Optimal Robust Double Auctions

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Abstract

In an exchange economy, we propose a novel double-auction format featuring two clock auctions, Vickrey-style payments, and carefully designed taxes. In the spirit of Ausubel (2004), we define a sincere ex-post perfect equilibrium and show that, under a certain disclosure policy, it is the only survivor of iterated elimination of weakly dominanted strategies. Furthermore, we show how the clocks can be adjusted dynamically to maximize disclosure. Finally, with private values, the auction implements an ex-ante optimal mechanism under the ex-post constraints. The associated tax is private and depends on the clock price. Further tractability is achieved given quadratic utilities.

JEL Classfication: D44, D47, D82

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In the 2017 FCC auction (the *Incentive Auction*) for spectrum licenses, the standard *forward auction* was combined with the novel *reverse auction* to acquire and repackage spectrum, historically dispersed over numerous small owners, see Ausubel et al. (2012, 2017) and Cramton et al. (2015). However, the success was tarnished by several instances of supply reduction due to the uniform-price nature of payments.[1] It was argued that this could lead to under-performance in terms of revenue, see Doraszelski et al. (2017).[2] These shortcomings set the stage for an auction format that would be strategy-proof and also revenue-maximizing, that is, robust and optimal.

In this paper, we devise a dynamic auction in a two-sided market with multi-unit demand and supply - a typical double-auction environment. This auction should promote sincere bidding and have the capacity to be optimal. Specifically, we are interested in optimality with independent private values and ex-post IC, IR, and market-clearing constraints. For simplicity, we model the trade of a perfectly divisible, homogeneous asset.[3] We break down this task into four steps.

In the first step, we lay down general auction rules. Since it is a dynamic auction, it will inevitably share multiple features with the Ausubel auction. To be precise, there will be two Ausubel auctions, forward and reverse, in which all agents participate. We describe activity rules, transactional prices, rules to clear the market, the clock policy, and the disclosure policy. Then, we introduce the per-unit (i.e., marginal) taxes paid on top of the baseline Vickrey-style payments.

[1]For example, OTA Broadcasting, a private equity firm, has sold less than half of its owned spectrum, some of which was acquired just before the Incentive Auction. See Ausubel et al. (2017) for details. The reduced revenue in the allocation stage of the Spectrum Auction should not be confused with additional revenue in the assignment stage, where a VCG mechanism was used. See Ausubel and Baranov (2023).

[2]Public officials were concerned about raising enough revenue, see Loertscher et al. (2015) and the 112th Congress hearing Keeping the New Broadband Spectrum Law on Track.

[3]Our auction only features the allocation stage, where volumes of the generic good are traded. In practice, Spectrum Auctions feature an additional assignment stage, where participants resolve the complementarities between ranges of spectrum by bidding for particular lots; see, among others, Ausubel and Baranov (2023) and Rostek and Yoder (2023).

In the second step, we analyze the strategic properties of the auction. We show the auction to have a sincere equilibrium in the spirit of Ausubel (2004, 2006), for any clock and disclosure policies. This equilibrium yields the efficient outcome in a fictitious economy that we refer to as *virtual*, which amounts to the equilibrium being ex-post perfect, see Proposition 1. This will allow us to implement a rich set of mechanisms, including optimal ones.

Furthermore, we study equilibrium refinements, similarly to Ausubel (2004). Due to the 2-sided nature of the auction, additional equilibria emerge that were not featured in the 1-sided version, see Examples 1 and 2. In particular, the order of the movements of the clocks and the information released to the participants matters. We show that, for any clock policy, the sincere equilibrium can become a unique survivor of iterated elimination of weakly dominated strategies, under full-support beliefs, see Proposition 2. The idea is that, by concealing certain information from the bidders, one can prevent the forward auction from informing the reverse auction about its future path, and vice versa. We refer to it as the *no-spoilers* disclosure policy.

In the third step, we study the additional degree of freedom that emerges in the 2-sided environment - the clock policy. For example, one could move the clocks to match supply with demand almost continuously, as in McAfee (1992). However, the objective that this approach minimizes - the temporary mismatch of supply and demand - has no direct connection to efficiency, optimality, or multiplicity of equilibria. Instead, we suggest paying attention to the flow of information between the two auctions.

To be precise, when one auction generates information that could be harmful to the other, we register an *informational spillover*. It turns moving the clock prices to balance spillovers on the two sides leads to a decrease in the total number of spillovers, see Proposition 3. As a result, the no-spoilers disclosure policy becomes less binding, and more information is released to the participants, without compromising the uniqueness of equilibrium. We refer to it as the *adaptive* clock policy.

In the fourth step, we study a broad class of piecewise-smooth optimization problems. This is equivalent to finding an efficient and robust direct mechanism in a fictitious economy, that we refer to as *virtual*, parametrized by one-dimensional private types. We call it a *v-optimal mechanism*. We show how and under what conditions our dynamic auction implements this direct mechanism, see Proposition 4. Next, we derive the original economy's optimal direct mechanism and, under mild regularity conditions, we show that it one of the v-optimal mechanisms. Thus, it can be implemented via our auction, see Proposition 5.

We pay special attention to the so-called worst-off types. Lu and Robert (2001), working on a similar mechanism with interim constraints, admits that two-sided trade creates difficulties beyond standard mechanism design. Indeed, the monotonicity of a trader's virtual valuation typically fails at the worst-off type, even if the distribution of types is regular. Moreover, with the ex-post constraints, the locus of the worst-off types is conditioned on other players' types, making it untractable.

Despite the apparent complexity of the optimal mechanism, the implementation is relatively simple - two Ausubel auctions with a marginal tax on top. We will refer to the anti-derivative of the marginal tax as the *integrated tax*.[4] The optimal integrated tax has four key features. First, it does not depend on the number of bidders. Second, it depends on the clock price due to the ex-post nature of the mechanism. Third, it generically has a kink at zero. Finally, it is typically concave (but less concave than the utility) on each side of the kink. See Figure 1.

The cusp shape of the integrated tax comes from two conflicting ideas. On the one hand, the auctioneer wants to exclude the traders whose contribution to exchange is minimal to exert pressure on the rest of the traders. A convex kink "in the middle" guarantees precisely that. On the other hand, the auctioneer wants to minimize the distortion among the strongest buyers and sellers. Thus,

[4] A buyer is a bidder who increases his clinched position and pays the marginal tax. A seller is a bidder that decreases his clinched position, so he is being paid.

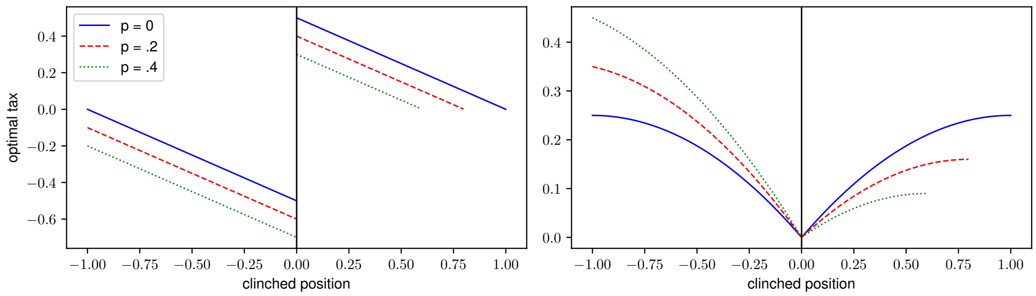

Figure 1: Marginal (left figure) and integrated (right figure) optimal tax at different levels of the clock price p, for a quadratic utility and uniform[-1,1] distribution.

the tax should flatten closer to the "shoulders."[5]

Lastly, a natural question is how much revenue the optimal mechanism yields compared to other robust mechanisms. To answer it, we focus on the special case of quadratic utility, which gives additional tractability to the model, see Proposition 6.

In Andreyanov and Sadzik (2021), two ad-hoc robust mechanisms were similarly implemented with taxes, albeit in a sealed-bid fashion. The first mechanism featured smooth, progressive (quadratic) per-unit taxes. The second one featured a flat per-unit tax.[6] We focus on two distributions of the hidden type: uniform and logistic; and measure their revenue and efficiency relative to the optimal mechanism, see Table 1. We find that the flat tax offers a better efficiency-revenue trade-off than progressive tax and, moreover, it almost reaches (but does not converge to) the Pareto frontier for certain distributions, see Figure 5. These findings were only made possible due to the characterization of the optimal robust mechanism. [7]

[5]This flattening happens for all distributions with a strictly positive density on compact support, such as uniform, because the $\frac{F}{f}$ and $\frac{1-F}{f}$ terms vanish on the boundary. However, for many distributions with full support, such as the logistic distribution, this does not happen.

[6]See Example 8 in Andreyanov and Sadzik (2021)

[7]For 100 bidders, simulations show that progressive taxation is also dominated in the following sense. For any quadratic tax, a flat tax can yield the same expected utility but higher expected revenue. Moreover, for the logistic distribution, a flat tax almost almost

I Literature

Our paper is linked to three strands of literature: robust mechanisms, optimal mechanisms, and practical auction rules.

The first strand is the classical literature on optimal mechanism design. The concept of virtualization, necessary for optimality, was developed independently by Mussa and Rosen (1978) and Myerson (1981). It was later generalized, among others, by Wilson (1985), Gresik and Satterthwaite (1989), Maskin and Riley (2000), and Lu and Robert (2001), to be used for two-sided and multi-unit environments.[8]

The second strand is the design of robust mechanisms. The concept of robust implementation is in the sense of Wilson (1987), Bergemann and Morris (2005), and Chung and Ely (2007), meaning that the mechanism should work for all information structures, distributions, and beliefs. Furthermore, in optimality, we can distinguish three approaches to robustness. The first classical approach is finding a mechanism with given properties, assuming the type distribution is known. The second approach is to estimate the properties of the distribution in a static environment, see Kojima and Yamashita (2017), or estimate it on the fly, see Loertscher and Marx (2020) and Loertscher and Mezzetti (2021). The third approach is to consider the worst-case, relative to the maximized objective, scenario. See Brooks and Du (2021) and Suzdaltsev (2022). Our paper belongs to the first approach, which can be justified by saying that the distribution can always be estimated using a small randomly sampled group of agents, which will be asymptotically negligible.

The third strand is the design of simple mechanisms when optimal mechanisms are impractical. For example, in Hart and Nisan (2017), it was argued that simple mechanisms for selling two goods could achieve a guaranteed fraction of the optimal revenue. In Andreyanov and Sadzik (2021), two families of

reaches the Pareto frontier.

[8]We add to this body of literature a non-linear utility and a small observation, see Lemma 2, that circumvents the non-monotonicity of virtual type.

simple mechanisms were suggested for an exchange environment with multi-unit demands. In this paper, we give the means to compare them to the optimal mechanism and find that they often capture a significant portion of optimal revenue.

Our numerical exercises contribute to the long ongoing debate over the efficiency-revenue tradeoff in two-sided markets with private information on both sides. One of the oldest results in this area is the impossibility of budget surplus under efficient trade by Myerson and Satterthwaite (1983), meaning that full ex-post efficiency is very costly in revenue.

Another argument was made by Gresik and Satterthwaite (1989) that optimal mechanisms converge to efficiency at a quadratic rate, and in Lu and Robert (2001), they converge to a simple bid-ask spread. Both results, however, rely on either unit demand or linear utility. With decreasing returns to scale, optimal mechanisms neither converge to efficiency nor to bid-ask spreads, which we confirm under quadratic utility.

Furthermore, Loertscher et al. (2015) argues that the efficiency-revenue tradeoff is steeper in markets with two-sided private information than those with one-sided, meaning that full optimality might be too costly in terms of efficiency. With our quadratic-utility model, we can reassess this claim by plotting the Pareto frontier. Interestingly, the simple mechanisms based on bid-ask spreads almost reach that frontier for the logistic distribution.

Our paper also contributes to the expanding literature studying uniform-price and pay-as-bid auctions; see Back and Zender (1993), Ausubel et al. (2014) and Wang and Zender (2002). One of the main takeaways is that demand reduction with multi-unit demands can severely impact auction revenues. We show that one possible solution to the problem is a combination of a per-unit tax with a bid-ask spread. However, in our numerical exercises, the latter is disproportionally more important. See Figure 5. Furthermore, Burkett and Woodward (2020) argues that there could also be low-revenue equilibria and suggests using reserve prices. Such "collusive-seeming" equilibria also emerge in our setting, but for a different reason: the inadvertent informational spillover between the two sides of the auction.

Finally, in the domain of robust auction design with private values, our paper is most similar to McAfee (1992) in its oral double-clock design and Ausubel (2004, 2006) in the clinching design of the payments. However, to our best knowledge, we are the first to characterize the optimal tax in the robust setting and to show how the price path can be guided to improve the strategic properties of the double auction.

II Dynamic auction

Our auction can be thought of as two copies: *forward* (i.e., ascending, buyers') and *reverse* (i.e., descending, sellers'); of the efficient dynamic auction of Ausubel (2004), with the clock prices running towards each other, and with carefully crafted per-unit taxes on top of the baseline Vickrey-style payments. These additional payments are necessary to implement mechanisms other than efficient ones, such as revenue-maximizing. [9]

A. Forward, reverse auctions and clinching

Two clock auctions run continuously or in discrete rounds. To distinguish between the two auctions, we will use superscript $+$ for the forward and $-$ for the reverse. We denote the *clock prices* in these auctions as p^+ and p^-.

Each player i participates in both auctions and, at any given pair of clock prices, submits a demand q_i^+ in the forward auction and q_i^- in the reverse auction. To be precise, in each auction round, the auctioneer first announces the clock price or a range of clock prices to be run. Bidders simultaneously and independently from each other respond with quantities or, in the latter case, demand schedules.

[9] The double-clock nature of the proposed auction resembles the recent Incentive Auction used for spectrum bandwidth reallocation.

The forward auction starts at a low price p_0^+ and gradually raises it. Likewise, the reverse auction starts at a high price p_0^- and gradually lowers it. The forward auction stops when the total demand becomes non-positive, while the reverse auction stops when the total demand becomes non-negative. We will refer to this pair of, possibly different, final clock prices as the *stop-off prices*.

There is much freedom in how the auctioneer can move the clock prices towards each other. The exact instructions would depend on the auctioning style (discrete or continuous clocks) and also on the objectives of the auctioneer, which we will discuss later.

Following Ausubel (2004), at any clock prices, we define *residual supply* (q_{-i}) and *clinched supply* ($q_{i,c}$) in the forward and reverse auctions correspondingly:

$$q_{-i}^+ := -\sum_{j\neq i} q_j^+, \quad q_{-i}^- := -\sum_{j\neq i} q_j^-, \quad q_{i,c}^+ := \max(0, q_{-i}^+), \quad q_{i,c}^- := \min(0, q_{-i}^-).$$

B. Activity and clearing rules

Buyers and sellers can submit demands satisfying two *activity rules*.

First, demands in both auctions are non-increasing in their respective prices, which we will refer to as *demand monotonicity*. Second, at any point, the agent's demand in the reverse auction is no greater than her demand in the forward auction, which we will refer to as *no-arbitrage*.

The *clearing rule* is a protocol for finalizing allocations and transfers at the stop-off prices. If everyone plays continuous demands, there will be an exact market clearing at one or several. However, because demands are allowed to jump, one can end up with a mismatch of supply and demand in the auction. If such a mismatch happens, some of the most recent demands might require rationing.

Luckily, it is always possible to put the final allocation "inside" the revealed demand of each agent, in order to incentivize them to submit truthfully.[10] One

[10]The definition of sincere demand in our paper will ensure that jumps in quantity de-

can easily see that the set of such allocations is convex and non-empty.[11] A member of this set can be selected, for example, by minimizing the sum of squared allocations. Other selections were proposed in Okamoto (2018).

C. Allocations, and payments

The final allocation is determined by the units clinched between the starting and the stop-off prices and the clearing rule.

Similar to Ausubel (2004), payments are only made for the incrementally clinched units. However, they consist of two parts. The first part is standard - each incrementally clinched unit costs exactly the clock price at which it was clinched in the corresponding auction. The second part consists of marginal taxes $m\tau$ that depend on the current clock price and the current position in clinched supplies. Namely, agent i pays $m\tau(p^+, q^+_{i,c})$ for the additional (positive) unit incrementally clinched in the forward auction and is paid $m\tau(p^-, q^-_{i,c})$ for an additional (negative) unit in the reverse auction.

Thus, agent i's total payment given final allocation q will be equal to

$$\int_0^q \left(p_{-i}(x) + m\tau_i(p_{-i}(x), x)\right) dx,$$

where $p_{-i}(\cdot)$ is the *inverse residual supply curve* facing agent i.

It is worth mentioning that agents do not have direct control over the units they clinch and the payments they make or receive. However, they can affect the stop-off price.

manded coincide with linear parts in the tax-adjusted utility of each agent, thus the jump in demand simply indicates a convex-valued demand correspondence.

[11]If, at a certain price, the sum of lower bounds to demand correspondences is non-positive, while the sum of upper bounds is non-negative, by Intermediate Value Theorem, there exists a convex combination of the upper and lower bounds such that it adds up to zero.

D. Clock and disclosure policies

In each round, the auctioneer determines which clock to advance, either the forward or reverse. Additionally, he has to determine how much information to reveal to the bidders. We will refer to it as the clock policy and the disclosure policy. In particular, we focus on two such policies.

- **Adaptive clock policy**: If the number of agents for whom $q_i^+ > q_{-i}^-$ is greater than the number of agents for whom $q_i^- < q_{-i}^+$ - move the forward clock. If the number of agents for whom $q_i^+ > q_{-i}^-$ is less than for whom $q_i^- < q_{-i}^+$ - move the reverse clock. Otherwise, move either clock.

- **No-spoilers disclosure policy**: Each bidder observes, apart from the clocks, two additional statistics:

  $$q_{i,d}^- = \max(q_i^+, q_{-i}^-), \quad q_{i,d}^+ = \min(q_i^-, q_{-i}^+).$$

  The idea is that this knowledge bears little consequence for the bidder. Indeed, conditional on $q_{i,d}^+$ and $q_{i,d}^-$, the possible range of final allocations for bidder i is $[q_i^+, q_i^-]$.

Combining the two policies above will allow us to deal with the potential 'void of incentives' when an agent learns about the outcome, regardless of his future play, see Example 1 and 2. This consideration is only valid it two-sided environments, as opposed to pure sales or procurement.

We will consider three additional policies for the exposition. With the *simple* clock policy, we fully advance the clock price in one of the two auctions: forward or reverse, until it hits the stop-off price. After that, we fully advance the clock price in the opposite auction. A *full-disclosure* policy informs the bidders about the most recent values of all forward and reverse auction demands. To the contrary, a *no-disclosure* policy does not share additional information with the bidders.[12]

[12] Even with the no-disclosure policy, bidders observe the switches between the forward and reverse auctions.

E. Structure of rounds

To summarise, our auction features forward and reverse clocks, with activity rules, disclosure and clock policies. It remains to define the structure of rounds, for which there are two approaches: continuous and discrete. Both are stylized representations of a dynamic (oral) auction and have unique strengths and weaknesses. For the purpose of the paper, we adopt a mixed approach by modeling discrete rounds but continuous demands.

We will refer to the position of the clocks at the beginning of round k as p_k^+ and p_k^-. The clocks start at round 0 at exogenous positions $p_0^+ < p_0^-$ and advance one at a time in discrete steps of size $(p_1^- - p_0^+)/M$. Thus, the auction ends in exactly $M-1$ rounds.

Once a round starts, each bidder submits a demand function in the range between that clock's most recent positions, constrained by the two activity rules. The clinches are then calculated in the respective range, with the exception of the round where the market clearing price was found. The information is revealed at the end of the round: whether the auction is over or not, which clock moves next according to the clock policy, and each bidder i is informed about the latest values of $q_{i,d}^+$ and $q_{i,d}^-$ according to the disclosure policy.

Special attention should be paid to the last rounds in the forward and reverse auctions because one of the two clock prices will likely overshoot, and the other will fall short of market clearing. During the round where overshooting happens, the clinches should be calculated only up to the market clearing price. Likewise, clinches must be added to the opposite side, as if the clock moved up to the market clearing.

This modeling approach retains most features of the oral design without the technical complications of switching between the two clocks continuously. [13]

[13] Electricitè de France (EDF) used a similar clock auction design for its power sales. The auction consists of several rounds; the clock price continuously advances during each round. Each round raises the price by some pre-determined amount unless the total demand decreases below the total supply. Bidders had to determine their bid plan before each round. Practically, this approach balances transparency with fast implementation of the auction.

III Strategic analysis

In this section, we introduce game-theoretic primitives, such as preferences, information, strategy, and equilibrium concepts, to set the basis for strategic analysis.

We then introduce the *sincere demand* - a stylized representation of the agent's tax-adjusted preferences. If all agents reveal their sincere demands during the double auction, the outcome will be efficient in the *virtual economy* populated by agents with tax-adjusted preferences.

We make three progressively narrow statements about this outcome.

First, it is an ex-post perfect equilibrium for all clock and disclosure policies. It is neither unique nor a dominant strategy equilibrium. However, using the no-spoilers disclosure policy, we ensure that it is the only survivor of equilibrium refinement - iterated elimination of weakly dominated strategies.

Finally, we show that the adaptive clock policy, in a certain sense, maximizes disclosure under the no-spoilers disclosure policy or, equivalently, maximizes the incentives to play sincerely under full disclosure.

A. Primitives and solution concepts

Agent i's preferences are represented by a quasilinear utility $u_i(q_i) - t_i$, where t_i is the transfer and $u_i(q_i)$ is agent i's utility from holding q_i units of asset. Let $\mathcal{U}_i$ denote the possible utility functions of agent i. Any $u_i \in \mathcal{U}_i$ is strictly concave and continuously differentiable so that it has a strictly decreasing derivative $mu_i(q) := \frac{\partial}{\partial q} u_i(q)$ for each $q \in \mathbb{R}$. Each agent i privately observes her utility u_i at the beginning and plays the double clock auction by the rules detailed in the previous section.

Assume, without loss of generality, that at round k, the forward clock moves from position p_k^+ to p_{k+1}^+. Agent i's strategy σ_i maps $h_{i,k}$ - the entire history

See pages 281-282 of Milgrom (2004) for more detail.

available to him at the beginning of the round, to a function on $[p_k^+, p_{k+1}^+]$, bound by the activity rules.

Following Ausubel (2004), we will consider two equilibrium concepts. The first one is ex-post equilibrium, extended naturally to the dynamic setting.

Definition 1. *A profile of strategies $(\sigma_i)_i$ is an ex-port perfect equilibrium if for every round k, following any history $h_{i,k}$, and for every realization of the profile of $(u_i)_i$, the profile of continuation strategies $(\sigma_i(\cdot|k, h_{i,k}, u_i))_i$ constitutes a Nash equilibrium of the game in which $(u_i)_i$ is common knowledge.*

The second one is more subtle. It is well-known that in the Vickrey auction, the additional equilibria can be discarded by eliminating weakly dominated strategies. In the dynamic setting, iterative elimination can be used. [14]

Definition 2. *A profile of strategies $(\sigma_i)_i$ is a unique survivor of iterated elimination of weakly dominated strategies if (i) there is an order of elimination such that every other strategy is weakly dominated, and (ii) there is no order of elimination, such that σ_i is weakly dominated, for some i.*

B. Sincere bidding

We aim to show that, in equilibrium, agents will behave similarly to a price-taking consumer. We describe such behavior by the sincere demand defined below. Note that sincere demand is not a strategy yet.

Definition 3. *The sincere demand $d_i(p)$ is defined as*

$$d_i(p) = \arg max_q \left[u_i(q) - \int_0^q m\tau_i(p,x)dx - pq \right]. \tag{1}$$

We will refer to $u_i(q) - \int_0^q m\tau_i(p,x)dx$ as *tax-adjusted utility.*

Sincere demand maximizes the agent's tax-adjusted utility, as if she were a

[14] Iterative elimination here entails comparing the preferred strategy to every other strategy, conditional on all histories that are consistent with the strategies that have not been eliminated yet.

price taker, given the wholesale price p. Thus, a market clearing price with sincere demands will achieve an efficient outcome in a fictitious economy with tax-adjusted utilities, evaluated at that price. However, these utilities are not private, as they depend on p.

For any utility u_i and tax $m\tau_i$, we wish to devise a private utility v_i that would replicate the same behavior as the tax-adjusted one, independently of the price. We will refer to these new utilities as *virtual*.

We can reverse engineer the virtual utility for agent i, up to a constant by solving the following system of first-order conditions,

$$p = mu_i(q) - m\tau_i(p, q) = mv_i(q_i).$$

In other words, $mv_i(q)$ is the graph of the set of points in the (q, p) space, where the first-order conditions are satisfied for the sincere demand. Furthermore, the v_i utilities should be strictly concave to validate the first-order approach. Thus, we introduce a joint restriction on the set of utilities $\mathcal{U}_i$ and the shape of the marginal tax.

Assumption 1. *For any i and any possible utility $u_i \in \mathcal{U}_i$,*

$$m\tau'_{i,q}(q) - mu'_{i,q}(q) > \varepsilon, \quad (1 + m\tau'_{i,p}(q,p))^{-1} > \delta,$$

for all $(p, q) \in \mathbb{R} \times (\mathbb{R} \backslash \{0\})$ and some $\varepsilon, \delta > 0$.

This assumption requires that the tax be *less concave than* any utility from $\mathcal{U}_i$ at all prices, and the marginal tax can not decrease too fast in the current price. These two properties ensure that the virtual utility is, indeed, concave. To see this point, linearize the first-order condition around (p, q) to obtain

$$mv'_{i,q}(q) = -\frac{m\tau'_{i,q}(p,q) - mu'_{i,q}(q)}{1 + m\tau'_{i,p}(p,q)} < -\varepsilon\delta,$$

at points of differentiability so that mv_i is strictly decreasing in q. We will refer to the economy with private utilities v_i as the *virtual economy*.

Thus, a market clearing price with sincere demands will achieve an efficient (i.e., Walrasian) outcome of the virtual economy. Coupled with Vickrey-style payments, the latter will amount to an ex-post equilibrium. It remains to find a strategy that will generate sincere demand along the equilibrium path.

Such a strategy exists. It entails playing sincere demand when not constrained by either activity rule, otherwise staying as close as possible to the demand.

Definition 4. *Agent i's strategy σ_i^* is said to be the sincere strategy if, at any round k after any history of play $h_{i,k}$ such that p_k^+ and p_k^- are the clock prices at the beginning of round k, her reporting plan for that round is*

$$\max(\min(d_i(p), q_i^+(p_k^+)), q_i^-(p_k^-))$$

in the forward auction, and

$$\min(\max(d_i(p), q_i^-(p_k^-), q_i^+(p_k^+)))$$

in the reverse auction.

We will refer to the profile of strategies $(\sigma_i^*)_i$ as *sincere bidding.* [15] We are now ready to state the first main result of our paper, see Appendix A for proof.

Proposition 1. *Consider any clock policy and disclosure policy, then*

1. *Sincere bidding yields the market clearing price and allocations of the Walrasian equilibrium in the virtual economy,*

2. *Sincere bidding is an ex-post perfect equilibrium,*

for any utilities and taxes satisfying Assumption 1.

[15]The domain on which the strategies are defined is unimportant since the sincere strategy is basically a constant function, constrained by the activity rules.

C. Weak dominance

It is worth noting that, with only a forward auction and no taxes, the sincere play is weakly dominant under no-disclosure, see Ausubel (2004), Theorem 1.

This is not true in the two-sided setting. The reason is that the auctioneer releases important information by merely switching between the forward and reverse auctions. This information can be used to manipulate the actions of other players in order to achieve certain results. We will use the simple clock policy to demonstrate it.

Example 1. *With the simple clock and no-disclosure policies, sincere bidding is not weakly dominant.*

Consider two players $i = 1, 2$ with sincere demands $d_1(p) = 2-p$, $d_2(p) = 1-2p$ that are common knowledge (i.e., $\mathcal{U}_i$ are singleton) and no additional taxes. Let the starting prices be $p_0^+ = 0$, $p_0^- = 2$, and let the forward clock advance first, to $p_1^+ = 1$.

Under sincere bidding, the stop-off price is found by the forward clock at the end of round 0, and it equals 1. The reverse clock then moves to confirm the same stop-off price. The first and second player's total clinches amount to 1 and -1. That is, the first player is the buyer. The average prices are 0.75 and 1.5 correspondingly, see Figure 2 (left).

Consider now a modified strategy for player 1. Namely, if the stop-off price after the forward auction is less than 1, she plays sincerely in the reverse auction. Otherwise, she plays $\tilde{d}_1(p) = d_1(2) = 0$, that is, she advances her demand in the reverse auction earlier than the sincere demand prescribes.

If player 2 proceeds with bidding sincerely, she will clinch everything at the stop-off price 1, see Figure 2 (right). Her loss due to the insidious actions of the first player in the reverse auction amounts to exactly 0.5. If, however, player 2 shifts her demand to $\tilde{d}_2(p) = d_2(p) - \varepsilon$, for a small $\varepsilon > 0$, the stop-off price in the forward auction will be equal to $1 - \frac{\varepsilon}{2}$ and player 1 will then play sincerely in the reverse auction.

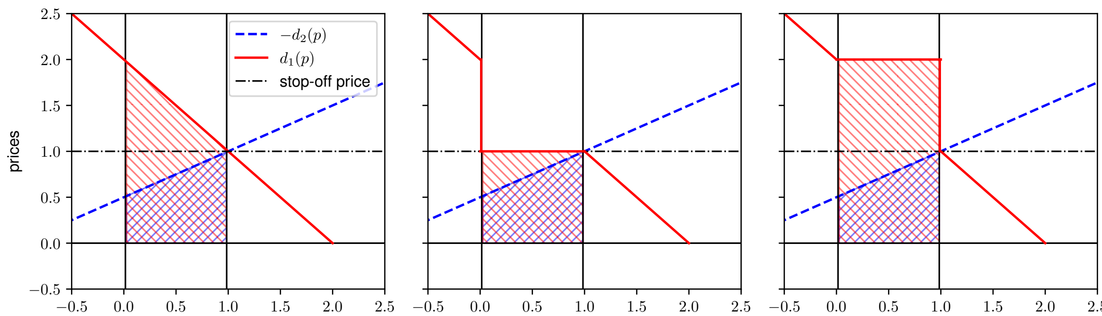

Figure 2: Payments when player 1 plays sincerely (left) when player 1 advances her demand late (middle) and early (right) in the reverse auction.

Thus, playing sincerely is not a dominant strategy for player 2.

D. Iterated elimination

Clearly, weak dominance is too strong an equilibrium concept. It is, however, possible to discard insincere strategies using iterated elimination of weakly dominated strategies. With only a forward auction and no taxes, sincere bidding was shown to be the unique survivor of such elimination, under full disclosure, see Ausubel (2004), Theorem 2.

Surprisingly, this is also not true in the two-sided setting. The reason is that the forward auction generates information that can be strategically used in the reverse auction and vice versa.

Example 2. *With the simple clock and full-disclosure policies, there are equilibria other than sincere bidding that survive iterated elimination of weakly dominated strategies.*

To build the counterexample, consider the same setting as in Example 1.

Consider now a modified strategy for player 1. Namely, if at the end of the first round, the stop-off price turns out to be 1 with final allocations 1 and -1, player 1 submits a flat demand of size 1 in the reverse auction. Otherwise, she plays sincerely. This can be thought of as dropping the demand “later” than the sincere strategy would prescribe, see Figure 2 (middle).

Since the non-standard strategy of player 1 is in the final round, it can not be eliminated in that subgame and thus can not be eliminated iteratively.

E. Full support beliefs

The reason why in the previous example, player 1 could deviate was that, by the end of round 0, her final allocation was known to be 1. Thus, she faced no consequence for changing her demand.

To keep the players from executing such deviations, one has to make sure that i) the information generated in the forward auction does not inform the players in the reverse auction about the potential range of allocations and vice versa, and ii) no matter what the players do, there is persisting uncertainty about the realization of the stop-off price and final allocations. The latter is typically called a full support assumption, see Ausubel (2004).

We will model this uncertainty by letting the auctioneer participate in the auction as a shill bidder, non-strategically and without taxes. In particular, she must be able to reduce her demand at any price and by any amount that does not violate the activity rules. Alternatively, we can interpret it as a population of noise traders.

Definition 5. *The double clock auction is said to satisfy the full support assumption if the auctioneer can play any demand that satisfies the activity rules.*

We are ready to state the second main result, see Appendix A for proof.

Proposition 2. *Consider any clock policy and no-spoilers disclosure policy, then sincere bidding is a unique survivor of iterated elimination of weakly dominated strategies under the full support assumption.*

IV Informational spillover and clock policy

The multiplicity of equilibria, discussed in the previous sections, is a consequence of a general phenomenon in double auctions: the inadvertent *informational spillover* between the forward and reverse auctions. In this section, we will attempt to create a language to describe and classify this phenomenon.

Furthermore, we will use a headcount of agents experiencing spillover to optimally guide the clock prices towards the stop-off price, which will justify the adaptive clock policy introduced in Section II.

Consider clock prices $p^+ \leqslant p^-$.

Definition 6. *For agent i, there is spillover into the forward auction if $q^-_{-i}(p^-) < q^+_i(p^+)$, and into the reverse auction if $q^-_i(p^-) < q^+_{-i}(p^+)$.* [16]

Imagine that at some point in time, agent i observes that the residual supply in the forward auction is ahead of agent i's sincere demand in the reverse auction, that is, $d_i(p^-) < q^+_{-i}(p^+)$, see Figure 6 (right). Then, i can reveal any value between $[d_i(p^-), q^+_{-i}(p^+)]$ in the reverse auction without risking changing the stop-off price. Alternatively, she can keep her demand unchanged for the range of prices $[d_i^{-1}(p^+), p^-]$ in the reverse auction. Thus, spillover allows supporting equilibria that are not sincere when agents observe the residual supply.

One can see that the no-spoilers disclosure policy, introduced in previous section, was designed precisely to conceal spillover. Indeed, spillover into the reverse auction happens if and only if the disclosure policy binds, that is, $q^+_{i,d} = \min(q^-_i, q^+_{-i}) = q^-_i$, making $q^+_{i,d}$ uninformative about the distance between her demand q^-_i and the residual supply in the opposite market q^+_{-i}. A similar argument applies to the forward auction, which allowed us to eliminate non-sincere equilibria under the no-spoilers policy in Proposition 2.

The question that we want to answer is whether there exists a clock policy that, in some sense, minimizes spillovers and thus maximizes disclosure under

[16] See Appendix B for an alternative definition of spillover.

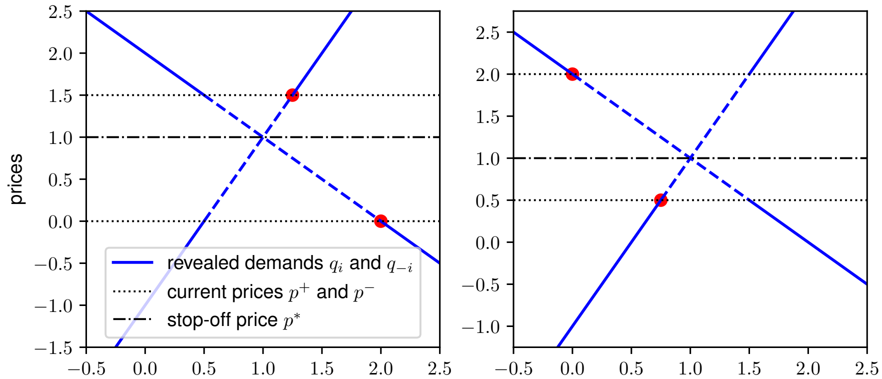

Figure 3: spillover into the forward auction (left figure) and into the reverse auction (right figure).

the no-spoilers disclosure policy.

The simple clock policy can not give us this property. Indeed, after fully advancing the clock in the forward auction, the residual supply there is no less than the sincere demand in the reverse auction for every player, and strictly so for strictly decreasing demands. Moreover, for $n = 2$, it is simply impossible to rule out spillovers with generic demands. However, we can try to make the number of agents that experience spillover as small as possible. This approach is motivated by the following lemma:

Lemma 1. *For any clock prices $p^+ \leqslant p^-$: if there is type-1 into both auctions (forward and reverse), then there is type-1 spillover for exactly one agent.*

According to this lemma, the number of agents experiencing type-1 spillovers at any time is far from arbitrary. Represented by a pair of numbers, it can only be one of the following: $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, $(2+, 0)$, $(0, 2+)$; where $x+$ stands for "x and more", see Figure 4 for an illustration. Moreover, when the numbers are $(1, 1)$, the same agent experiences spillover on both sides.

With this structure at hand, we can show that, for any collection of well-behaved sincere demands, a price path exists with special properties. Namely, along this path, the number of agents with spillovers monotonically decreases

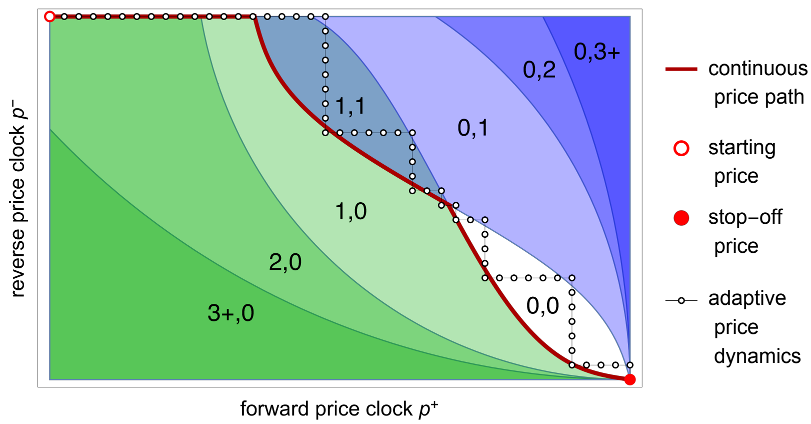

Figure 4: Stylized illustration of the price path associated with adaptive clock policy, and a continuous price path in Proposition 3.

until there is at most one such agent, and it stays that way, see Appendix B for formal proof. To make the result sharp, we put a few technical assumptions on the sincere demands and treat them as known.

Proposition 3. *Let agents play continuous and weakly monotone sincere demands, and there exists a stop-off price* p^* *such that the market clears. Then, for any starting prices* $p_0^+ \leqslant p^* \leqslant p_0^-$*, there exist a weakly monotone path* $p_+(t), p_-(t)$ *connecting* (p_0^+, p_0^-) *with* (p^*, p^*) *continuously.*

The path consists of two parts. In the first part, the number of agents experiencing type-1 spillover decreases monotonically until there is, at most, one such agent. In the second part, there is still at most one such agent.

How does this help with the design of the auction? If we could find a realistic clock policy that mimics the aforementioned path, it could be considered superior to other clock policies, as it minimizes spillovers and thus maximizes disclosure under the no-spoilers policy.

Now, recall the adaptive clock policy. If the number of agents for whom there is spillover into the forward auction is greater than the number of agents for whom there is spillover into the reverse auction, we move the forward clock. If the number of agents for whom there is spillover into the reverse auction is greater than the number of agents for whom there is spillover into the forward auction, we move the reverse clock. Otherwise, we move either clock.[17]

In other words, adaptive clock policy synchronizes the clocks in a way that balances the number of type-1 spillovers in the forward and reverse auctions, eventually localizing them to at most one agent, see Figure 4. [18]

V Direct mechanisms

In this section we derive the optimal tax for the double auction. To achieve this feat, we add more structure to the agents' preferences: private values, single-dimensional types and single-crossing preferences. We consider a flexible class of designer objectives, which covers expected revenue maximization and near-efficiency as special cases. First, we derive an optimal direct mechanism for this class of objectives. Then, we derive the taxation function that achieves the same allocation and payoffs in the sincere equilibrium of our dynamic auction.

A. Single-dimensional types

We model agent's preferences as a single-dimensional, private type $\theta_i \in \Theta_i \subset \mathbb{R}$. Thus, agent's payoff with type θ_i, allocation $q_i \in \mathbb{R}$ and transfer $t_i \in \mathbb{R}$, is

$$u_i(\theta_i, q_i) - t_i.$$

We will refer to the whole profile of types as θ and the profile of types other than agent i as θ_{-i}. For agent i, the domain of types, Θ_i, can be either a segment or the real line. For simplicity, we will refer to the domains of θ_i, θ_{-i} and θ as *the support*.

[17] A numerical example of how adaptive policy works is presented in Appendix E.

[18] There is no guarantee that the set of prices for which there are no spillovers is connected nor that it reaches the stop-off price. Thus, we can monotonically reduce the number of agents for whom spillover takes place to at most 1, but not 0. In Appendix B we present a slightly different definition of spillover, for which the set of prices for which there are no spillovers contains the stop-off price.

We begin with a minimal set of assumptions that are typically used in the mechanism design literature.

Assumption 2. *θ_i is independently distributed with CDF F_i and a strictly positive density f_i, $u_i(\theta_i, q_i)$ is twice continuously differentiable and strictly single crossing ($\frac{\partial^2}{\partial\theta_i \partial q_i} u_i(\theta_i, q_i) > 0$), for all i, $q_i \in \mathbb{R}$ and θ_i in the support.*

We focus on direct mechanisms with a truth-telling equilibrium, invoking the revelation principle. A direct mechanism (q, t) consists of an allocation rule $q : \mathbb{R}^n \to \mathbb{R}^n$ and a transfer rule $t : \mathbb{R}^n \to \mathbb{R}^n$. A direct mechanism must satisfy the incentive compatibility (IC) and individual rationality (IR) constraints so that the agents play a truth-telling equilibrium. In this paper, we require that both constraints are satisfied ex-post, that is, at each type profile on the type space, as in Andreyanov and Sadzik (2021), rather than on average, as in Lu and Robert (2001). Formally, they are defined as below.

Definition 7. *A direct mechanism (q, t) satisfies the ex-post IC and IR constraint if it satisfies the following inequalities.*

IC: $u_i(\theta_i, q(\theta_i, \theta_{-i})) - t(\theta_i, \theta_{-i}) \geqslant u_i(\theta_i, q(\theta_i', \theta_{-i})) - t(\theta_i', \theta_{-i})$,

IR: $u_i(\theta_i, q(\theta_i, \theta_{-i})) - t(\theta_i, \theta_{-i}) \geqslant u_i(\theta_i, 0)$.

for all i and all θ in the support.

Denote by $\tilde{u}_i(\theta_i, q_i)$ the net (i.e. relative to the autarky), utility of agent i:

$$\tilde{u}_i(\theta_i, q_i) = u_i(\theta_i, q_i) - u_i(\theta_i, 0).$$

A standard mechanism-design argument tells, see e.g. Milgrom and Shannon (1994); Milgrom and Segal (2002); Sinander (2022), that under strict single crossing, a direct mechanism (q, t) is ex-post IC if and only if: $q_i(\theta_i, \theta_{-i})$ is non-decreasing in θ_i (*monotonicity* constraint) and the *envelope conditions* hold:

$$t_i(\theta_i, \theta_{-i}) - t_i(\theta_i', \theta_{-i}) = \tilde{u}_i(\theta_i, q_i(\theta_i, \theta_{-i})) - \tilde{u}_i(\theta_i', q_i(\theta_i', \theta_{-i})) - \int_{\theta_i'}^{\theta_i} \frac{\partial}{\partial \theta_i} \tilde{u}_i(x, q_i(x, \theta_{-i}))dx, \tag{2}$$

for all i and $\theta_i, \theta_i', \theta_{-i}$ in the support.

B. Net surplus and worst-off-types

Another convenient way to describe the envelope conditions - is in terms of the agent's equilibrium payoff (added relative to the autarky), $\tilde{s}_i(\theta_i, \theta_{-i})$ that we refer to as her *net surplus*:

$$\tilde{s}_i(\theta_i, \theta_{-i}) = \tilde{u}_i(\theta_i, q_i(\theta_i, \theta_{-i})) - t_i(\theta_i, \theta_{-i}) = \max_{\theta_i' \in \Theta_i} \{\tilde{u}_i(\theta_i, q_i(\theta_i', \theta_{-i})) - t_i(\theta_i', \theta_{-i})\}.$$

This allows us to recast the IR constraint as

$$\inf_{\theta' \in \Theta_i} \tilde{s}_i(\theta_i', \theta_{-i}) \geqslant 0 \tag{3}$$

Furthermore, consider a candidate mechanism with the net surplus functions $\tilde{s}_i$, and the allocation functions q_i. Let $wot_i(\theta_{-i})$ denote the set of *worst-off types*, and $tet_i(\theta_{-i})$ denote the set of *types excluded from trade* of agent i:

$$wot_i(\theta_{-i}) = arg \min_{\theta_i' \in \Theta_i} \tilde{s}_i(\theta_i', \theta_{-i}), \quad tet_i(\theta_{-i}) = \{\theta_i' \in \Theta_i : q_i(\theta_i', \theta_{-i}) = 0\}.$$

Lemma 2. *Under Assumption 2, in an ex-post IC direct mechanism, any type excluded from trade is a worst-off type (i.e., $tet_i(\theta_{-i}) \subset wot_i(\theta_{-i})$).*

The above lemma allows us to recast the envelope conditions (2) as follows:

$$\tilde{s}_i(\theta_i, \theta_{-i}) = \inf_{\theta' \in \Theta_i} \tilde{s}_i(\theta_i', \theta_{-i}) + \int_{\theta_i^*}^{\theta_i} \frac{\partial}{\partial \theta_i} \tilde{u}_i(x, q_i(x, \theta_{-i}))dx, \tag{4}$$

where θ_i^* is one of the types excluded from trade if $tet_i(\theta_{-i})$ is non-empty in the candidate mechanism. If $tet_i(\theta_{-i})$ is empty, we will also have to consider

θ_i^* to be at the end of the support if it is compact, and $\pm\infty$ otherwise. [19]

Thus, we can replace the IR and IC constraints with (3) and (4). This is without loss of generality since any of the envelope conditions (2) can be obtained by combining two envelope conditions (4).

C. v-optimality

We are interested in a broad class of mechanisms, which we will refer to as *v-optimal*. Consider functions $v_i(\theta_i, q)$, which can be interpreted as contributions of each agent to a certain social utility. We wish to maximize it subject to the *market clearing* constraint $\sum_{i=1}^n q_i = 0$, ex-post. Additionally, we normalize agent's payoff at the worst-off type to be equal her payoff in the autarky.

Definition 8. *A v-optimal direct mechanism* (q, t) *maximizes*

$$\iiint\limits_{\mathbb{R}^n} \left[\sum_{i=1}^n v_i(\theta_i, q_i) \right] \prod_j dF_j(\theta_j) \tag{5}$$

subject to: monotonicity constraints, envelope conditions (4)*, market clearing and* $\inf_{\theta_i' \in \Theta_i} \tilde{s}_i(\theta_i', \theta_{-i}) = 0$*, for all* i *and* θ_{-i} *in the support.*

While not fully general, this formulation covers a number of important families of mechanisms. In particular, three such families have been studied before. The first family, studied in Gresik and Satterthwaite (1989), Lu and Robert (2001), in the context of Bayesian IC and IR constraints, can be informally defined via $v_i = (1-\alpha)u_i + \alpha t_i$, and can be thought of as a convex combination of efficient and revenue-maximizing mechanisms. The second and third families, studied in Andreyanov and Sadzik (2021), are $v_i = u_i - \sigma q_i^2/2$ and $v_i = u_i - \delta|q|$. They can be thought of as nearly efficient mechanisms capable of balancing the budget ex-post through controlled demand reduction. By coincidence, if the utility is quadratic: $u_i(\theta_i, q) = \theta_i q - \mu q^2/2$, the second family also contains (for $\sigma = \frac{\mu}{n-2}$) the uniform-price double auction, studied, among

[19] In the paper, we also provide conditions for the type excluded from trade to exist.

others, in Kyle (1989) and Rostek and Weretka (2012).

We place a few technical assumptions on the auctioneer's objective v, which ensure that the v-optimal mechanism is a solution to a smooth (with a notable exception of $q = 0$) and convex optimization problem.

Assumption 3. *$v_i(\theta_i, q_i)$ is twice continuously differentiable, strictly concave in q_i and strictly single crossing, for all i, $q_i \neq 0$, and θ_i in the support.*

With a slight abuse of notation, let $mv_i(\theta_i, q_i)$ denote $\frac{\partial}{\partial q_i} v_i(\theta_i, q_i)$ at points of differentiability, and the sub-gradient of v_i otherwise. Likewise, let $mu_i(\theta_i, q_i)$ denote $\frac{\partial}{\partial q_i} u_i(\theta_i, q_i)$.

D. Relaxed problem and monotonicity of allocation

As is common in the literature, we now relax the problem. Namely, we drop the monotonicity constraint, as well as all the remaining conditions stemming from the IR and IC constraints, with only the market clearing constraint remaining. This allows us to optimize (5) only over the allocation functions q_i, pointwise, and then check that all the ignored constraints can be satisfied.

Since the relaxed problem is convex, the method of Lagrange multipliers can be applied to characterize the optimum when it exists.[20] The first-order conditions for the saddle point of the Lagrangian are:

$$p(\theta) \in mv_i(\theta_i, q_i(\theta)), \qquad \sum_{i=1}^{n} q_i(\theta) = 0, \tag{6}$$

where $p(\theta) \in \mathbb{R}$ is the Lagrange multiplier. By strict concavity of the v_i functions, $q_i(\theta)$ is also single-valued.

Below, we verify that the solution to the relaxed problem is indeed monotonous. For convenience, let $mv'_{i,\theta}, mv'_{i,q}$ denote $\frac{\partial mv_i}{\partial \theta_i}(\theta_i, q_i)$, $\frac{\partial mv_i}{\partial q_i}(\theta_i, q_i)$ respectively. For

[20] For convex-constrained optimization problems, the saddle-point condition is both necessary and sufficient for optimality, see Theorem 2 in Luenberger 1969, 221p.

any $q \neq 0$, we may linearize (6) around (p, q) as below

$$mv'_{i,\theta} + mv'_{i,q} \cdot \frac{\partial}{\partial \theta_i} q_i(\theta) = \frac{\partial}{\partial \theta_i} p(\theta), \quad mv'_{j,q} \cdot \frac{\partial}{\partial \theta_i} q_j(\theta) = \frac{\partial}{\partial \theta_i} p(\theta), \quad j \neq i. \quad (7)$$

We can then solve for the slopes of allocation and price using market clearing

$$\frac{\partial}{\partial \theta_i} p(\theta) = \frac{mv'_{i,\theta}}{mv'_{i,q}} \frac{1}{\sum 1/mv'_{k,q}}, \; \frac{\partial}{\partial \theta_i} q_j(\theta) = \frac{mv'_{i,\theta}}{mv'_{j,q}} \left(\frac{1/mv'_{j,q}}{\sum 1/mv'_{k,q}} - \mathbb{I}(j = i) \right). \quad (8)$$

Clearly, under strict single crossing and strict concavity of v_i, the allocation of any agent is strictly increasing in her type, which is sufficient for the existence of transfers that satisfy all the IC and IR constraints that were omitted from the relaxed problem.[21] We would like to strengthen the properties of the allocation function further by uniformly bounding the slopes of mv_i.

Assumption 4. *$mv'_{i,\theta} \geqslant \varepsilon$, and $-1/mv'_{i,q} \geqslant \delta$, for all i and $\theta_i, \theta_{-i}, q_i$ in the support, and some $\varepsilon, \delta > 0$.*

We can bound the slope of the allocation function from below:

$$\frac{\partial}{\partial \theta_i} q_i(\theta) = mv'_{i,\theta} \cdot \frac{(-1/mv'_{i,q}) \cdot (\sum_{k \neq i} -1/mv'_{k,q})}{(-1/mv'_{i,q}) + (\sum_{k \neq i} -1/mv'_{k,q})} \geqslant \frac{n-1}{n} \varepsilon \delta. \quad (9)$$

Consequently, one can invert the allocation function with respect to own type everywhere except $q = 0$. We will refer to it as an *inverse allocation function* $q_i^{-1}(x, \theta_{-i})$, defined on the $(\mathbb{R} \backslash \{0\}) \times \mathbb{R}^{n-1}$ domain.

E. Existence of the type excluded from trade

To increase tractability, we would like to ensure the existence of the type excluded from trade.[22] Together with the *taxation principle*, see Rochet (1985), this will allow us to once again recast the envelope conditions using the inverse

[21] The allocation function is also strictly decreasing in types of others, and the market clearing price is strictly increasing in all types.

[22] This requirement can be dropped, but then, in the optimal mechanism, the designer will have to charge entry fees, conditional on θ_{-i}.

allocation function.

Lemma 3. *Under Assumptions 2 to 4, in a v-optimal mechanism (q,t), $tet_i(\theta_{-i})$ is nonempty, and the transfers can be written as:*

$$\tilde{t}_i(q,\theta_{-i}) = \int_0^q mu_i(q_i^{-1}(x,\theta_{-i}),x)dx, \tag{10}$$

for all i and θ_{-i} in the support.

We also provide two alternative versions of Assumption 4 that would ensure the existence of types excluded from trade, see Section .

F. Taxation scheme

We are now ready to derive the taxation scheme associated with the auction described in Section II, which would match the one in our v-optimal mechanism. According to the auction rules of the auction, the payments consist of two parts: the Vickrey-style payments and the integrated (along the residual supply curve) marginal taxes

$$\tilde{\tilde{t}}_i(q,\theta_{-i}) = \int_0^q m\tau(p_{-i}(x),\theta_{-i}) + p_{-i}(x)dx, \tag{11}$$

where $p_{-i}(x)$ is the residual supply curve facing agent i.

If we could set the marginal tax equal to the wedge between mu_i and mv_i at the desired allocation, the agents would essentially perceive v_i as their true utility. It only remains to do it for every realization of types.

Definition 9. *Set the marginal tax $m\tau_i(p,q) = x$, where $(x,\hat{\theta})$ solves*

$$x = mu_i(\hat{\theta},q) - mv_i(\hat{\theta},q), \quad p = mv_i(\hat{\theta},q), \tag{12}$$

for all p,q in the support.

We refer to the solution $\hat{\theta}_i(p,q)$ to the system of equations (12), as the *fixed-point type*. It is correctly defined on the $\mathbb{R}\times(\mathbb{R}\backslash\{0\})$ domain, and so is the marginal tax, when the support of θ_i is the real line.

We are ready to formulate our second main result.

Proposition 4. *Under Assumptions 2 to 4, the sincere equilibrium in the double clock auction with the marginal tax $m\tau(p,q)$ defined by (12) achieves the same allocation and transfer as in the v-optimal mechanism.*

The proof proceeds by observing, quite mechanically, the equivalence between the transfers $\tilde{t}_i(\theta_i, q_i)$ and $\tilde{\tilde{t}}_i(\theta_i, q_i)$, see Appendix C.

Finally, by linearizing (12) around (p, q), we can derive the slopes of the fixed-point type and the marginal tax

$$\hat{\theta}'_p = 1/mv'_{i,\theta}, \quad \hat{\theta}'_q = -mv'_{i,q}/mv'_{i,\theta}, \tag{13}$$

$$m\tau'_{i,p} = mu'_{i,q}\hat{\theta}'_p - 1, \quad m\tau'_{i,q} = mu'_{i,\theta}\hat{\theta}'_q + mu'_{i,q}. \tag{14}$$

Corollary 1. *Under Assumptions 2 to 4, the marginal tax $m\tau_i$ is continuously differentiable for all $q \neq 0$ and satisfies*

$$m\tau'_{i,q} - mu'_{i,q} > 0, \quad m\tau'_{i,p} + 1 > 0.$$

In other words, the (integrated) tax is less concave than the utility, and the marginal tax can not respond to the change in price too fast.

VI Revenue maximization

Our special case of interest is revenue maximization. Ignoring the monotonicity constraint, we will attempt to maximize the average transfer

$$\iint\limits_{\mathbb{R}^{n-1}} \sum_{i=1}^{n} \left[\int_{\mathbb{R}} \left(\tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i}) \right) dF_i(\theta_i) \right] \prod_{j \neq i} dF_j(\theta_j) \tag{15}$$

subject to market clearing and envelope conditions. Naturally, in the revenue-maximizing mechanism, any leftover surplus can be extracted via translation of monetary transfers, therefore, $\inf_{\theta_i'} \tilde{s}_i(\theta_i', \theta_{-i}) = 0$. We aim to get rid of the net surplus in (15), similarly to Myerson (1981) in a one-sided setting, via integration by parts.

Before we proceed, there is one more assumption that we have to make related to the integrability of the net surplus of a candidate mechanism, which is necessary for integration by parts on the real line. [23]

Assumption 5. *$\tilde{u}_i(\theta_i, q_i) \leqslant C(\theta_i)$ for any $q : \sum_{j=1}^n q_j = 0$ and some function $C(x)$, such that $\int C(x) dF_i(x) < \infty$.*

Although this assumption is very weak, it follows that the expected net surplus in the exchange economy is finite. To see the importance of this observation, note that even with simple quadratic models as in Section VII, the utility is not bounded on $\mathbb{R}$, and thus the expected net surplus is not obviously bounded.

Lemma 4. *Under Assumptions 2 and 5: $\int_{-\infty}^{\infty} \tilde{s}_i(z, \theta_{-i}) dF_i(z) < \infty$ for all θ_{-i} in the support.*

With this at hand, we split the integral of the net surplus at the type excluded from trade (when this type exists) and apply integration by parts to each of the two halves. Otherwise, we apply the integration by parts to the whole integral. This gives us the following equivalence

$$\int_{\mathbb{R}} (\tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i}))\, dF_i(\theta_i) = \int_{\mathbb{R}} J_i(\theta_i, q_i) dF_i(\theta_i) \tag{16}$$

$$J_i(\theta_i, q_i) = \tilde{u}(\theta_i, q_i) - \frac{\mathbb{I}(q_i > 0) - F(\theta_i)}{f(\theta_i)} \frac{\partial}{\partial \theta} \tilde{u}(\theta_i, q). \tag{17}$$

We will refer to J_i as the *virtual utility*. It is worth noting that the virtual utility, in this particular form, is continuous in both allocation and type. Indeed, the only potential source of discontinuity is the indicator function $\mathbb{I}(q_i > 0)$ multiplied by $\frac{\partial}{\partial \theta} \tilde{u}(\theta_i, q)$, which is zero at $q = 0$. Thus, there is no jump at

[23] Riemann integration if F is continuous, or, more generally, Stiltjes.

$q_i = 0$. Instead, there is a concave kink. [24]

It remains to check whether the premise of the Proposition 4 is satisfied so that we can also claim the implementation of the revenue-maximizing mechanism. One way to achieve this - is to put more assumptions on the true utilities u_i.

Assumption 6. *F_i is log-concave and both $mu'_{i,q} \cdot sgn(q_i(\theta))$ and $-mu'_{i,\theta} \cdot sgn(q_i(\theta))$ are nondecreasing in θ_i, for all i and θ in the support, such that $q_i(\theta) \neq 0$.*

This assumption guarantees that Assumption 3 is satisfied for $v_i = J_i$, which leads to the following proposition. See Appendix D for formal proof.

Proposition 5. *Under Assumptions 2 and 5, the revenue-maximizing mechanism is v-optimal with v_i equal to the virtual utility J_i.*

Corollary 2. *Under Assumptions 2, 5 and 6, the virtual utility J_i is twice continuously differentiable, strictly concave in q_i and strictly single crossing, for all i, $q_i \neq 0$, and θ_i in the support.*

Likewise, even stronger restrictions on the utility u_i can guarantee that Assumption 4 is satisfied for $v_i = J_i$.[25]

Assumption 7. *Assumption 6 hold and $mu'_{i,\theta} \geqslant \varepsilon$, $-mu'_{i,q} \leqslant \frac{2}{\delta}$, $\frac{1-F}{f} mu''_{i,\theta q} \leqslant \frac{2}{\delta}$ for some $\varepsilon, \delta > 0$.*

Corollary 3. *Under Assumptions 2, 5 and 7, the virtual utility satisfies all properties in Corollary 2 and also $mv'_{i,\theta} \geqslant \varepsilon$, and $-1/mv'_{i,q} \geqslant \delta$, for all i and $\theta_i, \theta_{-i}, q_i$ in the support.*

[24] It is also possible to derive a virtual utility without applying Lemma 2, in that case, it would be discontinuous (downward jump) in own type.

[25] These assumptions are automatically satisfied is the types are distributed on a segment.

	revenue from optimal		surplus from efficient	
distribution \ tax	optimal quad.	optimal flat	optimal quad.	optimal flat
uniform	50%	88.8%	75%	74.1%
logistic	64.8%	99.7%	75%	74.4%

Table 1: Percentage of optimal revenue and efficient surplus, achieved by the optimal quadratic-tax and flat-tax mechanisms, for a quadratic utility and in the large economy limit.

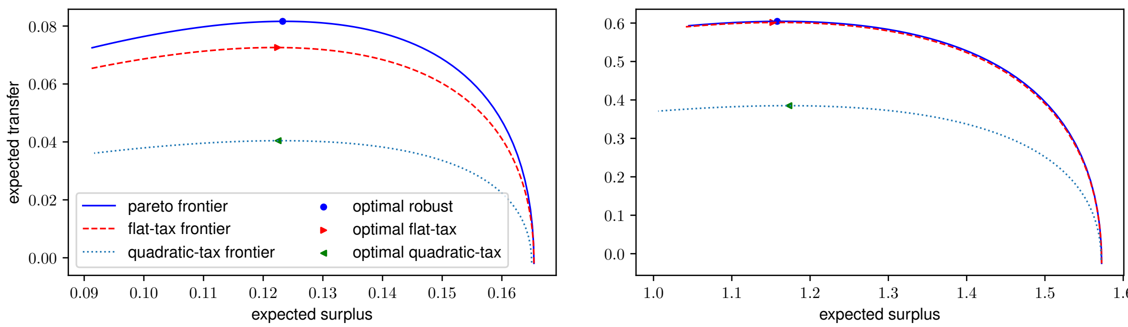

Figure 5: Comparison of flat-tax and quadratic-tax mechanisms relative to the Pareto frontier, for $n = 100$ agents with quadratic utility and either uniform[-1,1] (left figure) or logistic (right figure) distribution of θ. See Section for the description of simulations.

VII Symmetric quadratic model

This section illustrates our methodology in a symmetric model where each agent i has the following quadratic utility function.

$$u_i(\theta_i, q_i) = \theta_i q_i - \frac{\mu}{2} q_i^2,$$

for some known $\mu > 0$. We consider two log-concave distributions of private types θ_i: uniform on the $[-1, 1]$ interval and logistic (i.e., with full support). The above specification provides additional tractability and allows for comparison across different mechanisms, in large, as well as finite economies, see Table 1, Figure 5 and also Appendix E.

A. Pareto frontier

We now solve for the mechanism that maximizes a linear combination of expected transfer and surplus, in other words, finds the Pareto frontier. Following the arguments in Section VI, we have to maximize $\sum_i J_{\alpha,i}(\theta_i, q_i)$ over q, subject to the market clearing constraint $\sum q_i = 0$, where

$$J_{\alpha,i}(\theta_i, q_i) = q_i \left[\varphi_\alpha(\theta_i) \cdot \mathbb{I}(q_i > 0) + \psi_\alpha(\theta_i) \cdot \mathbb{I}(q_i \leqslant 0)\right] - \frac{\mu}{2} q_i^2,$$

pointwise, where $\varphi_\alpha(\theta_i) = \theta_i - \alpha \frac{1-F(\theta_i)}{f(\theta_i)}$ and $\psi_\alpha(\theta_i) = \theta + \alpha \frac{F(\theta_i)}{f(\theta_i)}$. These functions are monotone, as long as F is log-concave.

To identify the optimal allocation, we must find a Lagrange multiplier $p(\theta)$ such that the market clears and the first-order conditions hold. This leads to the following solution

$$d_i(p|\theta_i) = \mu^{-1} \left[\min(0,\ \psi_\alpha(\theta_i) - p) + \max(0,\ \varphi_\alpha(\theta_i) - p)\right],$$

for each agent i, which will also be her sincere demand in the auction implementation. The Lagrange multiplier $p(\theta)$ is then the root of $\sum_{i=1}^n d_i(p|\theta_i)/n = 0$, that is, the price at which the average sincere demand equals zero.

Finally, the marginal tax $m\tau$ and the fixed-point type $\hat{\theta}$ solve the system of equations (12) and thus

$$\hat{\theta}(p, q) = \varphi_\alpha^{-1}(\mu q + p) \cdot \mathbb{I}(q > 0) + \psi_\alpha^{-1}(\mu q + p) \cdot \mathbb{I}(q < 0) \tag{18}$$

$$m\tau(p, q) = \hat{\theta}(p, q) - (\mu q + p) \tag{19}$$

Proposition 6. *In the symmetric quadratic model with a log-concave distribution F with a strictly positive density f and a finite second moment, the optimal mechanism exists and is implemented via marginal taxes* (19).

Since the worst-off types are in the interior of the type space, the transfers can be formally written out, conditional on the shape of the residual supply curve.

$$t_i(q_i|\theta_{-i}) = \begin{cases} \int_0^q (\varphi_\alpha^{-1}(\mu z + p_{-i}(z|\theta_{-i})) - \mu z)\, dz, & q > 0 \\ \int_0^q (\psi_\alpha^{-1}(\mu z + p_{-i}(z|\theta_{-i})) - \mu z)\, dz, & q < 0 \end{cases}, \tag{20}$$

where $p_{-i}(z|\theta_{-i})$ is the inverse residual supply curve. Despite a relatively simple mechanism implementation, further characterization is rather difficult in a finite economy, even for standard distributions. [26]

B. Quadratic-tax mechanisms

Our first benchmark is smooth nearly-efficient mechanisms in Andreyanov and Sadzik (2021) called σ-VCG mechanisms, which can be thought of as an attempt to control demand reduction explicitly via quadratic (integrated) tax. One way to define this mechanism is the maximizer of $\sum_i J_{\sigma,i}(\theta_i, q_i)$ over q, subject to the market clearing constraint $\sum q_i = 0$, where

$$J_{\sigma,i}(\theta_i, q_i) = \theta_i q_i - \frac{\mu + \sigma}{2} q_i^2.$$

The ex-post allocation and transfer in this mechanism can be derived:

$$q_i = \frac{n-1}{n} \frac{\theta_i - \bar{\theta}_{-i}}{\mu + \sigma}, \quad t_i(q_i) = \bar{\theta}_{-i} q + \frac{\mu + n\sigma}{2(n-1)} q^2,$$

where $\bar{\theta}_{-i} = \frac{1}{n-1} \sum_{j \neq i} \theta_j$ is the average type other than agent i's type.

Since both transfer and utility are quadratic in types, we can compute their expected values given the variance of the type distribution:

$$\mathbb{E} t_i = \frac{(n-2)\sigma - \mu}{2n(\mu + \sigma)^2} \mathbb{V}\theta_i, \quad \mathbb{E} u_i = \frac{(n-1)(\mu + 2\sigma)}{2n(\mu + \sigma)^2} \mathbb{V}\theta_i,$$

[26] For $n = \infty$, the average demand curve converges pointwise to a monotone function, which has a root at 0 in a symmetric model. Since the slope of that function is strictly positive at 0, the root itself converges in the probability limit, which means that in the large economy limit, $p = 0$ effectively.

since $\mathbb{E}\bar{\theta}_{-i} = \mathbb{E}\theta_i$, $\mathbb{E}(\bar{\theta}_{-i})^2 = \frac{\mathbb{E}\theta_i^2+(n-2)(\mathbb{E}\theta_i)^2}{n-1}$ and $\mathbb{V}\theta = \mathbb{E}\theta_i^2 - (\mathbb{E}\theta_i)^2$. Naturally, for a uniform-price double auction ($\sigma = \frac{\mu}{n-2}$), the expected payment is equal to zero, while for the efficient mechanism ($\sigma = 0$), it is negative.

Finally, the maximum expected transfer over σ-VCG mechanisms is attained at $\sigma = \frac{n\mu}{n-2}$, and is equal to $\frac{(n-2)^2}{n(n-1)} \cdot \frac{\mathbb{V}\theta}{8\mu}$, while the utility is equal to $\frac{(n-2)(3n-2)}{n(n-1)} \cdot \frac{\mathbb{V}\theta}{8\mu}$.

C. Flat-tax mechanisms

Our second benchmark is non-smooth nearly-efficient mechanisms in Andreyanov and Sadzik (2021), which can be thought of as a bid-ask spread of size 2δ. One way to define this mechanism is the maximizer of $\sum_i J_{\delta,i}(\theta_i, q_i)$ over q, subject to the market clearing constraint $\sum q_i = 0$, where

$$J_{\delta,i}(\theta_i, q_i) = q_i \left[\varphi_\delta(\theta_i) \cdot \mathbb{I}(q_i > 0) + \psi_\delta(\theta_i) \cdot \mathbb{I}(q_i \leqslant 0)\right] - \frac{\mu}{2} q_i^2,$$

pointwise, where $\varphi_\delta(\theta) = \theta_i - \delta$ and $\psi_\delta(\theta) = \theta + \delta$. The rest of the algorithm is identical to the one used for revenue maximization, so we have to rely on Monte Carlo simulations for finite economies.

In the limit economy, however, there is no supply reduction, so the agent's demand is equal to $(\theta_i - \delta)/\mu$ if he turns out to be a buyer, and $(\theta_i + \delta)/\mu$ if he turns out to be a seller. Moreover, for symmetric distributions, the limit of the equilibrium price will be equal to 0, so buyers will pay a per-unit price of δ, while sellers will get a per-unit price of $-\delta$. Thus, we can compute the expected payment and utility:

$$\mathbb{E}t_i = 2\delta \int_\delta^{F^{-1}(1)} \left[\frac{x-\delta}{\mu}\right] dF(x), \quad \mathbb{E}u_i = 2 \int_\delta^{F^{-1}(1)} \left[x\frac{x-\delta}{\mu} - \frac{\mu}{2}(\frac{x-\delta}{\mu})^2\right] dF(x)$$

which can be easily maximized over δ, for any given distribution.

VIII Conclusion

The two-sided nature of exchange combined with robustness introduces challenges for auction designers that were not featured in the 1-sided setting.

The auctioneer has to decide on the order in which the clocks move – the clock policy – and which information to reveal to the bidders – the disclosure policy. In this paper, we show that certain combinations these policies can be used to balance the openness of the auction with the incentives to play sincerely.

Another big challenge is to extract maximal revenue. Due to the endogeneity of the worst-off type, the optimal direct mechanism is very complicated. However, the associated implementation is simple - two Ausubel auctions, forward and reverse, with the clock prices running towards each other.

Finally, with the optimal robust mechanism at hand, we are able to re-assess some of the ad-hoc robust mechanisms introduced in Andreyanov and Sadzik (2021). Numerical analysis shows that, for simple distributions, a flat tax captures the bulk of the optimal revenue and offers a better revenue-efficiency trade-off than a progressive (namely, quadratic) differential tax.

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Optimal Robust Double Auctions

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Online Appendix.

Appendix A Proofs for Section III

Proof of Proposition 1

Observe first that, in every subgame, along the conjectured equilibrium path, the revealed demands in the sincere ex-post equilibrium coincide with the sincere demands $\{d_i(p)\}_{i=1}^n$ which, together with the market-clearing condition fully characterize the outcome (price and allocation) of the game.

We will now prove that *the market clearing price and allocation under sincere bidding coincide with the Walrasian equilibrium in the virtual economy.*

Suppose that players bid by the sincere strategy in the clock auction. Then auction outcomes are characterized by the first-order condition for the sincere demands

$$p = mu(q_i) - m\tau(p, q_i), \quad i = 1, \ldots, n.$$

By definition of mv_i, the first-order condition above can also be expressed as

$$p = mv_i(q_i), \quad i = 1, \ldots, n,$$

which are the first order conditions for Walrasian demands in the economy with utilities v_i. Since the second-order conditions in both cases are satisfied by Assumption 1, the first-order conditions show that the market clearing price and allocation in both equilibria are the same.

We will now prove that *sincere bidding is an ex-post perfect equilibrium.* The proof considers two cases regarding the prior history of play in the clock auction.

First, we examine agents' incentives on the equilibrium path of play, where the demands revealed before the subgame was all sincere.

Assuming that all but player j continue to play sincerely, bidder j's payoff is path-independent in the following sense. At any counterfactual allocation q, the stop-off price $p_{-j}(q)$ is uniquely defined by the sincere demands of other players. Moreover, her payment to the auctioneer is equal to the marginal tax $m\tau_i(p_{-j}(x), x)$ plus $p_{-j}(x)$, integrated over $x \in [0, q]$. The optimality condition is, therefore

$$\begin{cases} mu_j(q) = m\tau_j(p, q) + p \\ p = mv_i(q_i), \quad i = 1, \ldots, n, \quad i \neq j \end{cases}$$

which yields $p = mv_i(q_j)$ by the definition of the v_i functions. Player j's payoff is aligned with the social surplus in the virtual economy, which is maximized by playing sincerely. Thus, sincere play is a Nash equilibrium of the subgame.

Second, we consider incentives off the equilibrium path, where traders reported non-sincerely before this subgame.

Assuming that all but player j continue to play sincerely, bidder j's payoff is path-independent, but her actions are constrained by the demands revealed before the subgame.

These payoffs are monotonically decreasing in the distance from the conjectured allocation. Thus, she finds it optimal to play as close to the sincere demand as possible. Therefore, sincere play is a Nash equilibrium of the subgame.

Proof of Proposition 2

Observe first that, in every subgame, and independently of the history of play, the no-spoilers policy with the full support assumption implies that, from the bidder's perspective, the stop-off price can be anywhere in the currently played range of prices.

To be precise, if the reported demand differs from the sincere at any price p^*

in this range, for at least one "strategy" of the auctioneer, consistent with the history, p^* becomes the stop-off price.[27] The payoff of the bidder would then be strictly smaller than if she was playing sincerely. Thus, insincere bidding can not weakly dominate.

We will now prove that *there is an order of elimination that yields sincere bidding.*

The proof is by induction over the auction rounds, counting from the final round.

Assume that, in the final round, the strategy of agent i differs from her sincere strategy at a price p^*. Since the stop-off price can be anywhere, this strategy is dominated in the event that the stop-off price is equal to exactly p^*. Thus, all strategies other than sincere ones are weakly dominated in the final round.

Consider a non-final round, and suppose we eliminated insincere bidding for all subsequent rounds. The same argument then applies to the range of prices in the current round as for the final round.

Thus, all strategies other than sincere ones have been eliminated.

We will now prove that *no order of elimination eliminates the sincere strategy.*

Assume that there exists an order of elimination that eliminates a sincere strategy. Consider the earliest instance in this order and the first round (starting from the end of the auction) of such elimination. By construction, in the current round and in the rounds that follow, no sincere strategies have been eliminated yet.[28]

Consider now the sincere strategy in this subgame. Since the stop-off price can be anywhere, the sincere strategy dominates all remaining strategies again. Thus, the sincere strategy can not be dominated in this subgame.

[27] With the full support assumption, the auctioneer participates, as if he was a bidder, but he does it non-strategically.

[28] If sincere bidding constitutes an ex-post equilibrium in the subgame, it can not be eliminated strictly but still could be eliminated weakly.

Thus, no order of elimination eliminates the sincere strategy.

Appendix B Proofs for Section IV

We will use two different notions of spillover. We will refer to the spillover defined in the main body of the paper as type-1 spillover. Type-2 spillover is a special case of type-1 spillover.

Definition 10. *For agent i, there is type-2 spillover into the forward auction if, additionally to being type-1 spillover, $q^-_{-i}(p^-) = q^-_{i,c}(p^-)$, and into the reverse auction if, additionally to being type-1 spillover, $q^+_{-i}(p^+) = q^+_{i,c}(p^+)$.*

While the type-1 spillover focuses on the informational content of the residual supply curves in the opposite market, the type-2 also checks whether these residual supply curves are equal to cliched ones. In the latter case, spillover will be accompanied by a transaction at the current price. However, if the auctioneer chooses to conceal the spillover, the agent will not be informed about this transaction in a timely fashion, which undermines the credibility of the auction. Thus, type-2 spillover can be considered more serious and harder to conceal than type-1 spillover.

We begin with the analog of Lemma 1 for type-2 spillovers.

Lemma 5. *For any clock prices $p^+ \leqslant p^-$: there can be type-2 spillover into at most one auction (forward or reverse).*

Since type-2 spillover is a special case of type-1 spillover, by Lemma 1 and Lemma 5, the number of agents experiencing type-2 can only be one of the following: $(0, 0)$, $(1, 0)$, $(0, 1)$, $(2+, 0)$, $(0, 2+)$, but not $(1, 1)$ as with type-1 spillovers, see Appendix B for details.

Proposition 7. *Let agents play continuous and weakly monotone sincere demands, and there exists a stop-off price p^* such that the market clears. Then, for any starting prices $p^+_0 \leqslant p^* \leqslant p^-_0$, there exist a weakly monotone path $p_+(t), p_-(t)$ connecting (p^+_0, p^-_0) with (p^*, p^*) continuously.*

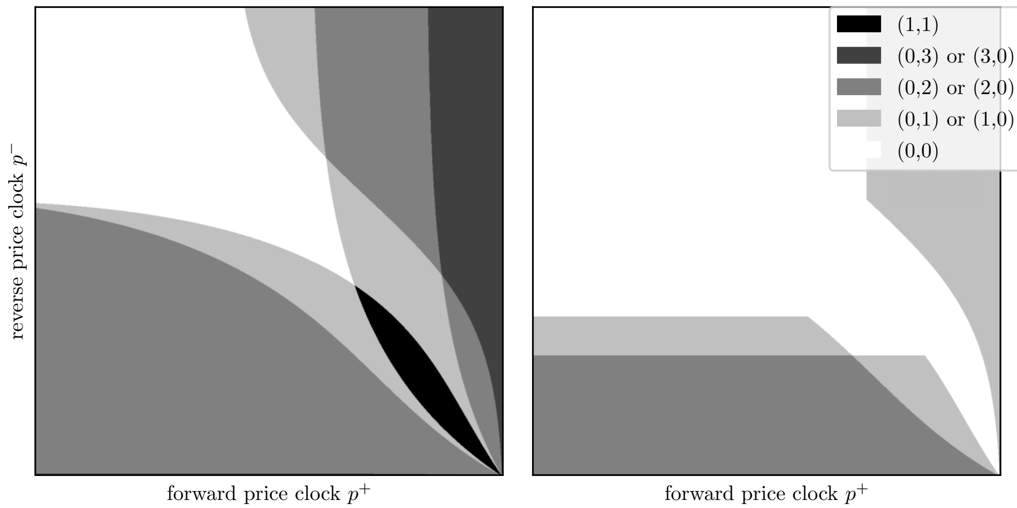

Figure 6: Illustration of the number of type-1 (left figure) and type-2 (right figure) spillovers, for three agents with revealed demands equal to $\sigma(4-p), \sigma(2-2p), \sigma(5-p)$; where $\sigma(x) = \frac{1}{1+\exp^{-x}} - \frac{1}{2}$. The stop-off price is in the bottom-right corner.

The path consists of two parts. In the first part, the number of agents experiencing type-2 spillover decreases monotonically to zero. In the second part, there is still no such agents.

The price path described in Proposition 7 is the same as the one used for Proposition 3 and illustrated in Figure 4. Because type-2 spillovers can not happen simultaneously in both auctions, when the path reaches the boundary of the region with $(1,1)$ type-1 spillovers, it automatically eliminates all type-2 spillovers. This means that, unlike type-1 spillovers, type-2 spillovers can always be decreased monotonically along the price path, which is a consequence of the fact that the latter are less frequent. Another way of thinking about it is that the set of prices with no type-2 spillovers is guaranteed to contain the stop-off price, which was not true for type-1 spillovers. See Figure 6 for an illustration.

On the practical side, when the price is in the region with (1,1) type-1 spillovers, we can refine the adaptive clock policy, which is otherwise indifferent between the two clocks. Namely, while we can not minimize type-1 spillovers, we can minimize type-2 spillovers by moving the clock that did not produce the type-2 spillover.

Finally, if the starting prices are sufficiently far away from the stop-off price, a continuous price path exists that has no type-2 spillovers. Such prices exist under mild conditions.

Corollary 4. *Let agents play continuous sincere demands that are unbounded (from both below and above) on the real line, and there exists a stop-off price* p^* *such that the market clears. Then, there exist starting prices* $p_0^+ \leqslant p^* \leqslant p_0^-$*, and a weakly monotone path* $p_+(t), p_-(t)$ *connecting* (p_0^+, p_0^-) *with* (p^*, p^*) *continuously and without type-2 spillovers.*

Proof of Lemma 1

Proof. To the contrary, assume that at some prices $p^+ \leqslant p^-$, there is type-1 spillover into both auctions, and, at the same time, there is type-1 spillover for more than one agent. This means that there exist two agents $i \neq j$ such that:

$$q_{-i}^-(p^-) < q_i^+(p^+), \quad q_j^-(p^-) < q_{-j}^+(p^+).$$

Using the definition of the residual supply, we can pair these inequalities:

$$-\sum_{k \neq i,j} q_k^-(p^-) < q_i^+(p^+) + q_j^-(p^-) < -\sum_{k \neq i,j} q_k^+(p^+)$$

which contradicts the fact that $q_k^-(p^-) \leqslant q_k^+(p^+)$ for all k. □

Proof of Lemma 5

Proof. To the contrary, assume that at some prices $p^+ \leqslant p^-$, there is type-2 spillover into both auctions. Then, it is also a type-1 spillover and, by

Lemma 1, it is a spillover for the same agent. However, no agent can clinch in both auctions simultaneously, which is a contradiction. □

Proof of Proposition 3

Proof. Consider the domain of prices $(p^+, p^-) \in [p_0^+, p^*] \times [p^*, p_0^-]$ and denote the subset of prices that have x type-1 spillovers into the forward and y spillovers into the reverse auctions by $S_{x,y}$.

Observe first that Lemma 1 implies that $S_{1+,2+} = \varnothing$ and $S_{2+,1+} = \varnothing$. Thus, any point in the price domain belongs to either $S_{0,1+}$, $S_{1+,0}$, $S_{1,1}$ or $S_{0,0}$. For simplicity, assume that the starting prices do not belong to $S_{0,0}$, in other words, there is at least one spillover.

The price path connecting (p_0^+, p_0^-) with (p^*, p^*) will have two parts. The first part is a straight line, and the second part goes along the boundary of either $S_{0,1+}$ or $S_{1+,0}$, see Figure 4. To construct the price path, consider three cases.

Case 1: If the starting prices are in $S_{1+,0}$, we first advance the forward clock till it reaches the boundary of $S_{1+,0}$. After that we move along the path $(\tilde{p}^+(p), p)$ where

$$\tilde{p}^+(p) = \sup_{x \in [p_0^-, p^*]} \{x : (x, p) \in S_{1+,0}\}.$$

Case 2: If the starting prices are in $S_{0,1+}$, we first advance the reverse clock till it reaches the boundary of $S_{0,1+}$. After that we move along the path $(p, \tilde{p}^-(p))$ where

$$\tilde{p}^-(p) = \inf_{x \in [p^*, p_0^+]} \{x : (p, x) \in S_{0,1+}\}.$$

Case 3: If the starting prices (p_0^+, p_0^-) are in $S_{1,1}$, any of the aforementioned trajectories will work.

We argue that along the first part of the trajectory, the number of agents experiencing spillovers is weakly decreasing. Indeed, on the one hand, advancing the forward (reverse) clock does not increase the number of spillovers in the forward (reverse) auction. On the other hand, the number of spillovers in the reverse (forward) auction is fixed at 0 by construction.

The function $\tilde{p}^+(.)$ does not have to be continuous. However, if it is monotone, we can connect the (at most countably many) points of discontinuity to obtain a monotone and continuous path $p^+(t), p^-(t)$. It remains to show that $\tilde{p}^+(.)$ is weakly monotone and that, along this path, the number of agents that experience spillover is at most one.

Monotonicity: Assume that $\tilde{p}^+(p_1^-) = p_1^+$, that is, (p_1^+, p_1^-) belongs to the closure of $S_{1+,0}$. Now, pick any $p_2^- < p_1^-$. When the clock prices move from (p_1^+, p_1^-) to (p_1^+, p_2^-), the number of spillovers in the reverse auction can not increase, while the number of spillovers in the forward auction is already at 0. Thus, (p_1^+, p_2^-) belongs to the closure of $S_{1+,0}$ as well, thus $\tilde{p}^+(p_2^-) \geqslant p_1^+$. Consequently, $\tilde{p}^+(p)$ is weakly monotone.

Finally, observe that $S_{1+,0}$ does not intersect with $S_{0,2}$ by Lemma 1. Thus, it can only share a boundary with $S_{1,1}$, $S_{0,1}$ and $S_{0,0}$. In either case, the number of agents experiencing spillovers is at most one. □

Proof of Proposition 7 and Corollary 4

The same exact path constructed in the proof of Proposition 3 works here, because when on the boundary between $S_{1,1}$ and $S_{2+,0}$, or $S_{1,1}$ and $S_{0,2+}$, there are no type-2 spillovers.

Finally, since q_i^+, q_i^- are unbounded, there exist starting prices such $q_i^+(p_0^+) > 0$ and $q_i^-(p_0^-) < 0$ for all i. Then $q_{i,c}^+ = q_{i,c}^- = 0$ for all i, that is, there are no type-2 spillovers.

Appendix C Proofs for Section V

Proof of Lemma 2

Fix θ_{-i} and consider two mutually exclusive cases. Suppose first that the set of types excluded from trade is empty. Then the claim holds trivially.

Suppose that it is not empty. Let $\hat{\theta}_i$ be a type excluded from trade. By definition of net utility, $\frac{\partial}{\partial \theta_i}\tilde{u}_i(\hat{\theta}_i, q_i(\hat{\theta}_i, \theta_{-i})) = 0$.

Next, the net surplus functions $\tilde{s}_i$ are absolutely continuous, a.e. differentiable and

$$\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i-, \theta_{-i}) \leqslant \frac{\partial}{\partial \theta_i}\tilde{u}_i(\hat{\theta}_i, q_i(\hat{\theta}_i, \theta_{-i})) \leqslant \frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i+, \theta_{-i}),$$

where $\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i-, \theta_{-i})$ and $\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i+, \theta_{-i})$ are left-hand and right-hand partial derivatives respectively, see Theorems 1,2 in Milgrom and Segal (2002).

Next, at points of differentiability, we can write:

$$\frac{\partial}{\partial \theta_i}\tilde{s}_i(\theta) = \frac{\partial}{\partial \theta_i}\tilde{u}_i(\theta_i, q_i(\theta)) = \int_0^{q(\theta)} \frac{\partial}{\partial \theta_i \partial q_i} u_i(\theta, x) dx,$$

thus $\tilde{s}_i$ is convex in θ_i by monotonicity of q_i in θ_i and single-crossing of u_i.

Finally, since $[\frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i-, \theta_{-i}), \frac{\partial}{\partial \theta_i}\tilde{s}_i(\hat{\theta}_i+, \theta_{-i})]$ contains 0 at the type excluded from trade, by the necessary first-order conditions, $\hat{\theta}_i$ is also the worst-off type.

Proof of Lemma 3

Equation (9) shows that $q_i(\theta_i, \theta_{-i})$ is continuous in θ_i and bounds it's slope away from zero. Thus, $q_i(\theta_i, \theta_{-i})$ is guaranteed to cross 0 at some type $\theta_i \in \mathbb{R}$, in other words, $tet(\theta_{-i})$ is non-empty, for any θ_{-i} in the support. Next, by formula (4)

$$t_i(\theta) = \tilde{u}(\theta_i, q(\theta)) - \tilde{s}(\theta_i, q(\theta)) =$$

$$= \tilde{u}(\theta_i, q(\theta)) - \inf_{\theta_i \in \Theta_i} \tilde{s}_i(\theta_i, \theta_{-i}) - \int_{\theta*}^{\theta_i} \frac{\partial}{\partial \theta}\tilde{u}(x, q(x, \theta_{-i}))dx$$

where $\theta^* \in tet(\theta_{-i})$. Recalling that, in a v-optimal mechanism, $\inf_{\theta' \in \Theta_i} \tilde{s}_i(\theta'_i, \theta_{-i}) = 0$

$$t_i(\theta) = \int_{\theta^*}^{\theta_i} \frac{d}{dx}\tilde{u}(x, q(x, \theta_{-i}))dx - \int_{\theta^*}^{\theta_i} \frac{\partial}{\partial \theta}\tilde{u}(x, q(x, \theta_{-i}))dx =$$
$$= \int_{\theta^*}^{\theta_i} \frac{\partial}{\partial q}\tilde{u}(x, q_i(x, \theta_{-i}))dq_i(x, \theta_{-i}) = \int_{\theta^*}^{\theta_i} mu_i(x, q_i(x, \theta_{-i}))dq_i(x, \theta_{-i}).$$

Finally, we get formula (10) via monotone change of variables from x to $q_i^{-1}(x, \theta_{-i})$.

Proofs of Lemma 3 with alternative versions of Assumption 4

Version 1: v_i are identical, F_i are identical.

Proof. To the contrary, assume that for some realization of types θ_{-i}, trader i only trades strictly positive quantities. Define a type $z = \min_{j \neq i} \theta_j$, and observe that it belongs to the support of each agent. Consequently, we can say that $q_i(z, \theta_{-i}) > 0$.

Furthermore, the allocation can not decrease if we lower the types of traders $j \neq i$. Consequently, we can say that $q_i(z, \ldots, z) > 0$. But this can not be true because any $p \in mv_i(z, 0)$ solves the first-order conditions in the symmetric case. □

Version 2: for any i and $p \in \mathbb{R}$, there exist a type z in the support such that $p \in mv_i(z, 0)$.

Proof. Pick a trader i, and fix a profile of types θ_{-i}. Next, consider the economy without trader i, that is, solve a system of first-order conditions

$$mv_j(\theta_j, \tilde{q}_j) = \tilde{p}, \quad \forall j \neq i, \quad \sum_{j \neq i} \tilde{q}_j = 0.$$

This solution exists for some $\tilde{p}$.

Next, pick a type z in the support, such that $\tilde{p} = mv_i(z, 0)$. By construction, i is excluded from trade in the original economy with the profile of types (z, θ_{-i}). $\square$

Proof of Proposition 4 and Corollary 1

We want to prove that agents face the same menus $t_i(q)$ in both the auction and the optimal mechanism. For that, it suffices to show that the integrand in (10) coincides with the one in (11) for any $q(\theta) \neq 0$.

Using the left-hand side of (12) we first write that

$$m\tau(p_{-i}(x), x) + p_{-i}(x) = mu_i(\hat{\theta}_i(p_{-i}(x), x), x).$$

Second, we combine the right-hand side of (12) with the definition of the residual supply curve

$$\begin{cases} p_{-i}(x) = mv_i(\hat{\theta}_i(p_{-i}(x), x), x) \\ p_{-i}(x) = mv_j(\theta_j, q_j), \quad \forall j \neq i \\ x + \sum_{j \neq i} q_j = 0. \end{cases}$$

The latter can be recognized as the system of first-order conditions for the optimal mechanism, given that x is the allocation of agent i and θ_{-i} are the types of others. Thus $\hat{\theta}_i(p_{-i}(x), x)$ and $q^{-1}(x, \theta_{-i})$ coincide and, therefore,

$$mu_i(\hat{\theta}_i(p_{-i}(x), x), x) = mu_i(q^{-1}(x, \theta_{-i}), x),$$

which completes the proof. Finally, Corollary 1 follows from formulas (13) and (14).

Appendix D Proofs for Section VI

Proof of Lemma 4

The boundedness of the expected net surplus comes from the fact that, on the one hand, the net surplus is nonnegative by IR, and on the other hand, the sum of net surpluses can not exceed the sum of net utilities at the efficient allocation

$$\tilde{s}_i(\theta_i, \theta_{-i}) \geqslant 0, \quad \sum_j \tilde{s}_j(\theta_j, \theta_{-j}) \leqslant \sum_j \tilde{u}_j(\theta_j, \theta_{-j}) \leqslant \sum_j C(\theta_j),$$

therefore $\tilde{s}_i(\theta_i, \theta_{-i}) \leqslant \sum_j C(\theta_j)$ for any θ in the support, and thus $\int \tilde{s}_i(z, \theta_{-i}) dF_i(z)$ is majorized by $\int C(z) dF_i(z) + \sum_{j \neq i} C(\theta_j) < \infty$ for all θ_{-i} in the support.

Proof of Proposition 5, Corollary 2 and Corollary 3

Recall that our objective is

$$\iint\limits_{\mathbb{R}^{n-1}} \sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} \left(\tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i}) \right) dF_i(\theta_i) \right] \prod_{j \neq i} dF_j(\theta_j), \tag{21}$$

where $\tilde{s}_i(\theta_i, \theta_{-i}) = \int_{\theta_i^*}^{\theta_i} \tilde{u}_1'(x, q(x, \theta_{-i})) dx$ and θ_i^* is either one of the types excluded from trade, or the end of support, or $\pm\infty$.[29]

For exposition, we will only consider the most difficult case with full support and when, conditional on some θ_{-i}, agent i can be both a buyer and a seller, depending on the realization of θ_i, in other words, θ_i^* is the type excluded from trade.

[29] At this point, we can not guarantee the existence of the type excluded from trade for any candidate mechanism.

By Lemma 4, the improper integral of $\tilde{s}_i(.,\theta_{-i})f_i(.)$ exists, thus

$$\int_{-\infty}^{\infty}\tilde{s}_i(\theta_i,\theta_{-i})dF_i(\theta_i) = \lim_{N\to\infty}\int_{-N}^{N}\tilde{s}_i(\theta_i,\theta_{-i})dF_i(\theta_i) =$$
$$\lim_{N\to\infty}[\int_{-N}^{\theta_i^*}\tilde{s}(\theta_i,\theta_{-i})dF_i(\theta_i) + \int_{\theta_i^*}^{N}\tilde{s}(\theta_i,\theta_{-i})d(F_i(\theta_i)-1)].$$

Integrating by parts, we get that

$$\int_{-N}^{\theta_i^*}\tilde{s}(\theta_i,\theta_{-i})dF_i(\theta_i) + \int_{\theta_i^*}^{N}\tilde{s}(\theta_i,\theta_{-i})d(F_i(\theta_i)-1) =$$
$$= \int_{-N}^{N}\frac{\mathbb{I}(q_i>0)-F_i(\theta_i)}{F_i(\theta_i)}\tilde{u}_1'(\theta_i,q)dF_i(\theta_i) + \tilde{s}_i(-N,\theta_{-i})F_i(-N) + \tilde{s}_i(N,\theta_{-i})(F_i(N)-1)$$

The remainder term $\tilde{s}_i(-N,\theta_{-i})F_i(-N)$ is majorized for N large enough:

$$\tilde{s}_i(-N,\theta_{-i})F_i(-N) = \tilde{s}_i(-N,\theta_{-i})\int_{-\infty}^{-N}dF_i(z) \leqslant$$
$$\leqslant \int_{-\infty}^{-N}\tilde{s}_i(z,\theta_{-i})dF_i(z) \leqslant \int_{-\infty}^{\infty}\tilde{s}_i(\theta_i,\theta_{-i})dF_i(\theta_i) < \infty,$$

and similarly for $\tilde{s}_i(N,\theta_{-i})(F_i(N)-1)$, because the net surplus $\tilde{s}_i(.,\theta_{-i})$ is positive and convex. Thus,

$$\int_{-\infty}^{\infty}\tilde{s}_i(\theta_i,\theta_{-i})dF_i(\theta_i) = \int_{-\infty}^{\infty}\frac{\mathbb{I}(q_i>0)-F_i(\theta_i)}{F_i(\theta_i)}\tilde{u}_1'(\theta_i,q)dF_i(\theta_i).$$

We next need to show that the virtual value J is concave and single-crossing to use the first-order approach.

$$\begin{aligned}
\frac{\partial^2 J}{\partial\theta\partial q} =& \frac{\partial^2\tilde{u}}{\partial\theta\partial q} - \frac{\partial}{\partial\theta}\left(\frac{\mathbb{I}(q_i>0)-F_i(\theta_i)}{F_i(\theta_i)}\frac{\partial^2\tilde{u}}{\partial\theta\partial q}\right) = \\
=& \frac{\partial^2\tilde{u}}{\partial\theta\partial q} - \frac{\partial}{\partial\theta}\left(\frac{\mathbb{I}(q_i>0)-F_i(\theta_i)}{F_i(\theta_i)}\right)\frac{\partial^2\tilde{u}}{\partial\theta\partial q} - \left(\frac{\mathbb{I}(q_i>0)-F_i(\theta_i)}{F_i(\theta_i)}\right)\frac{\partial^3\tilde{u}}{\partial\theta\partial^2 q} > 0 \\
\frac{\partial^2 J}{\partial q^2} =& \frac{\partial^2\tilde{u}}{\partial q^2} - \frac{\mathbb{I}(q_i>0)-F_i(\theta_i)}{F_i(\theta_i)}\frac{\partial^3}{\partial\theta\partial q^2}\tilde{u} < 0.
\end{aligned}$$

Both properties are guaranteed by Assumption 6. Finally, Corollary 2 and Corollary 3 follow directly from the formulas above.

Appendix E Proofs for Section VII

Proof of Proposition 6

Observe first that Assumptions 2 and 6 are satisfied for the quadratic utility and the log-concave distribution with strictly positive density. Assumption 5 requires more work. Observe that the utility of agent i is satiated at point $q_i = \frac{\theta_i}{\mu}$ yielding $\frac{\theta_i^2}{2\mu}$. Thus, we can take $C(x) := \frac{x^2}{2\mu}$. Since the distribution of θ_i has a finite second moment, $\int C(x)dF_i(x) < \infty$.

We have proved in Proposition 5 and Corollary 2 that under Assumptions 2, 5 and 6, the profit-maximizing mechanism is v-optimal with v_i equal to the virtual utility (17), such that the premise of Proposition 4 is satisfied. Thus, the sincere equilibrium in the double clock auction with the marginal tax $m\tau(p, q)$ defined by (12) achieves the same allocation and transfer as in the v-optimal mechanism.

Numerical simulations

To evaluate the welfare in large markets, we simulate 1000 draws of types θ_{-i} for $n = 100$ agents, from two distributions: uniform with support on [-1,1], and truncated logistic distribution with support on [-7,7]. The utility is quadratic with $\mu = 1$.

For every type θ_i on a grid, the demands are calculated for each draw of θ_{-i}, according to the formula below

$$d_i(p|\theta_i) = \left[\min(0,\ \psi_{\alpha,\delta}(\theta_i) - p) + \max(0,\ \varphi_{\alpha,\delta}(\theta_i) - p)\right],$$

$$\psi_{\alpha,\delta} = (1-\alpha)\theta + \alpha(\theta - \frac{1 - F_i(\theta_i)}{F_i(\theta_i)}) + \delta, \quad \varphi_{\alpha,\delta} = (1-\alpha)\theta + \alpha(\theta + \frac{F_i(\theta_i)}{F_i(\theta_i)}) - \delta$$

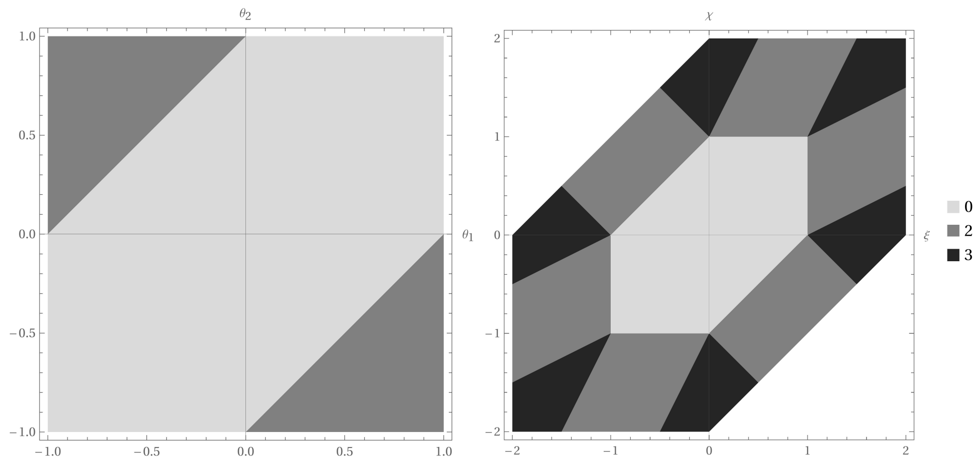

Figure 7: Exclusion region for two (left) players and three (right) players for a uniform[-1,1] distribution of types. The latter is in the coordinates $\xi = \theta_1 - \theta_3$, $\chi = \theta_2 - \theta_3$, independent from the value of θ_3.

where α is the parameter spanning the Pareto frontier (for $\delta = 0$), while δ is the parameter spanning the flat-tax family (for $\alpha = 0$). The worst-off types, transfers and surplus are calculated for each draw of θ_{-i} and each θ_i. The results are then weighted by the marginal density of θ_i.

The values for the expected utility and profit for the quadratic-tax family are computed from analytical formulas in Section B..

Optimal mechanisms for small and large n.

In this section, we attempt to characterize the optimal robust mechanism and plot the exclusion regions for $n = 2$ and $n = 3$, as well as $n = \infty$.

Uniform distribution

When the distribution is uniform, $\varphi_1(\theta) = 2\theta - 1$, $\psi_1(\theta) = 2\theta + 1$, thus

$$d_i(p|\theta_i) = \mu^{-1}\left[\min(0,\ 2\theta_i + 1 - p) + \max(0,\ 2\theta_i - 1 - p)\right],$$

and the marginal tax is $m\tau(p,q) = \frac{-\mu q - p - 1}{2} + \mathbb{I}(q > 0)$. Note that the number of agents excluded from trade depends on the location of the root of the average

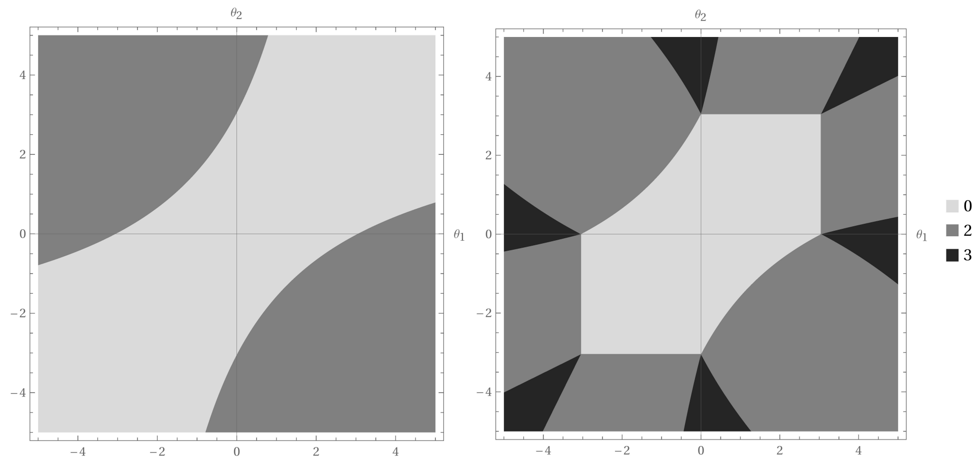

Figure 8: Exclusion region for two (left figure) players and three (right figure) players for a logistic distribution of types. The latter is in the coordinates θ_1, θ_2 for a fixed $\theta_3 = 0$.

demand curve and thus can not be explicitly characterized.

For example, for just $n = 3$ agents, the exclusion region follows an elaborate pattern, see Figure 7. When all three types are close to each other (a light grey area), nobody is trading. Next, with two significantly opposing types and a third in the middle (a dark grey area), only opposing types are trading with each other. Finally, when two types oppose the third, all three players are trading (black area).

When the number of players grows, the pattern becomes more complicated. However, the root of the average demand curve will converge in the probability limit, which is equal to 0. Thus, the limit exclusion region will simply be $\theta_i \in [-1/2, 1/2]$.

Logistic distribution

For a logistic distribution, $\varphi_1(\theta) = \theta - 1 - e^{-\theta}$, $\psi_1(\theta) = \theta + 1 + e^{\theta}$, thus

$$d_i(p|\theta_i) = \mu^{-1}\left[\min(0,\ \theta_i + 1 + e^{\theta} - p) + \max(0,\ \theta_i - 1 - e^{-\theta} - p)\right],$$

and the marginal tax is $m\tau(p, q) = sgn(q) \cdot \left[1 + \omega(e^{-1-sgn(q)\cdot(\mu q + p)})\right]$. The exclusion region for $n = 3$ follows a pattern similar to that of the uniform distribution, see Figure 8. In the limiting economy, again, the root of the average demand curve will be equal to 0. Thus the limit exclusion region is simply $\theta_i \in [-1 - \omega(1/e), 1 + \omega(1/e)]$, where $\omega(z)$ is the product-logarithm function.

Illustrative Example with Taxes

Suppose that two agents have the following linear-quadratic utility function $u_i(q) = \theta_i q - \frac{\mu}{2}q^2$ and marginal utilities $mu_i(q) = \theta_i - \mu q$. The auctioneer puts the following marginal tax $m\tau(p, q)$ in addition to the Vickrey price:

$$m\tau(p,q) = \begin{cases} -\frac{1}{2} - \frac{1}{2}(p + \mu q), & q < 0 \text{ and } -1 \leqslant p + 2q \leqslant 3, \\ \frac{1}{2} - \frac{1}{2}(p + \mu q), & q > 0 \text{ and } -3 \leqslant p + 2q \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

This tax function implements an expected-profit maximizing direct mechanism. The optimal direct mechanism shows the sincere demand to be

$$d(p, \theta) = \min\{0, \varphi(\theta) - p\} + \max\{0, \varphi(\theta) - p\},$$
$$\varphi(\theta) = \theta + \frac{F_i(\theta)}{F_i(\theta)} = 2\theta + 1, \quad \varphi(\theta) = \theta - \frac{1 - F_i(\theta)}{F_i(\theta)} = 2\theta - 1.$$

The residual supply $q_{-i}(p)$ is given by $q_{-i}(p) = -d(p, \theta_{-i})$ and its inverse by

$$p_{-i}(q) = \begin{cases} 2\theta - 1 + \mu q & q < 0, \\ 2\theta + 1 + \mu q & q > 0. \end{cases}$$

The marginal tax evaluated at the residual supply is then

$$m\tau(p_{-i}(q), q) = -\mu q - \theta_{-i} \quad q \neq 0.$$

If an agent has clinched x units until some round and clinches y more during some round, she pays

$$\int_x^y \{m\tau_i(p_{-i}(z), z) + p_{-i}(z)\}\, dz = \begin{cases} (y-x)(\theta_{-i}+1) & q > 0, \\ (y-x)(\theta_{-i}-1) & q < 0, \end{cases}$$

from that round.

Let the types of agents be $\theta_1 = 0.9$ and $\theta_2 = -0.7$. Thus, in each round, agent 1 pays $0.05 \cdot (\theta_2 + 1) = 0.05 \cdot 0.3$ and agent 2 pays $0.05 \cdot (\theta_1 - 1) = 0.05 \cdot (-0.1)$. The outcome of sincere bidding is described in Table 2.

Round	**Ends at Price**	**Demand at** p^+	**Demand at** p^-	**Agent**	**Clinches**	**Pays**	**Switching Decisions**
0	(-0.5,0.9)	(0.65,0)	(0,-0.65)	-	-	-	-
1	(**-0.4**,0.9)	(0.6,0)	-	-	-	-	$p^+ \uparrow$
2	(**-0.3**,0.9)	(0.55,-0.05)	-	1	0.05	0.015	$p^+ \uparrow$
3	(-0.3,**0.8**)	-	(0,-0.6)	-	-	-	$p^- \downarrow$
4	(-0.3,**0.7**)	-	(0.05,-0.55)	2	-0.05	-0.01	$p^- \downarrow$
5	(-0.3,**0.6**)	-	(0.1,-0.5)	2	-0.05	-0.01	$p^- \downarrow$
6	(**-0.2**,0.6)	(0.5,-0.1)	-	1	0.05	0.015	$p^+ \uparrow$
7	(-0.2,**0.5**)	-	(0.15,-0.45)	2	-0.05	-0.01	$p^- \downarrow$
8	(**-0.1**,0.5)	(0.45,-0.15)	-	1	0.05	0.015	$p^+ \uparrow$
9	(**0**,0.5)	(0.4,-0.2)	-	1	0.05	0.015	$p^+ \uparrow$
10	(0,**0.4**)	-	(0.2,-0.4)	2	-0.05	-0.01	$p^- \downarrow$
11	(**0.1**,0.4)	(0.35,-0.25)	-	1	0.05	0.015	$p^+ \uparrow$
12	(0.1,**0.3**)	-	(0.25,-0.35)	2	-0.05	-0.01	$p^- \downarrow$
13	(**0.2**,0.3)	(0.3,-0.3)	-	1	0.05	0.015	$p^+ \uparrow$
14	(**0.3**,0.3)	(0.3,-0.3)	-	2	-0.05	-0.01	-

Table 2: Summary of auction rounds: active side is boldfaced.