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Order of play in sequential network formation*

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Abstract

Network formation games that require pairs of agents to pay a sunk cost in order to form a link assume the cost as fixed. We assume this cost to be increasing in the agent's degree, and find that the order in which agents sequentially form and delete links (order of play) determines the equilibrium structure of the network. In particular, we find that only certain orders of play can explain the formation of networks composed of complete bipartite components and circle networks. We also give conditions for equilibrium uniqueness, and characterize the equilibrium network when the condition holds. We finally consider farsighted agents, and find that the equilibrium network is always composed of only regular components.

JEL classification: C72, D85
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1 Introduction

In certain network formation games such as Joshi et al. (2023) or Joshi et al. (2020), a pair of agents agrees on forming a link if the marginal utility they generate by forming the link exceeds some sunk cost. The interpretation of this cost is that forging a relationship "requires investment in time and resources to build the necessary trust" (Joshi et al., 2023). Building this trust allows agents to shield themselves against possible malicious intentions of the agent they are forming the link with. The value of this sunk cost is assumed to be fixed. However, malicious intentions of one's neighbors do not only affect oneself, but one's neighbors as well through spillover effects. A negative shock in one's exerted effort or production caused by a malicious act of a neighbor might lower one's reputation with respect to other neighbors, or lower the trust these neighbors have with respect to oneself in other networks. Therefore, agents need to build higher trust as the number of neighbors they have increases. In this spirit, we consider a network formation model in which sunk costs of link formation are increasing in one's own degree. In this class of games, the order in which agents sequentially form and delete links (order of play) determines the final structure of the network. Our main result is that networks composed of complete bipartite components and circle networks can only be explained by one order of play each, out of the four we consider. To the best of our knowledge, there is yet no network formation game in which the order of play has an impact on the final structure of the network.

The network formation game we consider is heavily inspired by Joshi et al. (2023) and Joshi et al. (2020), which are themselves inspired by Aumann and Myerson (2003), Jackson and Watts (2002) and König et al. (2014). We consider a dynamic network formation game in which agents can sequentially delete any subset of own links and propose at most a link to another agent. The decision to delete links is unilateral, while the formation to form a link is bilateral. A link is formed between two agents if the incremental utility they generate is higher than some sunk cost, which is increasing in the degree of the agent. We consider a linear-quadratic utility function (Ballester et al., 2006), and thus, the utility that an agent generates by forming a link is proportional to the increase in Bonacich centrality generated by the formation of the link.

The game we consider presents some differences with respect to Joshi et al. (2023) and Joshi et al. (2020). *First*, the sunk cost that agents pay when forming a link is increasing in the agent's degree. In Joshi et al. (2023, 2020), costs of link formation are fixed. Apart from the motivation given above for introducing costs of link formation increasing in degree, there is a technical reason as well. Due to the second and third differences of this model with respect to Joshi et al. (2023, 2020), which we next state, considering fixed costs of link formation leads to only complete or empty equilibrium networks. Considering costs of link formation increasing in one's own degree allows for richer equilibrium network structures. *Second*, the network starts as empty. In Joshi et al. (2023, 2020), there already is an exogenous network with a NSG architecture before the game starts. In the framework we study, the empty network can become a NSG as the network starts forming. *Third*, we consider the formation of only one network. Joshi et al. (2023, 2020) consider the formation of multiple networks which are interdependent. Because the formation of links in a network depends on the structure of the other other networks, Joshi et al. (2023, 2020) can explain the formation of rich architectures with a fixed cost of link formation, which is not the case in this game. *Fourth*, we consider myopic, as well as farsighted agents, whereas Joshi et al. (2023) consider only myopic agents.

Because the cost of link formation is increasing in the agent's degree, it can be profitable for agents to delete links, in order to reduce their cost of link formation and link with someone else with whom a link generates higher utility. In Joshi et al. (2023, 2020), agents can delete links, but it is never profitable for them do so, since the cost of link formation is fixed. In the game we consider, multiple components can arise in equilibrium. In Joshi et al. (2023, 2020), only one component can arise in equilibrium.

We consider four orders of play. We find that networks composed of complete bipartite components and circle networks can only be explained by one order of play each. We also give conditions for equilibrium uniqueness for each one of the orders of play, and study the equilibrium network given that the condition holds. Even though the equilibrium structure of the network cannot be known with certainty before the game starts, the network can reach a structure at which only one equilibrium structure can arise. We find that, under a certain condition, we arrive earlier to this structure with certain orders of play than with others. Finally, we consider farsighted agents and find that the equilibrium is composed of only regular components, i.e., components in which every agent has the same number of neighbors.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 presents the results. Section 4 concludes.

2 The model

2.1 Network, utility function and Bonacich centrality

We consider a set $\mathcal{N} = \{1, ..., N\}$ of agents who can exert effort in an activity. Each agent $i \in \mathcal{N}$ exerts effort $x_i \geq 0$ in the activity. Every pair of agents $i, j \in \mathcal{N}$ can either share a link (in which case, $g_{ij} = 1$) or not (in which case, $g_{ij} = 0$), i.e., links are unweighted. We call *network*, or *adjacency matrix*, the matrix $\boldsymbol{G}$ such that entry of the i^{th} row and j^{th} column corresponds to g_{ij}. We consider that $g_{ij} = g_{ji}$ for any pair of agents $i, j \in \mathcal{N}$, i.e., links are undirected. We define $L(\boldsymbol{G}) = \mathbf{1}^T\boldsymbol{G}\mathbf{1}$, where $\mathbf{1}$ is a one column vector of one's, which corresponds to the total number of links in network $\boldsymbol{G}$. The *neighborhood* of agent i is $\boldsymbol{N}_i = \{j \in \mathcal{N} \setminus \{i\} : g_{ij} = 1\}$, and its cardinality $|\boldsymbol{N}_i| = d_i$ is the degree of agent i. A *walk* from agent i to agent j is a sequence of agents $\{i, i+1, \cdots, j-1, j\}$ and links $\{g_{i,i+1,}, \cdots, g_{j-1,j}\}$ such that $g_{mn} = 1$ for all $m \in \{i, i+1, \cdots, j-1\}$ and $n = m+1$. A component of a network is a set $\boldsymbol{C}$ of nodes such that there exists a walk from any node $i \in \mathcal{C}$ to any node $j \in \mathcal{C}$, but not to any node outside $\mathcal{C}$. We denote by $\boldsymbol{C}_i$ the component to which node i belongs. Each agent i is assigned the same utility function U_i, defined in (1).

$$U_i(x_i, \boldsymbol{x_{-i}}) = x_i + \alpha \sum_{j \neq i} g_{ij} x_i x_j - \frac{1}{2} x_i^2, \qquad (1)$$

given $\boldsymbol{x_{-i}}$ and α, where vector $\boldsymbol{x_{-i}} = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_N \end{pmatrix}$ denotes the

effort exerted by all agents other than i, and $0 < \alpha < \frac{1}{(N-1)}$ is a parameter that amplifies the utility gained by a pair of agents when they exert effort together in the activity, i.e. agents benefit from strategic complementarity in the effort exerted.[1] We denote by $U_i(\boldsymbol{G})$ and by $x_i(\boldsymbol{G})$ the utility function of agent i in network $\boldsymbol{G}$ and the effort exerted by agent i in network $\boldsymbol{G}$ respectively. We also denote by $U_i(\boldsymbol{G_{+mn}})$ and by $x_i(\boldsymbol{G_{+mn}})$ the utility of agent i and the effort exerted by agent i, respectively, after the introduction of a link between agents m and n in network $\boldsymbol{G}$. We define $\Delta U_i(\boldsymbol{G}_{+mn}) = U_i(\boldsymbol{G}_{+mn}) - U_i(\boldsymbol{G})$ and $\Delta x_i(\boldsymbol{G}_{+mn}) = x_i(\boldsymbol{G}_{+mn}) - x_i(\boldsymbol{G})$. We denote by x_i^* the equilibrium value of agent i, which is the value x_i that maximizes her utility. The vector $\boldsymbol{x^*} = \begin{pmatrix} x_1^* & \cdots & x_N^* \end{pmatrix}$ is the Nash equilibrium of the game, in which no agent has a profitable unilateral deviation from her equilibrium value. If we let x_i' be any value x_i, and $\boldsymbol{x^*_{-i}}$ be the set of equilibrium values of all agents other than i, then the Nash equilibrium is such that, for all agents $i \in \mathcal{N}$, $U(x_i^*, \boldsymbol{x^*_{-i}}) \geq U(x_i', \boldsymbol{x^*_{-i}})$ for all x_i'.
As follows from Ballester et al. (2006), the Nash equilibrium of the game is given by vector $\boldsymbol{X}$, composed of one column and N rows.

$$\boldsymbol{X} = [\boldsymbol{I}_N - \alpha \boldsymbol{G}]^{-1}\mathbf{1}, \tag{2}$$

where $\boldsymbol{I}_N$ is the identity matrix of dimension N. The i^{th} entry of vector $\boldsymbol{X}$ is commonly called the Bonacich centrality of agent i (Bonacich, 1987), which can also be computed using equation (3),

$$\boldsymbol{X} = \sum_{k=0}^{\infty} \alpha^k \boldsymbol{G}^k \mathbf{1}, \tag{3}$$

where $\mathbf{1}$ is vector of one's. Differently from other centrality measures, Bonacich centrality sums the number of walks for each length k emanating from the node whose Bonacich centrality is computed, through term $\sum_{k=0}^{\infty} \boldsymbol{G}^k \mathbf{1}$, and discounts, through term $\sum_{k=0}^{\infty} \alpha^k$, the weight that walks of length k have on Bonacich centrality as length k increases. Recall that $0 < \alpha < \frac{1}{(N-1)}$, which implies that, the larger the value of walk's length k, the lower the value of α^k.

[1] Condition $0 < \alpha < \frac{1}{(N-1)}$ ensures that Bonacich centrality is well defined (Jackson, 2008).

We say that $\boldsymbol{G'}$ is *adjacent* to $\boldsymbol{G}$ if $\boldsymbol{G'} = \boldsymbol{G_{+ij}}$ or $\boldsymbol{G'} = \boldsymbol{G_{-ij}}$ for some ij. We say that two agents i and j are *permutable* in network $\boldsymbol{G}$ if $x_i(\boldsymbol{G}) = x_j(\boldsymbol{G})$. We say that $\boldsymbol{G}$ is *denser* than $\boldsymbol{G'}$ if $\boldsymbol{G'} \subseteq \boldsymbol{G}$.

2.2 Timing of events

The game is dynamic. Time is discrete and represented by set $\boldsymbol{T}$. After agent $i \in \mathcal{N}$ has played in time $t \in \boldsymbol{T}$, it is some agent $j \in \mathcal{N}$, designated by function $\mathcal{P} : \mathcal{N} \times \boldsymbol{T} \Rightarrow \mathcal{N}$, defined in Section 3, that plays at time $t + 1$. Notation $\Omega(t)$ denotes Ω at time t, where Ω is either a set, a matrix, the degree of an agent, or a link indicator g_{ij} between any pair of agents $i, j \in \mathcal{N}$. Agents are myopic in the sense that they seek to generate the highest immediate incremental utility from the deletion or the formation of a new link. In Section 3, we propose a model in which agents are farsighted. The structure of the network is common knowledge. The timing of events is the following:

- The network starts as empty, i.e. $g_{ij}(0) = 0$ for all $i, j \in \mathcal{N}$.
- A randomly chosen agent, denoted by 1, plays first at time $t = 1$. Agent 1 can delete any existing links with neighbors in $\boldsymbol{N}_1(1)$ (at time $t = 1$, the neighborhood of agent 1 is empty), and propose at most one link to one agent $j \notin \boldsymbol{N}_1(1)$.
- All then adjust the effort they exert in the activity. They play their best-response, given the new structure of the network.
- An agent will only propose a link to another agent if the incremental utility generated by the newly formed link is strictly larger than some cost she incurs from link formation. The cost she incurs from link formation is given by function $c(d_i)$, where d_i is her degree before the proposition of the link. We define continuous and differentiable function $c : \mathbb{R}^+ \Rightarrow \mathbb{R}^+$, with $c(0) = 0$ and $c'(d_i) > 0$.[2] If a link is formed between two agents, the cost corresponding to

[2]By defining $c(0) = 0$, we do not consider networks composed of only singletons, which are less interesting to study.

their degree before the formation of the link is incurred by both agents, and sunk. If an agent deletes a subset of own links with agents $\lambda_1, \lambda_2, \cdots, \lambda_r$, then the incremental utility generated by a newly formed link with agent j is $\Delta U_i(\boldsymbol{G}_{-i\lambda_1 - i\lambda_2 \cdots - i\lambda_r + ij}) = U_i(\boldsymbol{G}_{-i\lambda_1 - i\lambda_2 \cdots - i\lambda_r + ij}) - U_i(\boldsymbol{G})$, and the cost of link formation $c(d_i)$ agent i incurs considers the degree of agent i after the deletion of links with agents $\lambda_1, \lambda_2, \cdots, \lambda_r$.

- Out of all the agents who i can select to propose a link to, i will select the agent with whom the formation of a new link generates the highest incremental utility. If there exist two or more agents $j \notin \boldsymbol{N}_1(1)$ who generate the highest incremental utility, agent i will randomly select whom to propose the link to. The agent who proposes the link (in this case, agent i) is called the *sender*, and the agent to whom the link is proposed to (in this case, agent j) is called the *receiver*.

- Receiver j can either accept or decline the link proposed by sender i. If j accepts, then the link between i and j is formed. If j declines, then the link between i and j is not formed. Agent j accepts if and only if $\Delta U_j(\boldsymbol{G}_{+ij}) > c(d_j)$, and declines if and only if $\Delta U_j(\boldsymbol{G}_{+ij}) \leq c(d_j)$. A link is formed between two agents only when the incremental utility they both receive from the formation of the link is strictly larger than the cost they incur, which is defined by their respective degree and the shape of $c(d_i)$.

- Once all agents have adjusted their exerted effort, function $\mathcal{P} : \mathcal{N} \times \boldsymbol{T} \Rightarrow \mathcal{N}$, defined in Section 3, designates which agent plays next, at time $t = 2$.

- The process repeats itself starting from the second bullet point, where it is the agent designated by $\mathcal{P}$ that plays at the next period. We call any sequence of networks generated between times r and s, $\{\boldsymbol{G}(t = r), ..., \boldsymbol{G}(t = s)\}$, an *improving path*. When no agent has an incentive to *(i)* delete any subset of own links, and *(ii)* form any new link, the game ends. We denote by $\boldsymbol{G}^*$ such a resulting network.

Network $\boldsymbol{G}^*$ is commonly defined as the *pairwise stable (Nash) equilibrium*. Formally, $\boldsymbol{G}^*$ is a pairwise stable equilibrium if

$$U_i(\boldsymbol{G^*}) \geq U_i(\boldsymbol{G^*}_{-i\lambda_1 - i\lambda_2 \cdots - i\lambda_r}), \tag{4}$$

for each $i \in \mathcal{N}$ and any $\{\lambda_1, \lambda_2, \cdots, \lambda_r\} \subseteq \boldsymbol{N}_i(\boldsymbol{G^*})$, and

$$U_i(\boldsymbol{G^*}_{-i\lambda_1, \cdots, -i\lambda_r, +ij}) - c(d_i) > 0 \implies U_j(\boldsymbol{G^*}_{-i\lambda_1, \cdots, -i\lambda_r, +ij}) - c(d_j) < 0, \tag{5}$$

for each $i, j \in \mathcal{N}$ and any $\{\lambda_1, \lambda_2, \cdots, \lambda_r\} \subseteq \boldsymbol{N}_i(\boldsymbol{G^*})$. Equation (4) indicates that, in $\boldsymbol{G^*}$, no agent has an incentive to delete any subset of own links. Equation (5) indicates that, in $\boldsymbol{G^*}$, no agent has an incentive to delete any subset of own links and form a new link.

We say that a network $\boldsymbol{G}$ is *reachable* (from network $\boldsymbol{G_k}$) if there is an improving path (in which the first network in the sequence is $\boldsymbol{G_k}$) which generates network $\boldsymbol{G}$. We say that network $\boldsymbol{G}$ is *unreachable* (from network $\boldsymbol{G_k}$) if it is not reachable (from network $\boldsymbol{G_k}$). Given any α and $c(d_i)$, we denote by d_{max} the highest degree among all agents in all reachable networks.

2.3 Other useful notation

We denote by $|\boldsymbol{C}_{max}|$ the maximum number of nodes in a component during the link formation process. We denote by $\mathbf{GL}(t)$ the set of agents that have an incentive to form or delete links at time t. We denote by $\boldsymbol{GLC}_i(t) = \boldsymbol{GL}(t) \cap \{j \in \boldsymbol{C_i}(t)\}$ the subset of $\boldsymbol{GL}(t)$ in which all nodes belong to the component of i. We denote by $\boldsymbol{GLNC}_i(t) = \boldsymbol{GL}(t) \setminus \boldsymbol{GLC}_i(t)$ the subset of $\boldsymbol{GL}(t)$ in which all nodes do not belong to the component of i. Time $\tau \in \boldsymbol{T}$ is the first time in which $\mathcal{P}(i, \tau) = j \in \boldsymbol{GLNC}_i(\tau)$. We denote by $\boldsymbol{GP}(t)$ the set of agents who have played at least once in the time interval $\{1, ..., t\}$. $\mathcal{U}(\cdot)$ is the discrete uniform distribution, where $\cdot$ is the support from which elements are drawn. $L^t_{i \to j}$ is a binary variable that equals 1 if agent i proposes a link to agent j at time t and the link is formed at time t, and equals 0 otherwise. We denote by $\underline{c(1)}$ and $\underline{\underline{c(1)}}$ some arbitrary values of $c(1)$, by $\underline{c(2)}$ and $\underline{\underline{c(2)}}$ some arbitrary values of $c(2)$, and by $\underline{c(3)}$, $\underline{\underline{c(3)}}$ and by $\underline{\underline{\underline{c(3)}}}$ some arbitrary values of $c(3)$.

3 Network formation analysis

We first give some preliminary results regarding the incentives of players and the existence of a pairwise stable (Nash) equilibrium, in Section 3.1. We next characterize network structures in equilibrium by considering different orders of play, which will be formally defined below. First, we consider a order of play in which the sender $i \in \mathcal{N}$ of a link plays next, and the agents belonging to the component of i play afterwards, in Section 3.2. We then consider a order of play in which the sender $i \in \mathcal{N}$ of a link plays next, and the agents that do not belong to the component of i play afterwards, in Section 3.3. We next consider a order of play in which the receiver $j \in \mathcal{N}$ of a link plays next, and the agents that do not belong to the component of j play afterwards, in Section 3.4. In Appendix A, we consider a order of play in which the receiver $j \in \mathcal{N}$ of a link plays next, and the agents belonging to the component of j play afterwards. Results given in Appendix A are similar to the ones given in Section 3.2. We then study the periods at which some network structures become unreachable, and compare them across orders of play, in Section 3.5. In this section, we denote by $\mathcal{P}_{RI}$ the order of play presented in Appendix A. Finally, we consider a framework in which agents are farsighted in Section 3.6, and compare the formed network at equilibrium with the preferred network of a social planner who wishes to maximize the sum of utilities of agents in the network.

3.1 Preliminary results

Similarly to Joshi et al. (2023), we can express the incremental utility that an agent i receives when she deletes a subset of own links with agents $\lambda_1, \lambda_2, \cdots, \lambda_r$, and forms a new link with agent j.[3]

$$\Delta U_i(\boldsymbol{G}_{-i\lambda_1,\cdots,-i\lambda_r,+ij}) = \Delta x_i(\boldsymbol{G}_{-i\lambda_1,\cdots,-i\lambda_r,+ij}) \cdot (\frac{1}{2}\Delta x_i(\boldsymbol{G}_{-i\lambda_1,\cdots,-i\lambda_r,+ij}) + x_i(\boldsymbol{G})). \tag{6}$$

Equation (6) is particularly important in the sense that it sheds light on the incentives of agents. Notice that, if an agent becomes less Bonacich central by deleting

[3]Note that equation (6) holds as well when $\{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ is an empty set.

a subset of own links and forming a new one, i.e. $\Delta x_i(\boldsymbol{G}_{-i\lambda_1,\cdots,-i\lambda_r,+ij}) < 0$, then her utility decreases, i.e. $\Delta U_i(\boldsymbol{G}_{-i\lambda_1,\cdots,-i\lambda_r,+ij}) < 0$. It follows, as stated in Lemma 1(a) below, that an agent deletes a subset of own links and forms a new one if and only if it allows her to become more Bonacich central. Our framework differs from Joshi et al. (2023) in the sense that it can be profitable for agents to delete a subset of own links.

Two other results, stated in Lemma 1(b) and 1(c) below, follow from Equation (6). We now give an intuition of Lemma 1(b). First consider network $\boldsymbol{G}$ represented in Figure 1a, and suppose that it is profitable for agent 2 to form a link with agent 4, i.e., $\Delta U_2(\boldsymbol{G}_{+24}) > c(2)$. Now consider network $\tilde{\boldsymbol{G}}$ represented in Figure 2b. Because agent 2 is now in a denser component $\tilde{\boldsymbol{C}}_{\boldsymbol{2}}$, the Bonacich centrality of agent 2 in network $\tilde{\boldsymbol{G}}$, $x_2(\tilde{\boldsymbol{G}})$, is strictly larger than her Bonacich centrality in network $\boldsymbol{G}$, $x_2(\boldsymbol{G})$. The incremental Bonacich centrality of agent 2 generated by linking with agent 4 in network $\tilde{\boldsymbol{G}}$, $\Delta x_2(\tilde{\boldsymbol{G}}_{+24})$, is also strictly larger than the incremental Bonacich centrality of agent 2 generated by linking with agent 4 in network $\boldsymbol{G}$, $\Delta x_2(\boldsymbol{G}_{+24})$. It follows, from equation (6), that the incremental utility of agent 2 generated by linking with agent 4 in network $\tilde{\boldsymbol{G}}$, $\Delta U_2(\tilde{\boldsymbol{G}}_{+24})$, is strictly larger than the incremental utility of agent 2 generated by linking with agent 4 in network $\boldsymbol{G}$, $\Delta U_2(\boldsymbol{G}_{+24})$. This result is very helpful in the sense that the formation of one link between some agent i in a component $\boldsymbol{C}_{\boldsymbol{i}}$ and some agent j, e.g the formation of link g_{15} in network $\boldsymbol{G}$ of Figure 1a, implies that the formation of one link between some agent $\tilde{i}$ in a component $\tilde{\boldsymbol{C}}_{\boldsymbol{2}}$ denser than $\boldsymbol{C}_{\boldsymbol{2}}$, who is permutable with agent i in component $\boldsymbol{C}_{\boldsymbol{2}}$, and some agent who is as least as Bonacich central as j, generates at least as much incremental utility, e.g. the formation of link g_{24} in network $\tilde{\boldsymbol{G}}$ of Figure 1b. It is important that agents i and $\tilde{i}$ are permutable in $\boldsymbol{C}_{\boldsymbol{2}}$. Otherwise, we cannot infer the profitability of the formation of one link from the profitability of the formation of another link. For instance, agents 1 and 3 are permutable in component $\boldsymbol{C}_{\boldsymbol{1}}$ in network $\boldsymbol{G'}$ of Figure 1c. If it is profitable for agent 1 to form a link with agent 5 at time t, then we can infer that it is profitable as well for agent 3 to form a link with agent 5 at time t. We cannot infer, however, that it is profitable for agents 2 and 4 to form a link with agent 5, since their Bonacich centralities $x_2(\boldsymbol{G'})$ and $x_4(\boldsymbol{G'})$ are lower than the Bonacich centralities

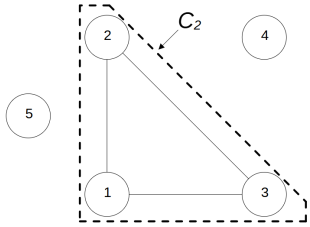

Figure 1a: Network $\boldsymbol{G}$

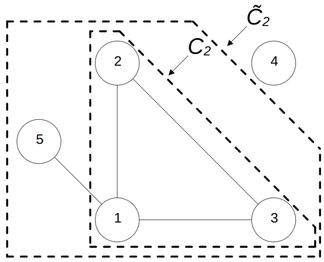

Figure 1b: Network $\tilde{\boldsymbol{G}}$

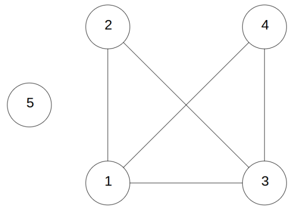

Figure 1c: Network $\boldsymbol{G'}$

$x_1(\boldsymbol{G'})$ and $x_3(\boldsymbol{G'})$ of agents 1 and 3. Finally, we give an intuition of Lemma 1(c) below. Consider again network $\boldsymbol{G}$ represented in Figure 1a, and suppose that it is profitable for agent 1 to form a link with agent 5, so that network $\tilde{\boldsymbol{G}}$ represented in Figure 1b arises. It follows that it is profitable for agents 2 and 3 to form a link with agent 4 as well. However, they generate more incremental utility by forming a link with agent 5, since agent 5's Bonacich centrality is larger than agent 4's. More generally, an agent i generates more incremental utility by forming a link with an agent $\tilde{j}$ than to an agent j who belongs to a less dense component $\boldsymbol{C_j} \subseteq \tilde{\boldsymbol{C}}_{\tilde{\boldsymbol{j}}}$ and who is permutable with $\tilde{j}$ in $\boldsymbol{C_j}$.

Lemmas 1(b) and 1(c) are slightly different from Lemma 1(a) of Joshi et al. (2023), since, in their framework, network $\boldsymbol{G}$ is composed of at most one non-singleton component during all the linking process. In the framework we study, $\boldsymbol{G}$ can be composed of more than one non-singleton component during the linking process, and so, it is not sufficient that $\boldsymbol{G} \subseteq \tilde{\boldsymbol{G}}$ and $ij \notin \tilde{\boldsymbol{G}}$ for $\Delta U_i(\tilde{\boldsymbol{G}}_{+ij}) \geq \Delta U_i(\boldsymbol{G}_{+ij})$ to be true, which is sufficient in the framework of Joshi et al. (2023). In the framework we study, it is necessary that $\tilde{\boldsymbol{G}}$ is denser than $\boldsymbol{G}$ because the component that i belongs to is denser, for $\Delta U_i(\tilde{\boldsymbol{G}}_{+ij}) \geq \Delta U_i(\boldsymbol{G}_{+ij})$ to be true.

Lemma 1. *(a) For any* $\boldsymbol{G}$ *where* $ij \notin \boldsymbol{G}$*:*

$$\Delta U_i(\boldsymbol{G}_{-i\lambda_1,\cdots,-i\lambda_r,+ij}) > 0 \textit{ if and only if } \Delta x_i(\boldsymbol{G}_{-i\lambda_1,\cdots,-i\lambda_r,+ij}) > 0.$$

(b) Suppose $\{\boldsymbol{C}_i \subseteq \boldsymbol{G}\} \subseteq (\subset)\{\tilde{\boldsymbol{C}}_i \subseteq \tilde{\boldsymbol{G}}\}$ *and* $ij \notin \tilde{\boldsymbol{C}}_{\boldsymbol{i}}$*. Then, for all* $\boldsymbol{G}$*:*

$$\Delta U_i(\tilde{\boldsymbol{G}}_{+ij}) \geq (>) \Delta U_i(\boldsymbol{G}_{+ij}).$$

(c) Suppose $\{\boldsymbol{C}_j \subseteq \boldsymbol{G}\} \subseteq (\subset)\{\tilde{\boldsymbol{C}}_j \subseteq \tilde{\boldsymbol{G}}\}$ *and* $ij \notin \tilde{\boldsymbol{C}}_{\boldsymbol{j}}$*. Then, for all* $\boldsymbol{G}$*:*

$$\Delta U_i(\tilde{\boldsymbol{G}}_{+ij}) \geq (>) \Delta U_i(\boldsymbol{G}_{+ij}).$$

Because it can be profitable for agents to delete a subset of own links, the existence of a pairwise stable equilibrium is less apparent than in Joshi et al. (2023). We prove its existence by first combining Lemma 1(a) and Theorem 1 of Harkins (2020), from which we can deduce that, when an agent plays, total effort exerted increases. We next reason by the absurd, and assume that a pairwise stable equilibrium does not exist, from which follows that there exist two different periods t_1 and t_2 such that $\boldsymbol{G}(t_1) = \boldsymbol{G}(t_2)$ (Jackson and Watts, 2001). Because at least one agent has played between t_1 and t_2, total effort exerted is strictly larger in $\boldsymbol{G}(t_2)$ than in $\boldsymbol{G}(t_1)$, and so a contradiction arises.

Proposition 1. *Given any order of play* $\mathcal{P}$*, a pairwise stable equilibrium exists.*

We also provide a simple insight in Lemma 2, which will be useful to prove other results.

Lemma 2. *Consider any two components* $\boldsymbol{C_i}$ *and* $\boldsymbol{C_j}$ *which are either both complete components or both complete bipartite components. If* $|\boldsymbol{C_i}| > |\boldsymbol{C_j}|$*, then* $x_i > x_j$ *for all* $i \in \boldsymbol{C_i}$ *and* $j \in \boldsymbol{C_j}$*.*

If some complete (bipartite) component $\boldsymbol{C_i}$ is composed of more agents than some other complete (bipartite) component $\boldsymbol{C_j}$, then, for any length k, there exist more walks emanating from any node $i \in \boldsymbol{C_i}$ than from any node $j \in \boldsymbol{C_i}$. Because Bonacich centrality sums, for each walk length, the number of walks emanating from a node, the Bonacich centrality x_i of any agent $i \in \boldsymbol{C_i}$ is strictly larger than the Bonacich centrality x_j of any agent $i \in \boldsymbol{C_j}$.

3.2 The sender-inside case

We first consider a order of play in which the sender of a link is the one playing next, and in which agents belonging to the component of the sender play afterwards. Formally, $\mathcal{P} = \mathcal{P}_{SI}$ where $\mathcal{P}_{SI}$ is defined in (7). Recall that, at time $t = 1$, an agent denoted by 1 is randomly selected to play first. At time $t = 1$, function $\mathcal{P}(1 \in \boldsymbol{G}, 1 \in \boldsymbol{T})$ designates which agent plays at time $t = 2$.

$$\mathcal{P}_{SI}(i \in \boldsymbol{G}, t \in \boldsymbol{T}) = \begin{cases} i \in \boldsymbol{G} & \text{if } L^t_{i \to j} = 1 \text{ and } i \in \mathbf{GL}(t) \\ j \sim \mathcal{U}(\boldsymbol{GLC}_i(t)) & \text{if } L^t_{i \to j} = 1, i \notin \mathbf{GL}(t) \text{ and } \exists j \in \boldsymbol{GLC}_i(t) \\ j \sim \mathcal{U}(\boldsymbol{GLNC}_i(t)) & \text{if } L^t_{i \to j} = 1, i \notin \mathbf{GL}(t) \text{ and } \nexists j \in \boldsymbol{GLC}_i(t) \end{cases} . \quad (7)$$

Function $\mathcal{P}_{SI}$ maps an agent $i \in \boldsymbol{G}$ and a time $t \in \boldsymbol{T}$ to an agent $j \in \mathcal{N}$. We distinguish three cases which lead to different identities of agent j:

- If an agent i successfully links with an agent j at time t, i.e. $L^t_{i \to j} = 1$, and agent i is able to form another link, i.e. $i \in \boldsymbol{GL}(t)$, then it is agent i that plays at period $t + 1$, i.e. $\mathcal{P}_{SI}(i, t) = i$.

- If an agent i successfully links with an agent j at time t, i.e. $L^t_{i \to j} = 1$, but agent i is not able to form another link, i.e. $i \notin \boldsymbol{GL}(t)$, and there exists some agent j belonging to the component of i who can form a link, i.e. $\exists j \in \boldsymbol{GLC}_i(t)$, then an agent j is randomly selected out of these agents who belong to the same component of i at time t and can form a new link, i.e. $\mathcal{P}_{RI}(i, t) = j \sim \mathcal{U}(\boldsymbol{GLC}_i(t))$.

- If an agent i successfully links with an agent j at time t, i.e. $L^t_{i \to j} = 1$, but agent i is not able to form another link, i.e. $i \notin \boldsymbol{GL}(t)$, and there is no another agent j belonging to the component of i who can form a link, i.e. $\nexists j \in \boldsymbol{GLC}_i(t)$, then an agent j is randomly selected out of the agents who do not belong to the component of agent i at time t and can form a new link, i.e. $\mathcal{P}_{SI}(i, t) = j \sim \mathcal{U}(\boldsymbol{GLNC}_i(t))$.

We now distinguish between two characterizations of $\boldsymbol{G}^*$ when $\mathcal{P} = \mathcal{P}_{SI}$. We first study the case in which there does not exist any time when two non-singleton components at time t belong to the same component at time $t + 1$. In this case scenario, there exists a unique pairwise stable equilibrium. We then consider the case in which there can exist times when two non-singleton components at time t

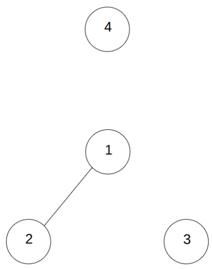

Figure 2a: $\boldsymbol{G}(t = 1)$

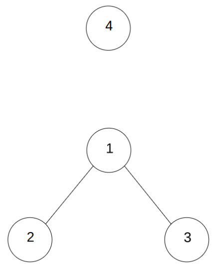

Figure 2b: $\boldsymbol{G}(t = 2)$

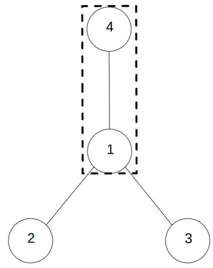

Figure 2c: $\boldsymbol{G}(t = 3)$

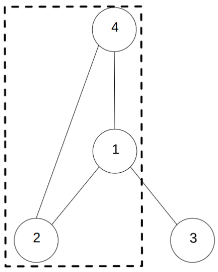

Figure 2d: $\boldsymbol{G}(t = 4)$

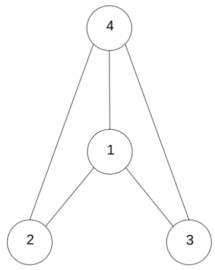

Figure 2e: $\boldsymbol{G}(t = 5)$

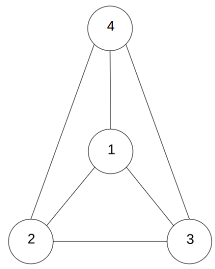

Figure 2f: $\boldsymbol{G}(t = 6)$

belong to the same component at time $t+1$. In this case scenario, there can exist multiple pairwise stable equilibria for a given α and a given cost function $c(d_i)$. We give additional conditions under which there exists a unique pairwise stable equilibrium for a given α and a given cost function $c(d_i)$.
When there does not exist any time when two non-singleton components at time t belong to the same component at time $t+1$, an agent successively forms links with other agents until it is not profitable for her to form a link with another agent, so that a star component forms. Then, agents with degree 1 in the formed star link together until the component becomes complete. An example is shown in Figures 2a, 2b, 2c, 2d, 2e and 2f which represent a network $\boldsymbol{G}$ composed of 4 agents at times 1, 2, 3, 4, 5 and 6 respectively, given some value of α and some cost of link formation function $c(d_i)$.

Example 1. Suppose $\alpha = \frac{1}{4}$, $c(1) = 0.2$ and $c(2) = 0.5$. At time $t = 1$, agent 1 is randomly selected to play first, and forms a link with agent 2. Given the values of α, $c(1)$ and $c(2)$, it is profitable for agent 1 to form a link with agent

3 at time $t = 2$, and with agent 4 at time $t = 3$. Notice that, at $t = 3$, the subgraph composed of agents 1 and 4, represented in the dashed frame in Figure 2c, has the same structure than $\boldsymbol{G}(t = 1)$. It follows that $\boldsymbol{G}(t = 1) \subseteq \boldsymbol{G}(t = 3)$. Agent 4 is randomly selected to play next. Because it was profitable for agent 1 in $\boldsymbol{G}(t = 1)$ to form a link with agent 3, and because $\boldsymbol{G}(t = 1) \subseteq \boldsymbol{G}(t = 3)$ and $d_4(t = 3) \leq d_1(t = 1)$, it is profitable for agent 4 to form a link with agent 2 at time $t = 4$, by Lemma 1(b). Notice that, at $t = 4$, the subgraph composed of agents 1, 2 and 4, represented in the dashed frame in Figure 2d, has a denser structure than $\boldsymbol{G}(t = 2)$, i.e., $\boldsymbol{G}(t = 2) \subseteq \boldsymbol{G}(t = 4)$. Because it was profitable for agent 1 in $\boldsymbol{G}(t = 2)$ to form a link with agent 4, and because $\boldsymbol{G}(t = 2) \subseteq \boldsymbol{G}(t = 4)$ and $d_4(t = 4) \leq d_1(t = 2)$, it is profitable for agent 4 to form a link with agent 3 at time $t = 5$, by Lemma 1(b). By following the same reasoning, we can deduce that link g_{23} is profitable as well. Because no agent has any incentive to form or delete a link at time $t = 6$, $\boldsymbol{G}(t = 6)$ is the pairwise stable equilibrium. □

Once the component becomes complete, it can be profitable for agents inside the component to link with a singleton. If it is the case, then an agent of the component successively links with isolated agents until it is not profitable for her to form another link. Agents in the newly formed component link together until the component becomes complete. This process can repeat itself until either the network becomes complete, or until it is too costly for agents in the component to form new links with isolated agents. In the latter case, —given that there does not exist any time when two non-singleton components at time t belong to the same component at time $t + 1$— more complete components emerge until the set of players is exhausted. When the number of players in the game is a multiple of the number of players inside each component, all components have the same number of players. Otherwise, the component which is formed last is composed by less players. In such case, players in the component which is formed last have a lower Bonacich centrality, by Lemma 2.

Formally, the characterization of $\boldsymbol{G}^*$ when $\mathcal{P} = \mathcal{P}_{SI}$ and when there does not exist any time in which two non-singleton components at time t belong to the same component at time $t + 1$, is given in Proposition 2.

Proposition 2. *Given* $\mathcal{P} = \mathcal{P}_{SI}$ *and* $g_{ij}(t) = 0$ *for* $\tau \leq t \leq 2\tau$ *for all* $i \in \mathcal{C}_1(\tau-1)$ *and all* $j \notin \mathcal{C}_1(\tau-1)$, $\boldsymbol{G}^*$ *is composed of complete components. Furthermore, there is equal effort exerted among players if and only if* N *is a multiple of* $(d_1(\tau)+1)$.

Recall that $\tau \in \boldsymbol{T}$, defined in Section 2.3, is the first time in which some agent $j \notin \boldsymbol{C}_i$ plays. Given $\mathcal{P} = \mathcal{P}_{SI}$ and assuming time τ exists, time interval $\tau \leq t \leq 2\tau$ corresponds to the time interval in which the second component is formed. If it is not profitable for any agent in the component that is first formed to link with any agent in the component that is formed second, i.e. $g_{ij}(t) = 0$ for $t \in \tau \leq t \leq 2\tau$ for all $i \in \mathcal{C}_1(\tau-1)$ and all $j \notin \mathcal{C}_1(\tau-1)$, then it is not profitable for any agent in any other non-singleton component to form a link with any non-singleton component, and so there does not exist any time when two non-singleton components at time t belong to the same component at time $t+1$. Proposition 2 also states that there is equal effort exerted among players if and only if N is a multiple of $(d_1(\tau)+1)$. Expression $(d_1(\tau)+1)$ corresponds to the number of agents in the complete component that is formed first. When N is not a multiple of $(d_1(\tau)+1)$, the complete component which is formed last is composed of less than $(d_1(\tau)+1)$ agents, which entails that they have a lower Bonacich centrality, and hence, that they exert less effort than the other agents.
We now characterize the set of pairwise stable equilibrium networks $\boldsymbol{G}^*$ when $\mathcal{P} = \mathcal{P}_{SI}$ and when there exists some time t in which two non-singleton components belong to the same component at time $t+1$. In such case, given α and cost function $c(d_i)$, network $\boldsymbol{G}^*$ can have different structures when components are composed of 6 agents or more at any time during the linking process. Depending on which player is randomly drawn when $\mathcal{P}_{SI} = j \sim \mathcal{U}(\boldsymbol{GLC}(t))$ or when $\mathcal{P}_{SI} = j \sim \mathcal{U}(\boldsymbol{GLNC}(t))$, or on which player the sender of a link randomly selects when there are multiple receivers that generate the highest incremental utility, one network structure may arise or not. An example is shown in Figures 3a, 3b, 3c and 3d which represent a network $\boldsymbol{G}'$ composed of 6 agents at times 6, 9, 10 and 11 respectively.

Example 2. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0105$, $c(3) = 0.0107985$ and $c(4) = 5$. At time $t = 1$, agent 1 plays first, and links with agent 2. Given the

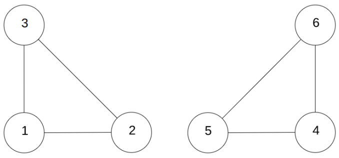

Figure 3a: $\boldsymbol{G'}(t = 6)$

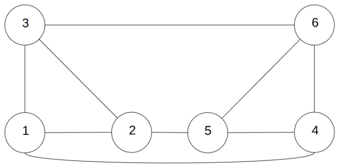

Figure 3b: $\boldsymbol{G'}(t = 9)$

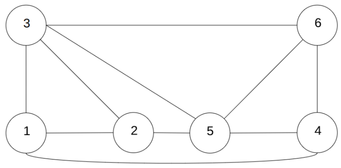

Figure 3c: $\boldsymbol{G'}(t = 10)$

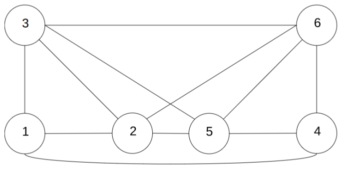

Figure 3d: $\boldsymbol{G'}(t = 11)$

values of α, $c(1)$, $c(2)$, $c(3)$ and $c(4)$, agent 1 plays at time $t = 2$ and links with agent 3. By Lemma 1(b), it is profitable for agents 2 and 3 to form a link at time $t = 3$. Agent 4 is next selected to play at time $t = 4$, and forms a second triad, with agents 5 and 6, at time $t = 6$. At time $t = 7$, agent 6 links with agent 3 at time $t = 7$. It is not profitable for the sender of the link, agent 6, to form or delete links at time $t = 8$.[4] Agent 4 is selected to play at time $t = 8$. It is too costly for agent 3 to accept a link from agent 4, and so agent 4 forms a link with agent 1. It is not profitable for the sender of the link, agent 4, to form or delete links. Agent 5 is selected to play at time $t = 9$. It is too costly for agents 1 and 3 to accept a link from agent 5, and so agent 5 forms a link with agent 2. At time $t = 10$, agent 5 links with agent 3. It is not profitable for agent 5 to form or delete links at time $t = 11$, and so agent 6 is selected to play at time $t = 11$ and links with agent 2, who generates the highest incremental utility. At time $t = 12$, agents 1 and 4 only share three links, but they cannot form any more links because it is not profitable for the other agents to form a fifth link. Because no agent has any incentive to form or delete a link at time $t = 12$, $\boldsymbol{G'}(t = 11)$ is the pairwise stable equilibrium. $\square$

[4] Agent 6 gains utility by deleting her link with either agent 4 or 5, and forming a link with either agent 1 or 2, at time $t = 8$. However, the marginal utility that she would generate would equal $\Delta U_6(\boldsymbol{G}_{-64+62}) = 0.00010650758$, which is lower than the cost $c(3) = 0.0107985$ she would need to incur.

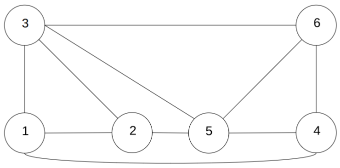

Figure 4a: $\boldsymbol{G''}(t = 10)$

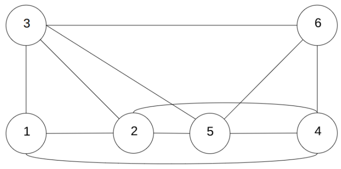

Figure 4b: $\boldsymbol{G''}(t = 11)$

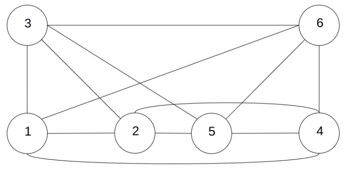

Figure 4c: $\boldsymbol{G''}(t = 12)$

We now consider a second example in which α and $c(d_i)$ have the same value and shape as in Example 2 respectively, and in which $\mathcal{P}_{RI}$ randomly selects agent 4 to play at time $t = 11$ instead of agent 6. This example is shown in Figures 5a, 5b and 5c which represent a network $\boldsymbol{G''}$ composed of 6 agents at times 10, 11 and 12 respectively.

Example 3. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0105$, $c(3) = 0.0107985$ and $c(4) = 5$. The link formation process is the same than the one presented in Example 2, up until $t = 10$. At time $t = 11$, it is not profitable for agent 5 to form or delete links, and so agent 4 is randomly selected to play instead of agent 6. At $t = 10$, agent 4 is already linked to all agents, except 2 and 3. Agent 3 would decline the link proposed by agent 4 because a fifth link is too costly. Therefore, agent 4 links with agent 2 at time $t = 11$. At time $t = 11$, all agents share four links, except agents 1 and 5 who share three links. Since it is too costly to form a fifth link, either agent 1 or 5 is selected to play at time $t = 12$ and links with the remaining node who shares three links. Therefore, network $\boldsymbol{G''}(t = 12)$ forms, and since it is too costly to form a fifth link, $\boldsymbol{G''}(t = 12)$ is the pairwise stable equilibrium. □

Even though α and $c(d_i)$ have the same value and shape respectively, in Examples 2 and 3, networks $\boldsymbol{G'^*}$ and $\boldsymbol{G''^*}$ have different structures. If we consider economies in which components can be composed of more than 6 agents, both $\boldsymbol{G'}(t = 11)$ and $\boldsymbol{G''}(t = 12)$ are structures that can temporarily arise before $\boldsymbol{G^*}$ forms, and so network $\boldsymbol{G^*}$ can have a different structure depending on whether $\boldsymbol{G'}(t = 11)$ or $\boldsymbol{G''}(t = 12)$ was formed during the linking process. If we consider economies in which components are composed of 5 agents or less, then $\boldsymbol{G^*}$ can only have

one possible structure given α and $c(d_i)$, as is shown by exhaustion of possible structures in the proof of Proposition 3.
Because, when components can be composed of 6 agents or more and two non-singleton components at time t can belong to the same component at time $t+1$, $\boldsymbol{G}^*$ can have multiple structures, we characterize $\boldsymbol{G}^*$ when components are composed of 5 agents or less at every period during the linking process. In such case, $\boldsymbol{G}^*$ is composed of either complete components, $\mathcal{C}_I$ components, or one $\mathcal{C}_{II}$, $\mathcal{C}_{III}$, or $\mathcal{C}_{IV}$ component, represented in Figures 5a, 5b, 5c and 5d respectively, and in the following matrixes:

$$\mathcal{C}_I = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mathcal{C}_{II} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \mathcal{C}_{III} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

$$\mathcal{C}_{IV} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Proposition 3. *Given $\mathcal{P} = \mathcal{P}_{SI}$, $c(d_i)$ and α, there can only exist multiple pairwise stable equilibria if $|\boldsymbol{C}_{max}| \geq 6$. If $|\boldsymbol{C}_{max}| \leq 5$, then $\boldsymbol{G}^*$ is composed of either*
(i) complete components,
(ii) $\mathcal{C}_I$ components and at most two complete components,
(iii) one $\mathcal{C}_{II}$ component and at most one singleton,
(iv) one $\mathcal{C}_{III}$ component and at most one singleton,
(v) one $\mathcal{C}_{IV}$ component and at most one complete component.

Differently from $\mathcal{C}_I$ components, only one $\mathcal{C}_{II}$, $\mathcal{C}_{III}$ or $\mathcal{C}_{IV}$ component can arise in $\boldsymbol{G}^*$ if we assume $|\boldsymbol{C}_{max}| \leq 5$. This is because, if either a $\mathcal{C}_{II}$, $\mathcal{C}_{III}$ or $\mathcal{C}_{IV}$ component is formed first, and a second component $\mathcal{C}_{II}$, $\mathcal{C}_{III}$ or $\mathcal{C}_{IV}$ component

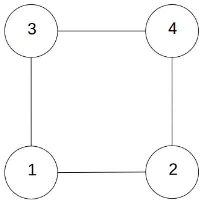

Figure 5a: C_I

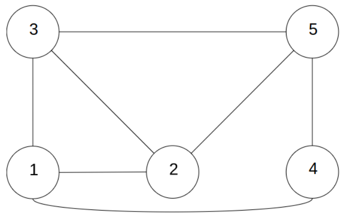

Figure 5b: C_{II}

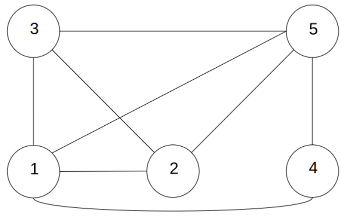

Figure 5c: C_{III}

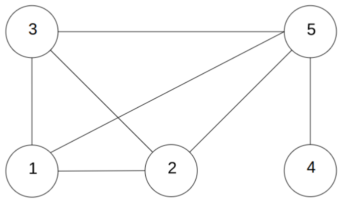

Figure 5d: C_{IV}

forms afterwards, then it is profitable for each agent with the lowest degree in each of the two components (this agent corresponds to agent 4 in Figures 5b, 5c and 5d) to link with each other, and so $|\boldsymbol{C}_{max}| \neq 5$.

3.3 The sender-outside case

We now consider a order of play in which the sender of a link is the one playing next, and in which agents which do not belong to the component of the sender play afterwards. Formally, $\mathcal{P} = \mathcal{P}_{SO}$ where $\mathcal{P}_{SO}$ is defined in (3.3).

$$\mathcal{P}_{SO}(i \in \boldsymbol{G}, t \in \boldsymbol{T}) = \begin{cases} i \in \boldsymbol{G} & \text{if } L^t_{i \to j} = 1 \text{ and } i \in \mathbf{GL}(t) \\ j \sim \mathcal{U}(\boldsymbol{GLNC}(t)) & \text{if } L^t_{i \to j} = 1, i \notin \mathbf{GL}(t) \text{ and } \exists j \in \boldsymbol{GLNC}(t) \\ j \sim \mathcal{U}(\boldsymbol{GLC}(t)) & \text{if } L^t_{i \to j} = 1, i \notin \mathbf{GL}(t) \text{ and } \nexists j \in \boldsymbol{GLNC}(t) \end{cases} .$$

(8)

Function $\mathcal{P}_{SO}$ maps an agent $i \in \boldsymbol{G}$ and a time $t \in \boldsymbol{T}$ to an agent $j \in \mathcal{N}$. We distinguish between three case scenarios which lead to different identities of agent

j.

- If an agent i successfully links with an agent j at time t, i.e. $L^t_{i \to j} = 1$, and agent i is able to form another link, i.e. $i \in \boldsymbol{GL}(t)$, then it is agent i that plays at period $t + 1$, i.e. $\mathcal{P}_{SO}(i, t) = i$.

- If an agent i successfully links with an agent j at time t, i.e. $L^t_{i \to j} = 1$, but agent i is not able to form another link, i.e. $i \notin \boldsymbol{GL}(t)$, and there exists at least another agent j who does not belong to the component of i who can form a link, i.e. $\exists j \in \boldsymbol{GLNC}_i(t)$, then an agent j is randomly selected out of these agents who do not belong to the same component of i at time t and can form a new link, i.e. $\mathcal{P}_{SO}(i, t) = j \sim \mathcal{U}(\boldsymbol{GLNC}_i(t))$.

- If an agent i successfully links with an agent j at time t, i.e. $L^t_{i \to j} = 1$, but agent i is not able to form another link, i.e. $i \notin \boldsymbol{GL}(t)$, and there does not exist at least another agent j who does not belong to the component of i who can form a link, i.e. $\nexists j \in \boldsymbol{GLC}_i(t)$, then an agent j is randomly selected out of the agents who do belong to the component of agent i at time t and can form a new link, i.e. $\mathcal{P}_{SO}(i, t) = j \sim \mathcal{U}(\boldsymbol{GLC}_i(t))$.

We also distinguish between two characterizations of $\boldsymbol{G}^*$. We first consider the case in which it is not profitable for agents who have played to link together. In this case scenario, there exists a unique pairwise stable equilibrium.

We then consider the case in which it can be profitable for agents who have played to link together. In this case scenario, there can exist multiple pairwise stable equilibria for a given α and a given cost function $c(d_i)$. We give additional conditions under which there exists a unique pairwise stable equilibrium for a given α and a given cost function $c(d_i)$.

When it is not profitable for agents who have played to link together, an agent successively forms links with other agents until it is not profitable for her to form a link with another agent, so that a star component forms. Then, agents outside the formed star link with agents who have degree 1 in the star link, and so a complete bipartite component forms. An example is shown in Figures 6a, 6b and 6c which represent a network $\boldsymbol{G}$ composed of 6 agents at times 3, 6 and 9 respectively, given

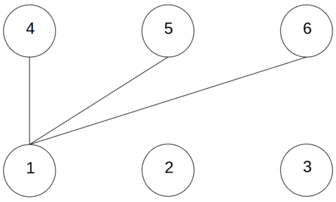

Figure 6a: $\boldsymbol{G}(t = 3)$

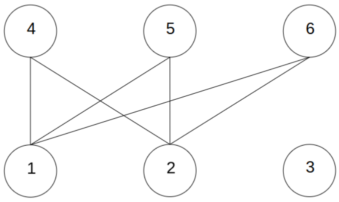

Figure 6b: $\boldsymbol{G}(t = 6)$

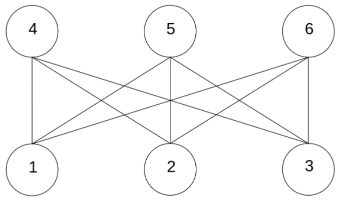

Figure 6c: $\boldsymbol{G}(t = 9)$

some value of α and cost function $c(d_i)$.

Example 4. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0103$ and $c(3) = 5$. At time $t = 1$, agent 1 plays and links with agent 4. Given the values of α, $c(1)$, $c(2)$ and $c(3)$, agent 1 links with agents 5 and 6 at times $t = 2$ and $t = 3$ respectively. At time $t = 4$, it is not profitable for agent 1 to form or delete links, and so agent 2 is selected to play at $t = 4$. By Lemma 1(b), it is profitable for agent 2 to link with agents 4, 5 and 6 as well, at times $t = 4$, $t = 5$ and $t = 6$ respectively. It is not profitable for agent 2 to form or delete links at time $t = 7$, and hence agent 3 plays at time $t = 7$. By Lemma 1(b), it is profitable for agent 3 to link with agents 4, 5 and 6, at times $t = 7$, $t = 8$ and $t = 9$ respectively. A fourth link is too costly to form. Since no agent has an incentive to delete or form a new link at time $t = 10$, $\boldsymbol{G}(t = 9)$ is the pairwise stable equilibrium. □

When $\mathcal{P} = \mathcal{P}_{SO}$, $\boldsymbol{G}^*$ can be composed of bipartite components of any size. When $\mathcal{P} = \mathcal{P}_{SI}$, or when the order of play is one of the two other orders we consider below, $\boldsymbol{G}^*$ cannot be composed of bipartite components of any size. In these orders, $\boldsymbol{G}^*$ can be composed of $\mathcal{C}_I$ components, which are a special case of complete bipartite components composed of 4 agents. This is the case because, in these other orders of play we consider, components composed of 5 agents or more necessarily contain at least one triad, which is absent from any bipartite component, by definition of a bipartite component. Bipartite networks are used to understand diverse matters such as European integration (Di Clemente et al., 2022), trade networks (Saracco et al., 2015), finance networks (Gualdi et al., 2016) or scientific competition of countries (Cimini et al., 2014). The order in which nodes form links determines whether these networks arise or not. We find that the sender-outside order, out of the four orders we consider, is the only one which can explain the existence of such networks.

When $\mathcal{P} = \mathcal{P}_{SO}$, complete multipartite components composed of more than two parts can also arise. Differently from complete bipartite components, there is no shape of cost function $c(d_i)$ that ensures complete multipartite graphs. Its formation depends on which agent is selected by $\mathcal{P}_{SO}(i, t) = j \sim \mathcal{U}(\boldsymbol{GLC}(t))$. An example is shown in Figures 7a, 7b and 7c that represent a network $\boldsymbol{G'}$ composed of 6 agents at times 8, 10 and 12 respectively, given some value of α and cost function $c(d_i)$.

Example 5. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0103$, $c(3) = 0.0104$ and $c(4) = 5$. At time $t = 1$, agent 1 plays and links with agent 3. Given the values of α, $c(1)$, $c(2)$ and $c(3)$, agent 1 links with agents 4, 5 and 6 at times $t = 2$, $t = 3$ and $t = 4$ respectively. At time $t = 5$, it is not profitable for agent 1 to form or delete links, and so agent 2 is selected to play at $t = 5$. By Lemma 1(b), it is profitable for agent 2 to link with agents 3, 4, 5 and 6 as well, at times $t = 5$, $t = 6$, $t = 7$ and $t = 8$ respectively. At time $t = 9$, it is not profitable for agent 2 to form or delete links, and so agent 6 is selected to play at $t = 9$. At times $t = 9$ and $t = 10$, agent 6 links with agents 3 and 4 respectively. At time $t = 11$, it is not profitable for agent 6 to form or delete links. If agent 5 is selected to play at $t = 11$, then she will link with agents 3 and 4 at times $t = 11$ and $t = 12$ respectively, and so $\boldsymbol{G'}(t = 12)$, represented in Figure 7c will be the pairwise stable equilibrium. If, however, agent 4 is selected to play at time $t = 11$, then she will link with agent 3 at time $t = 11$, and so a multipartite component does not arise at the pairwise stable equilibrium. □

Differently from Joshi et al. (2020) and Joshi et al. (2023), in which the network at the pairwise stable equilibrium is a nested-split graph with at most one (non-singleton) component, network $\boldsymbol{G^*}$ can be composed of multipartite components in this framework, due to the introduction of heterogeneity in costs of link formation.

Proposition 4. *Given $\mathcal{P} = \mathcal{P}_{SO}$ and $g_{ij}(t) = 0$ for $\tau \leq t \leq 2(\tau - 1)^2$ for all*

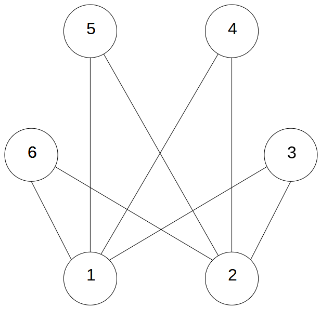

Figure 7a: $\boldsymbol{G'}(t = 8)$

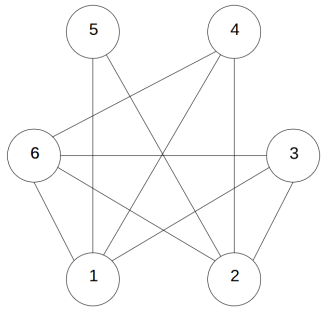

Figure 7b: $\boldsymbol{G'}(t = 10)$

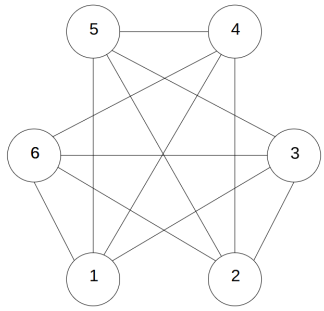

Figure 7c: $\boldsymbol{G'}(t = 12)$

$i, j \in \boldsymbol{GP}(t)$, $\boldsymbol{G^*}$ is composed of bipartite components. Furthermore, there is equal effort exerted among players if and only if N is a multiple of $2d_1(\tau - 1)$. $\boldsymbol{G^*}$ can also be composed of multipartite components.

Note that at $t = (\tau - 1)$, $\boldsymbol{G}$ is composed of a star in which the agent at the center has degree $(\tau - 1)$. It follows that every agent in the formed star accepts links from singletons until they reach degree $(\tau - 1)$. Therefore, each agent in the formed star accepts exactly $(\tau - 2)$ additional links. It is not profitable for them to accept more than $(\tau - 2)$ additional links because, if it were, it would entail that it is profitable for any agent that plays to form a link with agent 1, which contradicts with the assumption that $g_{ij}(t) = 0$ for $\tau \leq t \leq 2(\tau - 1)^2$ for all $i, j \in \boldsymbol{GP}(t)$. Because each of the $(\tau - 1)$ agents in the formed star forms exactly $(\tau - 2)$ additional links, it takes exactly $(\tau - 1)(\tau - 2) = \tau^2 - 2\tau - \tau + 2$ periods for the component formed first in $\boldsymbol{G}$ to transition from a star to a complete bipartite network. Since it took $(\tau - 1)$ periods for $\boldsymbol{G}$ to transition from an empty network to a being composed of a star, it takes $\tau^2 - 2\tau - \tau + 2 + \tau - 1 = \tau^2 - 2\tau + 1 = (\tau - 1)^2$ periods for $\boldsymbol{G}$ to transition from an empty network to being composed of a complete bipartite network. By following the same reasoning, a second complete bipartite component forms at $t = 2(\tau - 1)^2$. If it is not profitable for any agent to link with any agent who has already played, between time τ and time $2(\tau - 1)^2$, then it is not profitable for any agent in any formed complete bipartite component to link with any other agent in a different complete bipartite component. In such case, only one structure $\boldsymbol{G^*}$ can arise, and it is a network of composed of complete bipartite components. When it is not the case, $\boldsymbol{G^*}$ can have multiple structures for a given α and $c(d_i)$, as it is the case for order of play $\mathcal{P}_{SI}$. Proposition 4 also states that there is equal effort exerted among players if and only if N is a multiple of $2d_1(\tau-1)$. Expression $2d_1(\tau-1)$ corresponds to the number of agents in the complete bipartite component that is formed first. When N is not a multiple of $2d_1(\tau-1)$, the complete bipartite component which is formed last is composed of less than $2d_1(\tau-1)$ agents, which entails that they have a lower Bonacich centrality, and hence, that they exert less effort than the other agents.

We now characterize $\boldsymbol{G^*}$ when $\mathcal{P} = \mathcal{P}_{SO}$ and when there exists some time t in which an agent forms a link with another agent who has played at a previous period $t' < t$. In such case, given α and cost function $c(d_i)$, network $\boldsymbol{G^*}$ can have different structures. Differently from when $\mathcal{P} = \mathcal{P}_{SI}$, network $\boldsymbol{G^*}$ can have multiple structures, given $\mathcal{P} = \mathcal{P}_{SO}$, α and cost function $c(d_i)$, when components are composed of 5 agents or more at any time during the linking process. An example is shown in Figures 8a and 8b which represent a network $\boldsymbol{G''}$ composed of 5 agents at times 6 and 7 respectively.

Example 6. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0103$ and $c(3) = 0.0106$. At time $t = 1$, agent 1 plays and links with agent 2. Given the values of α, $c(1)$ and $c(2)$, agent 1 links with agents 3 and 4 at times $t = 2$ and $t = 3$ respectively. At time $t = 4$, it is not profitable for agent 1 to form or delete links, and so agent 5 is selected to play at $t = 4$. By Lemma 1(b), it is profitable for agent 5 to link with agents 3 and 4 as well, at times $t = 4$ and $t = 5$ respectively. At time $t = 6$, it is profitable for agent 5 to link with agent 1. At time $t = 7$, it is not profitable for agent 5 to form or delete links, and so agent 4 is selected to play at $t = 7$. By Lemma 1(b), it is profitable for agent 4 to link with agent 3 at time $t = 7$. It is not profitable for any agent in the network to link with agent 2 at time $t = 8$, and so $\boldsymbol{G''}(t = 7)$ is the pairwise stable equilibrium.[5] Suppose now, instead, that agent 2 is selected to play at $t = 7$. By Lemma 1(b), it is profitable for agent 2 to form a link with agents 3 and 4 at times $t = 7$ and $t = 8$ respectively. At time $t = 9$, it is

[5]Agent 1 was able to form a fourth link whereas agents 3, 4 and 5 were not, because agent 1 formed her fourth link with agent 5 at time $t = 6$, who had a higher Bonacich centrality than agent 2 at the pairwise stable equilibrium.

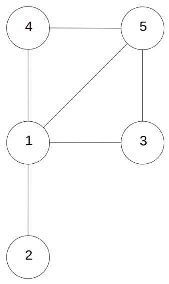

Figure 8a: $\boldsymbol{G''}(t = 6)$

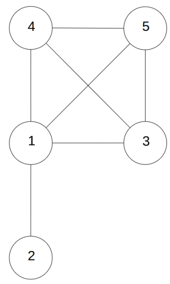

Figure 8b: $\boldsymbol{G''}(t = 7)$

profitable for agents 4 and 3 to form a link with each other, and for agents agents 5 and 2 to form a link with each other. Therefore, a pairwise stable equilibrium arises at $t = 11$, and is a complete component. □

Differently from when $\mathcal{P} = \mathcal{P}_{SI}$, $\boldsymbol{G^*}$ can have multiple structures given α and $c(d_i)$ when components can be composed of 5 or more agents at any period during the linking process. It is worth noting that we consider a network composed of 5 agents in Example 6. When $N \geq 6$, $\boldsymbol{G^*}$ can only have multiple structures given α and $c(d_i)$ when components can be composed of 6 or more agents at any period during the linking process.
Because, when components can be composed of 5 agents or more and it is profitable for any agent to link with any agent who has already played, $\boldsymbol{G^*}$ can have multiple structures given α and $c(d_i)$, we characterize $\boldsymbol{G^*}$ when components are composed of 4 agents or less at every period during the linking process. In such case, $\boldsymbol{G^*}$ is composed of either complete components or $\mathcal{C}_I$ components.

Proposition 5. *Given* $\mathcal{P} = \mathcal{P}_{SO}$, $c(d_i)$ *and* α, *there can only exist multiple pairwise stable equilibria if* $|\boldsymbol{C}_{max}| \geq 5$. *If* $|\boldsymbol{C}_{max}| \leq 4$, *then* $\boldsymbol{G^*}$ *is composed of either*
(i) complete components, or
(ii) $\mathcal{C}_I$ *components and one complete component at most.*

Differently from when $\mathcal{P} = \mathcal{P}_{SI}$, network $\boldsymbol{G^*}$ can be composed of $\mathcal{C}_I$ and at most one complete component. This is the case because, if $\mathcal{P} = \mathcal{P}_{SI}$ and $\mathcal{C}_I$ components successively form until there are exactly three singletons, then one dyad forms and it is not profitable for either agent in the dyad to form a link with the remaining singleton. If, however, $\mathcal{P} = \mathcal{P}_{SO}$ and $\mathcal{C}_I$ components successively form until there are exactly three singletons, then one line composed of three agents forms, and both degree-1 agents in the line form a link with each other at the next period, forming a triad in $\boldsymbol{G}^*$.

3.4 The receiver-outside case

We finally consider a order of play in which the receiver of the link is the one playing next, and in which agents which do not belong to the component of the receiver play afterwards. Formally $\mathcal{P} = \mathcal{P}_{RO}$ where $\mathcal{P}_{RO}$ is defined in (9).

$$\mathcal{P}_{RO}(i \in \boldsymbol{G}, t \in \boldsymbol{T}) = \begin{cases} j \in \boldsymbol{G}, & \text{if } L^t_{i \to j} = 1 \text{ and } j \in \boldsymbol{GL}(t) \\ l \sim \mathcal{U}(\boldsymbol{GLNC}_i(t)) & \text{if } L^t_{i \to j} = 1, j \notin \boldsymbol{GL}(t) \text{ and } \exists l \in \boldsymbol{GLNC}_i(t) \\ l \sim \mathcal{U}(\boldsymbol{GLC}_i(t)) & \text{if } L^t_{i \to j} = 1, j \notin \boldsymbol{GL}(t) \text{ and } \nexists l \in \boldsymbol{GLNC}_i(t) \end{cases}, \quad (9)$$

Function $\mathcal{P}_{RO}$ maps an agent $i \in \boldsymbol{G}$ and a time $t \in \boldsymbol{T}$ to an agent $j \neq i$. We distinguish between three case scenarios which lead to different identities of agent j:

- If an agent i successfully links with an agent j at time t, i.e. $L^t_{i \to j} = 1$, and agent j can form a new link, i.e. $j \in \boldsymbol{GL}(t)$, then it is agent j that plays at period $t + 1$, i.e. $\mathcal{P}_{RI}(i, t) = j$.

- If an agent i successfully links with an agent j at time t, i.e. $L^t_{i \to j} = 1$, but agent j is not able to form a new link, i.e. $j \notin \boldsymbol{GL}(t)$, and there exists at least another agent who belongs to the component of i who can form a link, i.e. $\exists l \in \boldsymbol{GLC}_i(t)$, then an agent l is randomly selected out of these agents who belong to the same component of i at time t and can form a new link, i.e. $\exists l \in \boldsymbol{GLC}_i(t)$.

- If an agent i successfully links with an agent j at time t, i.e. $L^t_{i \to j} = 1$, but agent j is not able to form a new link, i.e. $j \notin \boldsymbol{GL}(t)$, and there does not exist at least another agent who belongs to the component of i who can form a link, i.e. $\nexists l \in \boldsymbol{GLC}_i(t)$, then an agent l is randomly selected out of the

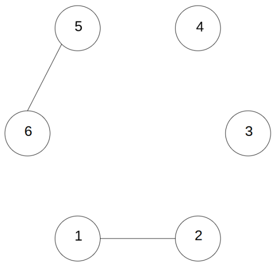

Figure 9a: $\boldsymbol{G}(t = 2)$

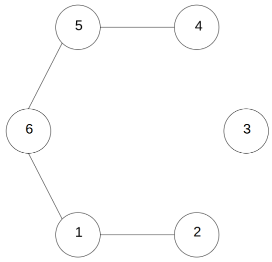

Figure 9b: $\boldsymbol{G}(t = 4)$

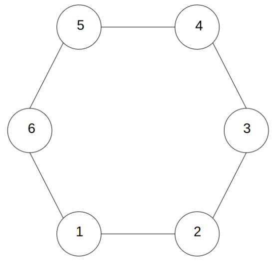

Figure 9c: $\boldsymbol{G}(t = 6)$

agents who do not belong to the same component of i at time t and can form a new link, i.e. $\exists l \in \boldsymbol{GLNC}_i(t)$.

We first show that, for certain shapes of cost function $c(d_i)$, $\boldsymbol{G}^*$ exists and is a circle. When the order of play is either of the other three we consider, $\boldsymbol{G}^*$ cannot be a circle. An example is shown in Figures 9a, 9b and 9c which represent a network $\boldsymbol{G}$ composed of 6 agents at times 2, 4 and 6 respectively.

Example 7. Suppose $\alpha = 0.01$, $c(1) = 0.01$ and $c(2) = 5$. Agent 1 plays at time $t = 1$ and links with agent 2. Given the values of α and $c(1)$, it is not profitable for agent 2 to form or delete links at time $t = 2$, and therefore, agent 5 is selected to play next. At time $t = 2$, agent 5 links with agent 6, and so network $\boldsymbol{G}(t = 2)$, represented in Figure 9a, arises. At time $t = 3$, agent 6 links with agent 1. At time $t = 4$, it is not profitable for agent 1 to form or delete links, and so agent 4 is selected to play next. At time $t = 4$, agent 4 links with agent 5, and so network $\boldsymbol{G}(t = 4)$, represented in Figure 9b, arises. At time $t = 5$, it is not profitable for agent 5 to form a new link, and so agent 3 is selected to play next. At time $t = 5$, agent 3 links with agent 4. At time $t = 6$, it is not profitable for agent 4 to form a new link, and so agent 2 is selected to play next. At time $t = 6$, agent 2 links with agent 3, and so network $\boldsymbol{G}(t = 6)$, represented in Figure 9c, arises. At time $t = 7$, no agent has an incentive to delete or form a link, and so $\boldsymbol{G}(t = 6)$ is the pairwise stable equilibrium. $\square$

The work of Cabrales and Hauk (2022) studies, in the context of leaders and followers interacting in a circle network, which agents should be leaders in order to

enhace payoff dominant play. We show that, in order to implement the presented policies that enhace payoff dominant play, the order in which the network forms matters. Order of play $\mathcal{P}_{RO}$ allows $\boldsymbol{G^*}$ to be a circle, while the other three orders of play presented do not.
When $\mathcal{P} = \mathcal{P}_{RO}$, network $\boldsymbol{G^*}$ can also be composed of lattice components. Similarly to when $\mathcal{P} = \mathcal{P}_{SO}$ and $\boldsymbol{G^*}$ can be composed of multipartite graphs with more than two parts, which element is selected by uniform distribution $l \sim \mathcal{U}(\boldsymbol{GLNC}_i(t))$ or $l \sim \mathcal{U}(\boldsymbol{GLC}_i(t))$ determines whether $\boldsymbol{G^*}$ is a lattice component or not. An example is shown in Figures 10a, 10b, 10c, 10d and 10e which represent a network $\boldsymbol{G'}$ composed of 6 agents at times 6, 7, 8, 10 and 12 respectively, and in Figures 11a, 11b and 11c which represent a network $\boldsymbol{G''}$ composed of 6 agents at times 6, 7, 8 and 9 respectively.

Example 8. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.01057280$, $c(3) = 0.010684$ and $c(4) = 5$. The linking process is the same than the one presented in Example 7, up until $t = 6$, so that $\boldsymbol{G'}(t = 6)$, represented in Figure 10a, arises. At time $t = 7$, agent 3 plays and links with agent 1, and so network $\boldsymbol{G'}(t = 7)$, represented in Figure 10b, arises. At $t = 8$, it is not profitable for agent 1 to form or delete links, and so agent 6 is selected to play next. At time $t = 8$, agent 6 links with agent 2, and so network $\boldsymbol{G'}(t = 8)$, represented in Figure 10c, arises. At time $t = 9$, agent 2 links with agent 4. At time $t = 10$, agent 4 links with agent 6, and so network $\boldsymbol{G'}(t = 10)$, represented in Figure 10d, arises. At time $t = 11$, it is not profitable for agent 6 to form or delete links, and so agent 1 is selected to play next. At time $t = 11$, agent 1 links with agent 5. At time $t = 12$, agent 5 links with agent 3, and so network $\boldsymbol{G'}(t = 12)$, represented in Figure 10e, arises. At time $t = 13$, no agent has an incentive to delete or form a link, and so $\boldsymbol{G'}(t = 12)$ is the pairwise stable equilibrium. Suppose now that agent 5 is selected to play at time $t = 7$ instead of agent 6. At time $t = 8$, agent 5 links with agent 2, and so network $\boldsymbol{G''}(t = 8)$, represented in Figure 11b, arises. At time $t = 9$, it is not profitable for agent 2 to form a new link, and so agent 6 is selected to play next. At time $t = 9$, agent 6 links with agent 4, and so network $\boldsymbol{G''}(t = 9)$, represented in Figure 11c, arises. At time $t = 10$, no agent has an incentive to delete or form

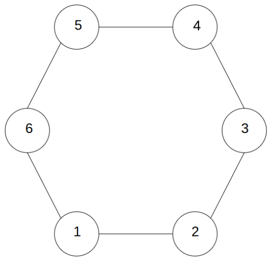

Figure 10a: $\boldsymbol{G'}(t = 6)$

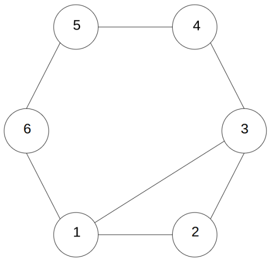

Figure 10b: $\boldsymbol{G'}(t = 7)$

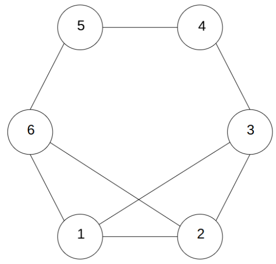

Figure 10c: $\boldsymbol{G'}(t = 8)$

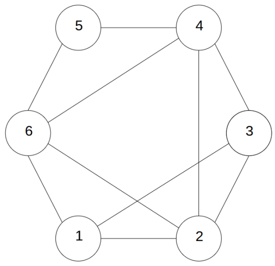

Figure 10d: $\boldsymbol{G'}(t = 10)$

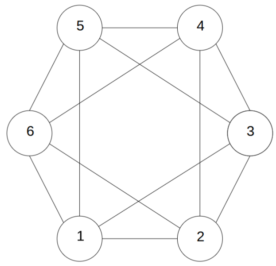

Figure 10e: $\boldsymbol{G'}(t = 12)$

a link, and so $\boldsymbol{G''}(t = 9)$ is the pairwise stable equilibrium. □

Differently from Joshi et al. (2020) and Joshi et al. (2023), in which the network at the pairwise stable equilibrium is a nested-split graph with at most one (non-singleton) component, network $\boldsymbol{G^*}$ can be composed of lattice components in this framework, due to the introduction of heterogeneity in costs of link formation.

Proposition 6. *Given $\mathcal{P} = \mathcal{P}_{RO}$, $\underline{\underline{c(1)}} \leq c(1) < \underline{c(1)}$ and $c(2) \geq \underline{c(2)}$ for some $\underline{\underline{c(1)}}$, $\underline{c(1)}$ and $\underline{c(2)}$, $\boldsymbol{G^*}$ is a circle. $\boldsymbol{G^*}$ can also be composed of lattice components.*

When $\boldsymbol{G^*}$ is a circle, every agent in the network has the same Bonacich centrality, and hence, exerts the same effort. For $\boldsymbol{G^*}$ to be a circle, it is necessary that $\underline{\underline{c(1)}} \leq c(1) < \underline{c(1)}$ is large enough so that it is not profitable for an agent in a dyad to link with a singleton, and low enough so that it is profitable for an agent in a dyad to link with another agent in a dyad. It is also necessary that $c(2) \geq \underline{c(2)}$ is large enough so that it is not profitable for agents to form a third link.

When $\mathcal{P} = \mathcal{P}_{RO}$, the linking process can be the same than the one presented in Example 8, and so, $\boldsymbol{G^*}$ can have multiple structures given α and cost function $c(d_i)$.

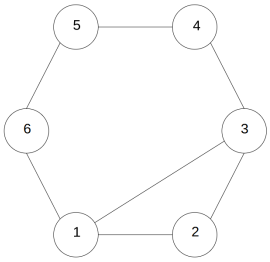

Figure 11a: $\boldsymbol{G''}(t = 7)$

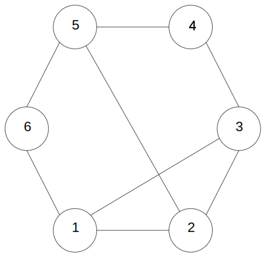

Figure 11b: $\boldsymbol{G''}(t = 8)$

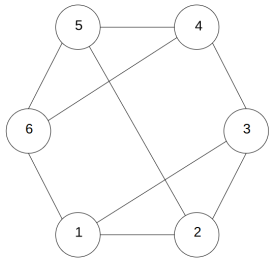

Figure 11c: $\boldsymbol{G''}(t = 9)$

Similarly to when $\mathcal{P} = \mathcal{P}_{SI}$, $\boldsymbol{G^*}$ can have multiple structures given α and $c(d_i)$ when $|\boldsymbol{C}_{max}| \geq 5$ given $N = 5$. When $N \geq 6$, $\boldsymbol{G^*}$ can only have multiple structures given α and $c(d_i)$ when components can be composed of 6 or more agents at any period during the linking process.
When $\mathcal{P} = \mathcal{P}_{RO}$, and $|\boldsymbol{C}_{max}| \geq 5$, $\boldsymbol{G^*}$ is composed of either complete components or one $\mathcal{C}_I$ component.

Proposition 7. *Given $\mathcal{P} = \mathcal{P}_{RO}$, $c(d_i)$ and α, there can only exist multiple pairwise stable equilibria if $|\boldsymbol{C}_{max}| \geq 5$. If $|\boldsymbol{C}_{max}| \leq 4$, then $\boldsymbol{G^*}$ is either*
(i) composed of complete components, or
(ii) one $\mathcal{C}_I$ component.

Differently from other orders of play, if there exists one $\mathcal{C}_I$ component in $\boldsymbol{G^*}$ when $\mathcal{P} = \mathcal{P}_{RO}$, then it is the only one. This is because, when $\mathcal{P} = \mathcal{P}_{RO}$ and $|\boldsymbol{C}_{max}| \leq 4$, a $\mathcal{C}_I$ component can only form if $N = 4$.

3.5 Time of viability

As explained throughout this paper, multiple pairwise stable equilibria can be reached given parameter α and cost function $c(d_i)$. This happens because there can exist periods in which function $\mathcal{P}$ randomly selects which agent plays at the next period, and, depending on which agent is selected, a different network structure at the pairwise stable equilibrium arises. We call the selection of some agent i by function $\mathcal{P}$ that ensures the existence of some pairwise stable equilibrium $\boldsymbol{G^*}$, the *viability* of $\boldsymbol{G^*}$ as the unique PS network. We call the period $\gamma_{\boldsymbol{G^*}} \in \boldsymbol{T}$ at which the viability of $\boldsymbol{G^*}$ as the unique PS network happens, the *time of viability* of $\boldsymbol{G^*}$

as the unique PS network. We call the structure of $\boldsymbol{G}$ at period γ_{G^*}, the *structure of viability* of $\boldsymbol{G^*}$ as the unique PS network. We denote by $\gamma^{SI}_{\boldsymbol{G^*}}$, $\gamma^{SO}_{\boldsymbol{G^*}}$, $\gamma^{RI}_{\boldsymbol{G^*}}$ and $\gamma^{RO}_{\boldsymbol{G^*}}$ the time of viability of $\boldsymbol{G^*}$ as the unique PS network, given that the order of play is $\mathcal{P}_{SI}$, $\mathcal{P}_{SO}$, $\mathcal{P}_{RI}$ and $\mathcal{P}_{RO}$ respectively.

Proposition 8. *If there exists a network $\boldsymbol{G}$ of viability of $\boldsymbol{G^*}$ as the unique PS network such that $L(\boldsymbol{G})$ is large enough, then $\gamma^{SI}_{G^*}(\gamma^{RI}_{G^*}) \leq \gamma^{SO}_{G^*}(\gamma^{RO}_{G^*})$.*

We now give an intuition of Proposition 8. Suppose order of play $\mathcal{P}_{SI}$ and some network $\boldsymbol{G^*}$ composed of 3 components $\boldsymbol{C^*_1}$, $\boldsymbol{C^*_2}$ and $\boldsymbol{C^*_3}$, and that it takes 12 periods for $\boldsymbol{C^*_1}$ to form, 12 periods for $\boldsymbol{C^*_2}$ to form, and 4 periods for $\boldsymbol{C^*_3}$ to form. Suppose further that $\boldsymbol{C^*_3}$ only has one possible structure, and that the structure $\boldsymbol{C}_1$ of viability of $\boldsymbol{C^*_1}$ as a component of the unique PS network arises at period $t = 11$, i.e., the total number of links $L(\boldsymbol{G})$ in network $\boldsymbol{G}$ of viability of $\boldsymbol{G^*}$ as the unique PS network is large. In that case, it takes 12 periods for $\boldsymbol{C^*_1}$ to form, and 11 additional periods to know with certainty the structure of $\boldsymbol{C^*_2}$. Thus, $\gamma^{SI}_{G^*} = 23$. Were the order of play $\mathcal{P}_{SO}$ instead, the formation of component $\boldsymbol{C^*_3}$ could have started before the viability of components $\boldsymbol{C^*_1}$ and $\boldsymbol{C^*_2}$ as components of the unique PS network, and thus, $\gamma^{SI}_{G^*} \leq \gamma^{SO}_{G^*}$.
Suppose instead that the structure $\boldsymbol{C}_1$ of viability of $\boldsymbol{C^*_1}$ as a component of the unique PS network arises at period $t = 7$, i.e., the total number of links $L(\boldsymbol{G})$ in network $\boldsymbol{G}$ of viability of $\boldsymbol{G^*}$ as the unique PS network is lower. If $\mathcal{P} = \mathcal{P}_{SO}$, then it is possible that the structure $\boldsymbol{C}_1$ of viability of $\boldsymbol{C^*_1}$ as a component of the unique PS network arises at period $t = 7$, the structure $\boldsymbol{C}_2$ of viability of $\boldsymbol{C^*_2}$ as a component of the unique PS network arises arises at period $t = 14$, and thus, $\gamma^{SO}_{G^*} = 14$, which is a lower time of viability of the uniqueness than $\gamma^{SO}_{G^*} = 23$.
Suppose instead that the structure $\boldsymbol{C}_1$ of viability of $\boldsymbol{C^*_1}$ as a component of the unique PS network arises at period $t = 7$, i.e., the total number of links $L(\boldsymbol{G})$ in network $\boldsymbol{G}$ of viability of $\boldsymbol{G^*}$ as the unique PS network is lower. If $\mathcal{P} = \mathcal{P}_{SO}$, then it is possible that the structure $\boldsymbol{C}_1$ of viability of $\boldsymbol{C^*_1}$ as a component of the unique PS network arises at period $t = 7$, the structure $\boldsymbol{C}_2$ of viability of $\boldsymbol{C^*_2}$ as a component of the unique PS network arises at period $t = 14$, and thus, $\gamma^{SO}_{G^*} = 14$, which is a lower time of viability of $\boldsymbol{G^*}$ as the unique PS network than $\gamma^{SO}_{G^*} = 23$.

Even though it is not always possible, *ex ante*, to know with certainty which network structure arises in the long run, it is possible to know it after the network starts forming, and it is possible to know it earlier for some orders of play than for others. Whether it is possible to know it earlier for orders of play $\mathcal{P}_{SI}$ and $\mathcal{P}_{RI}$ or $\mathcal{P}_{SO}$ and $\mathcal{P}_{RO}$ depends on the number of links of the structure of viability of the equilibrium network as the unique PS network. This result has implications regarding the efficiency of public policies. Some policymakers who propose policies in networks which are in the process of formation may have more information than others regarding the structure of the network in the long run. Therefore, part of the difference in efficiency of policies applied in any two different networks may be explained by asymmetry of information, which stems from the difference in orders of play, and not by nature of the policy itself.

3.6 Farsightedness and efficiency

We now consider a framework in which agents are farsighted, and can anticipate the possible final structures of $\boldsymbol{G}$. We first define the concepts related to farsightedness, and, next, the utility of the social planner. The equilibrium concept is *farsightedly pairwise stability*, as defined in Jackson (2008). Formally, a network $\boldsymbol{G'}$ is *improving* for a set of agents $\boldsymbol{S} \subseteq \mathcal{N}$ relative to $\boldsymbol{G}$ if it is weakly preferred by all agents in $\boldsymbol{S}$ to $\boldsymbol{G}$, with strict preference holding for at least one player in $\boldsymbol{S}$. A sequence of networks $\{\boldsymbol{G_1}, ..., \boldsymbol{G_K}\}$, and a corresponding sequence $\{\boldsymbol{S_1}, ..., \boldsymbol{S_{K-1}}\}$ such that $\boldsymbol{G_{k+1}}$ is reachable from $\boldsymbol{G_k}$ by deviations by $\boldsymbol{S_k}$, is a *farsightedly improving path* if, for each k, the ending network $\boldsymbol{G_K}$ is improving for $\boldsymbol{S_k}$ relative to $\boldsymbol{G_k}$. A network is *farsightedly pairwise stable* if there is no farsighted improving path from $\boldsymbol{G}$ to some other other network $\boldsymbol{G'}$ such that each pair of consecutive networks along the sequence are adjacent. By Theorem 1 of Herings et al. (2009), a set of farsightedly pairwise stable equilibria exists. We denote by d_{max} the maximum degree that can be attained in a network among all farsightedly pairwise stable networks. A *regular* component $\boldsymbol{\mathcal{C}}$ is a component such that, for any two nodes $i, j \in \boldsymbol{\mathcal{C}}, d_i = d_j$. We can now state Proposition 9.

Proposition 9. *Every farsightedly pairwise stable equilibrium is composed of only* *regular components.*

To see why there always is a farsightedly improving path which leads to a network composed of only regular components, suppose that for $N = 5$, some α and $c(d_i)$, the pairwise stable equilibrium is a $\boldsymbol{\mathcal{C}_{IV}}$ component, represented in Figure 5d. In order to reach a regular component, which is improving for the five agents relative to $\boldsymbol{\mathcal{C}_{IV}}$, agents can first delete links and arrive to a star network, in which agent 5 is at the center. Such a network, denoted by $\boldsymbol{G_1}$, is represented in Figure 12a. Because $\boldsymbol{\mathcal{C}_{IV}}$ is a pairwise stable equilibrium, there exists a network structure in which no agent has degree 2 or higher, and it is profitable for some agent i to form a second link with some agent j. The network structure in which i can generate the largest incremental utility through the formation of link g_{ij} is one in which i and j both form part of one dyad each. The incremental utility that agent 1 (agent 3) generates by linking with agent 2 (agent 4) in network 12a is strictly larger than the incremental utility that agent i generates by linking with agent j. Thus, network $\boldsymbol{G_2}$ represented in Figure 12b is reachable from network $\boldsymbol{G_1}$. Because $\boldsymbol{\mathcal{C}_{IV}}$ is a pairwise stable equilibrium, there exists a network structure in which no agent has degree 3 or higher, and it is profitable for some agent i to form a third link with some agent j. The network structure in which i can generate the largest incremental utility through the formation of link g_{ij} is one in which i and j both form part of the same component only composed of nodes with degree 2. The incremental utility that agent 1 (agent 2) generates by linking with agent 3 (agent 4) in network $\boldsymbol{G_2}$ is strictly larger than the incremental utility that agent i generates by linking with agent j. Thus, network $\boldsymbol{G_3}$ represented in Figure 12c is reachable from network $\boldsymbol{G_2}$. By following the same reasoning, we can show that network $\boldsymbol{G_4}$, represented in Figure 12d, is reachable as well. Thus, $\boldsymbol{G_4}$ is a farsightedly pairwise stable equilibrium.

Suppose now that $N = 6$, and for some α and $c(d_i)$, there exists a pairwise stable equilibrium, which we denote by $\boldsymbol{H^*}$, in which some agent, denoted by 6, has degree 5 instead of 4. In order to reach a regular component, agents can first delete links as well and arrive to a star network, in which agent 6 is at the center. Such a network, denoted by $\boldsymbol{H_1}$, is represented in Figure 13a. For the reason mentioned above, network $\boldsymbol{H_2}$, represented in Figure 13b, is reachable from network $\boldsymbol{H_1}$. It

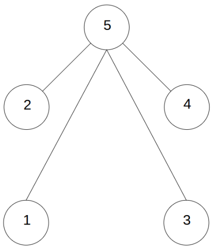

Figure 12a: $\boldsymbol{G_1}$

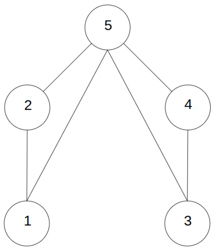

Figure 12b: $\boldsymbol{G_2}$

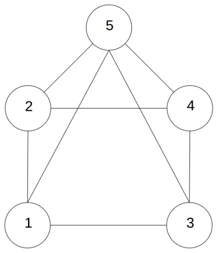

Figure 12c: $\boldsymbol{G_3}$

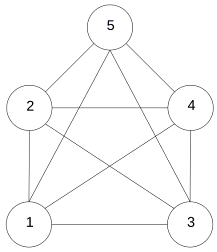

Figure 12d: $\boldsymbol{G_4}$

is possible that the improving path leading to $\boldsymbol{H^*}$ requires some agent forming a third link with an agent who has degree 2 or higher. In network $\boldsymbol{H_2}$, agents 1, 2, 3 and 4 may not be able to form a link with agent 5, since $d_5 = 1$. This happens because the number of agents in the component is even. One farsightedly improving path leading to a regular component first requires agents 1, 2, 3 and 4 to form links such that network $\boldsymbol{H_3}$, represented in Figure 12c, arises. This farsightedly improving path requires next that agents 1, 2, 3 and 4 delete links so that network $\boldsymbol{H_4}$, represented in Figure 12d, arises. It is then profitable for agents 1 and 5 to link together, and then for agents 2 and 3 to link together. Network $\boldsymbol{H_5}$, represented in Figure 13e, arises. Agent 6 can next delete links so that network $\boldsymbol{H_6}$, represented in Figure 13f, arises. It is then necessarily profitable for agents 1 and 3 to form a link, leading to network $\boldsymbol{H_7}$, represented in Figure 13g. Because all agents in $\boldsymbol{H_7}$ have degree 3 and the number of agents is even, a network in which every agent has degree 4 is reachable. Because all agents in this new network have degree 4 and the number of agents is even, the complete network $\boldsymbol{H_5}$, represented in Figure 13h, is reachable. Thus, $\boldsymbol{H_8}$ is the farsightedly pairwise stable equilibrium.

Note that in both intuitions above, the farsighted pairwise stable equilibrium is a complete network. To see why not all farsighted pairwise stable equilibria are composed of only complete components, consider network $\boldsymbol{G'}(t = 6)$, represented in Figure 10a. In any network in which $N = 6$, all agents are indifferent between $\boldsymbol{G'}(t = 6)$ and one network composed of two triads. Thus, if a third link is too costly, and there exists a farsightedly improving path to both networks, both networks are farsightedly pairwise stable equilibria.

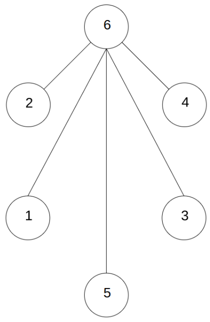

Figure 13a: $\boldsymbol{H_1}$

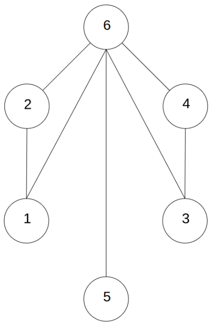

Figure 13b: $\boldsymbol{H_2}$

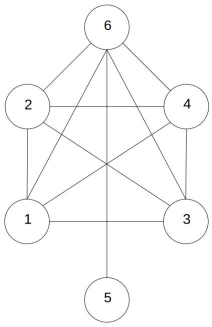

Figure 13c: $\boldsymbol{H_3}$

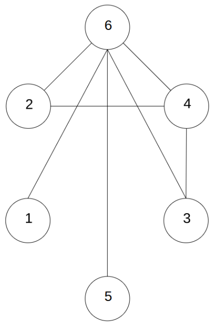

Figure 13d: $\boldsymbol{H_4}$

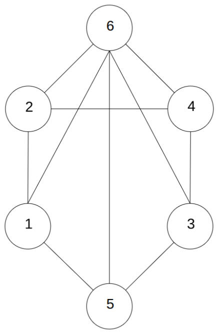

Figure 13e: $\boldsymbol{H_5}$

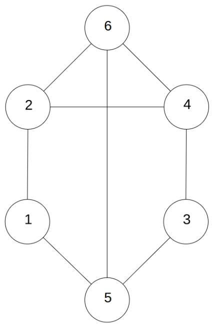

Figure 13f: $\boldsymbol{H_6}$

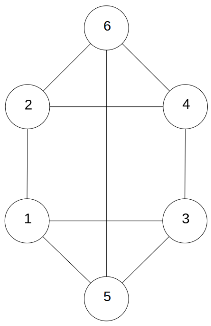

Figure 13g: $\boldsymbol{H_7}$

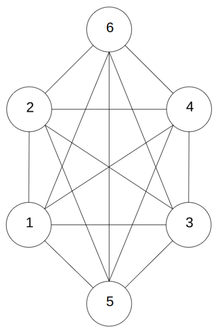

Figure 13h: $\boldsymbol{H_8}$

We now study the relationship between farsightedness of agents and network efficiency. We say that, given some α and $c(d_i)$, a network $\boldsymbol{G}$ which yields $\sum_{i \in \mathcal{N}} U_i(\boldsymbol{G})$ is *strongly efficient* if, given α and $c(d_i)$, there is no other improving path leading to some network $\boldsymbol{G'}$ yielding $\sum_{i \in \mathcal{N}} U_i(\boldsymbol{G'})$, such that $\sum_{i \in \mathcal{N}} U_i(\boldsymbol{G'}) > \sum_{i \in \mathcal{N}} U_i(\boldsymbol{G})$. We can now state Corollary 1.

Corollary 1. *Consider any α and $c(d_i)$. If N is a multiple of $(d_{max} + 1)$, then all farsightedly pairwise stable equilibria are strongly efficient.*

When N is a multiple of $(d_{max} + 1)$, every agent has degree d_{max}, which renders the farsightedly pairwise stable equilibrium strongly efficient. When N is not a multiple of $(d_{max}+1)$, farsightedly pairwise stable equilibria in which not all agents have degree d_{max} arise, such as network $\boldsymbol{I_1}$, represented in Figure 14a. There exists a farsightedly improving path from network $\boldsymbol{I_1}$ to network $\boldsymbol{I_2}$, represented in Figure 14b, since agents 1, 2, 3 and 4 are better off in network $\boldsymbol{I_2}$ than in network $\boldsymbol{I_1}$. For some values of α, e.g. $\alpha = 0.1$, $\sum_{i \in \mathcal{N}} U_i(\boldsymbol{I_1}) > \sum_{i \in \mathcal{N}} U_i(\boldsymbol{I_2})$. Therefore, when N

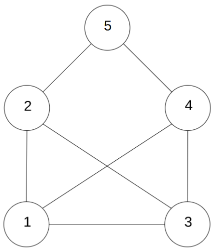

Figure 14a: $\boldsymbol{I_1}$

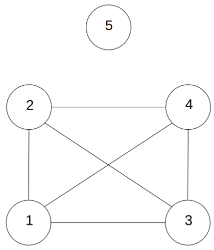

Figure 14b: $\boldsymbol{I_2}$

is not a multiple of $(d_{max} + 1)$, not every farsightedly pairwise stable equilibrium is strongly efficient.

4 Conclusion

Certain network formation models consider sunk costs of link formation (Joshi et al., 2020, 2023). The reason behind these costs is that building trust requires effort before forming the link. Building this trust allows agents to shield themselves against possible malicious intentions of the agent they are forming the link with. These sunk costs are fixed. However, malicious intentions of one's neighbors do not only affect oneself, but one's neighbors as well through spillover effects. A negative shock in one's exerted effort through a malicious act of a neighbor might lower one's reputation with respect to other neighbors, or lower the trust these neighbors have with respect to oneself in other networks. Therefore, agents need to build higher trust as the number of neighbors they have increases. In this spirit, we consider a network formation model in which sunk costs of link formation are increasing in degree. In this class of models, the order in which agents sequentially form and delete links (order of play) determines the final structure of the network. We consider four different orders of play. Our main result is that networks composed of complete bipartite components and circle networks can only be explained by one order of play each. We also give conditions for equilibrium uniqueness, and study the equilibrium structure when these conditions hold. Conditions for equilibrium uniqueness differ by order of play. Even though we cannot know with certainty, *ex ante*, which network structure arises at equilibrium, there exists a period of the linking formation process in which we can know with certainty which equilibrium arises. We give a condition under which it is possible to know at an earlier period which network arises at equilibrium. Finally, we study farsighted network formation. We find that networks composed of only regular components always arise at equilibrium, and give a condition under which the farsightedly pairwise equilibrium maximizes the sum of the utilities of agents in the network.

We believe that the class of network formation models considering costs of link formation increasing in degree opens up new possibilities of explaining richer network structures. We focus on the particular model presented in this paper by making certain assumptions. First, we consider four orders of play. There may exist other orders of play yielding other well-known structures. Second, we consider a cost function which is homogenous across agents. Some equilibrium network structures may only be explained by certain distributions of cost functions among agents. Third, the order of play is exogenous. It is possible to endogenize the order of play by selling in an auction the possibility to play next. Fourth, we follow Joshi et al. (2020, 2023) in the sense that only the sender of a link can delete links. If the receiver is able to delete links as well, different equilibrium structures than the ones we consider may arise. Fifth, we do not consider reputation shocks nor reduced trust in other networks due to malicious acts of a neighbor. We believe that future research on this class of network formation models incorporating these modifications can yield interesting results.

Appendix A The receiver-inside case

We consider a order of play in which the receiver of a link is the one playing next, and in which agents who belong to the component of the receiver play afterwards. Formally, $\mathcal{P} = \mathcal{P}_{RI}$ where $\mathcal{P}_{RI}$ is defined in (10).

$$\mathcal{P}_{RI}(i \in \boldsymbol{G}, t \in \boldsymbol{T}) = \begin{cases} j \in \boldsymbol{G}, & \text{if } L^t_{i\to j} = 1 \text{ and } j \in \boldsymbol{GL}(t) \\ l \sim \mathcal{U}(\boldsymbol{GLC}_i(t)) & \text{if } L^t_{i\to j} = 1, j \notin \boldsymbol{GL}(t) \text{ and } \exists l \in \boldsymbol{GLC}_i(t) \\ l \sim \mathcal{U}(\boldsymbol{GLNC}_i(t)) & \text{if } L^t_{i\to j} = 1, j \notin \boldsymbol{GL}(t) \text{ and } \nexists l \in \boldsymbol{GLC}_i(t) \end{cases}, \quad (10)$$

Function $\mathcal{P}_{RI}$ maps an agent $i \in \boldsymbol{G}$ and a time $t \in \boldsymbol{T}$ to an agent $j \neq i$. We distinguish between three case scenarios which lead to different identities of agent j:

- If an agent i successfully links with an agent j at time t, i.e. $L_{i \to j}^{t} = 1$, and j is able to form a new link, i.e. $j \in \boldsymbol{GL}(t)$, then it is agent j that plays at period $t+1$, i.e. $\mathcal{P}_{RI}(i,t) = j$.
- If an agent i successfully links with an agent j at time t, i.e. $L_{i \to j}^{t} = 1$, but agent j is not able to form a new link, i.e. $j \notin \boldsymbol{GL}(t)$, and there exists at least another agent who belongs to the component of i who can form a link, i.e. $\exists l \in \boldsymbol{GLC}_i(t)$, then an agent l is randomly selected out of these agents who belong to the same component of i at time t and can form a new link, i.e. $\exists l \in \boldsymbol{GLC}_i(t)$.
- If an agent i successfully links with an agent j at time t, i.e. $L_{i \to j}^{t} = 1$, but agent j is not able to form a new link, i.e. $j \notin \boldsymbol{GL}(t)$, and there does not exist at least another agent who belongs to the component of i who can form a link, i.e. $\nexists l \in \boldsymbol{GLC}_i(t)$, then an agent l is randomly selected out of the agents who do not belong to the same component of i at time t and can form a new link, i.e. $\exists l \in \boldsymbol{GLNC}_i(t)$.

Similarly to when $\mathcal{P}_{SI}$, we first study the case in which there does not exist any time when two non-singleton components at time t belong to the same component at time $t+1$. In this case scenario, there exists a unique pairwise stable equilibrium.
We then consider the case in which there can exist times when two non-singleton components at time t belong to the same component at time $t+1$. In this case scenario, there can exist multiple pairwise stable equilibria for a given α and a given cost function $c(d_i)$. We give additional conditions under which there exists a unique pairwise stable equilibrium for a given α and a given cost function $c(d_i)$. When there does not exist any time when two non-singleton components at time t belong to the same component at time $t+1$, a pair of agents successively forms triads until it is not profitable for either to form an additional triad. An example

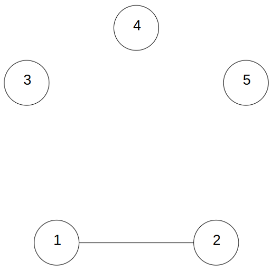

Figure 15a: $\boldsymbol{G}(t = 1)$

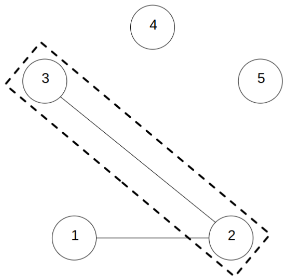

Figure 15b: $\boldsymbol{G}(t = 2)$

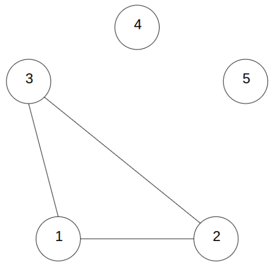

Figure 15c: $\boldsymbol{G}(t = 3)$

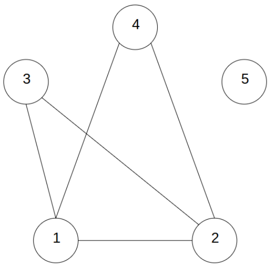

Figure 15d: $\boldsymbol{G}(t = 5)$

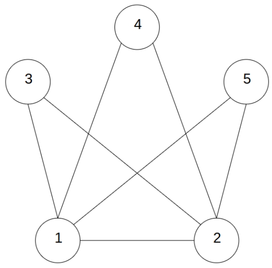

Figure 15e: $\boldsymbol{G}(t = 7)$

is shown in Figures 15a, 15b, 15c, 15d and 15e which represent a network $\boldsymbol{G}$ composed of 5 agents at times 1, 2, 3, 5 and 7 respectively, given some value of α and some cost of link formation function $c(d_i)$.

Example 9. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0103$ and $c(3) = 0.0104$. At time $t = 1$, agent 1 plays first and forms a link with agent 2. Given the values of α and $c(1)$, agent 2 plays and links with agent 3 at time $t = 2$. Notice that, at time $t = 2$, agent 3's dyad with agent 2, represented in the dashed frame in Figure 15b, has the same structure than $\boldsymbol{G}(t = 1)$, i.e., $\boldsymbol{G}(t = 1) \subseteq \boldsymbol{G}(t = 2)$. Because it was profitable for agent 2 with degree $d_2(t = 1) = 1$ to form a link with agent 3 at time $t = 2$, it is profitable for agent 3 with degree $d_3(t = 2) = 1$ to form a link with agent 1 at time $t = 3$. The linking process continues, and agents 1 and 2 form triads with agents 4 and 5 at time $t = 7$, at which they cannot form any additional link at $t = 7$. □

Once it is not profitable for the two agents that played first to form any more triads, it is the turn of agents inside the formed component to play. These agents link together until the component becomes complete. An example is shown in Figures 16a, 16b, 16c and 16d which represent network $\boldsymbol{G}$ at times 7, 8, 9 and 10 respectively.

Example 9 (continuation). Suppose that, at $t = 8$, agent 3 is selected to play. Notice that, at time $t = 7$, the subgraph composed of agents 1, 2 and 3, represented in the dashed frame in Figure 16a, has the same structure than $\boldsymbol{G}(t = 3)$, represented in Figure 15c, i.e., $\boldsymbol{G}(t = 3) \subseteq \boldsymbol{G}(t = 7)$. Because it was profitable for agent 1 with degree $d_1(t = 3) = 2$ to form a link with agent 4 at time $t = 4$, it is profitable for agent 3 with degree $d_3(t = 7) = 2$ to form a link with agent 4 at $t = 8$, by Lemma 1(b). Notice that, at time $t = 8$, the subgraph composed of agents 1, 2, 3 and 4, represented in the dashed frame in Figure 16b, has a denser structure than $\boldsymbol{G}(t = 5)$, represented in Figure 15d, i.e., $\boldsymbol{G}(t = 5) \subseteq \boldsymbol{G}(t = 8)$. Because it was profitable for agent 2 with degree $d_2(t = 5) = 3$ to form a link with agent 5 at time $t = 6$, it is profitable for agent 4 with degree $d_4(t = 8) = 3$ to form a link with agent 5 at $t = 9$, by Lemma 1(b). By following the same reasoning, we can deduce that it is profitable for agent 5 to form a link with agent 3 at time $t = 10$. At time $t = 11$, no agent has an incentive to form or delete a link, and so the complete network $\boldsymbol{G}(t = 10)$ is the pairwise stable equilibrium. □

Once the component becomes complete, it can be profitable for agents inside the component to link with a singleton. If it is the case, then a pair of agents of the component successively forms triads until it is not profitable for them to form another triad. Agents in the newly formed component link together until the component becomes complete. This process can repeat itself until either the network becomes complete, or until it is too costly for agents in the component to form new links with isolated agents. In the latter case, more complete components emerge until the set of players is exhausted. When the number of players in the game is a multiple of the number of players inside each optimal component, all components have the same number of players. Otherwise, the component which is formed last is composed by less players. In such case, players in the component which is formed

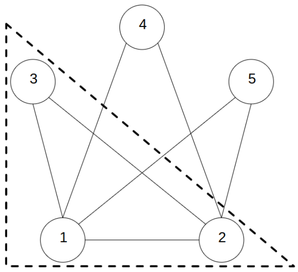

Figure 16a: $\boldsymbol{G}(t = 7)$

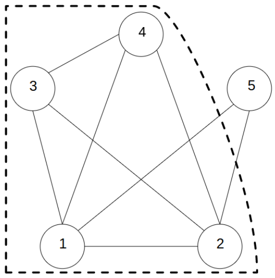

Figure 16b: $\boldsymbol{G}(t = 8)$

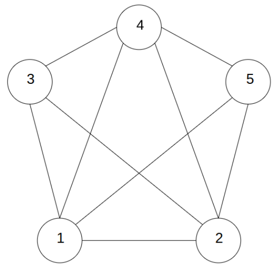

Figure 16c: $\boldsymbol{G}(t = 9)$

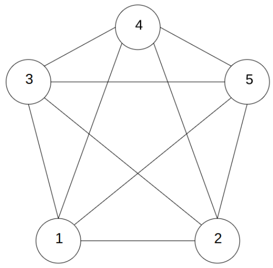

Figure 16d: $\boldsymbol{G}(t = 10)$

last have a lower Bonacich centrality, by Lemma 2.
Formally, the characterization of $\boldsymbol{G^*}$ when $\mathcal{P} = \mathcal{P}_{RI}$ and when there does not exist any time in which two non-singleton components at time t belong to the same component at time $t+1$, is given in Proposition 10.

Proposition 10. *Given $\mathcal{P} = \mathcal{P}_{RI}$ and $g_{ij}(t) = 0$ for $\tau \leq t \leq 2\tau$ for all $i \in \mathcal{C}_1(\tau-1)$ and all $j \notin \mathcal{C}_1(\tau-1)$, $\boldsymbol{G^*}$ is composed of complete components. Furthermore, there is equal effort exerted among players if and only if N is a multiple of $(d_1(\tau)+1)$.*

We now characterize $\boldsymbol{G^*}$ when $\mathcal{P} = \mathcal{P}_{RI}$ and when there exists some time t in which two non-singleton components belong to the same component at time $t+1$. In such case, network $\boldsymbol{G^*}$ can have different structures, depending on which player is randomly drawn when $\mathcal{P}_{RI} = j \sim \mathcal{U}(\boldsymbol{GLC}(t))$, and on which player the sender of a link randomly selects when there are multiple receivers that generate the highest incremental utility. Differently from when $\mathcal{P} = \mathcal{P}_{RI}$, network $\boldsymbol{G^*}$ can have different structures when components are composed of 5 agents or more at any time during the linking process. An example is shown in Figures 17a and 17b which represent

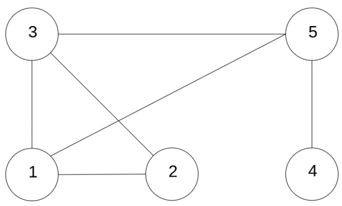

Figure 17a: $\boldsymbol{G'}(t = 6)$

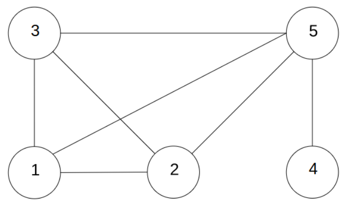

Figure 17b: $\boldsymbol{G'}(t = 7)$

a network $\boldsymbol{G'}$ composed of 5 agents at times 6 and 7 respectively.

Example 10. Suppose $\alpha = 0.01$, $c(1) = 0.01$, $c(2) = 0.0106$ and $c(3) = 0.01065$. At time $t = 1$, agent 1 plays first and forms a link with agent 2. Given the values of α and $c(1)$, agent 2 plays and links with agent 3 at time $t = 3$. At time $t = 3$, agent 3 plays and links with agent 1, by Lemma 1b. At time $t = 4$, it is not profitable for agent 1 to form or delete links, and so agent 4 is selected to play next. At $t = 4$, agent 4 links with agent 5. At time $t = 5$, agent 5 plays and links with agent 3. At time $t = 6$, it is not profitable for agent 3 to form or delete links, and so agent 5 is selected to play next. At time $t = 6$, agent 5 plays and links with agent 1, so that $\boldsymbol{G'}(t = 6)$, represented in Figure 17a, arises. At time $t = 7$, it is not profitable for agent 1 to form or delete links, and so agent 5 is selected to play. At time $t = 7$, agent 5 plays and links with agent 2, so that $\boldsymbol{G'}(t = 7)$, which is a $\mathcal{C}_{IV}$ component, represented in Figure 17b, arises. Since no agent has an incentive to form or delete a link, $\boldsymbol{G'}(t = 7)$ is the pairwise stable equilibrium. Suppose, instead, that agent 4 is selected to play at time $t = 7$. At time $t = 7$, agent 4 links with agent 2. At time $t = 8$, agent 2 plays and forms a link with agent 5. At times $t = 9$ and $t = 10$, links g_{14} and g_{34} form, and so the pairwise stable equilibrium is a complete network. □

Differently from when $\mathcal{P} = \mathcal{P}_{SI}$, $\boldsymbol{G^*}$ can have multiple structures given α and $c(d_i)$ when components can be composed of 5 or more agents at any period during the linking process. Differently from when $\mathcal{P} = \mathcal{P}_{SO}$, $\boldsymbol{G^*}$ can have multiple structures given α, $c(d_i)$ *and any number of agents* N, when components are composed of 5 or more agents at any period during the linking process.

Because, when components can be composed of 5 agents or more and it is profitable for any agent to link with any agent who has already played, $\boldsymbol{G}^{*}$ can have multiple structures given α and $c(d_i)$, we characterize $\boldsymbol{G}^{*}$ when components are composed of 4 agents or less at every period during the linking process. In such case, $\boldsymbol{G}^{*}$ is composed of either complete components or $\mathcal{C}_I$ components.

Corollary 2. *Given $\mathcal{P} = \mathcal{P}_{RI}$, $c(d_i)$ and α, there can only exist multiple pairwise stable equilibria if $|\boldsymbol{C}_{max}| \geq 5$. If $|\boldsymbol{C}_{max}| \leq 4$, then $\boldsymbol{G}^{*}$ is composed of either*
(i) complete components, or
(ii) $\mathcal{C}_I$ components and at most one complete component.

When $\mathcal{P} = \mathcal{P}_{RI}$ and $|\boldsymbol{C}_{max}| \leq 4$, the linking process is the same than when $\mathcal{P} = \mathcal{P}_{SI}$ and $|\boldsymbol{C}_{max}| \leq 4$, and so, the structure of $\boldsymbol{G}^{*}$ is the same in both case scenarios.

Appendix B Proofs

Proof of Lemma 1.
Step 1: Prove part (a).
Let us suppose that for any $\boldsymbol{G}$ where $ij \notin \boldsymbol{G}$, $\Delta U_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2,\cdots,-i\lambda_r,+ij}) > 0$ and, *ad absurdum*, that $\Delta x_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2,\cdots,-i\lambda_r,+ij}) \leq 0$. If $x_i(\boldsymbol{G}) = 0$ for any i and any $\boldsymbol{G}$, we have that $\Delta x_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2,\cdots,-i\lambda_r,+ij}) = 0$, because Bonacich centrality is non-negative. It follows that $\Delta U_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2,\cdots,-i\lambda_r,+ij}) = 0$, given equation (6), and a contradiction arises. If $x_i(\boldsymbol{G}) > 0$ for any i and any $\boldsymbol{G}$, we have that $\Delta x_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2,\cdots,-i\lambda_r,+ij}) \leq x_i(\boldsymbol{G})$, because Bonacich centrality is non-negative. It follows that $\Delta U_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2\cdots-i\lambda_r+ij}) \leq 0$, given equation (6), and a contradiction arises.
Let us next suppose that for any $\boldsymbol{G}$ where $ij \notin \boldsymbol{G}$, $\Delta x_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2,\cdots,-i\lambda_r,+ij}) > 0$ and, *ad absurdum*, that $\Delta U_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2,\cdots,-i\lambda_r,+ij}) \leq 0$. Since $x_i(\boldsymbol{G}) \geq 0$ for any i and any $\boldsymbol{G}$, we have that $x_i(\boldsymbol{G}) + \frac{1}{2}\Delta x_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2,\cdots,-i\lambda_r,+ij}) > 0$, and so that $\Delta U_i(\boldsymbol{G}_{-i\lambda_1,-i\lambda_2,\cdots,-i\lambda_r,+ij})$, given in equation (6), is strictly positive. A contradiction arises.

Step 2: Prove part (b).

Let us suppose $\boldsymbol{G} \subseteq (\subset)\tilde{\boldsymbol{G}}$ and $\boldsymbol{C}_i \subseteq \boldsymbol{G}, \tilde{\boldsymbol{C}}_i \subseteq \tilde{\boldsymbol{G}}$ are such that $\boldsymbol{C}_i \subseteq (\subset)\tilde{\boldsymbol{C}}_i$, and $ij \notin \tilde{\boldsymbol{C}}_{\boldsymbol{i}}$. Let us denote by $\Delta x_i(\boldsymbol{G}_{+ij}) = x_i(\boldsymbol{G}_{+ij}) - x_i(\boldsymbol{G})$ the incremental Bonacich centrality of agent i from linking with agent j in network $\boldsymbol{G}$, and by $\Delta x_i(\tilde{\boldsymbol{G}}_{+ij}) = x_i(\tilde{\boldsymbol{G}}_{+ij}) - x_i(\tilde{\boldsymbol{G}})$ the incremental Bonacich centrality of agent i from linking with agent j in network $\tilde{\boldsymbol{G}}$. We have that $\Delta x_i(\tilde{\boldsymbol{G}}_{+ij}) \geq (>)\Delta x_i(\boldsymbol{G}_{+ij})$ and $x_i(\tilde{\boldsymbol{G}}) \geq (>)x_i(\boldsymbol{G})$. It follows, by equation (6), that $\Delta U_i(\tilde{\boldsymbol{G}}_{+ij}) \geq (>)\Delta U_i(\boldsymbol{G}_{+ij})$.

Step 3: Prove part (c).
Let us suppose $\boldsymbol{G} \subseteq (\subset)\tilde{\boldsymbol{G}}$ such that $\boldsymbol{C}_j \subseteq \boldsymbol{G}, \tilde{\boldsymbol{C}}_j \subseteq \tilde{\boldsymbol{G}}$ are such that $\boldsymbol{C}_j \subseteq (\subset)\tilde{\boldsymbol{C}}_j$, and $ij \notin \tilde{\boldsymbol{C}}_{\boldsymbol{j}}$. Let us denote by $\Delta x_i(\boldsymbol{G}_{+ij}) = x_i(\boldsymbol{G}_{+ij}) - x_i(\boldsymbol{G})$ the incremental Bonacich centrality of agent i from linking with agent j in network $\boldsymbol{G}$, and by $\Delta x_i(\tilde{\boldsymbol{G}}_{+ij}) = x_i(\tilde{\boldsymbol{G}}_{+ij}) - x_i(\tilde{\boldsymbol{G}})$ the incremental Bonacich centrality of agent i from linking with agent j in network $\tilde{\boldsymbol{G}}$. We have that $\Delta x_i(\tilde{\boldsymbol{G}}_{+ij}) \geq (>)\Delta x_i(\boldsymbol{G}_{+ij})$ and $x_i(\tilde{\boldsymbol{G}}) \geq x_i(\boldsymbol{G})$. It follows, by equation (6), that $\Delta U_i(\tilde{\boldsymbol{G}}_{+ij}) \geq (>)\Delta U_i(\boldsymbol{G}_{+ij})$.

Proof of Lemma 2. Consider any two components $\boldsymbol{C_i}$ and $\boldsymbol{C_j}$ which are either both complete components or both complete bipartite components. Suppose that $|\boldsymbol{C_i}| > |\boldsymbol{C_j}|$, and, *ad absurdum*, that $x_i \leq x_j$ for some $i \in \boldsymbol{C_i}$ and some $j \in \boldsymbol{C_j}$. It follows that, for at least one walk length k, there exist more walks emanating from node j than from node i, or the same number of walks emanating from node i than from node j. This is impossible, because $\boldsymbol{C_i}$ and $\boldsymbol{C_j}$ are either both complete components or both complete bipartite components, and $|\boldsymbol{C_i}| > |\boldsymbol{C_j}|$. A contradiction arises. □

Proof of Proposition 1.
Step 1: Prove that, if an agent i deletes a subset of own links with agents $\{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ at time t, and forms a link with some agent j at time t, then $x_j > \sum_{\lambda \in \{\lambda_1, \lambda_2, \cdots, \lambda_r\}} x_\lambda$ at time $t-1$.
By Lemma 1(a), an agent i only deletes a subset of own links and forms a new link if it allows her to become more Bonacich central. Because $BC_i = 1 + \alpha \sum_{l \in \boldsymbol{N_l}} BC_l$, agent i becomes more Bonacich central by deleting links with agents λ_1, λ_2, ..., λ_r and forming a new link with agent j if and only if $x_j > \sum_{\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_r\}} x_\lambda$.

Step 2: Prove that total effort exerted increases at time t if and only if some agent plays at time t.

Let us denote by $\tilde{\boldsymbol{G}}$ and $\boldsymbol{G}$ the adjacency matrixes of two different networks. Formally, $\tilde{\boldsymbol{G}} = \boldsymbol{G} + \boldsymbol{P}$, where $\boldsymbol{P}$ is some $N \times N$ matrix. We denote by $\sum_{i=1}^{N} x_i$ the total effort exerted in network $\boldsymbol{G}$, and by $\sum_{i=1}^{N} \tilde{x}_i$ the total effort exerted in network $\tilde{\boldsymbol{G}}$. By Theorem 1 of Harkins (2020), if $\boldsymbol{x}^T \alpha \boldsymbol{P} \boldsymbol{x} > 0$, then $\sum_{i=1}^{N} \tilde{x}_i > \sum_{i=1}^{N} x_i$. When an agent i in network $\boldsymbol{G}$ plays at time t, she either deletes no link and forms a link, or deletes a subset of own links and forms a link, so that network $\tilde{\boldsymbol{G}}$ arises. If she deletes no link and forms a link, then it follows that $\sum_{i=1}^{N} \tilde{x}_i > \sum_{i=1}^{N} x_i$. If she deletes a subset of own links with agents $\lambda_1, \lambda_2, \cdots, \lambda_r$ and forms a link with agent j, then we have that $\boldsymbol{x}^T \alpha \boldsymbol{P} \boldsymbol{x} = \alpha 2 x_i (x_j - \sum_{\lambda \in \{\lambda_1, \lambda_2, \cdots, \lambda_r\}} x_\lambda)$. We have shown in Step 1 that $x_j - \sum_{\lambda \in \{\lambda_1, \lambda_2, \cdots, \lambda_r\}} x_\lambda$ is strictly positive, and so that $\boldsymbol{x}^T \alpha \boldsymbol{P} \boldsymbol{x} > 0$. We have shown that, if some agent plays at time t, then total effort increases at time t. If total effort increases at time t, then the structure of the network has changed at time t, and hence, an agent has played at time t.

Step 3: Prove that a pairwise stable equilibrium exists.

Let us suppose, *ad absurdum*, that a pairwise stable equilibrium does not exist. It follows, by Lemma 1 of Jackson and Watts (2001), that there exist two periods t_1 and t_2, with $t_1 < t_2$, such that $\boldsymbol{G}(t_1) = \boldsymbol{G}(t_2)$. Because total effort exerted increases at time t if and only if some agent plays at time t, it follows that total effort exerted in $\boldsymbol{G}(t_2)$ is strictly larger than total effort exerted in $\boldsymbol{G}(t_1)$. A contradiction arises and $\boldsymbol{G}(t_1) \neq \boldsymbol{G}(t_2)$. □

Proof of Proposition 2. Suppose that $\mathcal{P} = \mathcal{P}_{SI}$, that it is not profitable for any pair of agents $i \in \mathcal{C}_1(\tau - 1)$ and $j \notin \mathcal{C}_1(\tau - 1)$ to form a link during time interval $\tau \leq t \leq 2\tau$, and, *ad absurdum*, that $\boldsymbol{G^*}$ is not composed of complete components. Agent 1 successively links with x singletons until it is not profitable for her to form a new link, so that a star component forms. It follows that it is profitable for any agent i with degree $d_i \leq (x - 1)$ to link with any agent. Therefore, all agents inside the newly formed star link between themselves until the component becomes complete. If it is profitable for some agent i inside the formed complete component to successively form links with singletons, and attain some degree $\tilde{d}_i$, then it is profitable for any agent inside the component of i to attain degree $\tilde{d}_i$ as well, and form a complete component. By following the same reasoning and because $g_{ij}(t) = 0$ for $\tau \leq t \leq 2\tau$ for all $i \in \mathcal{C}_1(\tau - 1)$ and all $j \notin \mathcal{C}_1(\tau - 1)$, we can deduce that a second complete component forms in the time interval $\tau \leq t \leq 2\tau$. It is therefore not profitable either for any complete component formed afterwards to link with any other complete component at any time t. By induction of the argument, complete components form until the network attains a pairwise stable equilibrium, and a contradiction arises since we assumed that $\boldsymbol{G^*}$ is not composed of complete components.

Let us suppose that there is equal effort exerted among players, and, *ad absurdum*, that N is not a multiple of $(d_1(\tau) + 1)$. Term $(d_1(\tau) + 1)$ corresponds to the number of agents inside the component of agent 1 at time τ. If N is not a multiple of $(d_1(\tau)+1)$, then the component that is formed last is composed of less agents than $(d_1(\tau) + 1)$. It follows, by Lemma 2, that players do not exert the same effort, and so a contradiction arises.

Let us suppose that N is a multiple of $(d_1(\tau) + 1)$, and, *ad absurdum*, that there is not equal effort exerted among players. If there is not equal effort exerted among players, then there exists some agent who has a lower Bonacich centrality than the other agents. Because components that arise are complete, and there is the same number of agents in each component, each agent has the same Bonacich centrality, and hence, exerts the same amount effort. A contradiction arises. □

Proof of Proposition 3. Given $\mathcal{P} = \mathcal{P}_{SI}$, α and $|\boldsymbol{C}_{max}| \leq 5$, we give the unique pairwise stable equilibrium $\boldsymbol{G^*}$ for every shape of $c(d_i)$. Suppose first that no agent has an incentive to delete a link. If $|\boldsymbol{C}_{max}| = 2$, then $\boldsymbol{G^*}$ is composed of dyads, which are complete components. If N is not a multiple of 2, then $\boldsymbol{G^*}$ is composed of dyads, and one singleton, which corresponds to one complete component. If $|\boldsymbol{C}_{max}| = 3$, then $\boldsymbol{G^*}$ is composed of triads. Let us suppose, *ad absurdum*, that $\boldsymbol{G^*}$ is not composed of triads, which entails that it is composed of line components. It follows that it is not profitable for every pair of agents with degree 1 in the same line to link together. Because it was profitable for an agent with degree 1 to link with an isolated agent, and form a line of three agents, it is profitable for two agents with degree 1 in the same line to link together, and form a triad. A contradiction arises since we assumed that $\boldsymbol{G^*}$ is not composed of triads. If N is not a multiple of 3, then $\boldsymbol{G^*}$ is composed of triads, and at most one singleton or one dyad, which corresponds to one complete component.

Let us suppose now that $|\boldsymbol{C}_{max}| = 4$, that $\underline{c}(1) \leq c(1) \leq \underline{\underline{c}}(1)$ is high enough so that a dyad cannot link with a singleton, and low enough so that it can link with a dyad, and that $c(2) \geq \underline{c}(2)$ is high enough so that a third link is not profitable. If N is not a multiple of 4, then $\boldsymbol{G^*}$ is composed of $\mathcal{C}_I$ components, and either a singleton, a dyad, or a dyad and a singleton, which corresponds to two complete components at most. If $c(1) < \underline{c}(1)$ and $c(2) < \underline{c}(2)$, so that a dyad can link with a singleton, then a triad forms and each of the three agents in the triad links with the same isolated agent, and a complete component forms. If $c(1) < \underline{c}(1)$ and $c(2) \geq \underline{c}(2)$, then $|\boldsymbol{C}_{max}| \neq 4$. If $c(1) > \underline{\underline{c}}(1)$ and $c(2) < \underline{c}(2)$, or $c(1) > \underline{\underline{c}}(1)$ and $c(2) \geq \underline{c}(2)$, so that a dyad cannot link with a dyad, then $|\boldsymbol{C}_{max}| \neq 4$.

Let us suppose now that $|\boldsymbol{C}_{max}| = 5$, that $c(1) < \underline{c}(1)$ is low enough so that a dyad can link with a singleton, that $\underline{\underline{c}}(2) \leq c(2) < \underline{c}(2)$ is high enough so that an agent in a triad cannot link with a singleton and low enough so that an agent in a triad can link with an agent in a dyad, and that $c(3) \geq \underline{c}(3)$ is high enough so that a fourth link is not profitable. It is therefore profitable for an agent in a dyad to form a link with two agents in a triad, and the other agent in the dyad to form a link with the remaining agent in the triad, and so a $\mathcal{C}_{II}$ component forms. If $c(1) < \underline{c}(1)$, $c(2) < \underline{\underline{c}}(2)$ and $c(3) \geq c(3)$, then it is profitable for an agent in a triad to form a link with a singleton, and so a complete component forms. If $c(1) < \underline{c}(1)$, $c(2) \geq \underline{c}(2)$ and $c(3)$ takes any value, then it is not profitable for an agent in a triad to form a link with an agent in a dyad, and so $|\boldsymbol{C}_{max}| \neq 5$. If $c(1) \geq \underline{c}(1), c(2) < \underline{\underline{c}}(2)$ and $c(3)$ takes any value, a complete component forms. If $c(1) \geq \underline{c}(1), c(2) \geq \underline{c}(2)$ and $c(3)$ takes any value, then $|\boldsymbol{C}_{max}| \neq 5$. If $c(1) < \underline{c}(1)$, $c(2) \leq \underline{\underline{c}}(2)$ and $c(3) < \underline{c}(3)$, then a complete component forms. If $c(1) < \underline{c}(1)$, $\underline{\underline{c}}(2) \leq c(2) < \underline{c}(2)$ and $\underline{\underline{c}}(3) \leq c(3) < \underline{c}(3)$, which is low enough so that is profitable for an agent in a dyad to successively link with each of the three agents in a triad, and high enough so that any agent with degree 3 in the formed component forms a fourth link, then a $\mathcal{C}_{IV}$ component forms. If $c(1) < \underline{c(1)}$, $\underline{\underline{c(2)}} \leq c(2) < \underline{c(2)}$ and $\underline{\underline{\underline{c(3)}}} \leq c(3) < \underline{\underline{c(3)}}$, which is low enough so that two agents with degree 3 in the $\mathcal{C}_{IV}$ formed component link with each other, and high enough so that no other agent in the $\mathcal{C}_{IV}$ formed component forms a fourth link, then a $\mathcal{C}_{III}$ component forms. If $c(1) < \underline{c(1)}$, $\underline{\underline{c(2)}} \leq c(2) < \underline{c(2)}$ and $c(3) < \underline{\underline{\underline{c(3)}}}$, a complete component forms. If a $\mathcal{C}_{II}$ or a $\mathcal{C}_{III}$ component first forms, then it is profitable for the agent with degree 2 in the component to form a link with an agent in a dyad, since it was profitable for an agent in a triad to link with an agent in a dyad. It follows that, if $N \geq 7$ and a $\mathcal{C}_{II}$ or a $\mathcal{C}_{III}$ component first forms, then $|\boldsymbol{C}_{max}| \geq 6$. Therefore, a $\mathcal{C}_{II}$ or a $\mathcal{C}_{III}$ forms in $\boldsymbol{G}^*$ only if $N \in \{5, 6\}$, and in such case, $\boldsymbol{G}^*$ is also composed of one singleton at most. If a $\mathcal{C}_{IV}$ component first forms, then it is profitable for the agent with degree 1 in the component to form a link with an agent in a triad, since it was profitable for an agent in a triad to link with an agent in a dyad. It follows that, if $N \geq 8$ and a $\mathcal{C}_{IV}$ component first forms, then $|\boldsymbol{C}_{max}| \geq 6$. Therefore, a $\mathcal{C}_{IV}$ forms in $\boldsymbol{G}^*$ only if $N \in [5, 7]$, and in such case, $\boldsymbol{G}^*$ is also composed of one singleton or one dyad at most, which corresponds to two complete components at most.

Let us suppose now that agents can delete links. The only linking process in which an agent has an incentive to delete a link is the one in which $|\boldsymbol{C}_{max}| = 5$, some agent i who is in a dyad with some agent j successively links with two agents who are in a triad, and deletes her link with j at the next period to form a link with the other agent who is in the triad. It follows that a complete component of 4 agents first forms, and, since we assumed $|\boldsymbol{C}_{max}| = 5$, each agent of the complete component of 4 agents links with agent j, and so a complete component composed of 5 agents forms. □

Proof of Proposition 4. Suppose that $\mathcal{P} = \mathcal{P}_{SO}$, that it is not profitable for any pair of agents who have played during time interval $\tau \leq t \leq 2(\tau - 1)^2$ to form a link with each other during time interval $\tau \leq t \leq 2(\tau - 1)^2$, and, *ad absurdum*, that $\boldsymbol{G}^*$ is not composed of complete bipartite components. Agent 1 successively links with x isolated agents until it is not profitable for her to form a new link, so that a star component forms. Agent 2, chosen without loss of generality among the players which do not belong to the newly formed star component, successively links with all agents in the neighborhood of agent 1. Once agent 2 has linked with all the neighborhood of agent 1, it is not profitable for agent 2 to link with a singleton, since it is not profitable for agent 2 to link with agent 1. By induction of the argument, agents which do not belong to the component of agent 1 successively link with all agents in the neighborhood of agent 1, until a bipartite component forms. Because all agents in the formed bipartite component have the same Bonacich centrality, and because it is not profitable for agents outside the formed bipartite component to link with agents who have already played, it is not profitable for them either to link with any other agent inside the formed bipartite component. Therefore, a second complete bipartite component forms, and it forms at $t = 2(\tau - 1)^2$. By induction of the argument, complete bipartite components form until the network attains a pairwise stable equilibrium, and a contradiction arises since we assumed that $\boldsymbol{G^*}$ is not composed of complete bipartite components.

Let us suppose that there is equal effort exerted among players, and, *ad absurdum*, that N is not a multiple of $2d_1(\tau - 1)$. Term $2d_1(\tau - 1)$ corresponds to the number of agents inside the firstly formed complete bipartite component. If N is not a multiple of $2d_1(\tau - 1)$, then the component that is formed last is composed of less agents than $2d_1(\tau - 1)$. It follows, by Lemma 2, that players do not exert the same effort, and so a contradiction arises.

Let us suppose that N is a multiple of $2d_1(\tau - 1)$, and, *ad absurdum*, that there is not equal effort exerted among players. If there is not equal effort exerted among players, then there exists some agent who has a lower Bonacich centrality than the other agents. Because components that arise are complete bipartite components, and there is the same number of agents in each component, each agent has the same Bonacich centrality, and hence, exerts the same amount effort. A contradiction arises. □

Proof of Proposition 5. Given $\mathcal{P} = \mathcal{P}_{SO}$, α and $|\boldsymbol{C}_{max}| \leq 4$, we give the unique pairwise stable equilibrium $\boldsymbol{G^*}$ for every shape of $c(d_i)$. If $|\boldsymbol{C}_{max}| = 2$, then $\boldsymbol{G^*}$

is composed of dyads, which are complete components. If N is not a multiple of 2, then $\boldsymbol{G^*}$ is composed of dyads, and one singleton, which corresponds to one complete component. If $|\boldsymbol{C}_{max}| = 3$, then necessarily $N = 3$. Let us suppose that $|\boldsymbol{C}_{max}| = 3$, and, *ad absurdum*, that $N \geq 4$. It follows that, after a singleton successively linked with two singletons, an agent outside the formed line formed a link with another agent outside the formed line. A contradiction arises, since it was profitable for an agent in a dyad to link with a singleton, and so it was profitable as well for a degree 1-agent in a line to link with the other degree 1-agent in the line.

Let us suppose now that $|\boldsymbol{C}_{max}| = 4$, that $c(1) \leq \underline{c(1)}$ is low enough so that a dyad can link with a singleton, and that $c(2) \geq \underline{c(2)}$ is high enough so that a third link is not profitable. Then, a $\mathcal{C}_I$ component forms. By induction of the argument, $\mathcal{C}_I$ components form until $\boldsymbol{G}$ attains a pairwise stable equilibrium. If N is not a multiple of 4, then $\boldsymbol{G^*}$ is composed of $\mathcal{C}_I$ components, and either a singleton, a dyad, or a triad, which corresponds to one complete component at most. If $c(1) > \underline{c(1)}$ and $c(2) < \underline{c(2)}$, or $c(1) > \underline{c(1)}$ and $c(2) \geq \underline{c(2)}$, or $c(1) \leq \underline{c(1)}$ and $c(2) < \underline{c(2)}$, then $|\boldsymbol{C}_{max}| \neq 4$. $\square$

Proof of Proposition 6. Suppose that $\mathcal{P} = \mathcal{P}_{RO}$, that $c(1)$ is high enough so that it is not profitable for a dyad to link with an isolated agent, and low enough so that it is profitable for a dyad to link with a dyad, that $c(2)$ is high enough so that no agent can form a third link, and, *ad absurdum*, that $\boldsymbol{G^*}$ is not a circle. At time $t = 1$, agent 1 links with some agent 2. At time $t = 2$, it is not profitable for agent 2 to link with an isolated agent, and so some agent 3 plays and links with some agent 4. At time $t = 3$, agent 4 links with agent 2, chosen among agent 1 and agent 2 without loss of generality. It is not profitable for agent 2 to form a third link, and so some agent 5 plays at time $t = 4$, and links with agent 4, chosen among agent 1 and agent 4 without loss of generality. By induction of the argument, every agent who does not belong to the formed line component links with either of the two agents with degree 1 in the line, until $\boldsymbol{G}$ becomes a line. When $\boldsymbol{G}$ is a line, it is profitable for both agents with degree 1 to link with each other, and so $\boldsymbol{G^*}$ is a circle. A contradiction arises since we assumed that $\boldsymbol{G^*}$ is

not a circle.
It is proved in the main text that, given $\mathcal{P} = \mathcal{P}_{RO}$, $\underline{\underline{c(1)}} \leq c(1) < \underline{c(1)}$ and $\underline{\underline{c(2)}} \leq c(2) < \underline{c(2)}$ for some $\underline{\underline{c(2)}}$, $\boldsymbol{G^*}$ can be composed of lattice components. □

Proof of Proposition 7. Given $\mathcal{P} = \mathcal{P}_{RO}$, α and $|\boldsymbol{C}_{max}| \leq 4$, we give the unique pairwise stable equilibrium $\boldsymbol{G^*}$ for every shape of $c(d_i)$. If $|\boldsymbol{C}_{max}| = 2$, then $\boldsymbol{G^*}$ is composed of dyads, which are complete components. If N is not a multiple of 2, then $\boldsymbol{G^*}$ is composed of dyads, and one singleton, which corresponds to one complete component. If $|\boldsymbol{C}_{max}| = 3$, then necessarily $N = 3$. Let us suppose that $|\boldsymbol{C}_{max}| = 3$, and, *ad absurdum*, that $N \geq 4$. It follows that, after an agent in a dyad linked with a singleton, an agent outside the formed line formed a link with another agent outside the formed line. A contradiction arises, since it was profitable for an agent in a dyad to link with a singleton, and so it was profitable as well for a degree 1-agent in a line to link with the other degree 1-agent in the line.
Let us suppose now that $N = 4$, that $\underline{c(1)} \leq c(1) \leq \underline{\underline{c(1)}}$ is high enough so that a dyad cannot link with a singleton, and low enough so that it can link with a dyad and become a $\mathcal{C}_I$ component, and that $c(2) \geq \underline{c(2)}$ is high enough so that a third link is not profitable. If $c(1) < \underline{c(1)}$ and $c(2) < \underline{c(2)}$, so that a dyad can link with a singleton, then a triad forms and each of the three agents in the triad links with the same isolated agent, and a complete component forms. If $c(1) < \underline{c(1)}$ and $c(2) \geq \underline{c(2)}$, then $|\boldsymbol{C}_{max}| \neq 4$. If $c(1) > \underline{\underline{c(1)}}$ and $c(2) < \underline{c(2)}$, or $c(1) > \underline{\underline{c(1)}}$ and $c(2) \geq \underline{c(2)}$, so that a dyad cannot link with a dyad, then $|\boldsymbol{C}_{max}| \neq 4$. If $N \geq 5$ and $c(3) \geq \underline{c(3)}$ is such that a fourth link is not profitable, then every shape of $c(d_i)$ leads to the same $\boldsymbol{G^*}$ structure than when $N = 4$, except when $\underline{c(1)} \leq c(1) \leq \underline{\underline{c(1)}}$. In such case, two agents in a dyad each form a link with each other, and afterwards, an agent outside the formed component links with a degree-1 agent in the formed component, and so $|\boldsymbol{C}_{max}| \neq 4$. It follows that a $\mathcal{C}_I$ component can only form when $N = 4$. □

Proof of Proposition 8. We prove that, if there exists a network $\boldsymbol{G}$ of viability of $\boldsymbol{G^*}$ as the unique PS network such that $L(\boldsymbol{G})$is large enough, then $\gamma^{SI}_{G^*} \leq \gamma^{SO}_{G^*}$.

The same reasoning suffices to show $\gamma_{G^*}^{RI} \leq \gamma_{G^*}^{RO}$.
Let us suppose that, for a given α and $c(d_i)$, a network $\boldsymbol{G^*}$ composed of K components can arise, that the first component forms at period t, that its structure is known with certainty at period t, that the component that is formed last can only have one structure, and, *ad absurdum*, that $\gamma_{G^*}^{SI} > \gamma_{G^*}^{SO}$. Then, the earliest period at which the structure of $\boldsymbol{G^*}$ can be known with certainty is $(K-1)t$, which coincides with $\gamma_{\boldsymbol{G^*}}^{SI}$. A contradiction arises.
Let us now suppose that, for a given α and $c(d_i)$, a network $\boldsymbol{G^*}$ composed of K components can arise, that the first component forms at period t_1, that its structure is known with certainty at period t_1, that the component that is formed last can have multiple structures, that, if the first $(K-1)$ components have already formed, the component that is formed last is formed at period $(K-1)t_1 + t_2$ and its structure is known with certainty at period $(K-1)t_1 + t_2$, and, *ad absurdum*, that $\gamma_{G^*}^{SI} > \gamma_{G^*}^{SO}$. Then, the earliest period at which the structure of $\boldsymbol{G^*}$ can be known with certainty is $(K-1)t_1 + t_2$, which is $\gamma_{\boldsymbol{G^*}}^{SI}$. A contradiction arises □

Proof of Proposition 9. It is proved in the main text that, if any agent i can reach degree d_i, then there exists a farsightedly improving path leading i to being in a regular component $\boldsymbol{\mathcal{C}_1}$ in which each agent has degree d_i. If there exists a farsightedly improving path leading agents in $\boldsymbol{\mathcal{C}_1}$ and some other agents outside of the component to form a regular component $\boldsymbol{\mathcal{C}_2}$ in which agents have a degree $d_j > d_i$, then it is formed. Since agent i and component $\boldsymbol{\mathcal{C}_1}$ are chosen wlog, all components in the farsightedly pairwise stable network are regular. □

Proof of Proposition 10. Suppose that $\mathcal{P} = \mathcal{P}_{RI}$, that it is not profitable for any pair of agents $i \in \mathcal{C}_1(\tau-1)$ and $j \notin \mathcal{C}_1(\tau-1)$ to form a link during time interval $\tau \leq t \leq 2\tau$, and, *ad absurdum*, that $\boldsymbol{G^*}$ is not composed of complete components. At time $t = 1$, agent 1 links with some agent 2. At time $t = 2$, agent 2 links with some agent 3. At time $t = 3$, agent 3 links with agent 1, by Lemma 1, and so a triad forms. By induction of the argument, agents 1 and 2 successively form triads with other agents until it is not profitable for them to form a new triad. We denote this time by ψ. It follows that it is profitable for any agent who is part of x triads and with degree $(x+1)$ to form a link with any agent. At $t = \psi + 1$, agent 3 is part of 1 triad, and has degree 2, which implies that it is profitable for her to form a link with any agent inside the component. Agent 3, chosen among players inside the formed component wlog, links with agent 4, chosen among players inside the formed component wlog. Whenever agent 3 or 4 proposes a new link to an agent inside the component at any time t, they are in a new triad at time $t+1$, and so it is profitable for them to successively form triads until they link with all agents in the component. By induction of the argument, all pairs of agents successively form triads with all other agents in the component, until the component becomes complete. By following the same reasoning and because $g_{ij}(t) = 0$ for $\tau \leq t \leq 2\tau$ for all $i \in \mathcal{C}_1(\tau - 1)$ and all $j \notin \mathcal{C}_1(\tau - 1)$, we can deduce that a second complete component forms in the time interval $\tau \leq t \leq 2\tau$. It is therefore not profitable either for any complete component formed afterwards to link with any other complete component at any time t. By induction of the argument, complete components form until the network attains a pairwise stable equilibrium, and a contradiction arises since we assumed that $\boldsymbol{G^*}$ is not composed of complete components.

Let us suppose that there is equal effort exerted among players, and, *ad absurdum*, that N is not a multiple of $(d_1(\tau)+1)$. Term $(d_1(\tau)+1)$ corresponds to the number of agents inside the component of agent 1 at time τ. If N is not a multiple of $(d_1(\tau)+1)$, then the component that is formed last is composed of less agents than $(d_1(\tau)+1)$. It follows, by Lemma 2, that players do not exert the same effort, and so a contradiction arises.

Let us suppose that N is a multiple of $(d_1(\tau)+1)$, and, *ad absurdum*, that there is not equal effort exerted among players. If there is not equal effort exerted among players, then there exists some agent who has a lower Bonacich centrality than the other agents. Because components that arise are complete, and there is the same number of agents in each component, each agent has the same Bonacich centrality, and hence, exerts the same amount effort. A contradiction arises. □

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