed only with firms of the same type. Recall also that each type L connects with a larger number of C firms than an H type, since it has a lower cost of forming connections with C firms. This implies that the number of indirect C connections that are acquired when a linkage is formed with another B firm of the same type is larger for L firms. However it does not mean that the cost of establishing such a linkage is always higher for L types, since the cost of an indirect C connection is lower for L than for H firms, because c_H is larger than c_L. These two forces go in opposite directions. We then find that, for a range of parameter values, the total number of direct B connections formed in equilibrium by L type firms may now be smaller than the corresponding number for H firms, in contrast to the case of asymmetric information.

From Proposition 8 we know that L firms have more direct connections to other B firms under asymmetric information than under symmetric information, while the opposite is true for H firms. C connections are instead the same under symmetric and asymmetric information. It is then of interest to investigate what is the effect of asymmetric information on the total number of direct B connections and hence on the overall connectivity in the system. That is, the effect of asymmetric information on total risk-sharing between B firms, or equivalently on the extent of trading relationships among sustainable firms, or on social connections within groups.

To answer this question, we need to compare the relative size of the positive effect on connectivity we found for L firms with that of the negative effect obtained for H firms, solving so for the equilibrium total number of direct B connections. For this we focus again our attention on the specification of $g(.)$ given in (5), which allows us to derive explicit expressions for the number of B connections which are formed in equilibrium by the different types of B firms under different information regimes. On this basis we can then compare the total number of direct B connections which are formed in equilibrium in the system under symmetric (TC_{SI}) and asymmetric (TC_{AI}) information:

$$TC_{AI} - TC_{SI} = \pi_L n_L^{BD,AI} + \pi_H n_H^{BD,AI} - \left(\pi_L n_L^{BD,SI} + \pi_H n_H^{BD,SI}\right)$$

PROPOSITION 10. *When $g\left(n_r^{BD}\right) = A\left(n_r^{BD}\right)^{1-\gamma} / (1-\gamma)$, asymmetric information always reduces total connectivity among B firms in the system: $TC_{AI} - TC_{SI} < 0$.*

Proof. *See Appendix*

We should point out that the result holds for all values of γ, it is so quite general. The finding is somewhat reminiscent of the effect of private information in the Akerlof market for lemons. Connectivity decreases because H types restrain their connectivity with other B firms, fearing a high chance of being matched with L types, in which case they would bear a higher level of negative externalities. A difference with respect to Akerlof is that L types instead increase the number of direct B connections, due to the chance of being matched with some H type. This is turn further deters H types from forming connections. The overall effect as we show is that the total number of B connections in the system decreases.

It is useful to interpret these findings using our motivating stories. In the case of the production network with clean and brown firms, the L types could be clean firms with a lower cost of establishing relationships with brown firms (say, because they are more protected from climate shocks) than the H types. As a consequence, L types will want to form more of those C connections, which makes them a less desirable partner for other B firms, in particular the H types who are less able to stand climate shocks. On a similar vein, the L-type could be a healthier individual who is less susceptible to viral infections and will then interact with a larger number of partners outside his social group, thus becoming a more dangerous transmission vector for the other members of his group. In terms of our finance motivation, an $L-$type firm tends to have more lending relationships with C firms. It is so committed to provide a larger fraction of its own funds to C firms so that less funds will be available for its partner B firms when they are hit by a large shock, making this firm a less desireable partner. The findings in Propositions 8 and 9 then tell us that, when partner types are not observable, the clean firms with higher costs for interacting with less sustainable C firms, or the individuals who are more susceptible to infection, will be wary to form connections with firms/individuals in their own group. Under symmetric information the problem is different, as agents can select who they match with and by doing so can limit their indirect exposure to C partners. This fact will drive down the overall level of connectivity in the system.

6 Conclusions

We have analyzed a stylized model of link formation between agnts who engage in intermediation activity with others, and do not internalize default contagion externalities. The main results we obtain are as follows. The efficient structure cannot be obtained as a result of individual decisions, because those decisions lead to excessive intermediation activity and this in turn limits the extent of risk sharing in the system. When there is heterogeneous credit quality or heterogeneity in the ability to weather contagion risk (especially under asymmetric information) the inefficiency is amplified. These results suggest that regulatory interventions could be beneficial. It is unlikely that one can simply rely on contracts internalizing contracting externalities. A network is a very complex object and there is a limit to contract complexity.

Clearly, the model is simplified and it has limitations, but we see the simplicity as a virtue, as it allows to understand transparently the main forces in the model. We have examined a number of extensions, but there are other possibilities one could explore. For example, we have assumed for simplicity a centralized arrangement for the creation of links in the asymmetric information environment. There may be different results with a less centralized mechanism. In the same direction, we had to simplify the model to deal with asymmetric information environment. The problem there is that coalitional deviations are very difficult to analyze under asymmetric information. It might be perhaps possible to use some of the techniques in Kanishiro, Vohra and Serrano (2023) to address this challenging problem.

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A Appendix

Proof of Proposition 1

Within a component, given the form of $f(.)$ each C agent should be linked to only one B type. Moreover, it is immediate to see that any B type should be linked to the same number of C types. In this case we have so: $n_i^{CD} = n_k^{CD}$ for any pair i, k of B type firms lying in the same component. To establish the claim we proceed by contradiction: suppose that instead $n_i^{CD} > n_k^{CD}$. If so, it would be possible to increase the sum of the firms' payoff in the component by moving the linkage of a number of C firms from i to k. By so doing we do not affect the welfare of the B firms, while the sum of the payoff of the C firms that remain linked to i increases by $2cn_i^{CD}$,[25] the sum of the payoff of the n_k^{CD} firms of type C that were - and remain - directly linked to k decreases by $2cn_k^{CD}$ and the welfare of the individual moving increases by $c\left(n_i^{CD} - n_k^{CD}\right)$. Hence the total change in welfare is $2c(n_i^{CD} - n_k^{CD}) > 0$, thus total welfare strictly increases when $n_i^{CD} - n_k^{CD} > 0$.

With this in mind, recall the expression we derived in the text for the total welfare in a component where there are n_B B firms and each of them is directly connected to a number C_B of C firms:

$$W^{comp}(n_B, C_B) = n_B[g\left(n_B - 1\right) - cC_B n_B] + n_B C_B \left(K_B - c\left(C_B - 1\right)\right).$$

For any given n_B, the optimal number of C linkages, C_B^*, is such total welfare does not increase neither adding nor deleting a C linkage. We have so the two following conditions:

$$\begin{aligned}
W^{comp}(n_B, C_B^* + 1) - W^{comp}(n_B, C_B^*) &= n_B[g\left(n_B - 1\right) - c\left(C_B^* + 1\right) n_B] + n_B\left(C_B^* + 1\right)\left(K_B - cC_B^*\right) \\
&\quad - n_B[g\left(n_B - 1\right) - cC_B^* n_B] - n_B C_B^*\left(K_B - c\left(C_B^* - 1\right)\right) \\
&= -cn_B^2 + n_B K_B - n_B c\left(\left(C_B^* + 1\right) C_B^* - \left(C_B^* - 1\right) C_B^*\right) \\
&= -cn_B^2 + n_B K_B - n_B c 2C_B^* \leq 0 \\
&\iff -cn_B + K_B - c2C_B^* \leq 0 \\
& C_B^* \geq \frac{K_B - cn_B}{2c}
\end{aligned}$$

and

$$\begin{aligned}
W^{comp}(n_B, C_B^*) - W^{comp}(n_B, C_B^* - 1) &= n_B[g\left(n_B - 1\right) - c\left(C_B^*\right) n_B] + n_B\left(C_B^*\right)\left(K_B - c(C_B^* - 1)\right) \\
&\quad - n_B[g\left(n_B - 1\right) - c(C_B^* - 1)n_B] - n_B(C_B^* - 1)\left(K_B - c\left(C_B^* - 2\right)\right) \\
&= -cn_B^2 + n_B K_B - n_B c\left(\left(C_B^* - 1\right) C_B^* - \left(C_B^* - 1\right)\left(C_B^* - 2\right)\right) \\
&= -cn_B^2 + n_B K_B - n_B c2(C_B^* - 1) \geq 0 \\
&\iff -cn_B + K_B - c2(C_B^* - 1) \geq 0
\end{aligned}$$

[25] This is obtained by differentiating the total cost born by these firms, $-c\left(n_i^{CD}\right)^2$, with respect to n_i^{CD},

$$C_B^* \leq \frac{K_B - c(n_B - 2)}{2c}$$

We have so C_B^* being the integer satifying:

$$\max\left\{\frac{K_B - n_B c}{2c}, 0\right\} \leq C_B^* \leq \max\left\{\frac{K_B - c(n_B - 2)}{2c}, 0\right\}.$$

It is immediate to see that there is always a unique integers lying in the intreval above.□

Proof of derivation of (3)

Substituting the value of C_B^* into the expression of total welfare in a component $W^{comp}(n_B, C_B^*)$ and simplifying, yields:

$$\begin{aligned}
&\phi(n_B) = g\left(n_B - 1\right) n_B - c n_B^2 \left(\max\left\{\frac{K_B - n_B c}{2c}, 0\right\}\right) + \\
&n_B \max\left\{\frac{K_B - n_B c}{2c}, 0\right\}\left(K_B - c\left[\max\left\{\frac{K_B - n_B c}{2c}, 0\right\} - 1\right]\right) \\
= \;& g\left(n_B - 1\right) n_B - n_B\left(\max\left\{\frac{K_B - n_B c}{2c}, 0\right\}\right)\left(c n_B - K_B + c\left[\max\left\{\frac{K_B - n_B c}{2c}, 0\right\} - 1\right]\right) \\
= \;& g\left(n_B - 1\right) n_B + n_B\left(\max\left\{\frac{K_B - n_B c}{2c}, 0\right\}\right)\left(K_B - c n_B - c\max\left\{\frac{K_B - n_B c}{2c}, 0\right\} + c\right) \\
= \;& g\left(n_B - 1\right) n_B + \frac{n_B}{4c}\left(\max\left\{K_B - n_B c, 0\right\}\right)^2 + c n_B \max\left\{\frac{K_B - n_B c}{2c}, 0\right\}
\end{aligned}$$

□

Proof of Proposition 2

If $g\left(n_B - 1\right) = A\left(\left(n_B\right)^{1-\gamma} - 1\right) / (1 - \gamma)$,

$$\frac{\phi\left(n_B\right)}{n_B} = A\frac{\left(n_B\right)^{1-\gamma} - 1}{1 - \gamma} + \frac{1}{4c}\left(\max\left\{K_B - n_B c, 0\right\}\right)^2 + c\max\left\{\frac{K_B - n_B c}{2c}, 0\right\}$$

then the first order condition to maximize $\phi\left(n_B\right)/n_B$ is

$$A\left(n_B\right)^{-\gamma} - \frac{K_B - c(n_B - 1)}{2} = 0 \text{ when } K_B > n_B c. \tag{13}$$

Since $g'(0) = A$, when $A < \frac{K_B}{2}$ $\phi\left(n_B\right)/n_B$ is first decreasing and then increasing[26]; thus there exists one solution of equation (13), which is a minimum, hence the optimal component size is either 1 or N.

When instead $A > \frac{K_B}{2}$, $\phi\left(n_B\right)/n_B$ is increasing at $n_B = 1$ and again increasing as $n_B \to \infty$. Hence there are two or zero solutions of equation (13), depending on whether or

[26]This follows from the fact that $A\left(n_B\right)^{-\gamma}$, the first term in (13), is decreasing, convex and converging to 0 as $n_B \to \infty$, while the second term, $\frac{K_B - c(n_B - 1)}{2}$, is decreasing and linear. When $A < \frac{K_B}{2}$ the first term is smaller than the second at $n_B = 1$, so they cross only once, so (13) is first negative/decreasing and then increasing.

not $A(n)^{-\gamma} - \frac{K_B - c(n-1)}{2} < 0$ for some $1 \leq n \leq K_B/c$. Note that the function $A(n_B)^{-\gamma} + \frac{c(n_B-1)}{2} - \frac{K_B}{2}$ attains a minimum at

$$-\gamma A(n_B)^{-\gamma-1} + \frac{c}{2} = 0,$$

that is at

$$n_B = \left(\frac{2A\gamma}{c}\right)^{\frac{1}{1+\gamma}}.$$

Hence if

$$A\left(\frac{2A\gamma}{c}\right)^{\frac{-\gamma}{1+\gamma}} + \frac{c\left[\left(\frac{2A\gamma}{c}\right)^{\frac{1}{1+\gamma}} - 1\right]}{2} - \frac{K_B}{2} < 0$$

or equivalently[27],

$$A^{\frac{1}{1+\gamma}} c^{\frac{\gamma}{1+\gamma}} 2^{-\frac{\gamma}{1+\gamma}} \left(\gamma^{\frac{-\gamma}{1+\gamma}} + \gamma^{\frac{1}{1+\gamma}}\right) - \frac{K_B + c}{2} < 0 \tag{14}$$

equation (13) has two zeros, the first of which is a local maximum. Let us denote $\overline{n}_B$ the first zero. In that case the maximum of $\phi(n_B)/n_B$ is at $\overline{n}_B$, provided N is not too small ($\overline{n}_B < N$), nor too large, so that $\phi(\overline{n}_B)/\overline{n}_B > \phi(N)/N$, otherwise the maximum is at N. The two inequalities are always satisfied, for instance, when N is greater but close to $\bar{n}_B$ as $\phi(n_B)/n_B$ is decreasing immediately to the right of $\overline{n}_B$. If equation (14) is violated, the optimal component size is N because the expression in equation (13) is always positive and $\phi(n_B)/n_B$ is then always increasing.□

Proof of Proposition 3

The maximization of total welfare, given by the sum of the welfare of all the components in the system, requires finding the optimal division of agents into components of possibly different sizes.

Suppose first $\phi(n_B)$ is strictly concave everywhere, then there can be only one component size at the optimum. Suppose we had an allocation with two different component sizes, $n_1 < n_2$. In this case, $\phi(n_1) + \phi(n_2) < 2\phi((n_1 + n_2)/2)$ so it is better to replace each pair of these components into a pair of components of average size.

Suppose now $\phi(n_B)$ is strictly convex everywhere. To prove there is only one optimal component size, suppose at the optimum we have two different component sizes $1 < n_1 < n_2 < N$. Then we can have another allocation where each pair of components of these two sizes is replaced by a pair with sizes $n_1 - \varepsilon$ and $n_2 + \varepsilon$. We show in what follows that there

[27] The expression can be rewritten as follows:

$$\begin{aligned} A\left(\frac{2A\gamma}{c}\right)^{\frac{-\gamma}{1+\gamma}} + \frac{c\left(\frac{2A\gamma}{c}\right)^{\frac{1}{1+\gamma}}}{2} - \frac{K_B + c}{2} &< 0 \\ A^{\frac{1}{1+\gamma}} c^{\frac{\gamma}{1+\gamma}} (2\gamma)^{\frac{-\gamma}{1+\gamma}} + c^{\frac{\gamma}{1+\gamma}} 2^{\frac{-\gamma}{1+\gamma}} \gamma^{\frac{\gamma}{1+\gamma}} - \frac{K_B + c}{2} &< 0 \end{aligned}$$

thus yielding (14).

always exists some $\varepsilon > 0$ such that welfare is strictly higher at this other allocation. To see this, for an arbitrary $\lambda \in (0,1)$ set ε so as to satisfy:

$$\begin{aligned} \lambda n_2 + (1-\lambda)(n_1 - \varepsilon) &= n_1 \Leftrightarrow (1-\lambda)\varepsilon = \lambda(n_2 - n_1) \\ \lambda n_1 + (1-\lambda)(n_2 + \varepsilon) &= n_2 \Leftrightarrow (1-\lambda)\varepsilon = \lambda(n_2 - n_1) \end{aligned}$$

that is,

$$\varepsilon = \frac{\lambda}{1-\lambda}(n_2 - n_1).$$

Then by convexity

$$\begin{aligned} \lambda\phi(n_2) + (1-\lambda)\phi(n_1 - \varepsilon) &> \phi(n_1) \Longleftrightarrow (1-\lambda)\phi(n_1-\varepsilon) > \phi(n_1) - \lambda\phi(n_2) \\ \lambda\phi(n_1) + (1-\lambda)\phi(n_2 + \varepsilon) &> \phi(n_2) \Longleftrightarrow (1-\lambda)\phi(n_2+\varepsilon) > \phi(n_2) - \lambda\phi(n_1) \end{aligned}$$

and so

$$\begin{aligned} (1-\lambda)\left[\phi(n_1-\varepsilon) + \phi(n_2+\varepsilon)\right] &> (1-\lambda)\left[\phi(n_2) + \phi(n_1)\right] \\ &\Leftrightarrow \left[\phi(n_1-\varepsilon) + \phi(n_2+\varepsilon)\right] > \phi(n_2) + \phi(n_1), \end{aligned}$$

thus again it cannot be optimal to have two components of size n_1 and n_2.

The previous argument leaves open the possibility $n_1 = 1$, in which case the argument above does not apply. The case that is then left to consider is the one where we have an optimum with two component sizes $n_1 = 1$ and $n_2 > n_1$ when $\phi(n_B)$ is strictly convex everywhere. In that case, the following conditions must hold, for that to be an optimum: $\phi'(n_2) = \phi(n_2)/n_2 = \phi'(1)$, where the first equality comes from the first order condition for the optimal allocation of $N-1$ firms into equal components of size n_2 (analogous to the ones derived in what follows for the optimal allocation of N firms into equal components of size n_B), the second equality from the optimality of allocating n_2 firms to one component and one firm to another component. But this system of two equations is clearly over-determined, so generically it cannot be satisfied, thus in what follows we ignore this case.

We have thus established that if $\phi(.)$ is either everywhere convex or everywhere concave there is a single component of size n_B at the optimum, so that the number of components is N/n_B,and thus total welfare in all the economy is

$$\frac{N}{n_B}\phi(n_B) = N\frac{\phi(n_B)}{n_B}$$

so maximizing total welfare is equivalent to maximizing the *average* welfare within a component, subject to the constraint that the number of components is an integer. That implies the size of a component should be as close as possible to the *component-optimal* size, that is the size that maximizes per capita welfare in the component.□

PROPOSITION A1. *Suppose* $g(n_B - 1)$ *is as in (5), then* $g'(.)$ *is convex if* $\gamma > 0$ *and* *(i)* $\phi(n_B)$ *is uniformly concave if* $\gamma > 2$ *and*

$$A(\gamma-2) > \left(\frac{K_B}{2} - c\right)\left(\frac{K_B}{c}\right)^{\gamma} \tag{15}$$

is satisfied.

(ii) $\phi(n_B)$ *is uniformly convex if* $\gamma < 2$ *and, when* $\left(\frac{2\gamma(2-\gamma)A}{3c}\right)^{\frac{1}{\gamma+1}} \leq K_B/c$*, the inequalities*

$$(2-\gamma)\,A\left(\frac{3c}{2\gamma\,(2-\gamma)\,A}\right)^{\frac{\gamma}{\gamma+1}} + \frac{3c}{2}\left(\frac{2\gamma\,(2-\gamma)\,A}{3c}\right)^{\frac{1}{\gamma+1}} > K_B + c \tag{16}$$

and

$$-\gamma\,(2-\gamma)\,A + \frac{3}{2}c < 0 \tag{17}$$

also hold.

Proof of Proposition A1 Recall that

$$\phi\,(n_B) = g\,(n_B-1)\,n_B + \frac{n_B}{4c}\left(\max\{K_B - n_B c, 0\}\right)^2 + c n_B \max\left\{\frac{K_B - n_B c}{2c}, 0\right\}$$

$g'\,(n_B-1) = A\,(n_B)^{-\gamma}$; $g''\,(n_B-1) = -\gamma A\,(n_B)^{-\gamma-1}$; $g'''\,(n_B-1) = \gamma(1+\gamma)A\,(n_B)^{-\gamma-2} > 0$

When $K_B > n_B c$ we have:

$$\begin{aligned}
&\phi'\,(n_B) = g\,(n_B-1) + n_B g'\,(n_B-1) + \frac{1}{4c}\,(K_B - n_B c)^2 - \frac{n_B}{2}\,(K_B - n_B c) + \frac{K_B - n_B c}{2} - \frac{n_B}{2}c \\
= \;& g\,(n_B-1) + n_B g'\,(n_B-1) + \frac{1}{4c}\left(K_B^2 - 2K_B n_B c + n_B^2 c^2\right) - \frac{n_B}{2}\,(K_B - n_B c) + \frac{K_B - n_B c}{2} - \frac{n_B}{2}c \\
= \;& g\,(n_B-1) + n_B g'\,(n_B-1) + \frac{K_B^2}{4c} - \frac{K_B n_B}{2} + \frac{n_B^2 c}{4} - \frac{K_B}{2}n_B + \frac{n_B^2}{2}c + \frac{K_B - n_B c}{2} - \frac{n_B}{2}c \\
&\phi''\,(n_B) = n_B g''\,(n_B-1) + 2g'\,(n_B-1) - \frac{K_B}{2} + \frac{n_B c}{2} - \frac{K_B}{2} + n_B c - c \\
= \;& n_B g''\,(n_B-1) + 2g'\,(n_B-1) - (K_B + c) + \frac{3n_B c}{2}
\end{aligned}$$

In contrast, for $n_B \geq K_B/c$ we have:

$$\phi''\,(n_B) = n_B g''\,(n_B-1) + 2g'\,(n_B-1)$$

Subsituting the above expressions for $g(.)$ and its derivative, we obtain:

$$\begin{aligned}
\frac{\partial^2\phi\,(n_B)}{\partial n_B^2} &= -\gamma A\,(n_B)^{-\gamma} + 2A\,(n_B)^{-\gamma} - (K_B - c) + \frac{3n_B c}{2} \\
&= A(2-\gamma)\,(n_B)^{-\gamma} - (K_B + c) + \frac{3n_B c}{2} \text{ for } n_B < K_B/c \qquad (18) \\
\frac{\partial^2\phi\,(n_B)}{\partial n_B^2} &= A(2-\gamma)\,(n_B)^{-\gamma} \text{ for } n_B > K_B/c
\end{aligned}$$

Note that the function $\phi\,(n_B)$ is not differentiable at $n_B = K_B/c$. We see however from the above expressions that when $\gamma > 2$ the second derivative is negative for all $n_B > K_B/c$ and so is the right second derivative at $n_B = K_B/c$. Furthermore, we now verify that under the additional condition 15 stated in claim (i) of the proposition, the second derivative is also negative for all $n_B < K_B/c$ and so is the left second derivative at $n_B = K_B/c$.

$$\left.\frac{\partial^2 \phi\left(n_B\right)}{\partial n_B^2}\right|_{n_B=K_B/c} = A(2-\gamma)\left(\frac{K_B}{c}\right)^{-\gamma} - (K_B+c) + \frac{3K_B}{2} = A(2-\gamma)\left(\frac{K_B}{c}\right)^{-\gamma} - c + \frac{K_B}{2} < 0$$

$$\Leftrightarrow A(\gamma-2) > \left(\frac{K_B}{2} - c\right)\left(\frac{K_B}{c}\right)^{\gamma}$$

Therefore, for the assumed functional form of $g(.)$, the conditions stated in claim (i) ensure that $\partial^2\phi\left(n_B\right)/\partial n_B^2 < 0$ for all $n_B \leq N$.

Turning then to claim (ii), we see that if $\gamma < 2$, $\partial^2\phi\left(n_B\right)/\partial n_B^2 > 0$ for all $n_B \geq K_B/c$. Next note that, when $n_B < K_B/c$, $\partial^3\phi\left(n_B\right)/\partial n_B^3 = -\gamma\left(2-\gamma\right)A\left(n_B\right)^{-\gamma-1} + \frac{3}{2}c$. Hence we see that for $\gamma < 2$, $\partial^2\phi\left(n_B\right)/\partial n_B^2$ is always decreasing for low values of n_B when condition (17) is satisfied and could be increasing for higher values of n_B. More precisely, let $n_B^{\min}$ be the minimizer of $\partial^3\phi\left(n_B\right)/\partial n_B^3$ with respect to n_B, that is the unique value of n_B satisfying the following equation

$$\frac{\partial^3\phi\left(n_B\right)}{\partial n_B^3} = -\gamma\left(2-\gamma\right)A\left(n_B\right)^{-\gamma-1} + \frac{3}{2}c = 0$$

given by $n_B^{\min} = \left(\frac{2\gamma(2-\gamma)A}{3c}\right)^{\frac{1}{\gamma+1}}$. Thus a sufficient condition for $\partial^2\phi\left(n_B\right)/\partial n_B^2 > 0$ for all $n_B \leq K_B/c$ is that $\gamma < 2$ and $\partial^2\phi\left(n_B\right)/\partial n_B^2 > 0$ when evaluated at $n_B = \min\left\{K_B/c, n_B^{\min}\right\}$.

Hence evaluating (18) at $n_B = \min\left\{K_B/c, n_B^{\min}\right\}$ there are two possible cases

1. $\left(\frac{2\gamma(2-\gamma)A}{3c}\right)^{\frac{1}{\gamma+1}} > K_B/c$, so that $n_B^{\min} > K_B/c$: in this case $\partial^2\phi\left(n_B\right)/\partial n_B^2$ is decreasing for all $n_B \leq K_B/c$ and attains so its lowest value at $n_B = K_B/c$, given by

   $$A(2-\gamma)\left(K_B/c\right)^{-\gamma} + \frac{1}{2}K_B - c$$

   This expression is positive if

   $$A(\gamma-2) > \left(\frac{K_B}{2} - c\right)\left(\frac{K_B}{c}\right)^{\gamma}$$

   which is verified if $\gamma > 2$ and (15) is true.

2. $\left(\frac{2\gamma(2-\gamma)A}{3c}\right)^{\frac{1}{\gamma+1}} \leq K_B/c$, so that $n_B^{\min} \leq K_B/c$: in this case $\partial^2\phi\left(n_B\right)/\partial n_B^2$ is decreasing only for $n_B \leq n_B^{\min}$ and attains its lowest value at $n_B = n_B^{\min}$, given by:

$$\begin{aligned}\left.\frac{\partial^2\phi\left(n_B\right)}{\partial n_B^2}\right|_{n_B=n_B^{\min}} &= \left. A(2-\gamma)\left(n_B\right)^{-\gamma} - (K_B+c) + \frac{3n_Bc}{2}\right|_{n_B=n_B^{\min}} \\ &= \left(2-\gamma\right)A\left(\frac{3c}{2\gamma\left(2-\gamma\right)A}\right)^{\frac{\gamma}{\gamma+1}} + \frac{3c}{2}\left(\frac{2\gamma\left(2-\gamma\right)A}{3c}\right)^{\frac{1}{\gamma+1}} - K_B - c,\end{aligned}$$

strictly positive under condition (16). Hence when $\left(\frac{2\gamma(2-\gamma)A}{3c}\right)^{\frac{1}{\gamma+1}} \leq K_B/c$ we have $\partial^2\phi\left(n_B\right)/\partial n_B^2 > 0$ for all $n_B \leq N$ when $2 > \gamma$ and (16) holds.

□

Remark A1. *Note that uniform concavity or convexity is used to ensure, whenever the optimal component size n_B^* is interior (greater than 1 but smaller than N), that the efficient structure features components all of the same size. But the existence of an interior optimum $n_B^* \in (1, N)$ also requires some conditions, as we showed in Proposition 2, so the question arises if these conditions are compatible with the one for uniform concavity or convexity identified in Proposition A1. We now show this is indeed the case. The conditions in Proposition 2 can be rewritten as*

$$\left(\frac{c}{2\gamma}\right)^{\frac{\gamma}{1+\gamma}} A^{\frac{1}{1+\gamma}}\left(1+\gamma\right) - \frac{c}{2} < \frac{K_B}{2} < A \tag{19}$$

This is clearly compatible with the conditions for uniform concavity found above, $A(\gamma-2) > \left(\frac{K_B}{2} - c\right)\left(\frac{K_B}{c}\right)^{\gamma}$ and $\gamma > 2$, for A sufficiently large.

The case of uniform convexity is a bit less transparent, but if $K_B \simeq c\left(\frac{2\gamma(2-\gamma)A}{3c}\right)^{\frac{1}{\gamma+1}}$ (16) can be rewritten as follows

$$\left(2-\gamma\right) A \left(\frac{3c}{2\gamma\left(2-\gamma\right)A}\right)^{\frac{\gamma}{\gamma+1}} + \frac{c}{2}\left(\frac{2\gamma\left(2-\gamma\right)A}{3c}\right)^{\frac{1}{\gamma+1}} > c$$

which is easily satisfied for A large. Note this is also easily compatible with (19), even if $K_B \simeq c\left(\frac{2\gamma(2-\gamma)A}{3c}\right)^{\frac{1}{\gamma+1}}$, for c sufficiently small. A large also ensures (17) holds, since $\gamma < 2$.

Proof of Proposition 4

By the same argument as in the proof of Proposition 1, it follows the C types are equally distributed among the B types within any component (the C firms are willing to pay more to be attached to a less connected B firm). The fact that the equilibrium value of C linkages is $\bar{C}_B$ was then established in the main text.

Since there is a cost α of direct links with B agents, in any candidate equilibrium that is not a tree in a connected component, any individual with more than one link that is not essential for having a connected component will delete the link.

To show that the number of B maximizes (8), note that if $n > \overline{n}_B$ since all equilibrium components are trees, there must be some last node. Then, the last node of the component will be cut by the node preceding it in the tree. If $n < \overline{n}_B$ for more than one component, then a pair composed of nodes in two components that are not yet at $\overline{n}_B$ will create a new link, and one of them will destroy her link to the present component. □

Proof of Proposition 5

As shown above,

$$\overline{C}_B = \min\left\{F, \frac{K_B - c}{c}\right\}, \text{ while } C_B^* = \max\left\{\frac{K_B - n_B c}{2c}, 0\right\} < F.$$

Hence if $\frac{K_B - c}{c} \geq F$ we clearly have $\overline{C}_B > C_B^*$. The same is true also when $\frac{K_B - c}{c} < F$ since $\overline{C}_B > C_B^*$ is equivalent to

$$\frac{K_B - c}{c} > \frac{K_B - c}{2c} \geq \frac{K_B - n_B c}{2c}$$

Recall that

$$\bar{n}_B \in \arg\max_{n_B} \left\{g\left(n_B - 1\right) - c n_B \overline{C}_B\right\}$$

while, as shown in (4) above,

$$n_B^* \in \arg\max \frac{\phi\left(n_B\right)}{n_B} = g\left(n_B - 1\right) + \frac{1}{4c}\left(\max\left\{K_B - n_B c, 0\right\}\right)^2 + c \max\left\{\frac{K_B - n_B c}{2c}, 0\right\}.$$

The derivative of the two expressions with regard to n_B, using the above expression for C_B^*, is respectively:

$$g'\left(n_B - 1\right) - c\overline{C}_B$$

$$g'\left(n_B - 1\right) - cC_B^*(n_B) - \frac{c}{2} \text{ if } n_B \geq K_B/c, \text{ otherwise } g'\left(n_B - 1\right).$$

Hence the derivative in the first line is always smaller than the one in the second line if $\overline{C}_B - C_B^* > 1/2$, or

$$K_B - 3c + c n_B > 0,$$

always satisfied by Assumption 2. This then implies $\overline{n}_B < n_B^*$. □

Proof of Proposition 6

By the concavity of $g(.)$ it follows that the equilibrium value of n_B is obtained as a solution of the FOC's of (8):

$$A\left(n_B\right)^{-\gamma} - c n_B \overline{C}_B = 0.$$

Hence

$$\overline{n}_B = \left(\frac{A}{c\overline{C}_B}\right)^{\frac{1}{\gamma}}$$

□

Proof of Proposition 7

The surplus of a connection to be created between a C_1 type C firm and a B firm who is directly linked to $\widehat{C}_{B1}$ firms of type C_1 and $\widehat{C}_{B2}$ firms of type C_2 is

$$K_B - c_A - c(\widehat{C}_{B1} + \widehat{C}_{B2}).$$

The surplus in the case instead of a connection with a type C_2 firm is:

$$K_B - c_F - c(\widehat{C}_{B1} + \widehat{C}_{B2}).$$

Hence the equilibrium value of the connections to C_1 and C_2 firms is determined by the solution of the following system:

$$\begin{aligned} K_B - c_A - c(\bar{C}_{B1} + \bar{C}_{B2}) &\leq 0, \ = 0 \text{ if } \bar{C}_{B1} > 0 \\ K_B - c_F - c(\bar{C}_{B1} + \bar{C}_{B2}) &\leq 0, \ = 0 \text{ if } \bar{C}_{B2} > 0 \end{aligned}$$

Since $c_F > c_A$ we must have $\bar{C}_{B2} = 0$ and

$$\bar{C}_{B1} = \frac{1}{c}(K_B - c_A)$$

which is strictly positive under condition 2 of Assumption 3. The equilibrium number of connections among B firms is then obtained from:

$$\widehat{n}_B \in \arg\max_{n_B \geq 1} \left\{ g\left(n_B - 1\right) - c_A \frac{(K_B - c_A)}{c} - c_F \frac{(K_B - c_A)}{c}(n_B - 1) \right\}$$

In contrast, the socially optimal value of $C^*_{B1}(n_B), C^*_{B2}(n_B)$ for a fixed value of the number n_B of B firms in a component is obtained by maximizing the following expression:

$$\begin{aligned} SW &= n_B[g\left(n_B - 1\right) - c_A\left(C_{B1} + (n_B - 1)C_{B2}\right) - c_F\left(C_{B2} + (n_B - 1)C_{B1}\right)] \\ &+ n_B\left(C_{B1}\left(K_B - c(C_{B1} - 1 + C_{B2})\right) + C_{B2}\left(K_B - c(C_{B2} - 1 + C_{B1})\right)\right) \\ &= n_B[g\left(n_B - 1\right) - \left(c_A + c_F(n_B - 1)\right) C_{B1} - \left(c_F + c_A(n_B - 1)\right) C_{B2} \\ &+ \left(C_{B1} + C_{B2}\right)\left(K_B - c(C_{B1} + C_{B2} - 1)\right)] \end{aligned}$$

Since there is a quadratic term that depends on $(C_{B1} + C_{B2})$ and a linear term that depends separately on C_{B1} and C_{B2}, then it is clear that at the optimum either C_{B1} or C_{B2} have to be zero. Which one is zero depends on which of the linear terms is larger, and note that $c_A + c_F(n_B - 1) > c_F + c_A(n_B - 1)$ iff $c_F(n_B - 2) > c_A(n_B - 2)$, which in turn holds iff $n_B > 2$, in which case $C^*_2\left(n_B\right) = 0$.

Then in order to determine the optimal value $C^*_{B1}\left(n_B\right)$ when $C^*_2\left(n_B\right) = 0$, we need to find the value of C_{B1} at which social welfare is the same at C_{B1} and $C_{B1} + 1:$

$$\begin{aligned} & n_B[g\left(n_B - 1\right) - \left(c_A + c_F(n_B - 1)\right)\left(C_{B1} + 1\right) + \left(\left(C_{B1} + 1\right)\left(K_B - c(C_{B1})\right)\right)] \\ & -n_B[g\left(n_B - 1\right) - \left(c_A + c_F(n_B - 1)\right) C_{B1} + \left(C_{B1}\left(K_B - c(C_{B1} - 1)\right)\right)] \\ = & -\left(c_A + c_F(n_B - 1)\right) + K_B - cC_{B1} - cC_{B1} = 0 \end{aligned} \tag{20}$$

so that

$$C^*_{B1}\left(n_B\right) = \max\left\{\frac{K_B - \left(c_A + c_F(n_B - 1)\right)}{2c}, 0\right\}$$

And then when $n_B < 2$, $C^*_1\left(n_B\right) = 0$. Similarly as we did with equation (20) above, this leads to the expression

$$
\begin{aligned}
& n_B[g\,(n_B-1)-(c_F+c_A(n_B-1))\,(C_{B2}+1)+(C_{B2}+1)\,(K_B-c(C_{B2}))] \\
& -n_B[g\,(n_B-1)-(c_F+c_A(n_B-1))\,C_{B2}+(C_{B2})\,(K_B-c(C_{B2}-1))] \\
= & -(c_A+c_F(n_B-1))+K_B-cC_{B2}-cC_{B2}=0
\end{aligned}
$$

Which in turn implies

$$C^*_{B2}\,(n_B)=\max\left\{\frac{K_B-(c_F+c_A(n_B-1))}{2c},0\right\}$$

The component size n_B is then determined by:

$$n^*_B\in\arg\max_{n_B\geq 1}\left[g\,(n_B-1)+c_A\,[C^*_1(n_B)]^2+c_F\,[C^*_2(n_B)]^2\right].$$

□

Proof of proposition 9

From the FOC's we see that $n_L^{BD,AI}>n_H^{BD,AI}$ since $c_L<c_H$.
Remember $\overline{C}_{BL}$ and $\overline{C}_{BH}$ are the same under symmetric and asymmetric information.
With SI $n_L^{BD,SI}>n_H^{BD,SI}$ is true when $F\geq(K_B-c_L)/c$ if and only if

$$
\begin{aligned}
\overline{C}_{BL}c_L &< \overline{C}_{BH}c_H \\
&\Leftrightarrow\ c_L\frac{K_B-c_L}{c}<c_H\frac{K_B-c_H}{c} \\
&\Leftrightarrow\ c_H^2-c_L^2<K_B(c_H-c_L) \\
&\Leftrightarrow\ c_H+c_L<K_B
\end{aligned}
\tag{21}
$$

and $n_L^{BD,SI}>n_H^{BD,SI}$ when $F<(K_B-c_L)/c$ if and only if

$$
\begin{aligned}
\overline{C}_{BL}c_L &< \overline{C}_{BH}c_H \\
\left(\frac{c_H-c_L}{c}+\overline{C}_{BH}\right)c_L &< \overline{C}_{BH}c_H \\
\frac{c_L}{c} &< \overline{C}_{BH} \\
\frac{c_L}{c} &< F-\left(\frac{c_H-c_L}{c}\right)\frac{N_L}{N_H+N_L} \\
\frac{c_L}{c}+\left(\frac{c_H-c_L}{c}\right)\frac{N_L}{N_H+N_L} &< F \\
\frac{c_L}{c}\frac{N_H}{N_H+N_L}+\frac{c_H}{c}\frac{N_L}{N_H+N_L} &< F
\end{aligned}
$$

□

Proposition A2. $n_L^{BD,SI} = \left(\frac{1}{c_L\overline{C}_L}\right)^{1/\gamma}, n_H^{BD,SI} = \left(\frac{1}{c_H\overline{C}_H}\right)^{1/\gamma}$

$$n_L^{BD,AI} = \left(\frac{\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}}{c_L\left(\overline{C}_L\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}}\right)}\right)^{\frac{1}{\gamma}}, n_H^{BD,AI} = \left(\frac{\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}}{c_H\left(\overline{C}_L\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}}\right)}\right)^{\frac{1}{\gamma}}$$

Proof of Proposition A2 For symmetric information substituting in the FOC we have

$$n_L^{BD,SI} = \left(\frac{1}{c_L\overline{C}_L}\right)^{1/\gamma}$$
$$n_H^{BD,SI} = \left(\frac{1}{c_H\overline{C}_H}\right)^{1/\gamma}$$

The FOC's under asymmetric information

$$\left(n_r^{BD,AI}\right)^{-\gamma} = c_r\left(\overline{C}_L\frac{\pi_L n_L^{BD,AI}}{\pi_L n_L^{BD,AI} + \pi_H n_H^{BD,AI}} + \overline{C}_H\frac{\pi_H n_H^{BD,AI}}{\pi_L n_L^{BD,AI} + \pi_H n_H^{BD,AI}}\right)$$

so we have

$$\frac{\left(n_L^{BD,AI}\right)^{-\gamma}}{c_L} = \frac{\left(n_H^{BDA}\right)^{-\gamma}}{c_H}$$

$$\frac{n_H^{BD,AI}}{n_L^{BD,AI}} = \left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}}$$
$$n_H^{BD,AI} = \left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}} n_L^{BD,AI}$$

Substituting in the FOC we get

$$\left(n_L^{BD,AI}\right)^{-\gamma} = c_L\left(\overline{C}_L\frac{\pi_L n_L^{BD,AI}}{\pi_L n_L^{BD,AI} + \pi_H\left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}} n_L^{BD,AI}} + \overline{C}_H\frac{\pi_H\left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}} n_L^{BD,AI}}{\pi_L n_L^{BD,AI} + \pi_H\left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}} n_L^{BD,AI}}\right)$$

$$\left(n_L^{BD,AI}\right)^{-\gamma} = c_L\left(\overline{C}_L\frac{\pi_L}{\pi_L + \pi_H\left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}}} + \overline{C}_H\frac{\pi_H\left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}}}{\pi_L + \pi_H\left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}}}\right)$$

$$\left(n_L^{BD,AI}\right)^{-\gamma} = c_L\frac{\overline{C}_L\pi_L + \overline{C}_H\pi_H\left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}}}{\pi_L + \pi_H\left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}}}$$

$$n_L^{BD,AI} = \left(\frac{\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}}{c_L\left(\overline{C}_L\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H\frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}\right)}\right)^{\frac{1}{\gamma}}$$

and hence

$$
\begin{aligned}
n_H^{BD,AI} &= \left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}} \left(\frac{\pi_L + \pi_H \left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}}}{c_L \left(\overline{C}_L \pi_L + \overline{C}_H \pi_H \left(\frac{c_L}{c_H}\right)^{\frac{1}{\gamma}} \right)} \right)^{\frac{1}{\gamma}} \\
&= \left(\frac{\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}}{c_H \left(\overline{C}_L \frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}} \right)} \right)^{\frac{1}{\gamma}}
\end{aligned}
$$

□

Proof of proposition 10

$$
\begin{aligned}
\pi_L n_L^{BD,AI} + \pi_H n_H^{BD,AI} &= \pi_L \left(\frac{\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}}{c_L \left(\overline{C}_L \frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}} \right)} \right)^{\frac{1}{\gamma}} + \pi_H \left(\frac{\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}}{c_H \left(\overline{C}_L \frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}} \right)} \right)^{\frac{1}{\gamma}} \\
&= \frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} \left(\frac{\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}}{\overline{C}_L \frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}} \right)^{\frac{1}{\gamma}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}} \left(\frac{\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}}{\overline{C}_L \frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}}} \right)^{\frac{1}{\gamma}} \\
&= \frac{\left(\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}} \right)^{\frac{1}{\gamma}+1}}{\left(\overline{C}_L \frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}} \right)^{\frac{1}{\gamma}}}
\end{aligned}
$$

with symmetric information:

$$
\pi_L n_L^{BD,SI} + \pi_H n_H^{BD,SI} = \pi_L \left(\frac{1}{c_L \overline{C}_L} \right)^{\frac{1}{\gamma}} + \pi_H \left(\frac{1}{c_H \overline{C}_H} \right)^{\frac{1}{\gamma}}
$$

so we need to check whether the expression

$$
\frac{\left(\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}} \right)^{\frac{1}{\gamma}+1}}{\left(\overline{C}_L \frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} + \overline{C}_H \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}} \right)^{\frac{1}{\gamma}}} - \left(\pi_L \left(\frac{1}{c_L \overline{C}_L} \right)^{\frac{1}{\gamma}} + \pi_H \left(\frac{1}{c_H \overline{C}_H} \right)^{\frac{1}{\gamma}} \right) \leq 0
$$

To simplify notation let

$\frac{\pi_L}{(c_L)^{\frac{1}{\gamma}}} = x; \frac{\pi_H}{(c_H)^{\frac{1}{\gamma}}} = y; \overline{C}_L = A; \overline{C}_H = B$

$$
\frac{(x+y)^{\frac{1}{\gamma}+1}}{(Ax+By)^{\frac{1}{\gamma}}} - \left(x \left(\frac{1}{A} \right)^{\frac{1}{\gamma}} + y \left(\frac{1}{B} \right)^{\frac{1}{\gamma}} \right) \leq 0
$$

Another way for writing this is

$$x\left(\frac{\left(1+\frac{y}{x}\right)^{\frac{1}{\gamma}+1}}{\left(A+B\frac{y}{x}\right)^{\frac{1}{\gamma}}}-\left(\left(\frac{1}{A}\right)^{\frac{1}{\gamma}}+\frac{y}{x}\left(\frac{1}{B}\right)^{\frac{1}{\gamma}}\right)\right)\leq 0$$

setting $\frac{y}{x}=z$, since $x>0$ this is equivalent to:

$$\begin{aligned}
&\frac{(1+z)^{\frac{1}{\gamma}+1}}{(A+Bz)^{\frac{1}{\gamma}}}-\left(\left(\frac{1}{A}\right)^{\frac{1}{\gamma}}+z\left(\frac{1}{B}\right)^{\frac{1}{\gamma}}\right)\leq 0\\
\Leftrightarrow\quad&\left(\frac{1}{A}\right)^{\frac{1}{\gamma}}\left(\frac{(1+z)^{\frac{1}{\gamma}}}{\left(1+\frac{B}{A}z\right)^{\frac{1}{\gamma}}}(1+z)-\left(1+z\left(\frac{A}{B}\right)^{\frac{1}{\gamma}}\right)\right)\leq 0\\
=\quad&\left(\frac{1}{A}\right)^{\frac{1}{\gamma}}\left(1+z\left(\frac{A}{B}\right)^{\frac{1}{\gamma}}\right)\left(\frac{(1+z)^{\frac{1}{\gamma}}}{\left(1+\frac{B}{A}z\right)^{\frac{1}{\gamma}}}\frac{(1+z)}{\left(1+z\left(\frac{A}{B}\right)^{\frac{1}{\gamma}}\right)}-1\right)\leq 0
\end{aligned}$$

Now we take $\frac{A}{B}=u$ and we define $h(z,u)$

$$h(z,u)=\frac{(1+z)^{\frac{1}{\gamma}}}{\left(1+\frac{z}{u}\right)^{\frac{1}{\gamma}}}\frac{(1+z)}{\left(1+z(u)^{\frac{1}{\gamma}}\right)}$$

and show that $h(z,u)$ for every z has a max at $u=1$. And we know that $f(z,1)=1$, thus

$$\frac{(1+z)^{\frac{1+\gamma}{\gamma}}}{\left(1+\frac{z}{u}\right)^{\frac{1}{\gamma}}\left(1+z(u)^{\frac{1}{\gamma}}\right)}$$

$$\begin{aligned}
\frac{\partial h(z,u)}{\partial u}&=(1+z)^{\frac{1+\gamma}{\gamma}}\left[-\frac{1}{\gamma}\frac{(-z/u^2)}{\left(1+\frac{z}{u}\right)^{\frac{1}{\gamma}+1}\left(1+z(u)^{\frac{1}{\gamma}}\right)}-\frac{1}{\gamma}\frac{z(u)^{\frac{1}{\gamma}-1}}{\left(1+\frac{z}{u}\right)^{\frac{1}{\gamma}}\left(1+z(u)^{\frac{1}{\gamma}}\right)^2}\right]\\
&=\frac{(1+z)^{\frac{1+\gamma}{\gamma}}\frac{z}{u}}{\gamma\left(1+\frac{z}{u}\right)^{\frac{1}{\gamma}}\left(1+z(u)^{\frac{1}{\gamma}}\right)}\left[\frac{1/u}{\left(1+\frac{z}{u}\right)}-\frac{(u)^{\frac{1}{\gamma}}}{1+z(u)^{\frac{1}{\gamma}}}\right]\\
&=\frac{(1+z)^{\frac{1+\gamma}{\gamma}}\frac{z}{u}}{\gamma\left(1+\frac{z}{u}\right)^{\frac{1}{\gamma}}\left(1+z(u)^{\frac{1}{\gamma}}\right)}\left[\frac{\frac{1}{u}+z(u)^{\frac{1}{\gamma}-1}-(u)^{\frac{1}{\gamma}}-z(u)^{\frac{1}{\gamma}-1}}{\left(1+\frac{z}{u}\right)\left(1+z(u)^{\frac{1}{\gamma}}\right)}\right]\\
&=\frac{(1+z)^{\frac{1+\gamma}{\gamma}}\frac{z}{u}}{\gamma\left(1+\frac{z}{u}\right)^{\frac{1}{\gamma}}\left(1+z(u)^{\frac{1}{\gamma}}\right)}\left[\frac{\frac{1}{u}-(u)^{\frac{1}{\gamma}}}{\left(1+\frac{z}{u}\right)\left(1+z(u)^{\frac{1}{\gamma}}\right)}\right]
\end{aligned}$$

the sign of $\frac{\partial h(z,u)}{\partial u}$ is then the same as the sign of $1-u^{\frac{1}{\gamma}+1}$.

We cannot check $u=0$, at which above expressions are not defined, but as argued above we have $\overline{C}_L\geq\overline{C}_H$, hence $u=\overline{C}_L/\overline{C}_H\geq 1$ while z may take any positive value. Note that the bigger the adverse selection, the larger is u, which means the impact on connectivity is bigger. Note that for all z, $\frac{\partial h(z,0)}{\partial u}=0$ and $\frac{\partial h(z,0)}{\partial u}>0$ for $u<1$ and $\frac{\partial h(z,0)}{\partial u}<0$ for $u>1$, thus $h(z,1)=1$ for every z and $h(z,u)$ has a max at $u=1$. □

Matching for Section 5

In a group of $B=\{1,...,N\}$, player i request n_i matches. We can aggregate players into sets N_{n_i} where all members demand the same number of matches, n_j. There are $J\leq N$ such sets. Without loss of generality we have $n_1\leq n_2\leq ...\leq n_J$.

Step 1 Suppose first n_1 is even. Then all players in N are matched with n_1 players in the following way. Match each player with the next $(n_1+1)/2$ players, modulo N. When n_1 is odd, then match the player odd-numbered players with the next $\lceil(n_1+1)/2\rceil$, and the even players with the next $\lfloor(n_1+1)/2\rfloor$ (see illustrating table below). After this step, all players in N_{n_1} do not have capacity, and they are retired.

Step k At this step, $\sum_{i=1}^{k}|N_{n_i}|$ players have retired. The remaining number of players is thus $N-\sum_{i=1}^{k}|N_{n_i}|$. Then, if $n_{k+1}-n_k\leq N-\sum_{i=1}^{k}|N_{n_i}|-1$, all remaing players are matched with $n_{k+1}-n_k$ players. If this is not feasible, they are matched with $N-\sum_{i=1}^{k}|N_{n_i}|-1$ players and matching stops.

Step J Matching stops at this point, if it has not stopped before.

Clearly, if $N-\sum_{i=1}^{k}|N_{n_i}|-1\geq n_k-n_{k-1}$, for all $k\leq J$, all players get their desired number of matches.

All existing links are randomly rematched. Take two random links ij and kl and rematch to ik, and jl an arbitrary large number of times. In this way the number of links of all individuals remain the same and matching is random.

$$
\begin{aligned}
N &= 6, n_1 = 4 \\
& \quad 1 \leftrightarrow 2,3 \\
& \quad 2 \leftrightarrow 3,4 \\
& \quad 3 \leftrightarrow 4,5 \\
& \quad 4 \leftrightarrow 5,6 \\
& \quad 5 \leftrightarrow 6,1 \\
& \quad 6 \leftrightarrow 1,2
\end{aligned}
$$

$$
\begin{aligned}
N &= 6, n_1 = 3 \\
& \quad 1 \leftrightarrow 2,3 \\
& \quad 2 \leftrightarrow 4 \\
& \quad 3 \leftrightarrow 4,5 \\
& \quad 4 \leftrightarrow 6 \\
& \quad 5 \leftrightarrow 6,1 \\
& \quad 6 \leftrightarrow 2
\end{aligned}
$$

$$
\begin{aligned}
N &= 7, n_1 = 6 \\
& \begin{aligned}
1 &\leftrightarrow 2,3,4 \\
2 &\leftrightarrow 3,4,5 \\
3 &\leftrightarrow 4,5,6 \\
4 &\leftrightarrow 5,6,7 \\
5 &\leftrightarrow 6,7,1 \\
6 &\leftrightarrow 7,1,2 \\
7 &\leftrightarrow 1,2,3
\end{aligned}
\end{aligned}
$$

$$
\begin{aligned}
N &= 7, n_1 = 5 \\
& \begin{aligned}
1 &\leftrightarrow 2,3,4 \\
2 &\leftrightarrow 4,5 \\
3 &\leftrightarrow 4,5,6 \\
4 &\leftrightarrow 6,7 \\
5 &\leftrightarrow 6,7,1 \\
6 &\leftrightarrow 2,3 \\
7 &\leftrightarrow 1,2,3
\end{aligned}
\end{aligned}
$$