Overconfidence in Elimination Contests

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Abstract

This paper analyzes how overconfidence affects behavior in elimination contests. We focus on two-stage elimination contests with four players. In the first-stage, the players are matched pairwise, and each pair competes in one semifinal. In the second-stage, the first-stage winners compete in the final. An overconfident player overestimates his winning probability in each pairwise interaction. Our findings reveal a nuanced interplay between overconfidence and effort exertion. An overconfident player expends less effort in the final stage than a rational rival. However, this pattern can be inverted in the semifinal, where an overconfident player can exert more effort than a rational rival. We also uncover that an overconfident player can have the highest equilibrium probability of winning an elimination contest. Our results offer a novel perspective on the promotion of overconfident individuals to CEO positions. They also highlight that large increases in executive compensation can render the pursuit of CEO positions exceptionally appealing to overconfident individuals.

Keywords: Overconfidence, Elimination Contest, Encouragement Effect, Complacency Effect.

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1 Introduction

Elimination contests are a common feature in politics, organizations, sports, and academia. In politics, politicians compete for top positions in the party, and those who reach prominent positions in the party compete to become high-level government officials. In organizations, employees compete for promotion to manager, and managers compete for promotion to a chief executive officer position. In academia, PhDs compete to be hired as assistant professors, and assistant professors compete for tenure. In tennis and many other sports, players compete in elimination contests.

Overconfidence is one of the most widely documented biases in judgment and has been detected both in the laboratory and in the field.[1] Overconfidence has consequences for economic behavior in labor markets (Spinnewijn 2013, Spinnewijn 2015, Köszegi 2014, Santos-Pinto and de la Rosa 2020). A large proportion of CEOs is overconfident and CEO overconfidence affects corporate decisions (Malmendier and Tate, 2005, 2008, 2015). It remains an open research question why do these overconfident CEOs obtain their jobs in the first place.

In this paper we analyze how overconfidence, conceptualized as overestimation of ability, affects behavior in elimination contests. We are interested in finding answers to the following questions. How does overconfidence affect effort provision in the different stages of an elimination contest? Is an overconfident player more or less likely to win an elimination contest than a rational player? What are the welfare implications of overconfidence for the players and for the contest designer? How does overconfidence alter the optimal prize structure chosen by the contest designer?

To address these questions we consider a two-stage elimination contest with four players. In the first stage, the players are matched pairwise, and each pair competes in one semifinal. The first-stage winners go on to the second stage of the contest and compete against each other in the final. The winner receives prize w_1, the runner-up prize w_2, and the first-stage losers receive nothing, with $w_1 > w_2 \geqslant 0$.

In each pairwise interaction the players choose their efforts simultaneously to maximize their expected utilities. The probability of winning a pairwise interaction depends on the efforts of both players through Alcalde and Dahm's (2007) contest success function (CSF).[2] Player are homogeneous, except for their confidence levels. This allow us to zero in on the impact of overconfidence on players' incentives to exert effort. Finally, we assume an overconfident player has a correct perception of the prizes and cost of effort but overestimates the impact of his effort on his winning probability in each pairwise interaction. Furthermore, an overconfident player's bias is observable by his rivals.

Section 3 analyzes an elimination contest with four rational players. Proposition 1 characterizes its equilibrium which serves as a benchmark to which we compare all our results. When the four players are rational the elimination contest is symmetric and hence each player has 1/4 probability of being the winner.

Section 4 considers an elimination contest with one overconfident player and three rational players. This configuration describes scenarios where a minority of players is overconfident and allows us to study the effects of overconfidence in the simplest possible way. Proposition 2 demonstrates that an overconfident player always exerts less effort in a final than a rational rival. Intuitively, the (mis)perceived advantage of the overconfident

[1] Moore and Healy (2008) distinguish between three types of overconfidence: overestimation of one's skill (absolute overconfidence), overplacement (relative overconfidence), and excessive precision in one's beliefs (miscalibration or overprecision).

[2] This CSF allows us to derive closed form solutions for the players' efforts, winning probabilities, continuation values, and expected utilities at each stage of the contest. Section 2 explains Alcalde and Dahm's (2007) CSF in detail. Section 6 shows that our main results extend to Tullock's (1980) CSF.

player leads him to lower his effort. The rational player, anticipating the overconfident player will lower his effort, also lowers her effort but not as much. Hence, both players exert less effort in the final than if both were rational. Proposition 2 also shows that both players' perceived expected utility of the final increases in the bias of the overconfident player. Hence, the bias makes reaching the final more attractive for an overconfident player and for a rational rival.

Proposition 3 shows that an overconfident player can exert higher effort in a semifinal than a rational rival. In a semifinal, a player chooses the level of effort at which the perceived marginal benefit equals the marginal cost. The perceived marginal benefit is the product of the perceived marginal probability of winning the semifinal and the perceived continuation value (a player's perceived expected utility of the final before knowing the identity of his rival in the final). As Rosen (1986) first hinted at, overconfidence has two effects on a player's incentives to exert effort in a semifinal. On the one hand, the bias lowers the perceived marginal probability of winning the semifinal which motivates an overconfident player to lower effort. We label this the complacency effect of overconfidence. On the other hand, the bias raises the perceived continuation value which motivates an overconfident player to raise effort. We label this the encouragement effect of overconfidence. When the encouragement (complacency) effect dominates, the overconfident player exerts higher (lower) effort at equilibrium in the semifinal. Proposition 3 also shows that two conditions have to be met for the encouragement effect to dominate. First, the prize spread needs to be sufficiently large. Second, the overconfident player's bias cannot be too high. Propositions 2 and 3 generate the model's main testable prediction: an overconfident player always exerts less effort in a final than a rational rival but can exert higher effort in a semifinal than a rational rival.

Proposition 4 reveals that the presence of an overconfident player in one of the semifinals of an elimination contest has a spillover effect on the efforts in the other semifinal. In the semifinal with two rational players, each player has an incentive to exert higher effort than if all players in the contest were rational. This happens because the winner of this semifinal will face an overconfident rival in the final with positive probability. This raises the rational players' continuation value due to the higher expected utility of the final.

Proposition 5 shows that the overconfident player can be the one with the highest equilibrium probability of winning the elimination contest. Three conditions have to be met for this to be the case. First, the role that effort plays in determining the winning probabilities must be sufficiently high. Second, the prize spread needs to be sufficiently large. Third, the overconfident player's bias needs to be small.

Section 5 studies an elimination contest with two overconfident players and two rational players. We assume the overconfident players have different biases and consider two types of seedings: (i) the overconfident players are seeded in the same semifinal, and (ii) the overconfident players are seeded in different semifinals. Proposition 6 demonstrates that in a final featuring two overconfident players, the more confident player always exerts lower effort, and both players exert lower efforts than if both were rational. Proposition 7 reveals that when two overconfident players are seeded in the same semifinal, both can exert higher efforts than if both were rational. Proposition 8 shows that when two overconfident players are seeded in different semifinals, both can exert higher efforts than their rational rivals. Propositions 6, 7, and 8 collectively underscore that the findings derived for an elimination contest with one overconfident and three rational players extend to an elimination contest with two overconfident and two rational players, irrespective of the seeding.

Section 6 analyzes the welfare implications of overconfidence for the players and the contest designer. To do that we consider an elimination contest with one overconfident player and three rational players. Proposition 9 shows the overconfident player can gain from his bias. The rational player seeded in the same semifinal as the overconfident player losses (gains) when the overconfident player exerts higher (lower) effort. The rational players seeded in the same semifinal always gain. Proposition 10 shows that while overconfidence always lowers aggregate effort in the final stage it can either raise or lower aggregate effort in the semifinals stage. This result implies that we are unable to state general conditions under which overconfidence lowers or raises the contest designer's welfare measured as the players' aggregate effort in the two stages.

Section 7 considers how the contest designer should set prizes when players are overconfident and her goal is to maximize players' aggregate efforts in the two stages. To answer this question we assume players are homogeneous and the contest designer has a fixed prize budget she can allocate to either the winner or the runner-up. Proposition 11 shows that the contest designer chooses a winner-take-all prize structure when players are risk neutral. In contrast, when players are risk-averse, the optimal prize structure involves multiple prizes with the winner receiving most of the prize money and a smaller part being assigned to the runner-up. Finally, Proposition 11 shows that an increase in players' overconfidence, leads the contest designer to allocate an increasingly higher share of the prize budget to the winner.

Section 8 discusses three extensions of the model which demonstrate the robustness of our findings. It starts by showing that our main results extend to a two-stage elimination Tullock contest. Subsequently, it shows our main results also hold when the overconfident player's bias is not observable by the rational players. Next, it shows our main results extend to a three-stage elimination contest with eight players. Finally, it discusses the impact of underconfidence on a two-stage elimination contest with one underconfident and three rational players. It uncovers that the underconfident player exerts less effort than his rational rivals during the final and the semifinal stages of an elimination contest. Hence, the underconfident player is the one with the lowest equilibrium probability of winning the elimination contest.

Our study relates to four strands of literature. First, it contributes to the literature on CEO overconfidence. Empirical evidence documents that a substantial share of CEOs are overconfident (for a review see Malmendier and Tate, 2015). The seminal contribution to this literature is Malmendier and Tate (2005, 2008) who measure CEO overconfidence as the tendency to hold stock options longer before exercise. Malmendier and Tate (2015) use this measure together with additional controls and find that approximately 40 percent of CEOs of companies listed in the Standard & Poor's 1500 index are overconfident.

The literature on CEO overconfidence offers two main explanations for why overconfident managers are promoted to CEO positions. Goel and Thakor (2008) study tournaments where risk-averse managers compete for promotion to become CEO by choosing the level of risk of their projects. Some managers are rational while others are overconfident. An overconfident manager underestimates project risk which increases the propensity to take risky projects (e.g. R&D activities). Some of the more risky projects will be successful and hence, the higher risk taking of overconfident managers will improve their chances of promotion to CEO. According to Van den Steen (2005), CEO overconfidence can serve as a commitment device that helps attract and retain employees that share the same values as the CEO.

Our results provide a new explanation for why overconfident managers are promoted to CEO positions. One can think of reaching a CEO position as winning a series of labor promotions in which managers compete based on performance to move up the corporate ladder. Our model predicts that if the prize spread over the corporate ladder is large, then moderately overconfident managers can be more likely to be promoted to a CEO position than rational ones. This happens because moderately overconfident managers will exert more effort than rational ones (e.g., by working longer hours) in the early stages of their careers due to the encouragement effect of overconfidence. The larger the prize spread between the compensation of a low level manager and that of a CEO, the greater the encouragement effect. Hence, our results highlight the role that large increases in executive compensation (Murphy 2013) can have in rendering the pursuit of a CEO position extremely attractive to overconfident individuals.

Second, our study contributes to the large literature on gender gaps in the labor market. Empirical evidence documents gender gaps in wages and in top business positions. For instance, in 2022 women in the US earn 82 percent of their male counterparts (Kochhar 2023) and women represent only 6 percent of top business executives in the US (Keller et al. 2022). The wage gender gap is larger in high skilled work, and much of it seems to be caused by gaps in promotions (Blau and DeVaro 2007, Blau and Kahn 2017, Bronson and Thoursie 2020). Laboratory experiments show that gender differences in confidence and risk attitudes can account for gender gaps in behavior in tournaments and contests (Niederle and Vesterlund 2007, Kamas and Preston 2012, Gillen et al. 2019, Price 2020, Buser et al. 2021, van Veldhuizen 2022).

Our findings show that the large executive compensation spreads coupled with higher male confidence can make competing for a top business position much more attractive for male candidates. We also predict that much of the gender gap in promotions will take place early in workers' careers. This could place women at a further disadvantage besides the negative effects of childbirth and child-rearing (Bertrand et al. 2010, Goldin and Katz 2011, Goldin 2014).[3]

Third, our study also contributes to the theoretical literature on overconfidence, tournaments, and contests. Santos-Pinto (2010) shows how firms can optimally set tournament prizes to exploit workers' overconfidence, defined as overestimation of ability. Ludwig et al. (2011) show that an overconfident player, defined as someone who underestimates the cost of effort, exerts more effort than a rational player in a Tullock contest. Santos-Pinto and Sekeris (2023) study how confidence gaps affect effort provision and entry in Lazer-Rosen tournaments and Tullock contests. They find, among other things, that the more confident player exerts lower effort in a Tullock contest. All of these studies focus on one-shot tournaments and contests. To the best of our knowledge, ours is the first theoretical study on the impact of overconfidence on a two-stage elimination contest.

Finally, our study contributes to the literature on elimination contests. The seminal contribution is Rosen (1986) who shows how to optimally set prizes in a multiple stage elimination contest. There are many studies on elimination contests regarding different aspects, such as the discouragement effects in multi-stage contests (Konrad, 2012), optimal prize setting (Mago et al, 2013; Cheng et al, 2019; Coehn et al, 2018; Moldovanu and Sela, 2006), optimal contest structure (Gradstein and Konrad, 1999; Moldovanu and Sela, 2006; Fu and Lu, 2018; Hou and Zhang, 2021), heterogeneity in abilities (Rosen, 1986; Brown and Minor, 2014) and seeding (Groh et al, 2012). Our paper expands this strand of literature by considering a new dimension: heterogeneity in confidence levels.

The paper is organized as follows. Section 2 sets-up the model. Sections 3, 4, and 5 study an elimination contest with four rational players, one overconfident and three rational players, and two overconfident and two rational players, respectively. Section 6 considers the welfare implications of overconfidence. Section 7 shows how the con-

[3] Many studies suggest that gender gap varies with culture (Gneezy et al, 2003 and 2009; Booth and Nolen, 2009 and 2014). In societies where gender equality is more promoted, gender gaps become less significant in many areas, including entry and performance in a competitive environments. Differences in work environment, characteristics of professions, and education also affect the magnitude of gender gaps.

test designer should set prizes when players are overconfident. Section 8 discusses the extensions. Section 9 concludes the paper. All proofs are in the Appendix.

2 Set-up

Consider a two-stage elimination contest with four players. In the first stage, the players are matched pairwise, and each pair competes in one semifinal. The first-stage winners go on to the second stage of the contest and compete against each other in the final. The winner receives prize w_1, the runner-up receives prize w_2, and the first-stage losers receive nothing, with $w_1 > w_2 \geqslant 0$.

The players choose their efforts simultaneously to maximize their expected utilities in each pairwise interaction. The effort of player i is denoted by e_i. Players derive utility $u(w)$ from prize $w \geqslant 0$, where $u'(w) > 0$, $u''(w) \leqslant 0$, and $u(0) = 0$. Players have linear cost of effort, i.e., $c(e_i) = ce_i$, with $c \geq 1$, and $e_i \geqslant 0$. The probability of winning a pairwise interaction depends on the efforts of the two players through Alcalde and Dahm's (2007) contest success function. Letting p_{ij} denote player i's winning probability when paired with j, we have

$$p_{ij} = \begin{cases} 1 - \frac{1}{2}(\frac{e_j}{e_i})^\alpha & \text{if} \quad e_i \geqslant e_j \\ \frac{1}{2}(\frac{e_i}{e_j})^\alpha & \text{if} \quad e_i \leqslant e_j \end{cases}$$

Under this CSF a player's winning probability is increasing in effort. When players exert the same effort, each has a 50 percent chance of winning. The player who exerts more effort is the favorite, having a winning probability higher than 1/2 and his rival is the underdog. The parameter α determines how sensitive the CSF is to effort. When $\alpha = 0$ the CSF is completely insensitive to effort and we obtain the extreme case of a (fair) lottery. As α increases, the CSF becomes more sensitive to effort, and the contest becomes more deterministic until the extreme case of an all-pay auction is reached when $\alpha \to \infty$. We assume $0 < \alpha \leq 1$ which implies that each stage of the elimination contest has a unique pure strategy Nash equilibrium.

This CSF has several desirable properties. First, it is homogeneous of degree zero in players' efforts. Second, it is piecewise continuous which makes the first-order conditions solvable. Third, as mentioned by Alcalde and Dahm (2007), it yields a very tractable model for multi-stage games as the equilibrium efforts and payoffs of the subgames can be easily computed and plugged into earlier stages of the game. These properties, together with the linear cost function, allow us to derive close-form solutions for the equilibrium efforts and payoffs.

We assume an overconfident player has a correct perception of the prizes and cost of effort but overestimates the impact of his effort on his winning probability in each pairwise interaction. This definition of overconfidence is in line with Santos-Pinto (2008, 2010) and Santos-Pinto and Sekeris (2023). Letting λ_i denote player i's level of confidence and $\widetilde{p}_{ij}$ player i's perceived winning probability when paired with j we have:

$$\widetilde{p}_{ij} = \begin{cases} 1 - \frac{1}{2}\frac{e_j^\alpha}{\lambda_i e_i^\alpha} & \text{if} \quad \lambda_i e_i^\alpha \geqslant e_j^\alpha \\ \frac{1}{2}\frac{\lambda_i e_i^\alpha}{e_j^\alpha} & \text{if} \quad \lambda_i e_i^\alpha \leqslant e_j^\alpha \end{cases} \tag{1}$$

We assume $\lambda_i \geqslant 1$, meaning that a player can be either rational ($\lambda_i = 1$) or overconfident ($\lambda_i > 1$). We see from (1) that an overconfident player holds a higher perceived winning probability than his true winning probability for any given efforts. On Figure 1, we depict the true (solid blue curve) and the perceived (solid red curve) winning probabilities of an overconfident player with $\lambda_i = 1.5$. We set $\alpha = 0.9$ and $e_j = 1$. We see that the true and perceived winning probabilities are increasing in effort. Note that when player i's effort is identical to j's, i.e., $e_i = e_j = 1$, player i's true winning probability is $1/2$.

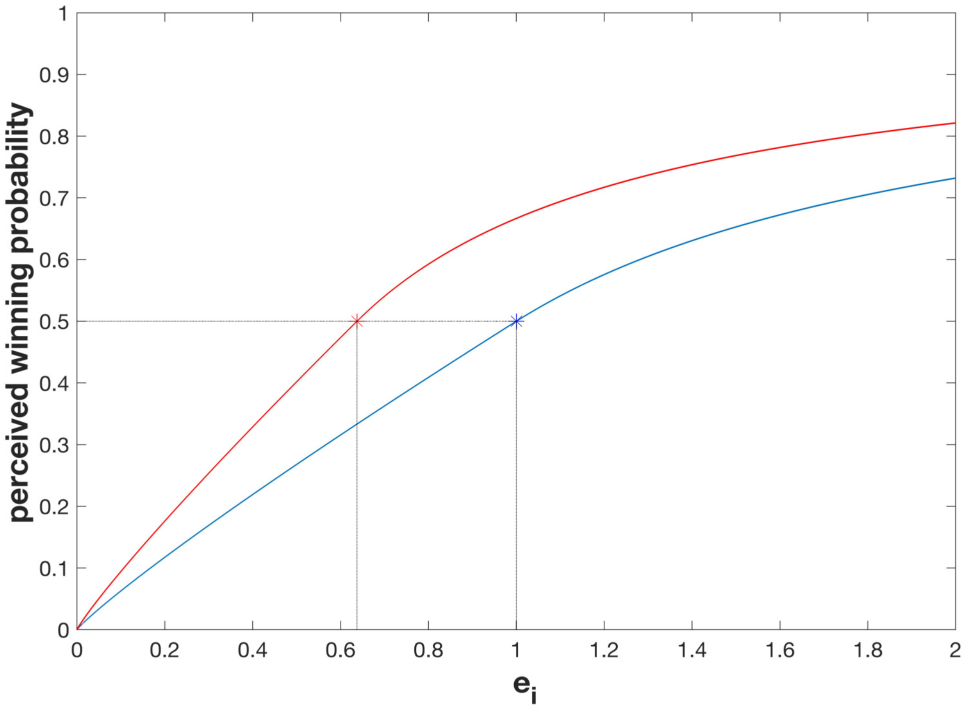

Figure 1: True and Perceived Winning Probabilities

The bias also affects an overconfident player's perceived marginal winning probability. Letting $mg\widetilde{p}_{ij} = \partial\widetilde{p}_{ij}/\partial e_i$ denote player i's perceived marginal winning probability when paired with j, it follows from (1) that

$$mg\widetilde{p}_{ij} = \begin{cases} \frac{\alpha}{2\lambda_i}\frac{e_j^\alpha}{e_i^{\alpha+1}} & \text{if} \quad \lambda_i e_i^\alpha \geqslant e_j^\alpha \\ \frac{\alpha}{2}\lambda_i\frac{e_i^{\alpha-1}}{e_j^\alpha} & \text{if} \quad \lambda_i e_i^\alpha \leqslant e_j^\alpha \end{cases} \tag{2}$$

On Figure 2, we depict the true (solid blue curve) and the perceived (solid red curve) marginal winning probabilities of an overconfident player with $\lambda_i = 1.5$. As before, we set $\alpha = 0.9$ and $e_j = 1$. We see that the true and perceived marginal winning probabilities are decreasing in effort. Furthermore, the perceived marginal winning probability of an overconfident player is less (greater) than his true marginal winning probability when his effort is high (low).

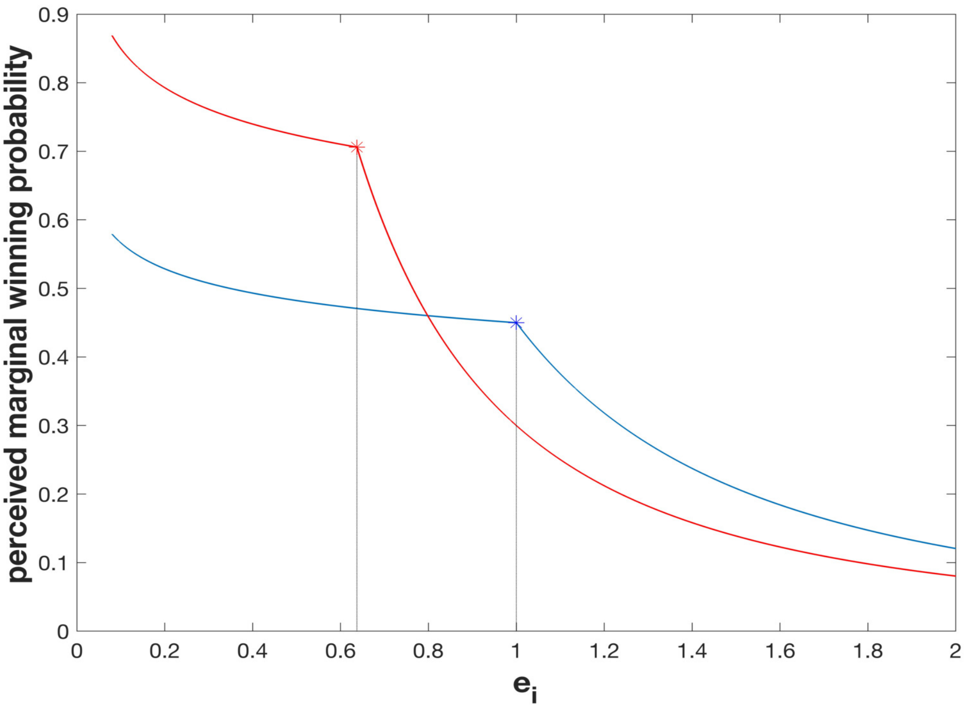

Figure 2: True and Perceived Marginal Winning Probabilities

The solution concept is Subgame Perfect Nash Equilibrium. We solve the elimination contest via backwards induction and determine the equilibrium of the second-stage (the final) before we determine equilibrium in the first-stage (the semifinals). To be able to compute the equilibrium taking into account that players can hold mistaken beliefs we assume: (i) a player who faces a biased opponent is aware that the latter's perception (and probability of winning) is mistaken, (ii) each player thinks that his own perception (and probability of winning) is correct, and (iii) both players have a common understanding of each other's beliefs, despite their disagreement on the accuracy of their opponent's beliefs. Hence, players agree to disagree about their perceptions (and winning probabilities). This approach follows Heifetz et al. (2007a, 2007b) for games with complete information, and Squintani (2006) for games with incomplete information.[4] Finally, we assume that each player not only knows the confidence level of his direct rival in the semifinal but also the confidence levels of the other two potential rivals in the other semifinal.[5]

In a final between players i and j, player i chooses the level of effort e_i that maximizes his perceived expected utility:

$$\widetilde{E}^f(U_{ij}) = \widetilde{p}_{ij}^f u(w_1) + (1 - \widetilde{p}_{ij}^f) u(w_2) - ce_i$$

subject to the participation constraint $\widetilde{E}^f(U_{ij}) \geq 0$, where $\widetilde{p}_{ij}^f$ is player i's perceived winning probability in a final against j. Hence, the first-order condition of player i in a final against j is given by:

$$mg\widetilde{p}_{ij}^f [u(w_1) - u(w_2)] = c. \tag{3}$$

Since

$$\frac{\partial^2 \widetilde{p}_{ij}^f}{\partial e_i^2} < 0,$$

[4]These assumptions are consistent with the psychology literature on the "Blind Spot Bias" according to which individuals believe that others are more susceptible to behavioral biases than themselves (Pronin et al. 2002, Pronin and Kugler 2007).

[5]In Section 8 we discuss what happens when overconfidence is unobservable.

we have

$$\frac{\partial^2 \widetilde{E}^f(U_{ij})}{\partial e_i^2} < 0,$$

and the second-order condition is satisfied.

Let $R_i^f(e_j)$ denote player i's best response in the final obtained from (3). Lemma 1 describes the shape of player i's best response in the final.

Lemma 1 *$R_i^f(e_j)$ is quasi-concave in e_j and reaches a maximum at $\lambda_i e_i^\alpha = e_j^\alpha$.*

Lemma 1 tells us that the players' best responses in the final are non-monotonic. Given high effort of the rival, a player reacts to an increase in effort of the rival by decreasing effort; given low effort of the rival, a player reacts to an increase in effort of the rival by increasing effort.

Lemma 2 describes how player i's best response in the final changes with his overconfidence parameter λ_i.

Lemma 2 *An increase in player i's overconfidence λ_i leads to a contraction of his best response in the final, $\partial R_i^f/\partial\lambda_i < 0$, for $e_j^\alpha < \lambda_i e_i^\alpha$, and to an expansion of his best response in the final, $\partial R_i^f/\partial\lambda_i > 0$, for $e_j^\alpha > \lambda_i e_i^\alpha$. Moreover, the maximum value of player i's best response in the final is independent of player i's overconfidence.*

Lemma 2 characterizes how overconfidence shifts a player's best response in the final. For a high effort of the rival, an increase in confidence raises player i's effort level, while for low effort of the rival, an increase in confidence lowers player i's effort level. Moreover, the maximal value taken by player i's best response in the final is independent of his overconfidence bias.

Now consider the semifinals stage. Let players i and h be paired up in one semifinal and players j and k be paired up in the other semifinal. If player i wins his semifinal, then i faces j in the final with probability p_{jk}^s and k in the final with probability $p_{kj}^s = 1 - p_{jk}^s$. Hence, player i's perceived expected utility of the final before knowing the identity of his rival in the final is $p_{jk}^s \widetilde{E}^f(U_{ij}) + p_{kj}^s \widetilde{E}^f(U_{ik})$. Therefore, player i's perceived benefit of winning his semifinal, or player i's perceived continuation value $\widetilde{v}_i$, is:

$$\widetilde{v}_i = p_{jk}^s \widetilde{E}^f(U_{ij}) + p_{kj}^s \widetilde{E}^f(U_{ik}).$$

In the semifinal between players i and h, player i chooses the level of effort e_i that maximizes his perceived expected utility:

$$\widetilde{E}^s(U_{ih}) = \widetilde{p}_{ih}^s \widetilde{v}_i - ce_i.$$

subject to the participation constraint $\widetilde{E}^s(U_{ih}) \geq 0$, where $\widetilde{p}_{ih}^s$ is player i's perceived winning probability in a semifinal against h. Hence, the first-order condition of player i in a semifinal against h is given by:

$$mg\widetilde{p}_{ih}^s \widetilde{v}_i = c. \tag{4}$$

Since

$$\frac{\partial^2 \widetilde{p}_{ih}^s}{\partial e_i^2} < 0,$$

we have

$$\frac{\partial^2 \widetilde{E}^s(U_{ih})}{\partial e_i^2} < 0,$$

and the second-order condition is satisfied.

Let $R_i^s(e_h)$ denote player i's best response in the semifinal obtained from (4). It can be inferred from equations (3) and (4) that the shape of player i's best response in the semifinal closely mirrors that of his best response in the final. However, Lemma 3 shows that the impact of overconfidence on player i's best response in the semifinal diverges from the final.

Lemma 3 *If player i's perceived continuation value $\widetilde{v}_i$ increases in his overconfidence λ_i, $\partial\widetilde{v}_i/\partial\lambda_i > 0$, then an increase in player i's overconfidence leads to a contraction of his best response in the semifinal, $\partial R_i^s/\partial\lambda_i < 0$, for $e_j^\alpha < \lambda_i e_i^\alpha$ and $\partial\widetilde{v}_i/\partial\lambda_i < \widetilde{v}_i/\lambda_i$, otherwise, it leads to an expansion of his best response in the semifinal, $\partial R_i^s/\partial\lambda_i > 0$. Moreover, the maximum value of player i's best response in the semifinal increases in player i's overconfidence.*

Lemma 3 characterizes how overconfidence shifts a player's best response in the semifinal. It shows that if player i's perceived continuation value increases in overconfidence, then an increase in player i's overconfidence contracts his best response in the semifinal for low effort of the rival when the elasticity of the perceived continuation value with respect to bias is smaller than 1. Otherwise, an increase in player i's overconfidence expands his best response in the semifinal. Moreover, if player i's perceived continuation value increases in his overconfidence, then the maximal value taken by player i's best response in the semifinal is increasing in his overconfidence bias.

Let P_i denote player i's probability of winning the elimination contest. P_i is the product of player i's probability of winning his semifinal and winning the final before knowing the identity of his rival in the final:

$$P_i = p_{ih}^s(p_{jk}^s p_{ij}^f + p_{kj}^s p_{ik}^f).$$

We denote the four players as 1, 2, 3, and 4 from now on. Players 1 and 2 are paired in one semifinal and players 3 and 4 are paired in the other semifinal.

3 Four Rational Players

This section characterizes the equilibrium of an elimination contest with four rational players. This serves as a benchmark to which we compare all our results.

Proposition 1 *In a final with two rational players, the equilibrium effort is*

$$\overline{e}^f = \frac{\alpha}{2c}[u(w_1) - u(w_2)],$$

the equilibrium winning probability is

$$\overline{p}^f = \frac{1}{2},$$

and the equilibrium expected utility is

$$\overline{E}^f(U) = \frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2).$$

In a semifinal of a two-stage elimination contest with four rational players, the equilibrium effort is

$$\overline{e}^s = \frac{\alpha}{2c}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right],$$

the equilibrium winning probability is

$$\overline{p}^s = \frac{1}{2},$$

and the equilibrium expected utility is

$$\overline{E}^s(U) = \frac{1-\alpha}{2}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right].$$

In a final featuring two rational players, equilibrium is reached when both exert the same effort, resulting in an equal winning probability for each player. The equilibrium effort in the final increases in $u(w_1) - u(w_2)$, in α, and decreases in c. Similarly, in a semifinal featuring two rational players, equilibrium is reached when both exert the same effort, resulting in an equal winning probability for each player. The equilibrium effort in the semifinal is smaller than the equilibrium effort in the final for all $\alpha \in (0, 1]$ when $u(w_1) > 3u(w_2)$. Finally, we have

$$\overline{P} = \overline{p}^s \overline{p}^f = \frac{1}{4}$$

When all players are rational the elimination contest is symmetric and hence each has 1/4 probability of being the winner.

4 One Overconfident Player and Three Rational Players

This section characterizes the equilibrium of an elimination contest with one overconfident player and three rational players. Throughout we assume player 1 is overconfident with $\lambda_1 > 1$ and players 2, 3, and 4 are rational with $\lambda_2 = \lambda_3 = \lambda_4 = 1$.

4.1 Final

We start by analyzing the impact of overconfidence on the final. Since players 3 and 4 are identical, we consider a final with an overconfident player 1 and a rational player 3 without loss of generality.

Proposition 2 *In a final between an overconfident player and a rational player, the equilibrium effort of the overconfident player is*

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)],$$

and the equilibrium effort of the rational player is

$$e_3^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)]$$

with $e_1^f < e_3^f < \overline{e}^f$. The perceived equilibrium winning probability of the overconfident player is

$$\widetilde{p}_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}},$$

and the true equilibrium winning probabilities are

$$p_{13}^f = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}$$

$$p_{31}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}$$

with $\widetilde{p}_{13}^f > p_{31}^f > 1/2 > p_{13}^f$. *The perceived equilibrium expected utility of the overconfident player is*

$$\widetilde{E}^f(U_{13}) = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]$$

and the equilibrium expected utility of the rational player is

$$E^f(U_{31}) = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)],$$

with $\widetilde{E}^f(U_{13}) > E^f(U_{31}) > \overline{E}^f(U)$.

Proposition 2 shows that an overconfident player always exerts less effort in a final than a rational rival. This result is intuitive. In the final, the overconfident player chooses the level of effort at which the perceived marginal benefit of effort equals the marginal cost and the rational player chooses the level of effort at which the marginal benefit of effort equals the marginal cost. Hence, in the final, the players' equilibrium efforts satisfy

$$mg\widetilde{p}_{13}^f[u(w_1) - u(w_2)] = c,$$

and

$$mgp_{31}^f[u(w_1) - u(w_2)] = c.$$

An increase in confidence leads to a drop in the overconfident player's perceived marginal probability of winning the final $mg\widetilde{p}_{13}^f$. As a result, he lowers effort to save on costs of effort. The rational player, anticipating the overconfident player will lower his effort, also lowers her effort but not as much. Hence, both players exert lower effort than if both were rational. At equilibrium, the perceived winning probability of the overconfident player is greater than 1/2 whereas his true winning probability is less than 1/2 given the lower equilibrium effort. Intuitively, the overconfident player, given his (mis)perceived advantage, thinks, mistakenly, he can reduce his effort without endangering his prospects of success.

Proposition 2 also shows that the overconfident player's perceived equilibrium expected utility of the final increases in his bias. This happens due to two channels. First, an increase in the bias raises the overconfident player's perceived probability of winning the final. Second, an increase in the bias lowers the overconfident player's cost of effort. Furthermore, Proposition 2 shows that the rational player's equilibrium expected utility of the final increases in the overconfident player's bias. An increase in the overconfident player's bias raises the rational player's probability of winning the final and lowers his cost of effort. Hence, an increase in an overconfident player's bias makes reaching the final more attractive not only for the overconfident player but also for a rational rival. Note also that, at equilibrium, the overconfident player's perceived expected utility of the final is greater than that of the rational player.[6]

Figure 3 illustrates Proposition 2. It depicts the best responses and equilibrium efforts in a final where $\alpha = 0.9$ and $[u(w_1) - u(w_2)]/c = 2$. The best response of a rational player 1 is depicted in solid red and the best response of a rational player 3 in solid blue. The equilibrium when players 1 and 3 are rational is depicted by point E at the 45 degree

[6] As the overconfident player's bias converges to infinity, the efforts of both players converge to zero, the overconfident player's perceived probability of winning the final converges to 1, his true probability of winning the final converges to zero, his perceived equilibrium expected utility of the final converges to the winner's prize $u(w_1)$ and so does the rational player's equilibrium expected utility of the final.

line. The best response of an overconfident player 1 with $\lambda_1 = 1.2$ is depicted in dotted red. The equilibrium when player 1 is overconfident and player 3 is rational is depicted by point E' above the 45 degree line. In the final, overconfidence shifts the best response of player 1 inwards for low values of effort of player 3 and outwards for high values of effort of player 3.

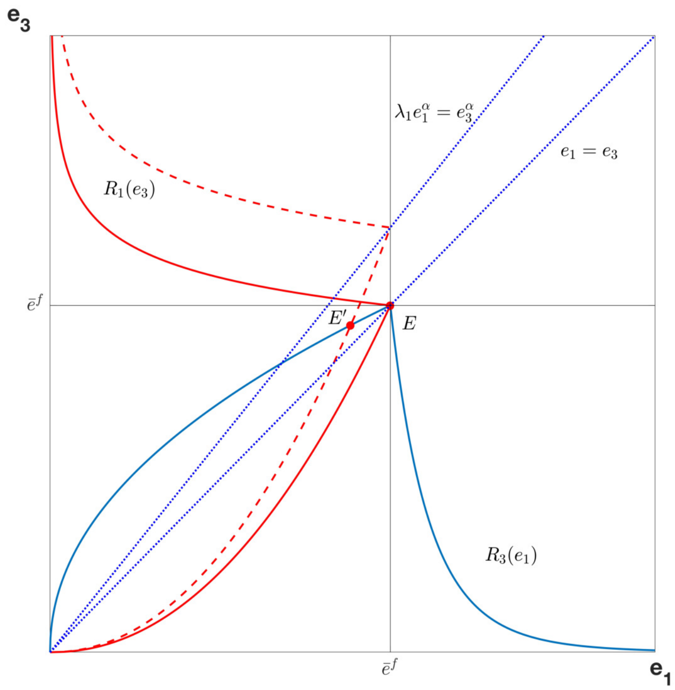

Figure 3: Best Responses and Equilibrium Efforts in the Final

4.2 Semifinals

We now analyze the impact of overconfidence on the two semifinals. We start with the semifinal with overconfident player 1 and rational player 2. Next, we consider the semifinal with rational players 3 and 4.

Proposition 3 *Consider a semifinal between an overconfident player and a rational player of a two-stage elimination contest where player 1 is overconfident and players 2, 3, and 4 are rational.*
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ *and* $\lambda_1 < \hat{\lambda}$ *where* $\hat{\lambda}$ *is given by* $\frac{1+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_1)} = \frac{\hat{\lambda}-1}{\hat{\lambda}-\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}}$, *then the equilibrium efforts and winning probabilities satisfy* $e_1^s > \bar{e}^s > e_2^s$ *and* $\widetilde{p}_{12}^s > p_{12}^s > 1/2 > p_{21}^s$.
(ii) If either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ *or* $\lambda_1 \geqslant \hat{\lambda}$, *then the equilibrium efforts and winning probabilities satisfy* $e_1^s \leqslant e_2^s \leqslant \bar{e}^s$ *and* $\widetilde{p}_{12}^s > p_{21}^s \geqslant \frac{1}{2} \geqslant p_{12}^s$.

Proposition 3 shows that an overconfident player exerts higher effort in a semifinal than a rational rival when the prize spread is sufficiently large, $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, and overconfidence is not too extreme, $\lambda_1 < \hat{\lambda}$. When either of these two conditions is not met, the overconfident player exerts less effort in the semifinal than a rational rival. This result, together with Proposition 2, provides the model's main testable implication, namely, that an overconfident player always exerts less effort in a final than a rational rival but can exert higher effort in semifinal than a rational rival.

In a semifinal, the overconfident player chooses the level of effort at which the perceived marginal benefit $mg\widetilde{p}_{12}^{s}\widetilde{v}_1$ equals the marginal cost c, and the rational player chooses the level of effort at which the marginal benefit $mgp_{21}^{s}v_2$ equals the marginal cost c. Hence, in a semifinal, the players' equilibrium efforts satisfy

$$mg\widetilde{p}_{12}^{s}\widetilde{v}_1 = c, \tag{5}$$

and

$$mgp_{21}^{s}v_2 = c. \tag{6}$$

Differentiating (5) and (6) and solving for $\partial e_1^s/\partial\lambda_1$ we find how the overconfident player's equilibrium effort changes in his bias

$$\frac{\partial e_1^s}{\partial\lambda_1} = \frac{mg\widetilde{p}_{12}^{s}}{\lambda_1}\frac{\frac{\partial mg\widetilde{p}_{12}^{s}}{\partial\lambda_1}\frac{\lambda_1}{mg\widetilde{p}_{12}^{s}} + \frac{\partial\widetilde{v}_1}{\partial\lambda_1}\frac{\lambda_1}{\widetilde{v}_1}}{-\frac{\partial mg\widetilde{p}_{12}^{s}}{\partial e_1} + \frac{\partial mg\widetilde{p}_{12}^{s}}{\partial e_2}\frac{\frac{\partial mgp_{21}^{s}}{\partial e_1}}{\frac{\partial mgp_{21}^{s}}{\partial e_2}}}, \tag{7}$$

where the sign of the denominator in equation (7) is positive.[7] In equation (7), the term

$$\varepsilon_{mg\widetilde{p}_{12}^{s},\lambda_1} = \frac{\partial mg\widetilde{p}_{12}^{s}}{\partial\lambda_1}\frac{\lambda_1}{mg\widetilde{p}_{12}^{s}}, \tag{8}$$

represents the elasticity of the perceived marginal probability of winning the semifinal with respect to the bias. Meanwhile, the term

$$\varepsilon_{\widetilde{v}_1,\lambda_1} = \frac{\partial\widetilde{v}_1}{\partial\lambda_1}\frac{\lambda_1}{\widetilde{v}_1}, \tag{9}$$

represents the elasticity of the perceived continuation value with respect to the bias. Hence, the sign of $\partial e_1^s/\partial\lambda_1$ depends on the signs of these two elasticities.

The proof of Proposition 3 shows that at equilibrium $\lambda_1 e_1^{\alpha} \geqslant e_3^{\alpha}$ which, together with (2), implies that the elasticity of the perceived marginal probability of winning the semifinal with respect to the bias is

$$\varepsilon_{mg\widetilde{p}_{12}^{s},\lambda_1} = -\frac{\alpha}{2\lambda_1^2}\frac{e_2^{\alpha}}{e_1^{\alpha+1}}\frac{\lambda_1}{\frac{\alpha}{2\lambda_1}\frac{e_2^{\alpha}}{e_1^{\alpha+1}}} = -1. \tag{10}$$

The overconfident player's perceived continuation value is

$$\widetilde{v}_1 = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]$$

Hence, the elasticity of the perceived continuation value with respect to the bias is

$$\varepsilon_{\widetilde{v}_1,\lambda_1} = \frac{(\alpha+1)^2}{2(2\alpha+1)}\frac{1}{\frac{u(w_1)}{u(w_1)-u(w_2)}\lambda_1^{\frac{\alpha+1}{2\alpha+1}} - \frac{1+\alpha}{2}}. \tag{11}$$

We see from (10) and (11) that the bias has two effects on the overconfident player's incentives to exert effort in a semifinal. On the one hand, an increase in the bias lowers the overconfident player's perceived marginal probability of winning the semifinal which motivates him to lower effort. We label this the complacency effect of overconfidence.

[7] In the Appendix we derive equation (7) and show that its denominator is positive.

On the other hand, an increase in the bias raises the overconfident player's perceived continuation value which motivates him to raise effort. We label this the encouragement effect of overconfidence. When $\varepsilon_{\widetilde{v}_1,\lambda_1} > (<) 1$ the encouragement (complacency) effect dominates, and the overconfident player exerts higher (lower) effort in the semifinal than the rational player.

Equation (10) shows that the size of the complacency effect is -1 and equation (11) shows that the size of the encouragement effect decreases in the bias and converges to zero when the bias converges to infinity (as $\lambda_1 \to \infty$ we have $\widetilde{v}_1 \to u(w_1)$). Hence, when the bias is small, i.e., λ_1 is close to 1, a necessary condition for the encouragement effect to dominate is that $\varepsilon_{\widetilde{v}_1,\lambda_1} > 1$ at $\lambda_1 = 1$. Setting $\lambda_1 = 1$ in (11) this is equivalent to $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$. Thus, the encouragement effect dominates when the prize spread is sufficiently large and the bias is close to 1. As the bias increases, the size of the encouragement effect decreases and converges to zero while the size of the complacency effect is fixed at -1. Hence, there exists an upper bound for the bias above which the complacency effect dominates. Figure 4 depicts in light green the values of $\left(\alpha, \frac{u(w_1)-u(w_2)}{u(w_2)}\right)$ that satisfy $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$.

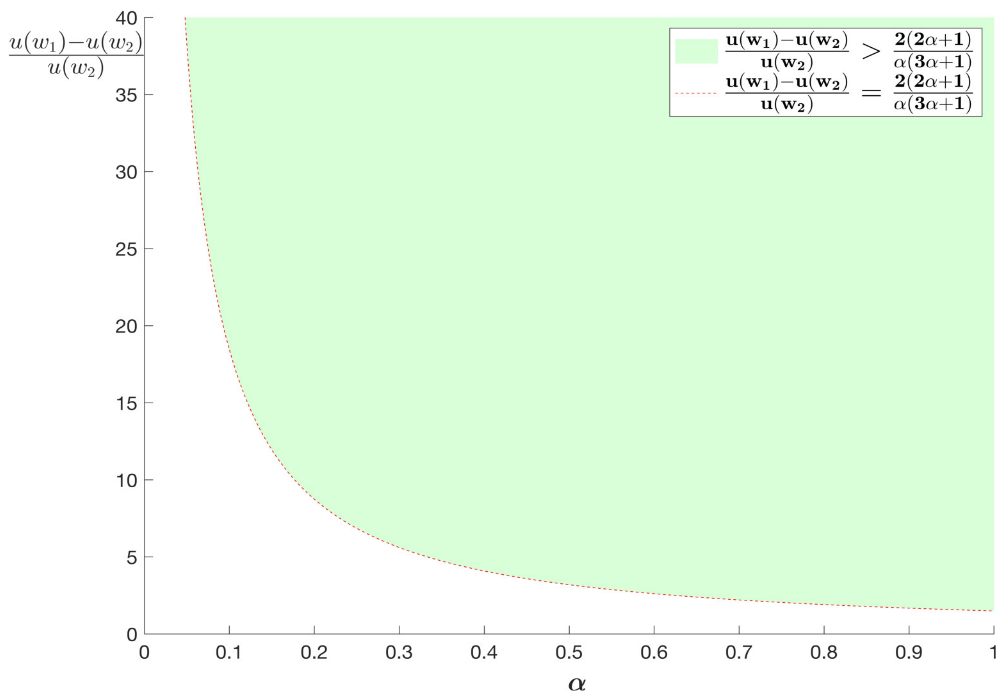

Figure 4: Values of $\left(\alpha, \frac{u(w_1)-u(w_2)}{u(w_2)}\right)$ where $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$

Figure 5 illustrates result (i) in Proposition 3. It depicts the best responses and equilibrium efforts in a semifinal of an elimination contest where $u(w_1) = 11.25$, $u(w_2) = 1.25$, $\alpha = 0.9$, and $c = 1$. Note that these parameter values imply $\hat{\lambda} = 4.4606$ and satisfy $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$. The best response of a rational player 1 is depicted in solid red and that of a rational player 2 in solid blue. Point E at the 45 degree line depicts the equilibrium when players 1 and 2 are rational. The best response of an overconfident player 1 with $\lambda_1 = 1.1$ is depicted in dashed red. Point point E' below the 45 degree line depicts the equilibrium when player 1 is overconfident with $\lambda_1 = 1.1$ and player

2 is rational. Since the encouragement effect dominates, overconfidence shifts the best response of player 1 outwards for all values of effort of player 2.

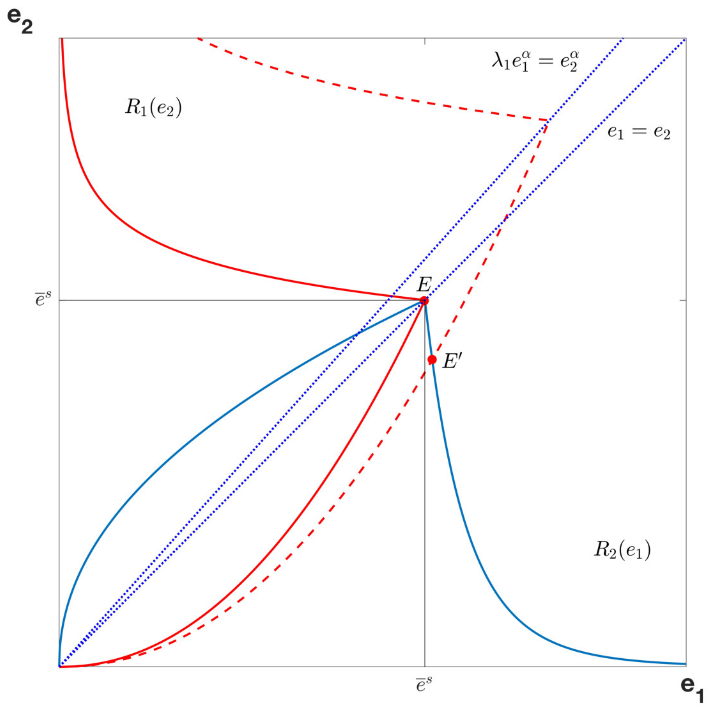

Figure 5: Best Responses and Equilibrium Efforts in a Semifinal where $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

Figure 6 illustrates result (ii) in Proposition 3. It depicts the best responses and equilibrium efforts in a semifinal of an elimination contest where $u(w_1) = 11.25$, $u(w_2) = 1.25$, $\alpha = 0.9$, and $c = 1$. The best response of an overconfident player 1 with $\lambda_1 = 7$ is depicted in dashed red. Point point E' above the 45 degree line depicts the equilibrium when player 1 is overconfident with $\lambda_1 = 7$ and player 2 is rational. Since the complacency effect dominates, overconfidence shifts the best response of player 1 inwards for low values of effort of player 2 and outwards for high values of effort of player 2.

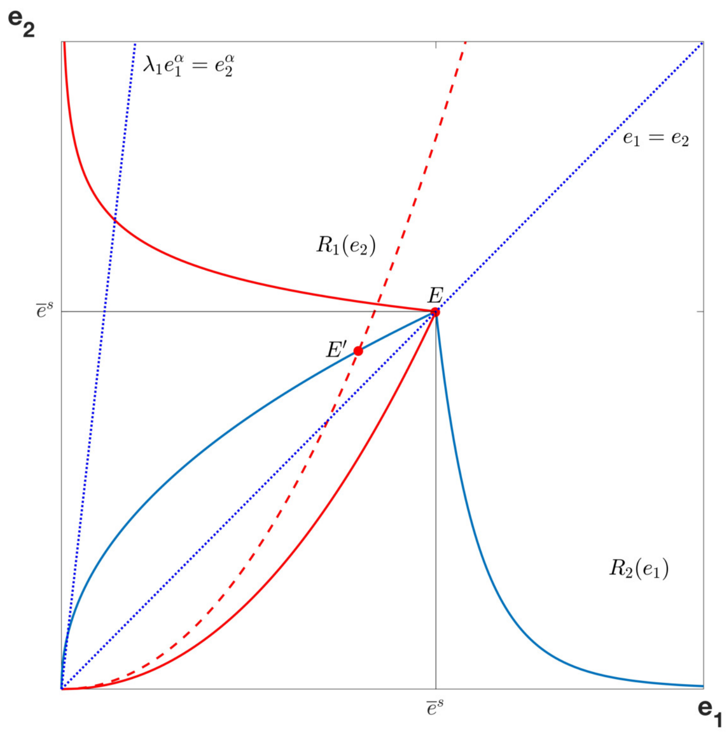

Figure 6: Best Responses and Equilibrium Efforts in a Semifinal where either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

Next, we characterize the equilibrium of the semifinal with rational players 3 and 4.

Proposition 4 *In a semifinal between two rational players of a two-stage elimination contest where player 1 is overconfident and players 2, 3, and 4 are rational, the equilibrium efforts and winning probabilities satisfy* $e_3^s = e_4^s > \overline{e}^s$ *and* $p_{34}^s = p_{43}^s = 1/2$.

Proposition 4 shows that the presence of an overconfident player in one of the semifinals of an elimination contest has a spillover effect on the equilibrium effort in the other semifinal. In the semifinal with two rational players, both players has an incentive to exert higher effort when player 1 is overconfident than if player 1 were rational. This happens because the winner of the semifinal with two rational players will face the overconfident player 1 in the final with probability p_{12}^s and this leads to a higher continuation value than if player 1 were rational.

4.3 Equilibrium Winning Probabilities

We now consider how overconfidence affects each player's equilibrium probability of winning the elimination contest. We are interested in knowing whether the overconfident player 1 can have the brighter prospects throughout the elimination contest. The winning probability of overconfident player 1 is

$$P_1 = p_{12}^s(p_{34}^s p_{13}^f + p_{43}^s p_{14}^f) = p_{12}^s p_{13}^f,$$

where the second equality follows from $p_{13}^f = p_{14}^f$. The winning probability of rational player 2 is

$$P_2 = p_{21}^s(p_{34}^s p_{23}^f + p_{43}^s p_{24}^f) = p_{21}^s \frac{1}{2},$$

where the second equality follow from $p_{34}^s + p_{43}^s = 1$ and $p_{23}^f = p_{24}^f = 1/2$. The winning probabilities of rational players 3 and 4 are

$$P_3 = P_4 = p_{34}^s(p_{12}^s p_{31}^f + p_{21}^s p_{32}^f) = \frac{1}{2}\left[p_{12}^s(1 - p_{13}^f) + (1 - p_{12}^s)\frac{1}{2}\right].$$

Since the overconfident player 1 has an equilibrium probability of winning the final p_{13}^f that is less than 1/2, a necessary condition for him to have the highest equilibrium winning probability is that his equilibrium probability of winning the semifinal p_{12}^s is greater than 1/2. In other words, the overconfident player must exert higher effort in his semifinal than the rational player 2. When this is the case, rational player 2 has an equilibrium probability of winning which is less than 1/4 since $p_{21}^s = 1 - p_{12}^s < 1/2$. Regardless of the identity of the winner of the semifinal between players 1 and 2, rational players 3 and 4 have an equilibrium winning probability greater than 1/4 since they have a positive probability of facing the overconfident player 1 in the final. Hence, in equilibrium, the overconfident player 1 has the highest winning probability when $P_1 > P_3 = P_4$ which, from the equations above, is equivalent to $6p_{12}^s p_{13}^f - p_{12}^s - 1 > 0$.

Proposition 5 *In a two-stage elimination contest where player 1 is overconfident and players 2, 3, and 4 are rational, if* $\alpha > \frac{\sqrt{97}-5}{12}$ *and* $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(4\alpha+5)}{6\alpha^2+5\alpha-3}$, *then there exist* $\lambda_1 \in (1, \hat{\lambda})$ *for which the overconfident player has the highest equilibrium winning probability, i.e.,* $P_1 > P_3 = P_4 > 1/4 > P_2$.

Proposition 5 shows that, in equilibrium, the overconfident player can have the highest probability of winning the elimination contest. Moreover, for this to be the case three conditions need to be met. First, the CSF's effort sensitivity parameter α must be greater than $\frac{\sqrt{97}-5}{12} \approx 0.404$. Hence, the role that effort plays in determining the winning probabilities must be sufficiently high. Second, the utility prize spread needs to be sufficiently large, $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(4\alpha+5)}{6\alpha^2+5\alpha-3}$. Third, the overconfidence bias must be low. Figure 7 depicts in light green the values of $\left(\alpha, \frac{u(w_1)-u(w_2)}{u(w_2)}\right)$ that satisfy the two inequalities in Proposition 5.

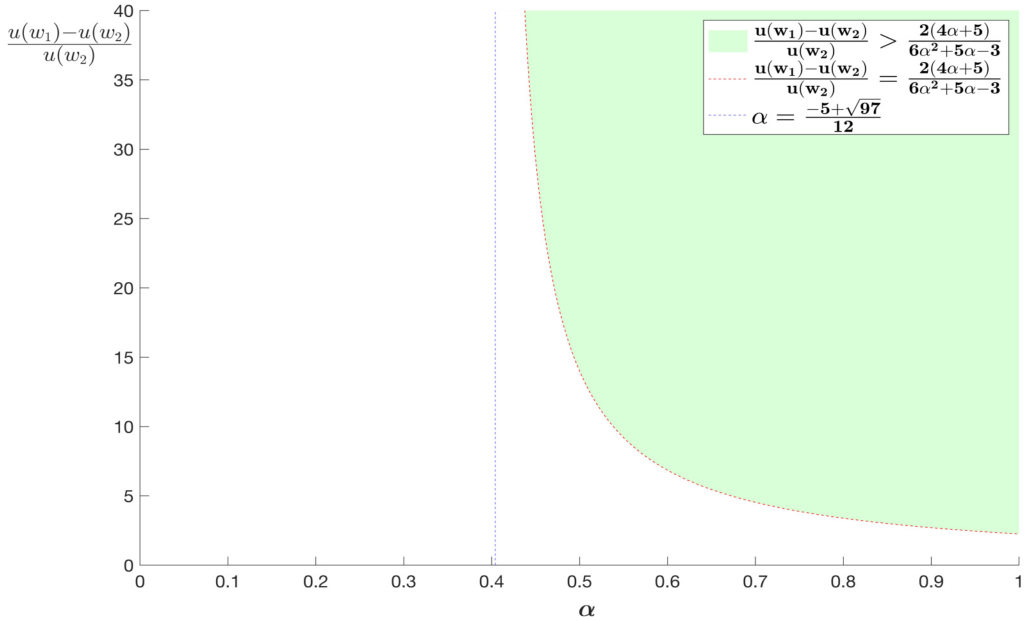

Figure 7: Values of $\left(\alpha, \frac{u(w_1)-u(w_2)}{u(w_2)}\right)$ where $\alpha > \frac{\sqrt{97}-5}{12}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(4\alpha+5)}{6\alpha^2+5\alpha-3}$

5 Two Overconfident Players and Two Rational Players

This section studies the equilibrium of an elimination contest with two overconfident players and two rational players. We assume the two overconfident players differ in their confidence levels. This setup enables us to assess if our prior findings remain applicable when two overconfident players encounter each other, either in the final or the semifinal.

Now, there are two possible seedings: (i) the overconfident players are seeded in the same semifinal, and (ii) the overconfident players are seeded in different semifinals. These two types of seeding induce different results and hence we study them separately.

5.1 Final

When the overconfident players are seeded in the same semifinal, the final will be played between an overconfident and a rational player and we can apply Proposition 2. In contrast, when the overconfident players are seeded in different semifinals, the final can have two overconfident players. Hence, we start by characterizing the equilibrium of a final with two overconfident players. Without loss of generality we consider a final between players 1 and 3 with $\lambda_1 > \lambda_3 > 1$.

Proposition 6 *In a final between two overconfident players, the equilibrium effort of the more overconfident player is*

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)]$$

and the equilibrium effort of the less overconfident player is

$$e_3^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]$$

with $e_1^f < e_3^f < \overline{e}^f$. *The perceived equilibrium winning probabilities are*

$$\widetilde{p}_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}$$

$$\widetilde{p}_{31}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}$$

with $\widetilde{p}_{13}^f > \widetilde{p}_{31}^f > 1/2$. *The equilibrium winning probabilities are*

$$p_{13}^f = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{\frac{\alpha}{2\alpha+1}}$$

$$p_{31}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{\frac{\alpha}{2\alpha+1}}$$

with $p_{13}^f < 1/2 < p_{31}^f$. *The perceived equilibrium expected utilities are*

$$\widetilde{E}^f(U_{13}) = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)],$$

$$\widetilde{E}^f(U_{31}) = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)],$$

with $\widetilde{E}^f(U_{13}) > \widetilde{E}^f(U_{31}) > \overline{E}^f(U)$.

Proposition 6 shows that in a final between two overconfident players, the more overconfident player exerts lower effort at equilibrium. As we have seen, the bias lowers an overconfident player's perceived marginal probability of winning the final. The more overconfident a player is, the higher is the drop in his perceived marginal probability of winning the final. Hence, the more overconfident player exerts lower effort at equilibrium. Both players exert lower effort than if both were rational. Each player perceives he has a winning probability greater than 1/2 but, in fact, only the less overconfident player has a true winning probability greater than 1/2. The perceived expected utility of each player is increasing in his own bias as well as in the rival's bias.

5.2 Overconfident players seeded in the same semifinal

Assume players 1 and 2, seeded in one semifinal, are overconfident with $\lambda_1 > \lambda_2 > 1$ and players 3 and 4, seeded in the other semifinal, are rational with $\lambda_3 = \lambda_4 = 1$. Note that, under this seeding, the final will involve an overconfident and a rational player and hence we can apply Proposition 2. Note also that since the two rational players are identical, they exert equal efforts in the semifinal and hence, each has an equal probability of winning it. This means that the identity of winner of the semifinal between two rational players does not affect the overconfident players' behavior in their semifinal. However, since the overconfident players' biases differ, the identity of winner of the semifinal between two overconfident players matters for the effort choices of the rational players in their semifinal. Taking this into account, we start by solving the equilibrium of the semifinal with two overconfident players.

Proposition 7 *Consider the semifinal between two overconfident players of a two-stage elimination contest where the overconfident players 1 and 2 are seeded in one semifinal, the rational players 3 and 4 are seeded in the other semifinal, and* $\lambda_1 > \lambda_2 > 1 = \lambda_3 = \lambda_4$.

(i) If $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$, *then the equilibrium efforts and winning probabilities satisfy* $e_1^s > e_2^s$, $e_1^s > \overline{e}^s$, *and* $\widetilde{p}_{12}^s > p_{12}^s > 1/2 \geqslant \widetilde{p}_{21}^s > p_{21}^s$.

(ii) If $\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}}-\frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}$*, then the equilibrium efforts and winning probabilities satisfy* $e_1^s > e_2^s > \overline{e}^s$ *and* $\widetilde{p}_{12}^s > p_{12}^s > 1/2 > p_{21}^s$.

(iii) If $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \max\left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}}-\frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}\right]$*, then the equilibrium efforts and winning probabilities satisfy* $e_1^s \leqslant e_2^s$ *and* $\widetilde{p}_{12}^s > \widetilde{p}_{21}^s > p_{21}^s \geqslant 1/2 \geqslant p_{12}^s$.

Proposition 7 reveals that in a semifinal with two overconfident players, both players can exert higher efforts than if both were rational. It also shows that the identity of the player who exerts the highest effort depends on the prize spread, on the confidence gap, $\lambda_1 - \lambda_2$, and the bias of the less overconfident player 2.

Part (i) tells us that the more overconfident player 1 exerts higher effort at equilibrium when the prize spread is large and the confidence gap is moderate.[8] Part (ii) tells us that the more overconfident player 1 exerts higher effort at equilibrium when the prize spread is moderate, the confidence gap is small, and the bias of the less overconfident player 2 is low. In this case both players exert higher effort than if both were rational.[9] Finally, part (iii) tells us that the less overconfident player 2 exerts higher effort at equilibrium when either the prize spread is small, or the confidence gap is large, or the confidence gap is small and the bias of the less overconfident player 2 is large.

Figure 8 illustrates result (ii) in Proposition 7. It depicts the best responses and equilibrium efforts in a semifinal of an elimination contest where $u(w_1) = 11$, $u(w_2) = 1$, $c = 1$, and $\alpha = 0.9$. Point E depicts the equilibrium when both players are rational. Point E' below the 45 degree line depicts the equilibrium when player 1 is overconfident with $\lambda_1 = 1.18$, and player 2 is overconfident with $\lambda_2 = 1.07$. These parameter values satisfy the two inequalities in (ii) and hence the more overconfident player 1 exerts higher effort at equuilibrium than the less overconfident player 2.

[8]When the confidence gap becomes increasingly large, i.e., $\lambda_1 \to \infty$, the right hand side of the inequality in part (i) converges to $\lambda_2^{-\frac{\alpha+1}{2\alpha+1}} < 1$. When the confidence gap becomes increasingly small, i.e., $\lambda_1 \to \lambda_2$, the right hand side of the inequality in part (i) also converges to $\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$ which is less than 1. Hence, since the left hand side of the inequality in part (i) is greater than 1, the inequality cannot be satisfied when the confidence gap is either too large or too small. These two limits are computed in Lemma 4 in the Appendix.

[9]When $\lambda_1 \to \infty$, the right hand side of the second inequality in part (ii) converges to $\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$ which is less than 1. Hence, since the left hand side of the second inequality in part (ii) is greater than 1, the second inequality in part (ii) cannot be satisfied when the confidence gap is large. When $\lambda_1 \to \lambda_2$, the left hand side of the first inequality in (ii) converges to $\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$ which is less than 1 and the right hand side of the second inequality converges to $\frac{3\alpha+2}{2\alpha+1}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$. Hence, the two inequalities in (ii) can be satisfied when the confidence gap becomes increasingly small as long as the bias of the less overconfident player 2 is low. These two limits are computed in Lemma 4 in the Appendix.

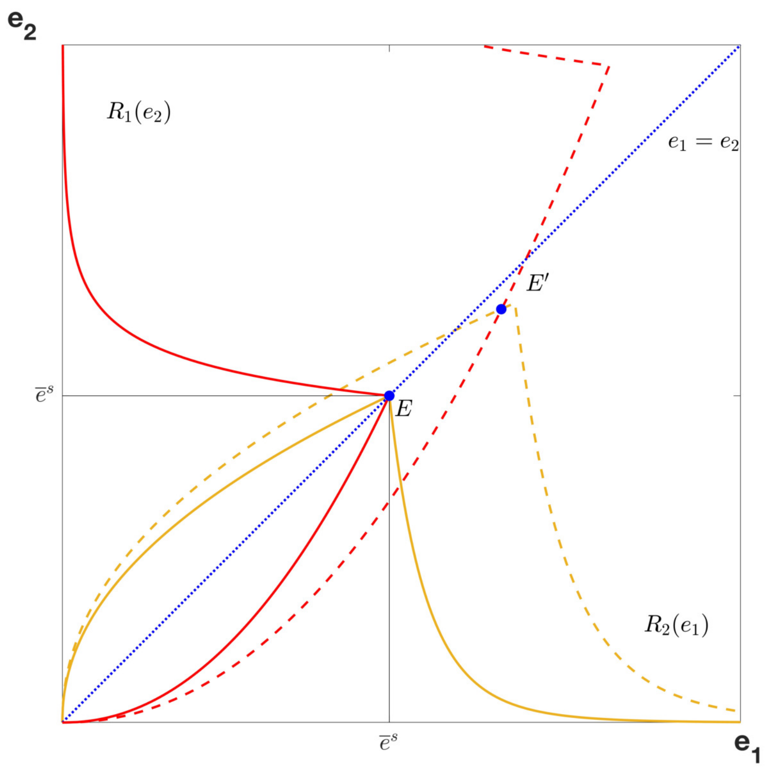

Figure 8: Best Responses and Equilibrium Efforts in a Semifinal with Two Overconfident Players

The equilibrium of the semifinal with two rational players is similar to that derived in Proposition 4 and is relegated to Lemma 5 in the Appendix. The main difference is that now the continuation value of the rational players takes into account the fact that the overconfident players exert different efforts and hence have different winning probabilities. Still, regardless of the identity of the winner of the semifinal between the two overconfident players, the rational players will have a higher continuation value than if all players were rational. Each rational player knows she will meet an overconfident player in the final which makes reaching the final more attractive. Hence, in the semifinal with two rational players, the equilibrium effort is higher than if all players were rational.

Proposition 7 also shows that, except for a knife-hedge parameter configuration, one of the two overconfident players has a probability of winning his semifinal that is greater than $1/2$. Moreover, some confidence gaps will generate quite large gaps between p^s_{12} and p^s_{21}. We also know that given the equal equilibrium effort, each rational player has an equal probability of winning his semifinal. This means that there will exist parameter configurations where an overconfident player is the one with the highest equilibrium probability of winning the elimination contest.

Hence, the findings obtained for an elimination contest with one overconfident and three rational players extend to an elimination contest where two overconfident players are seeded in one semifinal and two rational players are seeded in the other semifinal.

5.3 Overconfident players seeded in different semifinals

We continue to assume players 1 and 2 are seeded in one semifinal and players 3 and 4 are seeded in the other semifinal. However, we now assume players 1 and 3 are overconfident with $\lambda_1 > \lambda_3 > 1$, and players 2 and 4 are rational with $\lambda_2 = \lambda_4 = 1$.

In the semifinal between players 1 and 2, the continuation values are

$$\begin{aligned}\widetilde{v}_1 &= p_{34}^s \widetilde{E}^f(U_{13}) + (1-p_{34}^s)\widetilde{E}^f(U_{14})\\ &= u(w_1) - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1)-u(w_2)],\end{aligned}$$

and

$$\begin{aligned}v_2 &= p_{34}^s E^f(U_{23}) + (1-p_{34}^s)E^f(U_{24})\\ &= u(w_1) - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)[u(w_1)-u(w_2)].\end{aligned}$$

In the semifinal between players 3 and 4, the continuation values are

$$\begin{aligned}\widetilde{v}_3 &= p_{12}^s \widetilde{E}^f(U_{31}) + (1-p_{12}^s)\widetilde{E}^f(U_{32})\\ &= u(w_1) - \frac{1+\alpha}{2}\left(1 - p_{12}^s + p_{12}^s \lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1)-u(w_2)],\end{aligned}$$

and

$$\begin{aligned}v_4 &= p_{12}^s E^f(U_{41}) + (1-p_{12}^s)E^f(U_{42})\\ &= u(w_1) - \frac{1+\alpha}{2}\left(1 - p_{12}^s + p_{12}^s \lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)[u(w_1)-u(w_2)].\end{aligned}$$

The four expressions above shows us that the continuation values of players seeded in one semifinal depend on the equilibrium winning probabilities of players seeded in the other semifinal. As the equilibrium efforts of one semifinal cannot be solved separately from those of the other semifinal, the equilibrium efforts in the semifinals are jointly determined by the four first-order conditions $mg\widetilde{p}_{12}^s\widetilde{v}_1 = c$, $mgp_{21}^s v_2 = c$, $mg\widetilde{p}_{34}^s\widetilde{v}_3 = c$, and $mgp_{43}^s v_4 = c$.

Still, the findings in Proposition 3 can be applied to both semifinals. In other words, we know that in both semifinals there exist parameter configurations where the overconfident player exerts higher effort than the rational player. Our next result shows that this is indeed the case.

Proposition 8 *Consider the semifinals of a two-stage elimination contest where overconfident player 1 and rational player 2 are seeded in one semifinal, overconfident player 3 and rational player 4 are seeded in the other semifinal, and* $\lambda_1 > \lambda_3 > 1 = \lambda_2 = \lambda_4$. *If* $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ *and* $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-1}\left(\lambda_1 - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha}{\alpha+1}}$ *and* $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_3-1}\left(\lambda_3 - \lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha}{\alpha+1}}$, *then the equilibrium efforts and winning probabilities satisfy* $e_1^s > e_2^s$, $e_3^s > e_4^s$, $p_{12}^s > 1/2 > p_{21}^s$, *and* $p_{34}^s > 1/2 > p_{43}^s$.

Proposition 8 shows that in an elimination contest where two overconfident players are seeded in different semifinals, the overconfident players can exert higher effort at equilibrium than their rational rivals. This happens when the prize spread is sufficiently large and the overconfident players are not too confident. In this case, each overconfident player has a higher probability of winning his semifinal than his rational rival.

Hence, the results found for an elimination contest with one overconfident player and three rational players also extend to an elimination contest where one overconfident and one rational player are seeded in each semifinal.

6 Welfare

This section studies the effects of overconfidence on the welfare of the players and of the contest designer. The analysis focuses on a two-stage elimination contest with one overconfident and three rational players.

6.1 Players

To evaluate the impact of overconfidence on player i's welfare we consider how overconfidence changes player i's equilibrium expected utility in the semifinal with players i and h which is given by

$$E^s(U_{ih}) = p_{ih}^s \left[p_{ij}^f E^f(U_{ij}) + p_{ik}^f E^f(U_{ik}) \right] - c e_i^s,$$

where e_i^s is player i's equilibrium effort in the semifinal with h, p_{ih}^s is player i's equilibrium winning probability in the semifinal with h, and the term inside parenthesis is player i's equilibrium continuation value.

Proposition 9 *Consider a two-stage pairwise elimination contest with one overconfident player and three rational players.*
(i) If $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$*, then there exist* $\lambda_1 \in (1, \hat{\lambda})$ *for which* $E^s(U_{12}) > \overline{E}^s(U)$*.*
(ii) If $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ *and* $\lambda_1 < \hat{\lambda}$*, then* $E^s(U_{21}) < \overline{E}^s(U)$*, otherwise,* $E^s(U_{21}) \geqslant \overline{E}^s(U)$*.*
(iii) $E^s(U_{34}) = E^s(U_{43}) > \overline{E}^s(U)$*.*

Part (i) shows that if the prize spread is large enough, then there exist (small) overconfidence levels where the equilibrium expected utility of the overconfident player 1 in the semifinal with rational player 2 is higher than if all players were rational. Part (ii) shows that if the prize spread is large enough and overconfidence is not too extreme, then the equilibrium expected utility of the rational player 2 in the semifinal with the overconfident player 1 is lower than if all players were rational. Finally, part (iii) shows that the equilibrium expected utility of each of the rational players seeded in the same semifinal is higher than if all players are rational.

6.2 Contest Designer

We assume the welfare of the contest designer is increasing in the players' aggregate effort, the sum of the efforts in the two-stages of the contest.

Proposition 10 *Consider a two-stage pairwise elimination contest with one overconfident player and three rational players.*
(i) The equilibrium aggregate effort in a final with players i *and* j *satisfies* $e_i^f + e_j^f \leqslant 2\overline{e}^f$*.*
(ii) If $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ *and* $\alpha < \frac{1}{2}$*, then there exist* $\lambda_1 \in (1, \hat{\lambda})$ *such that the equilibrium aggregate effort in the semifinals stage satisfies* $\sum_{i=1}^{4} e_i^s > 4\overline{e}^s$*.*

Part (i) shows that the equilibrium aggregate effort in the final stage is less than or equal to the equilibrium aggregate effort in a final with two rational players. This result follows directly from Propositions 1 and 2. The impact of overconfidence on aggregate effort in the semifinals stage is harder to characterize. We know from Proposition 3 that in the semifinal with one overconfident and one rational player two situations can emerge. If the prize spread is high enough, $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, and overconfidence is not too extreme, $\lambda_1 < \hat{\lambda}$, the overconfident player's equilibrium effort is higher than if he were rational and the rational player's equilibrium effort is lower than if she were facing a rational rival. In all other cases, both players' equilibrium efforts are less than if all players were rational. We also know from Proposition 4 that in the semifinal with two rational players equilibrium efforts go up since both players have a higher continuation value of winning their semifinal. Still, part (ii) shows that overconfidence raises aggregate effort in the semifinals stage when the prize spread is high enough, $\alpha < 1/2$, and the overconfident player's bias is small. This result implies that it is unclear whether overconfidence lowers or raises players' aggregate efforts in the two stages. As a consequence, we are unable to provide general conditions under which overconfidence unambiguously lowers or raises the contest designer's welfare.

7 Optimal Prize Structure

This section analyzes how the contest designer should optimally choose the prize structure of the contest when players are overconfident. We assume the overall prize money is fixed and consider that the goal of the contest designer is the maximization of aggregate incentives, defined as the sum of efforts provided by all players across the two stages of the contest (this is a natural objective of the contest designer, see Sisak 2009). We also assume the four players are homogeneous, that is, all hold the same overconfidence bias. Finally, we assume W units are available as the total prize money, hence $W = w_1 + w_2$.

Proposition 11 *Consider a two-stage elimination contest with four identical players with an overconfidence bias equal to* $\lambda \geqslant 1$.
(i) If players are risk-neutral, then the optimal prize structure is the winner-take-all, i.e., $w_1 = W$ *and* $w_2 = 0$.
(ii) If players are risk-averse, then the optimal prize structure solves $W = w_1 + w_2$ *and*

$$\frac{u'(w_1)}{u'(w_2)} = \frac{\frac{\alpha}{\lambda}}{2 - \frac{\alpha}{\lambda}}. \tag{12}$$

Proposition 11 shows how the optimal prize structure depends on the players' risk attitudes and overconfidence. Part (i) shows that the contest designer allocates all of the prize money to the winner of the final when the players are rational and risk-neutral. This result is in line with Fu and Lu (2012) and Stracke et al. (2014) who show, in the context of a pyramid Tullock contest and of a two-stage elimination Tullock contest, respectively, that a winner-take-all prize structure maximizes aggregate effort when players are rational and risk-neutral. The novel result in part (i) is that the optimal prize structure is also the winner-take-all when players are overconfident and risk-neutral.

Part (ii) shows that when the players are rational and risk-averse, the optimal prize structure involves multiple prizes with the winner of the final receiving most of the prize money and a smaller part being assigned to the runner-up. This result is in line with Krishna and Morgan (1998) findings in the context of two-stage elimination Lazer-Rosen tournament with rational and risk-averse players. The novel result in part (ii) is that when players are overconfident and risk-averse, an increase in overconfidence leads the contest designer to allocate an increasingly larger share of the prize money to the winner of the final. The contest designer, exploits the players' overconfidence by shifting pay from w_2 to w_1 given that the bias leads the players to overestimate the probability of gaining prize w_1 and underestimate the probability of gaining prize w_2. This result is in line with Santos-Pinto (2010) who finds, in the context of a one-shot Lazear-Rosen tournament, that the tournament designer can exploit players' overconfidence by raising the prize spread.

8 Extensions

This section discusses four extensions of the model. It starts by showing that most of the results extend to a two-stage elimination Tullock contest. Next, it shows all of our results hold when the overconfident player's rivals cannot observe his bias. After that, it shows our results also extend to a three-stage elimination contest. Finally, it describes the impact of underconfidence on a two-stage elimination contest.

8.1 Two-Stage Elimination Contest with Tullock CSF

In a two-stage elimination contest where the players' winning probabilities are determined by Tullock's (1980) CSF, player i's probability of winning when paired with j is

$$p_{ij} = \begin{cases} \frac{q(e_i)}{q(e_i)+q(e_j)} & \text{if} \quad q(e_i) + q(e_j) > 0 \\ \frac{1}{2} & \text{if} \quad q(e_i) + q(e_j) = 0 \end{cases}$$

where the function $q(e_i)$, often referred to as the impact function (Ewerhart 2015), satisfies $q(0) \geqslant 0$, $q'(e_i) > 0$, and $q''(e_i) \leqslant 0$.[10] Following Santos-Pinto and Sekeris (2023), an overconfident player i's perceived probability of winning when paired with j is

$$\widetilde{p}_{ij} = \begin{cases} \frac{\lambda_i q(e_i)}{\lambda_i q(e_i)+q(e_j)} & \text{if} \quad \lambda_i q(e_i) + q(e_j) > 0 \\ \frac{1}{2} & \text{if} \quad \lambda_i q(e_i) + q(e_j) = 0 \end{cases}$$

where $\lambda_i > 1$. Hence, in a final between an overconfident player 1 and a rational player 3, the equilibrium efforts (e_1^f, e_3^f) satisfy

$$\frac{\lambda_1 q'(e_1^f) q(e_3^f)}{[\lambda_1 q(e_1^f) + q(e_3^f)]^2}[u(w_1) - u(w_2)] = c, \tag{13}$$

and

$$\frac{q'(e_3^f) q(e_1^f)}{[q(e_1^f) + q(e_3^f)]^2}[u(w_1) - u(w_2)] = c. \tag{14}$$

It is not possible to solve (13) and (14) explicitly for the equilibrium efforts in the final. Nevertheless, Santos-Pinto and Sekeris (2023) characterize the equilibrium of a one-shot Tullock contest between two overconfident players. In the Online Appendix we show that the results of Santos-Pinto and Sekeris (2023) also apply to a one-shot Tullock contest between one overconfident player and one rational player. Namely, the overconfident player exerts less effort than the rational player, and both players exert less effort than if both were rational. Furthermore, the equilibrium efforts of both players decrease in the overconfident player's bias. Hence, the equilibrium efforts in the final of a two-stage elimination Tullock contest satisfy the same qualitative properties as those in the final of a two-stage elimination Alcalde and Dham (2007) contest.

Let us now turn our attention to the semifinals of a two-stage elimination Tullock contest. In the semifinal between an overconfident player 1 and a rational player 2, the

[10]When $q(e_i) = e_i^\alpha$, where $0 < \alpha \leqslant 1$, the parameter α determines how sensitive the CSF is to effort.

equilibrium efforts (e_1^s, e_2^s) satisfy

$$\frac{\lambda_1 q'(e_1^s)q(e_2^s)}{[\lambda_1 q(e_1^s)+q(e_2^s)]^2}\widetilde{v}_1 = c,$$

and

$$\frac{q'(e_2^s)q(e_1^s)}{[q(e_1^s)+q(e_2^s)]^2}v_2 = c,$$

The Online Appendix shows that the overconfident player's perceived continuation value $\widetilde{v}_1$ increases in his bias. Hence, overconfidence has an encouragement effect in the semifinal of a two-stage elimination Tullock contest. The complacency effect is given by

$$\frac{\partial mg\widetilde{p}_{12}^s}{\partial\lambda_1}\frac{\lambda_1}{mg\widetilde{p}_{12}^s} = -q(e_2^s)q'(e_1^s)\frac{\lambda_1 q(e_1^s)-q(e_2^s)}{(\lambda_1 q(e_1^s)+q(e_2^s))^3}\frac{\lambda_1}{\frac{\lambda_1 q(e_2^s)q'(e_1^s)}{(\lambda_1 q(e_1^s)+q(e_2^s))^2}} = -\frac{\lambda_1 q(e_1^s)-q(e_2^s)}{\lambda_1 q(e_1^s)+q(e_2^s)}. \quad (15)$$

It follows from (15) that there is a complacency effect when $\lambda_1 q(e_1^s) > q(e_2^s)$. Hence, the effect of overconfidence on the equilibrium efforts in a semifinal of a two-stage elimination Tullock contest will depend on the sizes of the encouragement and complacency effects.

8.2 Unobservability of Overconfidence

Our results also extend to an elimination contest where the overconfident player's rivals cannot observe his bias. In the Online Appendix we characterize the equilibrium of a final between an overconfident player and a rational player who is unaware of the overconfident player's bias. In this case, the rational player exerts the benchmark equilibrium effort $\overline{e}^f$. The overconfident player chooses a best response to $\overline{e}^f$ given his mistaken beliefs, i.e., he chooses the level of effort e that solves $mg\widetilde{p}^f(e,\overline{e}^f)[u(w_2)-u(w_1)] = c$. In equilibrium, the overconfident player exerts less effort than the rational player in the final. Note that since the rational player does not lower her effort, the overconfident player lowers his effort by less than when the rational player is aware of the overconfident player's bias. An increase in the bias raises the overconfident player's perceived probability of winning the final and lowers his cost of effort. Hence, an increase in the bias raises the overconfident player's perceived equilibrium expected utility of the final. We also characterize the equilibrium of a semifinal between an overconfident player and a rational rival who is unaware of the overconfident player's bias. Once again, we show that if the prize spread is sufficiently large and the bias is not too high, the overconfident player exerts more effort than the rational player in the semifinal. In the semifinal between two rational players, overconfidence no longer leads to an increase in equilibrium effort due to the unobservability of bias. Finally, we show that there exists conditions under which the overconfident player has the highest equilibrium probability of winning the elimination contest.

8.3 Elimination Contest with Three Stages

Our results also hold in a three-stage elimination contest. In the Online Appendix we study a three-stage elimination contest with eight players. In the third-stage, the eight players are matched pairwise and each pair competes in one of the four quarterfinals. The third-stage winners move on to compete in the second-stage. The winner of the contest receives prize w_1, the runner-up prize w_2, the second-stage losers receive prize w_3, and the third-stage losers receive nothing, with $w_1 > w_2 > w_3 \geqslant 0$.

We fully characterize the equilibrium of the quarterfinal between an overconfident and a rational player in a three-stage elimination contest with one overconfident player and seven rational players. We find that, regardless of the overconfident player exerting more or less effort than a rational rival in the semifinals stage, an overconfident player's perceived expected utility of the semifinal $\widetilde{E}^s(U)$ is greater than the benchmark $\overline{E}^s(U)$. Hence, in a quarterfinal, an overconfident player has a higher perceived continuation value $\widetilde{v}^q$ than the benchmark $\overline{v}^q$. This implies that overconfidence always has an encouragement effect in the quarterfinals stage. Moreover, depending on the parameters of the model, the encouragement effect can dominate the complacency effect.[11]

8.4 Underconfidence

In the Online Appendix we characterize the equilibrium of a two-stage elimination contest with one underconfident player and three rational players. The underconfident player underestimates the impact of his effort on his winning probability in each pairwise interaction, i.e., he has a bias $\lambda \in (0, 1)$. We find that an underconfident player exerts less effort than a rational rival in the final. An increase in the underconfident player's bias (a decrease in λ) has two opposite effects on his perceived equilibrium expected utility of the final. On the one hand, it lowers the underconfident player's perceived probability of winning the final. On the other hand, it lowers the underconfident player's cost of effort. The former effect dominates and an underconfident player's perceived equilibrium expected utility of the final decreases in his bias. This, in turn, implies that underconfidence has a double negative effect in the semifinal. First, it lowers the underconfident player's perceived continuation value as it makes reaching the final less attractive. Second, it lowers the underconfident player's perceived marginal probability of winning the semifinal. Hence, an underconfident player exerts less effort than a rational rival in a semifinal. The presence of an underconfident player has a spillover effect on effort provision in the semifinal between the two rational players since the probability of facing an underconfident player in the final raises their continuation values. These results imply that the underconfident player has the lowest equilibrium probability of winning the elimination contest.

9 Conclusion

Our findings reveal a nuanced interplay between overconfidence and effort exertion in a two-stage elimination contest. An overconfident player expends less effort in the final stage than a rational rival. However, this pattern can be inverted in the semifinals stage, where an overconfident can exert more effort than a rational rival.

In the final stage, an overconfident player always exerts lower effort at equilibrium than a rational player. The (mis)perceived advantage of the overconfident player leads him to think, mistakenly, he can reduce his effort without endangering his prospects of success. The rational player, aware of the rival's bias, also lowers her effort but not as much. Hence, the bias unambiguously lowers the overconfident player's probability of winning the final.

Overconfidence also changes how the players' assess the attractiveness of the final. The overconfident player's perceived expected utility of the final increases in his bias. This

[11]The spillover effect is always present in the quarterfinals among the rational players who won't meet the overconfident player until the final. However, this is not true for the quarterfinal among the two rational players who have the chance of meeting the overconfident player in the semifinal. The continuation values of these two players compared to the benchmark depends on whether the overconfident player or the rational player exerts higher effort in the semifinal. The spillover effect only exists if the rational player has a higher equilibrium winning probability than the overconfident player in the semifinal.

happens due to two channels. First, the overconfident player overestimates his probability of winning the final. Second, the overconfident player anticipates, correctly, he needs to exert lower effort in the final. The rational player's expected utility of the final increases with the overconfident players' bias. She has a higher probability of winning the final and anticipates, correctly, she needs to exert lower effort in the final. Hence, overconfidence makes reaching the final more attractive not only for an overconfident player but also for a rational player.

In the semifinals stage, the bias has two opposite effects on an overconfident player's incentives to exert effort. On the one hand, it leads to a decrease in the overconfident player's perceived marginal probability of winning the semifinal. On the other hand, it raises the overconfident player's perceived continuation value due to the higher expected utility of moving on to the final. The first effect discourages an overconfident player from exerting effort whereas the second effect encourages him to exert effort. If the encouragement (complacency) effect dominates, an overconfident player exert higher (lower) effort at equilibrium in a semifinal than a rational player.

The encouragement effect dominates when two conditions are met. First, the prize spread needs to be sufficiently large. Second, the overconfident player's bias cannot be too high. The intuition behind these two conditions is as follows. The higher is the prize spread, the higher is the continuation value, and the higher is the encouragement effect. As the overconfident player's bias increases, the increase in the encouragement effect gets smaller whereas the increase in the complacency effect gets larger. Hence, there exists an upper bound for the bias above which the complacency effect dominates.

Our results also reveal that the presence of an overconfident player in an elimination contest has a spillover effect on the effort provision of the semifinal with rational players. This result is straightforward. The presence of an overconfident player in the contest makes reaching the final more attractive for rational players and hence they have incentives to exert higher effort in their semifinal.

Next, we consider how the overconfident player's bias affects his equilibrium probability of winning the elimination contest. We find that the overconfident player can be the one with the highest probability of winning the elimination contest. For this to be the case three conditions have to be met. First, the role that effort plays in determining the winning probabilities must be sufficiently high. Second, the prize spread needs to be sufficiently large. Third, the overconfident player's bias needs to be small.

Our results contribute to the literature on labor market promotions and overconfidence. More particularly, they contribute to the literature on CEO overconfidence. They provide a novel explanation for why overconfident individuals are promoted to CEO positions, namely, exerting higher efforts at the early stages of their careers. In addition, our results highlight the role that increases in executive compensation (interpreted as increases in the prize spread) can have in making elimination contests more attractive to overconfident individuals.

Future work might study elimination contests where players are not only heterogeneous in their confidence levels but also in terms of their abilities. When an overconfident player has a lower ability than a rational player, overconfidence might not only increase a player's perceived continuation value but also his perceived marginal probability of winning a final and a semifinal. Future work can also study how the contest designer should optimally set prizes in the presence of overconfident players.

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Appendix

Proof of Lemma 1

The best response of player i in the final, $R_i^f(e_j)$, is defined by (3). Hence, the slope of the best response of player i in the final is given by

$$-\frac{\frac{\partial R_i^f}{\partial e_j}}{\frac{\partial R_i^f}{\partial e_i}} = -\frac{\frac{\partial^2 \widetilde{E}^f(U_{ij})}{\partial e_i \partial e_j}}{\frac{\partial^2 \widetilde{E}^f(U_{ij})}{\partial e_i^2}} = -\begin{cases} \dfrac{\frac{\alpha^2}{2\lambda_i}\frac{e_j^{\alpha-1}}{e_i^{\alpha+1}}[u(w_1)-u(w_2)]}{-(1+\alpha)\frac{\alpha}{2\lambda_i}\frac{e_j^{\alpha}}{e_i^{\alpha+2}}[u(w_1)-u(w_2)]} = -\frac{\alpha}{1+\alpha}\frac{e_i}{e_j} & \text{if } \lambda_i e_i^\alpha > e_j^\alpha \\ \dfrac{-\frac{\alpha^2}{2}\lambda_i\frac{e_i^{\alpha-1}}{e_j^{\alpha+1}}[u(w_1)-u(w_2)]}{-(1-\alpha)\frac{\alpha}{2}\lambda_i\frac{e_i^{\alpha-2}}{e_j^{\alpha}}[u(w_1)-u(w_2)]} = \frac{\alpha}{1-\alpha}\frac{e_i}{e_j} & \text{if } \lambda_i e_i^\alpha < e_j^\alpha \end{cases}$$

Therefore, the sign of the slope of player i's best response in the final is positive for $\lambda_i e_i^\alpha > e_j^\alpha$ and negative for $\lambda_i e_i^\alpha < e_j^\alpha$. This implies that $R_i^f(e_j)$ increases in e_j for $\lambda_i e_i^\alpha > e_j^\alpha$, reaches the maximum at $\lambda_i e_i^\alpha = e_j^\alpha$, and decreases in e_j for $\lambda_i e_i^\alpha < e_j^\alpha$.

Proof of Lemma 2

Player i's best response in the final is defined by (3). Hence, we have

$$\frac{\partial R_i^f}{\partial \lambda_i} = \begin{cases} -\frac{\alpha}{2\lambda_i^2}\frac{e_j^\alpha}{e_i^{\alpha+1}}[u(w_1)-u(w_2)] & \text{if } \lambda_i e_i^\alpha > e_j^\alpha \\ \frac{\alpha}{2}\frac{e_i^{\alpha-1}}{e_j^\alpha}[u(w_1)-u(w_2)] & \text{if } \lambda_i e_i^\alpha < e_j^\alpha \end{cases}$$

We see that $\partial R_i^f/\partial\lambda_i < 0$ for $\lambda_i e_i^\alpha > e_j^\alpha$ and $\partial R_i^f/\partial\lambda_i > 0$ for $\lambda_i e_i^\alpha < e_j^\alpha$. Substituting $e_j^\alpha = \lambda_i e_i^\alpha$ into equation (3) and denoting the maximal effort that i is willing to invest in the final by e_i^{fmax} we obtain

$$\frac{\alpha}{2\lambda_i}\frac{\lambda_i(e_i^{fmax})^\alpha}{(e_i^{fmax})^{\alpha+1}}[u(w_1)-u(w_2)] = c$$

or

$$e_i^{fmax} = \frac{\alpha}{2c}[u(w_1)-u(w_2)].$$

This implies that the value of e_i corresponding to the maximum value of player i's best response in the final, e_i^{fmax}, does not depend on λ_i.

Proof of Lemma 3

Player i's best response in the semifinal is defined by (4). Hence, we have

$$\frac{\partial R_i^s}{\partial \lambda_i} = \begin{cases} -\frac{\alpha}{2\lambda_i^2}\frac{e_j^\alpha}{e_i^{\alpha+1}}\widetilde{v}_i + \frac{\alpha}{2\lambda_i}\frac{e_j^\alpha}{e_i^{\alpha+1}}\frac{\partial \widetilde{v}_i}{\partial \lambda_i} = \frac{\alpha}{2\lambda_i^2}\frac{e_j^\alpha}{e_i^{\alpha+1}}\left(-\widetilde{v}_i + \lambda_i\frac{\partial \widetilde{v}_i}{\partial \lambda_i}\right) & \text{if } \lambda_i e_i^\alpha > e_j^\alpha \\ \frac{\alpha}{2}\frac{e_i^{\alpha-1}}{e_j^\alpha}\widetilde{v}_i + \frac{\alpha}{2}\lambda_i\frac{e_i^{\alpha-1}}{e_j^\alpha}\frac{\partial \widetilde{v}_i}{\partial \lambda_i} = \frac{\alpha}{2}\frac{e_i^{\alpha-1}}{e_j^\alpha}\left(\widetilde{v}_i + \lambda_i\frac{\partial \widetilde{v}_i}{\partial \lambda_i}\right) & \text{if } \lambda_i e_i^\alpha < e_j^\alpha \end{cases} \tag{16}$$

It follows from (16) that if $\frac{\partial \widetilde{v}_i}{\partial \lambda_i} > 0$, then $\frac{\partial R_i^s}{\partial \lambda_i} > 0$ for $\lambda_i e_i^\alpha < e_j^\alpha$. That is, if the overconfident player's perceived continuation value increases in overconfidence, then an increase in overconfidence expands the overconfident player's best response in the semifinal for high effort of the rival.

It also follows from (16) that if $\frac{\partial \widetilde{v}_i}{\partial \lambda_i} > 0$ and $\frac{\partial \widetilde{v}_i}{\partial \lambda_i}\frac{\lambda_i}{\widetilde{v}_i} < 1 (> 1)$, then $\frac{\partial R_i^s}{\partial \lambda_i} < 0 (> 0)$ for $\lambda_i e_i^\alpha > e_j^\alpha$. That is, if the overconfident player's perceived continuation value increases in overconfidence and the elasticity of the perceived continuation value with respect to the bias is less (greater) than 1, then an increase in overconfidence contracts (expands) the overconfident player's best response in the semifinal for low effort of the rival.

Substituting $e_j^\alpha = \lambda_i e_i^\alpha$ into equation (4) and denoting the maximal effort that i is willing to invest in the semifinal by e_i^{smax} we obtain

$$\frac{\alpha}{2\lambda_i}\frac{\lambda_i (e_i^{smax})^\alpha}{(e_i^{smax})^{\alpha+1}}\widetilde{v}_i = c$$

or

$$e_i^{smax} = \frac{\alpha}{2c}\widetilde{v}_i.$$

Hence, if $\frac{\partial \widetilde{v}_i}{\partial \lambda_i} > 0$, then e_i^{fmax} increases in λ_i.

Proof of Proposition 1

The final stage

$$p_{13}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha} & \text{if}\, e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha} & \text{if}\, e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

$$p_{31}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha} & \text{if}\, e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha} & \text{if}\, e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

Rational player 1 $\quad max \quad E^f(U_{13}) = p_{13}^f[u(w_1) - u(w_2)] + u(w_2) - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\right)[u(w_1) - u(w_2)] + u(w_2) - ce_1 & \text{if} \quad e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}[u(w_1) - u(w_2)] + u(w_2) - ce_1 & \text{if} \quad e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

Rational player 3 $\quad max \quad E^f(U_{31}) = p_{31}^f[u(w_1) - u(w_2)] + u(w_2) - ce_3$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)[u(w_1) - u(w_2)] + u(w_2) - ce_3 & \text{if}\, e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}[u(w_1) - u(w_2)] + u(w_2) - ce_3 & \text{if}\, e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

There are 2 cases.

$$\begin{cases} e_1^\alpha \geqslant e_3^\alpha \\ e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

1. equilibrium efforts

(1) case 1: $e_1^\alpha \geqslant e_3^\alpha$

Player 1 $\quad max \quad E^f(U_{13}) = \left(1 - \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}\right)[u(w_1) - u(w_2)] + u(w_2) - ce_1$

Player 3 $\quad max \quad E^f(U_{31}) = \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}[u(w_1) - u(w_2)] + u(w_2) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2}\frac{e_3^\alpha}{e_1^{\alpha+1}}[u(w_1) - u(w_2)] - c = 0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}[u(w_1) - u(w_2)] - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2}(-\alpha - 1)\frac{e_3^\alpha}{e_1^{\alpha+2}}[u(w_1) - u(w_2)] < 0$

$[e_3] \quad \frac{\alpha}{2}(\alpha - 1)\frac{e_3^{\alpha-2}}{e_1^\alpha}[u(w_1) - u(w_2)] < 0$

Solve F.O.C , we get $e_1 = e_3 = \frac{\alpha}{2c}[u(w_1) - u(w_2)]$

(2) case 2: $e_1^\alpha \leqslant e_3^\alpha$

The same as the previous case.

Thus the unique equilibrium is $\overline{e}^f = e_1^f = e_3^f = \frac{\alpha}{2c}[u(w_1) - u(w_2)]$,

2. winning probabilities

The true winning probabilities are

$$\overline{p}^f = p_{13}^f = p_{31}^f = \frac{1}{2}$$

3. expected utilities of final

$$\begin{aligned}\overline{E}^f(U) = E^f(U_{13}) = E^f(U_{31}) &= \frac{1}{2}[u(w_1) + u(w_2)] - c\frac{\alpha}{2c}[u(w_1) - u(w_2)] \\ &= \frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\end{aligned}$$

Since $0 < \alpha \leqslant 1$, we have $\overline{E}^f(U) \geqslant 0$. The participation constraints are satisfied.

The semifinals stage

1. continuation values

Using the expected utility of the final, we can get the continuation values of each player in the semifinal.
The continuation value of player 1 is given by

$$\begin{aligned}v_1 &= p_{34}^s E^f(U_{13}) + p_{43}^s E^f(U_{14}) \\ &= \overline{E}^f(U) \\ &= \frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\end{aligned}$$

Since all 4 players are identical,

$$\overline{v} = v_1 = v_2 = v_3 = v_4 = \frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2).$$

2. equilibrium efforts in the semifinal

Using the extension of the equilibrium result in the final, we can get that

$$\overline{e}^s = e_1^s = e_2^s = e_3^s = e_4^s = \frac{\alpha}{2c}\overline{v} = \frac{\alpha}{2c}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]$$

3. true winning probabilities

$$\overline{p}^s = p_{12}^s = p_{21}^s = p_{34}^s = p_{43}^s = \frac{1}{2}$$

4. expected utilities of semifinal

$$\overline{E}^s(U) = \frac{1}{2}\overline{v} - c\frac{\alpha}{2c}\overline{v} = \frac{1-\alpha}{2}\overline{v} = \frac{1-\alpha}{2}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]$$

Since $0 < \alpha \leqslant 1$, we have $\overline{E}^s(U) \geqslant 0$. The participation constraints are satisfied.

5. the prize spread that satisfies $\overline{e}^s < \overline{e}^f$

$$\begin{aligned}
\overline{e}^s < \overline{e}^f &\iff \frac{\alpha}{2c}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right] < \frac{\alpha}{2c}[u(w_1) - u(w_2)] \\
&\iff (1-\alpha)u(w_1) + (1+\alpha)u(w_2) < 2[u(w_1) - u(w_2)] \\
&\iff (3+\alpha)u(w_2) < (1+\alpha)u(w_1) \\
&\iff \frac{u(w_1)}{u(w_2)} > \frac{3+\alpha}{1+\alpha}
\end{aligned}$$

Since $\alpha \in (0,1]$ this inequality is satisfied for all α when $u(w_1) > 3u(w_2)$.

Proof of Proposition 2

The perceived winning probabilities of the players are:

$$\widetilde{p}_{13}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha} & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha} & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

$$p_{31}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha} & \text{if} \quad e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha} & \text{if} \quad e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

Overconfident player 1 $\quad max \quad \widetilde{E}^f(U_{13}) = \widetilde{p}_{13}^f[u(w_1) - u(w_2)] + u(w_2) - ce_1$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)[u(w_1) - u(w_2)] + u(w_2) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}[u(w_1) - u(w_2)] + u(w_2) - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

Rational player 3 $\quad max \quad E^f(U_{31}) = p^f_{31}[u(w_1) - u(w_2)] + u(w_2) - ce_3$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)[u(w_1) - u(w_2)] + u(w_2) - ce_3 & \text{if} \quad e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}[u(w_1) - u(w_2)] + u(w_2) - ce_3 & \text{if} \quad e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

1. equilibrium efforts

   (1) case 1: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1$

   Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)[u(w_1) - u(w_2)] + u(w_2) - ce_1$

   Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)[u(w_1) - u(w_2)] + u(w_2) - ce_3$

   F.o.c

   $[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}[u(w_1) - u(w_2)] - c = 0$

   $[e_3] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}}[u(w_1) - u(w_2)] - c = 0$

   S.o.c

   $[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha - 1)\frac{e_3^\alpha}{e_1^{\alpha+2}}[u(w_1) - u(w_2)] < 0$

   $[e_3] \quad \frac{\alpha}{2}(-\alpha - 1)\frac{e_1^\alpha}{e_3^{\alpha+2}}[u(w_1) - u(w_2)] < 0$

   Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]$$

$$e_3 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)]$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$ and $e_3 \geqslant e_1$:

$$\begin{aligned}
\lambda_1 e_1^\alpha \geqslant e_3^\alpha &\iff \lambda_1 \left(\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^\alpha \geqslant \left(\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)^\alpha \\
&\iff \left(\frac{\alpha}{2c}\right)^\alpha \lambda_1^{-\frac{(\alpha+1)\alpha}{2\alpha+1}+1} \geqslant \left(\frac{\alpha}{2c}\right)^\alpha \lambda_1^{-\frac{\alpha^2}{2\alpha+1}} \\
&\iff \lambda_1^{-\frac{(\alpha+1)\alpha}{2\alpha+1}+1+\frac{\alpha^2}{2\alpha+1}} \geqslant 1 \\
&\iff \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \geqslant 1 \\
&\iff \frac{\alpha+1}{2\alpha+1} \geqslant 0
\end{aligned}$$

$$\begin{aligned}
e_3 \geqslant e_1 &\iff \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \geqslant \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \\
&\iff \lambda_1^{-\frac{\alpha}{2\alpha+1}} \geqslant \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \\
&\iff \lambda_1^{\frac{1}{2\alpha+1}} \geqslant 1 \\
&\iff \frac{1}{2\alpha+1} \geqslant 0
\end{aligned}$$

The conditions are always satisfied.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad e_3 \leqslant e_1$

Player 1 $\quad max \quad \left(1-\frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)[u(w_1)-u(w_2)]+u(w_2)-ce_1$

Player 3 $\quad max \quad \frac{1}{2}\frac{e_3^\alpha}{e_1^\alpha}[u(w_1)-u(w_2)]+u(w_2)-ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}[u(w_1)-u(w_2)]-c=0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_3^{\alpha-1}}{e_1^\alpha}[u(w_1)-u(w_2)]-c=0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1}=\lambda_1>1$$

which contradicts the condition that $e_3 \leqslant e_1$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad e_3 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}[u(w_1)-u(w_2)]+u(w_2)-ce_1$

Player 3 $\quad max \quad \left(1-\frac{1}{2}\frac{e_1^\alpha}{e_3^\alpha}\right)[u(w_1)-u(w_2)]+u(w_2)-ce_3$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_3^\alpha}[u(w_1)-u(w_2)]-c=0$

$[e_3] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_3^{\alpha+1}}[u(w_1)-u(w_2)]-c=0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \frac{1}{\lambda_1} < 1$$

which contradicts the condition that $e_3 \geqslant e_1$

Thus the unique equilibrium is

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]$$
$$e_3^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)]$$

where $\lambda_1 e_1^\alpha > e_3^\alpha$ and $e_3 > e_1$.

We show that $e_1^f < \overline{e}^f$ and $e_3^f < \overline{e}^f$:

$$e_1^f < \overline{e}^f \iff \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)] < \frac{\alpha}{2c}[u(w_1) - u(w_2)]$$
$$\iff \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} < 1$$

$$e_3^f < \overline{e}^f \iff \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)] < \frac{\alpha}{2c}[u(w_1) - u(w_2)]$$
$$\iff \lambda_1^{-\frac{\alpha}{2\alpha+1}} < 1$$

2. equilibrium winning probabilities

The true winning probabilities are

$$p_{13}^f = \frac{1}{2}\left(\frac{e_1^f}{e_3^f}\right)^\alpha$$
$$= \frac{1}{2}\left(\lambda_1^{-\frac{1}{2\alpha+1}}\right)^\alpha$$
$$= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}$$

$$p_{31}^f = 1 - p_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}$$

The overconfident player 1's perceived winning probabilities are

$$\widetilde{p}_{13}^f = 1 - \frac{1}{2}\frac{(e_3^f)^\alpha}{\lambda_1(e_1^f)^\alpha}$$
$$= 1 - \frac{1}{2}\frac{\left(\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)]\right)^\alpha}{\lambda_1\left(\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]\right)^\alpha}$$
$$= 1 - \frac{1}{2}\frac{\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)^\alpha}{\lambda_1\left(\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^\alpha}$$
$$= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}$$

We show that $\widetilde{p}^f_{13} > p^f_{31} > \frac{1}{2} > p^f_{13}$:

$$\begin{aligned}
\widetilde{p}^f_{13} > p^f_{31} &\iff 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} > 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \\
&\iff \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} > \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \\
&\iff \lambda_1^{\frac{1}{2\alpha+1}} > 1
\end{aligned}$$

$$\begin{aligned}
p^f_{31} > \frac{1}{2} &\iff 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} > \frac{1}{2} \\
&\iff \lambda_1^{-\frac{\alpha}{2\alpha+1}} < 1
\end{aligned}$$

$$\begin{aligned}
p^f_{13} < \frac{1}{2} &\iff 1 - p^f_{31} < \frac{1}{2} \\
&\iff p^f_{31} > \frac{1}{2}
\end{aligned}$$

3. expected utilities of final

$$\begin{aligned}
\widetilde{E}^f(U_{13}) &= \widetilde{p}^f_{13} u(w_1) + (1 - \widetilde{p}^f_{13}) u(w_2) - ce^f_1 \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) u(w_1) + \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} u(w_2) - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)] \\
&= u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]
\end{aligned}$$

$$\begin{aligned}
E^f(U_{31}) &= p^f_{31} u(w_1) + (1 - p^f_{31}) u(w_2) - ce^f_3 \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right) u(w_1) + \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} u(w_2) - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)] \\
&= u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)]
\end{aligned}$$

We show that $\widetilde{E}^f(U_{13}) > E^f(U_{31}) > \overline{E}^f(U)$:

$$\begin{aligned}
&\widetilde{E}^f(U_{13}) > E^f(U_{31}) \\
&\iff u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)] > u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)] \\
&\iff -\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)] > -\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)] \\
&\iff \lambda_1^{-\frac{\alpha}{2\alpha+1}} > \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}
\end{aligned}$$

$$\begin{aligned}
&E^f(U_{31}) > \overline{E}^f(U) \\
&\iff u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)] > u(w_1) - \frac{1+\alpha}{2}[u(w_1) - u(w_2)] \\
&\iff -\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)] > -\frac{1+\alpha}{2}[u(w_1) - u(w_2)] \\
&\iff 1 > \lambda_1^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}$$

Since $\overline{E}^f(U) \geqslant 0$, the participation constraints of both players are satisfied.

Proof of Proposition 3

1. Continuation values

   Using Proposition 2, we can get the continuation values of each player.

   Overconfident player 1:

   $$\widetilde{v}_1 = p_{34}^s \widetilde{E}^f(U_{13}) + p_{43}^s \widetilde{E}^f(U_{14})$$

   Since player 3 and 4 are identical, $\widetilde{E}^f(U_{13}) = \widetilde{E}^f(U_{14})$

   $$\widetilde{v}_1 = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]$$

   Since $\widetilde{E}^f(U_{13}) > \overline{E}^f(U)$, we can get $\widetilde{v}_1 > \overline{v}$.

   Rational player 2:

   $$v_2 = p_{34}^s E^f(U_{23}) + p_{43}^s E^f(U_{24})$$

   Since players 3 and 4 are identical, $E^f(U_{23}) = E^f(U_{24})$

   $$v_2 = \frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2) = \overline{v}$$

2. The equilibrium efforts and winning probabilities

   Player 1 $\quad max \quad \widetilde{E}^s(U_{12}) = \widetilde{p}_{12}^s \widetilde{v}_1 - ce_1$

   $$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}$$

   Player 2 $\quad max \quad E^s(U_{21}) = p_{21}^s v_2 - ce_2$

   $$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2 - ce_2 & \text{if}\, e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2 - ce_2 & \text{if}\, e_2 \leqslant e_1 \end{cases}$$

   There are 4 cases.

   $$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha & \textit{and} \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha & \textit{and} \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha & \textit{and} \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha & \textit{and} \quad e_2 \leqslant e_1 \end{cases}$$

   Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$, which corresponds to proposition 3 (i).

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha}v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}v_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha-1)\frac{e_2^{\alpha-2}}{e_1^\alpha}v_2 < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(v_2)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(v_2)^{\alpha+1}$$

$$\frac{e_2}{e_1} = \lambda_1\frac{v_2}{\widetilde{v}_1}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \leqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied.

② $e_2 \leqslant e_1$

$$\begin{aligned}
e_1 \geqslant e_2 &\Longleftrightarrow \frac{e_1}{e_2} \geqslant 1 \\
&\Longleftrightarrow \lambda_1^{-\frac{1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{1}{2\alpha+1}}(v_2)^{-\frac{1}{2\alpha+1}} \geqslant 1 \\
&\Longleftrightarrow \left(\frac{\widetilde{v}_1}{\lambda_1 v_2}\right)^{\frac{1}{2\alpha+1}} \geqslant 1 \\
&\Longleftrightarrow \frac{\widetilde{v}_1}{\lambda_1 v_2} \geqslant 1 \\
&\Longleftrightarrow \frac{\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)[u(w_1)-u(w_2)]}{\lambda_1\left[\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)[u(w_1)-u(w_2)]\right]} \geqslant 1 \\
&\Longleftrightarrow \frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \geqslant \lambda_1\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)
\end{aligned}$$

Let

$$f(\lambda_1) = \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \lambda_1\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$$

we can easily get that $f(\lambda_1 = 1) = 0$ and $f(\lambda_1 \to \infty) < 0$.

$$f'(\lambda_1) = \frac{(1+\alpha)^2}{2(2\alpha+1)}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$$

$$\begin{aligned}
f'(\lambda_1) \lesseqqgtr 0 &\iff \frac{(1+\alpha)^2}{2(2\alpha+1)}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} \lesseqqgtr \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\iff \frac{(1+\alpha)^2}{2\alpha+1}\frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)} \lesseqqgtr \lambda_1^{\frac{\alpha+1}{2\alpha+1}+1} \\
&\iff \left[\frac{(1+\alpha)^2}{2\alpha+1}\frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \lesseqqgtr \lambda_1
\end{aligned}$$

Let $g(\alpha) = \left[\frac{(1+\alpha)^2}{2\alpha+1}\frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}}$

a) $g(\alpha) \leqslant 1$

if $g(\alpha) \leqslant 1$, then $f'(\lambda_1) < 0$ always holds. Which means $f(\lambda_1) < 0$ always holds, thus $\frac{e_1}{e_2} < 1$ always holds.

$$\begin{aligned}
g(\alpha) \leqslant 1 &\iff \frac{(1+\alpha)^2}{2\alpha+1}\frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)} \leqslant 1 \\
&\iff \frac{(1+\alpha)^2}{2\alpha+1} \leqslant \frac{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}{u(w_1)-u(w_2)} \\
&\iff (1+\alpha)^2[u(w_1)-u(w_2)] \leqslant (2\alpha+1)[(1-\alpha)u(w_1)+(1+\alpha)u(w_2)] \\
&\iff (\alpha+3\alpha^2)u(w_1) \leqslant (2+5\alpha+3\alpha^2)u(w_2) \\
&\iff \frac{u(w_1)}{u(w_2)} \leqslant \frac{2+5\alpha+3\alpha^2}{\alpha(1+3\alpha)} \\
&\iff \frac{u(w_1)}{u(w_2)} - 1 \leqslant \frac{2+5\alpha+3\alpha^2}{\alpha(1+3\alpha)} - 1 \\
&\iff \frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}
\end{aligned}$$

When $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, the condition $e_1 \geqslant e_2$ is never satisfied given that $\lambda_1 > 1$.

b) $g(\alpha) > 1$

if $g(\alpha) > 1$, then

$$f'(\lambda_1)\begin{cases} > 0 & \text{when } \lambda_1 < \left[\frac{(1+\alpha)^2}{2\alpha+1}\frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \\ < 0 & \text{when } \lambda_1 > \left[\frac{(1+\alpha)^2}{2\alpha+1}\frac{u(w_1)-u(w_2)}{(1-\alpha)u(w_1)+(1+\alpha)u(w_2)}\right]^{\frac{1}{\frac{\alpha+1}{2\alpha+1}+1}} \end{cases}$$

We now show that if $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, then there exists a unique threshold $\hat{\lambda} > 1$ where $f(\lambda_1) = 0$, that is,

$$\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} = \lambda_1\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$$

which is equivalent to

$$u(w_1) - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1)-u(w_2)] = \hat{\lambda}\left[\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)[u(w_1)-u(w_2)]\right].$$

To see this is the case, we rearrange the equality as

$$u(w_1) - \frac{1+\alpha}{2}\hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1)-u(w_2)] = \hat{\lambda}\left[u(w_1) - \frac{1+\alpha}{2}[u(w_1)-u(w_2)]\right],$$

or

$$\frac{1+\alpha}{2}[\hat{\lambda} - \hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}][u(w_1)-u(w_2)] = (\hat{\lambda}-1)u(w_1),$$

or

$$\frac{1+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_1)} = \frac{\hat{\lambda}-1}{\hat{\lambda} - \hat{\lambda}^{-\frac{\alpha+1}{2\alpha+1}}}. \tag{17}$$

Since $\alpha \in (0,1]$ and $u(w_1) > u(w_2)$, the left-hand side of (17) takes a value in the interval $(0,1)$. The right-hand side of (17) is increasing in $\hat{\lambda}$ for $\lambda > 1$, its limit when $\hat{\lambda} \to 1$ is $\frac{2\alpha+1}{3\alpha+2}$, and its limit when $\hat{\lambda} \to \infty$ is 1. Hence, the threshold $\hat{\lambda}$ exists and is unique provided that

$$\frac{1+\alpha}{2}\frac{u(w_1)-u(w_2)}{u(w_1)} > \frac{2\alpha+1}{3\alpha+2}.$$

It is easy to show that this inequality is equivalent to

$$\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}.$$

Therefore, if $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$, then there exists a unique value for $\hat{\lambda}$, greater than 1, that satisfies (17). This, in turn, implies:

$$f(\lambda_1)\begin{cases} > 0 & \text{when } \lambda_1 < \hat{\lambda} \\ = 0 & \text{when } \lambda_1 = \hat{\lambda} \\ < 0 & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

$$e_1 - e_2\begin{cases} > 0 & \text{when } \lambda_1 < \hat{\lambda} \\ = 0 & \text{when } \lambda_1 = \hat{\lambda} \\ < 0 & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

The condition $e_1 \geqslant e_2$ is only satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 \leqslant \hat{\lambda}$. And $e_1 > e_2$ is only satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$.

Therefore the solution

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(v_2)^{\alpha}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(v_2)^{\alpha+1}$$

only applies when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$.

(2) case 2: $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha}$ *and* $e_2 \geqslant e_1$, which corresponds to proposition 3 (ii).

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^{\alpha}}{\lambda_1 e_1^{\alpha}}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^{\alpha}}{e_2^{\alpha}}\right)v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^{\alpha}}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^{\alpha}}{e_2^{\alpha+1}}v_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^{\alpha}}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^{\alpha}}{e_2^{\alpha+2}}v_2 < 0$

Solve F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(v_2)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1)^{-\frac{1}{2\alpha+1}}(v_2)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha}$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha}$

$$\lambda_1 e_1^{\alpha} \geqslant e_2^{\alpha} \Longleftrightarrow \frac{\lambda_1 e_1^{\alpha}}{e_2^{\alpha}} \geqslant 1$$
$$\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1$$

Since $\lambda_1 > 1$ and $\widetilde{v}_1 > v_2$, the inequality is always satisfied. Therefore $\lambda_1 e_1^{\alpha} > e_2^{\alpha}$ always holds when $\lambda_1 > 1$.

② $e_2 \geqslant e_1$

We have already seen in case (1) that $e_2 \geqslant e_1$ is satisfied when either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$.

Therefore the solution

$$e_1 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(v_2)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{\frac{\alpha+1}{2\alpha+1}}$$

only applies when either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha$ *and* $e_2 \geqslant e_1$

Player 1 max $\frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1$

Player 2 max $[1-\frac{1}{2}(\frac{e_1}{e_2})^\alpha]v_2 - ce_2$

F.o.c

$[e_1]$ $\frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1 - c = 0$

$[e_2]$ $\frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2 - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{v_2}{\lambda_1\widetilde{v}_1} < 1$$

which contradicts the condition that $e_2 \geqslant e_1$

Therefore, the equilibrium in this semi-final:

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$, which corresponds to Proposition 3 (i)

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}$$
$$\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}[u(w_1)-u(w_2)]$$

$$e_2^s = \frac{\alpha}{2c}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}$$
$$\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{1+\alpha}[u(w_1)-u(w_2)]$$

where $e_1^s > e_2^s$.

$$\begin{aligned} p_{21}^s &= \frac{1}{2}\left(\frac{e_2^s}{e_1^s}\right)^\alpha \\ &= \frac{1}{2}\left(\frac{\lambda_1 v_2}{\widetilde{v}_1}\right)^\alpha \\ &= \frac{1}{2}\lambda_1^\alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^\alpha \end{aligned}$$

$$\begin{aligned}
p_{12}^s &= 1 - p_{21}^s \\
&= 1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}
\end{aligned}$$

$$\begin{aligned}
\widetilde{p}_{12}^s &= 1 - \frac{1}{2}\frac{(e_2^s)^\alpha}{\lambda_1 (e_1^s)^\alpha} \\
&= 1 - \frac{1}{2}\lambda_1^{-1}\left[\lambda_1\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)\right]^{\alpha} \\
&= 1 - \frac{1}{2}\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}
\end{aligned}$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$, which corresponds to Proposition 3 (ii).

$$\begin{aligned}
e_1^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}[u(w_1)-u(w_2)]
\end{aligned}$$

$$\begin{aligned}
e_2^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\frac{\alpha+1}{2\alpha+1}}[u(w_1)-u(w_2)]
\end{aligned}$$

where $\lambda_1 (e_1^s)^\alpha > (e_2^s)^\alpha$ and $e_1^s \leqslant e_2^s$.

$$\begin{aligned}
p_{12}^s &= \frac{1}{2}\left(\frac{e_1^s}{e_2^s}\right)^{\alpha} \\
&= \frac{1}{2}\left[\lambda_1^{-\frac{1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{1}{2\alpha+1}}\right]^{\alpha} \\
&= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}$$

$$\begin{aligned}
p_{21}^s &= 1 - p_{12}^s \\
&= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}$$

$$
\begin{aligned}
\widetilde{p}_{12}^{s} &= 1-\frac{1}{2}\frac{\left(e_2^s\right)^\alpha}{\lambda_1\left(e_1^s\right)^\alpha} \\
&= 1-\frac{1}{2}\lambda_1^{-1}\lambda_1^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&= 1-\frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

3. equilibrium efforts compared to benchmark

   (1) Proposition 3 (i): when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

   We show that $e_1^s > \overline{e}^s > e_2^s$:

$$
\begin{aligned}
e_1^s > \overline{e}^s &\iff \frac{e_1^s}{\overline{e}^s} > 1 \\
&\iff \frac{\frac{\alpha}{2c}\lambda_1^{\alpha-1}\widetilde{v}_1^{\,1-\alpha}v_2^\alpha}{\frac{\alpha}{2c}\overline{v}} > 1 \\
&\iff \left(\frac{\widetilde{v}_1}{\lambda_1 v_2}\right)^{1-\alpha} > 1
\end{aligned}
$$

$$
\begin{aligned}
\frac{e_2^s}{\overline{e}^s} &\iff \frac{e_2^s}{\overline{e}^s} > 1 \\
&\iff \frac{\frac{\alpha}{2c}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{-\alpha}\left(v_2\right)^{\alpha+1}}{\frac{\alpha}{2c}\overline{v}} > 1 \\
&\iff \left(\frac{\lambda_1 v_2}{\widetilde{v}_1}\right)^{\alpha} > 1
\end{aligned}
$$

   Since $\frac{\widetilde{v}_1}{\lambda_1 v_2} > 1$, we can get $e_1^s > \overline{e}^s$ and $e_2^s < \overline{e}^s$.

   (2) Proposition 3 (ii): when either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

   We show that $e_1^s \leqslant e_2^s \leqslant \overline{e}^s$:

   Since we already showed that $e_1^s \leqslant e_2^s$ is satisfied under this condition, we only have to show $e_2^s \leqslant \overline{e}^s$.

$$
\begin{aligned}
e_2^s \leqslant \overline{e}^s &\iff \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\widetilde{v}_1^{\frac{\alpha}{2\alpha+1}}v_2^{\frac{\alpha+1}{2\alpha+1}} \leqslant \frac{\alpha}{2c}\overline{v} \\
&\iff \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\widetilde{v}_1^{\frac{\alpha}{2\alpha+1}}v_2^{\frac{\alpha+1}{2\alpha+1}} \leqslant \frac{\alpha}{2c}v_2 \\
&\iff \widetilde{v}_1^{\frac{\alpha}{2\alpha+1}} \leqslant \lambda_1^{\frac{\alpha}{2\alpha+1}}v_2^{\frac{\alpha}{2\alpha+1}} \\
&\iff \widetilde{v}_1 \leqslant \lambda_1 v_2
\end{aligned}
$$

   which always holds, thus $e_1^s \leqslant e_2^s \leqslant \overline{e}^s$ always holds.

4. perceived and true winning probabilities compared to benchmark

(1) Proposition 3 (i): when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

We show that $p_{21}^s < \frac{1}{2}$ and $\widetilde{p}_{12}^s > p_{12}^s > \frac{1}{2}$:

$$\begin{aligned} p_{21}^s < \frac{1}{2} &\iff \frac{1}{2}\left(\frac{e_2^s}{e_1^s}\right)^{\alpha} < \frac{1}{2} \\ &\iff e_2^s < e_1^s \end{aligned}$$

$$p_{12}^s = 1 - p_{21}^s > \frac{1}{2}$$

$$\begin{aligned} \widetilde{p}_{12}^s > p_{12}^s &\iff 1 - \frac{1}{2}\left(\frac{e_2^s}{\lambda_1 e_1^s}\right)^{\alpha} > 1 - \frac{1}{2}\left(\frac{e_2^s}{e_1^s}\right)^{\alpha} \\ &\iff \lambda_1 > 1 \end{aligned}$$

(2) Proposition 3 (ii): when either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

We show that $p_{12}^s \leqslant \frac{1}{2}$, $p_{21}^s \geqslant \frac{1}{2}$ and $\widetilde{p}_{12}^s > \frac{1}{2}$:

$$\begin{aligned} p_{12}^s \leqslant \frac{1}{2} &\iff \frac{1}{2}\left(\frac{e_1^s}{e_2^s}\right)^{\alpha} \leqslant \frac{1}{2} \\ &\iff e_1^s \leqslant e_2^s \end{aligned}$$

$$p_{21}^s = 1 - p_{12}^s \geqslant \frac{1}{2}$$

$$\begin{aligned} \widetilde{p}_{12}^s > \frac{1}{2} &\iff 1 - \frac{1}{2}\left(\frac{e_2^s}{\lambda_1 e_1^s}\right)^{\alpha} > \frac{1}{2} \\ &\iff \frac{e_2^s}{\lambda_1 e_1^s} < 1 \\ &\iff \frac{\lambda_1^{-\frac{\alpha}{2\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}} (v_2)^{\frac{\alpha+1}{2\alpha+1}}}{\lambda_1^{\frac{\alpha}{2\alpha+1}} (\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}} (v_2)^{\frac{\alpha}{2\alpha+1}}} < 1 \\ &\iff \lambda_1^{-\frac{2\alpha}{2\alpha+1}} (\widetilde{v}_1)^{-\frac{1}{2\alpha+1}} (v_2)^{\frac{1}{2\alpha+1}} < 1 \\ &\iff \lambda_1^{-\frac{2\alpha}{2\alpha+1}} \left(\frac{v_2}{\widetilde{v}_1}\right)^{\frac{1}{2\alpha+1}} < 1 \end{aligned}$$

5. Participation constraints

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

$$\begin{aligned}
\widetilde{E}^s(U_{12}) &= \widetilde{p}^s_{12}\widetilde{v}_1 - ce^s_1 \\
&> p^s_{12}\widetilde{v}_1 - ce^s_1 \\
&= \left(1-\frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{-\alpha}\left(v_2\right)^{\alpha}\right)\widetilde{v}_1 - c\frac{\alpha}{2c}\lambda_1^{\alpha-1}\left(\widetilde{v}_1\right)^{1-\alpha}\left(v_2\right)^{\alpha} \\
&= \widetilde{v}_1 - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(v_2\right)^{\alpha} - \frac{\alpha}{2}\lambda_1^{\alpha-1}\left(\widetilde{v}_1\right)^{1-\alpha}\left(v_2\right)^{\alpha} \\
&> \widetilde{v}_1 - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(v_2\right)^{\alpha} - \frac{1}{2}\lambda_1^{\alpha-1}\left(\widetilde{v}_1\right)^{1-\alpha}\left(v_2\right)^{\alpha} \\
&> \widetilde{v}_1 - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(v_2\right)^{\alpha} - \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(v_2\right)^{\alpha} \\
&= \widetilde{v}_1 - \lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{1-\alpha}\left(v_2\right)^{\alpha} \\
&= \left(\widetilde{v}_1\right)^{1-\alpha}\left[\left(\widetilde{v}_1\right)^{\alpha} - \lambda_1^{\alpha}\left(v_2\right)^{\alpha}\right] \\
&> 0
\end{aligned}$$

$$\begin{aligned}
E^s(U_{21}) &= p^s_{21}v_2 - ce^s_2 \\
&= \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{-\alpha}\left(v_2\right)^{\alpha}v_2 - c\frac{\alpha}{2c}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{-\alpha}\left(v_2\right)^{\alpha+1} \\
&= \frac{1}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{-\alpha}\left(v_2\right)^{1+\alpha} - \frac{\alpha}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{-\alpha}\left(v_2\right)^{\alpha+1} \\
&= \frac{1-\alpha}{2}\lambda_1^{\alpha}\left(\widetilde{v}_1\right)^{-\alpha}\left(v_2\right)^{1+\alpha} \\
&\geqslant 0
\end{aligned}$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$\widetilde{E}^s(U_{12}) = \widetilde{p}^s_{12}\widetilde{v}_1 - ce^s_1$$

Since $\widetilde{p}^s_{12} > \frac{1}{2}$, $\widetilde{v}_1 > \overline{v}$ and $e^s_1 < \overline{e}^s$, we can get that $\widetilde{E}^s(U_{12}) > \overline{E}^s(U) \geqslant 0$.

$$\begin{aligned}
E^s(U_{21}) &= p^s_{21}v_2 - ce^s_2 \\
&= \left(1-\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2\right)^{-\frac{\alpha}{2\alpha+1}}\right)v_2 - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= v_2 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= v_2 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}}\left(\left(v_2\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\right) \\
&\geqslant 0
\end{aligned}$$

Derivation of equation (7)

The two FOCs in the semifinal are

$$\frac{\partial \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1}\widetilde{v}_1(\lambda_1)=c$$

$$\frac{\partial p^s_{21}(e_1,e_2)}{\partial e_2}v_2=c$$

Total differentiation gives us

$$\left(\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1^2}de_1+\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial e_2}de_2+\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial \lambda_1}d\lambda_1\right)\widetilde{v}_1(\lambda_1)$$
$$+\frac{\partial \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1}\frac{\partial \widetilde{v}_1(\lambda_1)}{\partial \lambda_1}d\lambda_1=0$$

$$\left(\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2\partial e_1}de_1+\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2^2}de_2\right)v_2=0$$

or

$$\left(\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1^2}\frac{de_1}{d\lambda_1}+\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial e_2}\frac{de_2}{d\lambda_1}+\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial \lambda_1}\right)\widetilde{v}_1(\lambda_1)+\frac{\partial \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1}\frac{\partial \widetilde{v}_1(\lambda_1)}{\partial \lambda_1}=0$$

$$\left(\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2\partial e_1}\frac{de_1}{d\lambda_1}+\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2^2}\frac{de_2}{d\lambda_1}\right)v_2=0$$

Solving the second equation for $\frac{de_2}{d\lambda_1}$ we obtain

$$\frac{de_2}{d\lambda_1}=-\frac{\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2\partial e_1}}{\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2^2}}\frac{de_1}{d\lambda_1}$$

Replacing in the first equation we obtain

$$\left(\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1^2}\frac{de_1}{d\lambda_1}-\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial e_2}\frac{\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2\partial e_1}}{\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2^2}}\frac{de_1}{d\lambda_1}+\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial \lambda_1}\right)\widetilde{v}_1(\lambda_1)$$
$$+\frac{\partial \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1}\frac{\partial \widetilde{v}_1(\lambda_1)}{\partial \lambda_1}=0$$

Let's solve this equation for $\frac{de_1}{d\lambda_1}$

$$\frac{de_1}{d\lambda_1}\left(\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1^2}-\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial e_2}\frac{\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2\partial e_1}}{\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2^2}}\right)\widetilde{v}_1(\lambda_1)$$
$$=-\frac{\partial \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1}\frac{\partial \widetilde{v}_1(\lambda_1)}{\partial \lambda_1}-\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial \lambda_1}\widetilde{v}_1(\lambda_1)$$

or

$$\frac{de_1}{d\lambda_1}=-\frac{\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial \lambda_1}\widetilde{v}_1(\lambda_1)+\frac{\partial \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1}\frac{\partial \widetilde{v}_1(\lambda_1)}{\partial \lambda_1}}{\left(\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1^2}-\frac{\partial^2 \widetilde{p}^s_{12}(e_1,e_2,\lambda_1)}{\partial e_1\partial e_2}\frac{\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2\partial e_1}}{\frac{\partial^2 p^s_{21}(e_1,e_2)}{\partial e_2^2}}\right)\widetilde{v}_1(\lambda_1)}$$

$$=-\frac{\frac{\partial mg\widetilde{p}^s_{12}}{\partial \lambda_1}\widetilde{v}_1(\lambda_1)+mg\widetilde{p}^s_{12}\frac{\partial \widetilde{v}_1(\lambda_1)}{\partial \lambda_1}}{\left(\frac{\partial mg\widetilde{p}^s_{12}}{\partial e_1}-\frac{\partial mg\widetilde{p}^s_{12}}{\partial e_2}\frac{\frac{\partial mgp^s_{21}}{\partial e_1}}{\frac{\partial mgp^s_{21}}{\partial e_2}}\right)\widetilde{v}_1(\lambda_1)}$$

The denominator has to be negative, that is,

$$D^s = \frac{\partial mg\widetilde{p}^s_{12}}{\partial e_1} - \frac{\partial mg\widetilde{p}^s_{12}}{\partial e_2}\frac{\frac{\partial mgp^s_{21}}{\partial e_1}}{\frac{\partial mgp^s_{21}}{\partial e_2}} < 0$$

Note that in a semifinal where the overconfident player exerts more effort than the rational player we have $\lambda_1 e_1^\alpha > e_2^\alpha$ and $e_2 < e_1$. This implies that in the semifinal we have

$$mg\widetilde{p}^s_{12} = \frac{\partial \widetilde{p}^s_{12}(e_1, e_2, \lambda_1)}{\partial e_1} = \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}$$

$$mgp^s_{21} = \frac{\partial p^s_{21}(e_1, e_2)}{\partial e_2} = \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}$$

This implies

$$\frac{\partial mg\widetilde{p}^s_{12}}{\partial e_1} = \frac{\partial^2 \widetilde{p}^s_{12}(e_1, e_2, \lambda_1)}{\partial e_1^2} = -(\alpha+1)\frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+2}}$$

$$\frac{\partial mg\widetilde{p}^s_{12}}{\partial e_2} = \frac{\partial^2 \widetilde{p}^s_{12}(e_1, e_2, \lambda_1)}{\partial e_1 \partial e_2} = \frac{\alpha^2}{2\lambda_1}\frac{e_2^{\alpha-1}}{e_1^{\alpha+1}}$$

$$\frac{\partial mgp^s_{21}}{\partial e_2} = \frac{\partial^2 p^s_{21}(e_1, e_2)}{\partial e_2^2} = (\alpha-1)\frac{\alpha}{2}\frac{e_2^{\alpha-2}}{e_1^\alpha}$$

$$\frac{\partial mgp^s_{21}}{\partial e_1} = \frac{\partial^2 p^s_{21}(e_1, e_2)}{\partial e_2 \partial e_1} = -\frac{\alpha^2}{2}\frac{e_2^{\alpha-1}}{e_1^{\alpha+1}}$$

Hence, we have

$$\begin{aligned}
D^s &= \frac{\partial mg\widetilde{p}^s_{12}}{\partial e_1} - \frac{\partial mg\widetilde{p}^s_{12}}{\partial e_2}\frac{\frac{\partial mgp^s_{21}}{\partial e_1}}{\frac{\partial mgp^s_{21}}{\partial e_2}} \\
&= -(\alpha+1)\frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+2}} - \frac{\alpha^2}{2\lambda_1}\frac{e_2^{\alpha-1}}{e_1^{\alpha+1}}\frac{-\frac{\alpha^2}{2}\frac{e_2^{\alpha-1}}{e_1^{\alpha+1}}}{(\alpha-1)\frac{\alpha}{2}\frac{e_2^{\alpha-2}}{e_1^\alpha}} \\
&= -(\alpha+1)\frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+2}} + \frac{\alpha^3}{2\lambda_1(\alpha-1)}\frac{e_2^{\alpha-1}}{e_1^{\alpha+1}}\frac{\frac{e_2^{\alpha-1}}{e_1^{\alpha+1}}}{\frac{e_2^{\alpha-2}}{e_1^\alpha}} \\
&= -(\alpha+1)\frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+2}} + \frac{\alpha^3}{2\lambda_1(\alpha-1)}\frac{e_2^\alpha}{e_1^{\alpha+2}} \\
&= \frac{\alpha}{2\lambda_1}\left[-(\alpha+1) + \frac{\alpha^2}{(\alpha-1)}\right]\frac{e_2^\alpha}{e_1^{\alpha+2}} \\
&= -\frac{\alpha}{2\lambda_1}\frac{1}{1-\alpha}\frac{e_2^\alpha}{e_1^{\alpha+2}} \\
&< 0
\end{aligned}$$

In a semifinal where the overconfident player exerts less effort than the rational player we have $\lambda_1 e_1^\alpha > e_2^\alpha$ and $e_2 > e_1$. This implies that in the semifinal we have

$$mg\widetilde{p}^s_{12} = \frac{\partial \widetilde{p}^s_{12}(e_1, e_2, \lambda_1)}{\partial e_1} = \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}$$

$$mgp^s_{21} = \frac{\partial p^s_{21}(e_1, e_2)}{\partial e_2} = \frac{\alpha}{2} \frac{e_1^\alpha}{e_2^{\alpha+1}}$$

$$\frac{\partial mg\widetilde{p}^s_{12}}{\partial e_1} = \frac{\partial^2 \widetilde{p}^s_{12}(e_1, e_2, \lambda_1)}{\partial e_1^2} = -(\alpha+1)\frac{\alpha}{2\lambda_1} \frac{e_2^\alpha}{e_1^{\alpha+2}}$$

$$\frac{\partial mg\widetilde{p}^s_{12}}{\partial e_2} = \frac{\partial^2 \widetilde{p}^s_{12}(e_1, e_2, \lambda_1)}{\partial e_1 \partial e_2} = \frac{\alpha^2}{2\lambda_1} \frac{e_2^{\alpha-1}}{e_1^{\alpha+1}}$$

$$\frac{\partial mgp^s_{21}}{\partial e_2} = \frac{\partial^2 p^s_{21}(e_1, e_2)}{\partial e_2^2} = -(\alpha+1)\frac{\alpha}{2} \frac{e_1^\alpha}{e_2^{\alpha+2}}$$

$$\frac{\partial mgp^s_{21}}{\partial e_1} = \frac{\partial^2 p^s_{21}(e_1, e_2)}{\partial e_2 \partial e_1} = \frac{\alpha^2}{2} \frac{e_1^{\alpha-1}}{e_2^{\alpha+1}}$$

Hence, we have

$$\begin{aligned}
D^s &= \frac{\partial mg\widetilde{p}^s_{12}}{\partial e_1} - \frac{\partial mg\widetilde{p}^s_{12}}{\partial e_2} \frac{\frac{\partial mgp^s_{21}}{\partial e_1}}{\frac{\partial mgp^s_{21}}{\partial e_2}} \\
&= -(\alpha+1)\frac{\alpha}{2\lambda_1} \frac{e_2^\alpha}{e_1^{\alpha+2}} - \frac{\alpha^2}{2\lambda_1} \frac{e_2^{\alpha-1}}{e_1^{\alpha+1}} \frac{\frac{\alpha^2}{2} \frac{e_1^{\alpha-1}}{e_2^{\alpha+1}}}{-(\alpha+1)\frac{\alpha}{2} \frac{e_1^\alpha}{e_2^{\alpha+2}}} \\
&= -(\alpha+1)\frac{\alpha}{2\lambda_1} \frac{e_2^\alpha}{e_1^{\alpha+2}} + \frac{\alpha^3}{2\lambda_1(\alpha+1)} \frac{e_2^{\alpha-1}}{e_1^{\alpha+1}} \frac{\frac{e_1^{\alpha-1}}{e_2^{\alpha+1}}}{\frac{e_1^\alpha}{e_2^{\alpha+2}}} \\
&= -(\alpha+1)\frac{\alpha}{2\lambda_1} \frac{e_2^\alpha}{e_1^{\alpha+2}} + \frac{\alpha^3}{2\lambda_1(\alpha+1)} \frac{e_2^\alpha}{e_1^{\alpha+2}} \\
&= \frac{\alpha}{2\lambda_1} \left[-(\alpha+1) + \frac{\alpha^2}{(\alpha+1)} \right] \frac{e_2^\alpha}{e_1^{\alpha+2}}
\end{aligned}$$

This is negative as long as

$$\alpha + 1 > \frac{\alpha^2}{\alpha+1}$$

or

$$(\alpha+1)^2 > \alpha$$

which is true.

Proof of Proposition 4

1. Continuation values:

   Rational player 3:

$$\begin{aligned}
v_3 &= p^s_{12} E^f(U_{31}) + p^s_{21} E^f(U_{32}) \\
&= p^s_{12} \left[u(w_1) - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha}{2\alpha+1}} [u(w_1) - u(w_2)] \right] + p^s_{21} \left[\frac{1-\alpha}{2} u(w_1) + \frac{1+\alpha}{2} u(w_2) \right]
\end{aligned}$$

Rational player 4:

$$\begin{aligned} v_4 &= p^s_{12} E^f(U_{41}) + p^s_{21} E^f(U_{42}) \\ &= p^s_{12}\left[u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1)-u(w_2)]\right] + p^s_{21}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right] \end{aligned}$$

Note that since $E^f(U_{31}) = E^f(U_{41}) > \overline{E}^f(U) = E^f(U_{32}) = E^f(U_{42})$, we have $v_3 = v_4 > \overline{v}$.

2. The equilibrium

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

$$\begin{aligned} e^s_3 = e^s_4 &= \frac{\alpha}{2c}v_3 \\ &= \frac{\alpha}{2c}\left[p^s_{12}\left[u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1)-u(w_2)]\right] + p^s_{21}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]\right] \end{aligned}$$

where

$p^s_{12} = 1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}.$

$$p^s_{34} = p^s_{43} = \frac{1}{2}$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$\begin{aligned} e^s_3 = e^s_4 &= \frac{\alpha}{2c}v_3 \\ &= \frac{\alpha}{2c}\left[p^s_{12}\left[u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1)-u(w_2)]\right] + p^s_{21}\left[\frac{1-\alpha}{2}u(w_1) + \frac{1+\alpha}{2}u(w_2)\right]\right] \end{aligned}$$

where

$p^s_{12} = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right]^{-\frac{\alpha}{2\alpha+1}}.$

$$p^s_{34} = p^s_{43} = \frac{1}{2}$$

We show that $e^s_3 = e^s_4 > \overline{e}^s$ holds in both (1) and (2):

$$\begin{aligned} e^s_3 = e^s_4 > \overline{e}^s &\Longleftrightarrow \frac{\alpha}{2c}v_3 > \frac{\alpha}{2c}\overline{v} \\ &\Longleftrightarrow v_3 > \overline{v} \end{aligned}$$

3. Participation constraint

$$\begin{aligned} E^s(U_{34}) = p^s_{34}v_3 - ce^s_3 &= \frac{1}{2}v_3 - c\frac{\alpha}{2c}v_3 \\ &= \frac{1-\alpha}{2}v_3 \\ &\geqslant 0 \end{aligned}$$

$$E^s(U_{43}) = E^s(U_{34}) \geqslant 0$$

Proof of Proposition 5

1. When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ and $\lambda_1 < \hat{\lambda}$

   (1) P_1

$$P_1 = p_{13}^f p_{12}^s$$
$$= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right]$$

Let
$$f(\lambda_1) = P_1 - \frac{1}{4}.$$

We can get
$$f(\lambda_1 = 1) = \frac{1}{2} \times \frac{1}{2} - \frac{1}{4} = 0$$
$$f(\lambda_1 = \hat{\lambda}) = \frac{1}{2}\hat{\lambda}^{-\frac{\alpha}{2\alpha+1}} \times \frac{1}{2} - \frac{1}{4} < 0$$

$f(\lambda_1)$ can also be written as the following:

$$f(\lambda_1) = \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}$$
$$- \frac{1}{4}\lambda_1^{\alpha-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha} - \frac{1}{4}$$

Taking derivative of $f(\lambda_1)$ we obtain

$$f'(\lambda_1) = -\frac{1}{2}\frac{\alpha}{2\alpha+1}\lambda_1^{-\frac{\alpha}{2\alpha+1}-1} - \frac{1}{4}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}$$
$$\left[\left(\alpha - \frac{\alpha}{2\alpha+1}\right)\lambda_1^{\alpha-\frac{\alpha}{2\alpha+1}-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\right.$$
$$+ \lambda_1^{\alpha-\frac{\alpha}{2\alpha+1}}(-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha-1}$$
$$\left.\left(-\frac{\alpha+1}{2}\right)\left(-\frac{\alpha+1}{2\alpha+1}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right]$$

$$
\begin{aligned}
f'(\lambda_1 = 1) = &-\frac{1}{2}\frac{\alpha}{2\alpha+1} - \frac{1}{4}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\left[\left(\alpha - \frac{\alpha}{2\alpha+1}\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)}\right.\right. \\
&\left.\left. - \frac{1+\alpha}{2}\right)^{-\alpha} - \alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha-1}\frac{\alpha+1}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
= &-\frac{1}{2}\frac{\alpha}{2\alpha+1} - \frac{1}{4}\left[\alpha - \frac{\alpha}{2\alpha+1} - \alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
= &-\frac{1}{2}\frac{\alpha}{2\alpha+1} - \frac{1}{4}\left[\alpha\left(\frac{2\alpha}{2\alpha+1} - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}\frac{\alpha+1}{2\alpha+1}\right)\right] \\
= &-\frac{1}{2}\frac{\alpha}{2\alpha+1} - \frac{1}{4}\left[\frac{\alpha}{2\alpha+1}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right)\right] \\
= &-\frac{1}{2}\frac{\alpha}{2\alpha+1}\left[1 + \frac{1}{2}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right)\right]
\end{aligned}
$$

$f'(\lambda_1 = 1)$ and $1+\frac{1}{2}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right)$ has the opposite sign.

When $1+\frac{1}{2}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right) < 0,\ f'(\lambda_1 = 1) > 0.$

And since

$$
\begin{aligned}
&1+\frac{1}{2}\left(2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha)\right) < 0 \\
\iff &2\alpha - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha) < -2 \\
\iff &2\alpha + 2 < \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2}(1+\alpha) \\
\iff &2 < \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{\alpha+1}{2} \\
\iff &4\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) < 1+\alpha \\
\iff &4 + \frac{4u(w_2)}{u(w_1)-u(w_2)} < 3(1+\alpha) \\
\iff &\frac{4u(w_2)}{u(w_1)-u(w_2)} < 3(1+\alpha) - 4 \\
\iff &\frac{4u(w_2)}{u(w_1)-u(w_2)} < 3\alpha - 1
\end{aligned}
$$

$f'(\lambda_1 = 1) > 0$ is only satisfied when $\alpha > \frac{1}{3}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{4}{3\alpha-1}$.

We show $\frac{4}{3\alpha-1} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$:

$$\begin{aligned}
\frac{4}{3\alpha-1} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)} &\Longleftrightarrow 4\alpha\left(3\alpha+1\right) > 2(2\alpha+1)\left(3\alpha-1\right) \\
&\Longleftrightarrow 12\alpha^2 + 4\alpha > 12\alpha^2 + 2\alpha - 2 \\
&\Longleftrightarrow 2\alpha + 2 > 0
\end{aligned}$$

Thus we know that under the conditions $\alpha > \frac{1}{3}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{4}{3\alpha-1}$, $f(\lambda_1)$ is positive when λ_1 is small and close to 1. Therefore there exist parameter configurations where the overconfident player's equilibrium winning probability P_1 is higher than the benchmark.

(2) P_2

We show that $P_2 < \frac{1}{4}$:

$$\begin{aligned}
P_2 &= p_{23}^f p_{21}^s \\
&= \frac{1}{2} p_{21}^s \\
&< \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}
\end{aligned}$$

(3) P_3 and P_4

We show that $P_3 = P_4 > \frac{1}{4}$:

$$\begin{aligned}
P_3 = P_4 &= p_{12}^s p_{31}^f p_{34}^s + p_{21}^s p_{32}^f p_{34}^s \\
&= p_{12}^s p_{31}^f \frac{1}{2} + p_{21}^s \frac{1}{2}\frac{1}{2} \\
&= p_{12}^s p_{31}^f \frac{1}{2} + (1 - p_{12}^s)\frac{1}{2}\frac{1}{2} \\
&= p_{12}^s \left(p_{31}^f \frac{1}{2} - \frac{1}{4} \right) + \frac{1}{4} \\
&> p_{12}^s \left(\frac{1}{2}\frac{1}{2} - \frac{1}{4} \right) + \frac{1}{4} = \frac{1}{4}
\end{aligned}$$

(4) compare P_1 and P_3

$$\begin{aligned}
P_1 - P_3 &= p_{13}^f p_{12}^s - p_{12}^s p_{31}^f p_{34}^s - p_{21}^s p_{32}^f p_{34}^s \\
&= p_{13}^f p_{12}^s - p_{12}^s p_{31}^f p_{34}^s - \left(1 - p_{12}^s\right) p_{32}^f p_{34}^s \\
&= p_{13}^f p_{12}^s - \frac{1}{2} p_{12}^s p_{31}^f - \frac{1}{2}\frac{1}{2}\left(1 - p_{12}^s\right) \\
&= p_{13}^f p_{12}^s - \frac{1}{2} p_{12}^s \left(1 - p_{13}^f\right) - \frac{1}{4}\left(1 - p_{12}^s\right) \\
&= \frac{3}{2} p_{13}^f p_{12}^s - \frac{1}{4} p_{12}^s - \frac{1}{4}
\end{aligned}$$

The sign of $P_1 - P_3$ is the same as the sign of $6p_{13}^f p_{12}^s - p_{12}^s - 1$

Let $f(\lambda_1) = 6p_{13}^f p_{12}^s - p_{12}^s - 1$

$$f(\lambda_1 = 1) = 6 \times \frac{1}{2} \times \frac{1}{2} - \frac{1}{2} - 1 = 0$$

$$f(\lambda_1 = \hat{\lambda}) = 6 \times \frac{1}{2}\hat{\lambda}^{-\frac{\alpha}{2\alpha+1}} \times \frac{1}{2} - \frac{1}{2} - 1 < 0$$

$$\begin{aligned}
f(\lambda_1) = 3\lambda_1^{-\frac{\alpha}{2\alpha+1}} & \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right] \\
& - \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right] - 1 \\
= 3\lambda_1^{-\frac{\alpha}{2\alpha+1}} & - \frac{1}{2}\lambda_1^{\alpha}\left(3\lambda_1^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha} \\
& \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} - 2
\end{aligned}$$

$$\begin{aligned}
f'(\lambda_1) = & -3\frac{\alpha}{2\alpha+1}\lambda_1^{-\frac{\alpha}{2\alpha+1}-1} - \frac{1}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \\
& \left[\left(3\left(-\frac{\alpha}{2\alpha+1} + \alpha\right)\lambda_1^{-\frac{\alpha}{2\alpha+1}+\alpha-1} - \alpha\lambda_1^{\alpha-1}\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\right. \\
& + \lambda_1^{\alpha}\left(3\lambda_1^{-\frac{\alpha}{2\alpha+1}} - 1\right)(-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha-1} \\
& \left.\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right]
\end{aligned}$$

$$
\begin{aligned}
f'(\lambda_1 = 1) &= -3\frac{\alpha}{2\alpha+1} - \frac{1}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \\
&\quad \left[\left(3\left(-\frac{\alpha}{2\alpha+1} + \alpha\right) - \alpha\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\right. \\
&\quad \left. - 2\alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
&= -3\frac{\alpha}{2\alpha+1} - \frac{1}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \\
&\quad \left[\left(2\alpha - 3\frac{\alpha}{2\alpha+1}\right)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha}\right. \\
&\quad \left. - 2\alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-\alpha-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
&= -3\frac{\alpha}{2\alpha+1} - \frac{1}{2}\left(2\alpha - 3\frac{\alpha}{2\alpha+1}\right) \\
&\quad + \alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
&= -\frac{3}{2}\frac{\alpha}{2\alpha+1} - \alpha + \alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
&= \alpha\left(-\frac{3}{2}\frac{1}{2\alpha+1} - 1 + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
f'(\lambda_1 = 1) > 0 &\iff -\frac{3}{2}\frac{1}{2\alpha+1} - 1 + \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} > 0 \\
&\iff \frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{-1} > \frac{3}{2}\frac{1}{2\alpha+1} + 1 \\
&\iff \frac{\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}}{\frac{3}{2}\frac{1}{2\alpha+1} + 1} > \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\iff \frac{\frac{(\alpha+1)^2}{2(2\alpha+1)}}{\frac{3+2(2\alpha+1)}{2(2\alpha+1)}} > \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\iff \frac{(\alpha+1)^2}{(4\alpha+5)} > \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&\iff \frac{(\alpha+1)^2}{(4\alpha+5)} + \frac{\alpha-1}{2} > \frac{u(w_2)}{u(w_1)-u(w_2)} \\
&\iff \frac{2(\alpha+1)^2 + (\alpha-1)(4\alpha+5)}{2(4\alpha+5)} > \frac{u(w_2)}{u(w_1)-u(w_2)} \\
&\iff \frac{6\alpha^2 + 5\alpha - 3}{2(4\alpha+5)} > \frac{u(w_2)}{u(w_1)-u(w_2)}
\end{aligned}
$$

This is satisfied when $\alpha > \frac{-5+\sqrt{97}}{12}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(4\alpha+5)}{6\alpha^2+5\alpha-3}$.

We show $\frac{2(4\alpha+5)}{6\alpha^2+5\alpha-3} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$:

$$\begin{aligned}
\frac{2\left(4\alpha+5\right)}{6\alpha^2+5\alpha-3} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)} &\iff \frac{4\alpha+5}{6\alpha^2+5\alpha-3} > \frac{2\alpha+1}{\alpha(3\alpha+1)} \\
&\iff \left(4\alpha+5\right)\alpha(3\alpha+1) > \left(2\alpha+1\right)\left(6\alpha^2+5\alpha-3\right) \\
&\iff 12\alpha^3+19\alpha^2+5\alpha > 12\alpha^3+16\alpha^2-\alpha-3 \\
&\iff 3\alpha^2+6\alpha+3 > 0
\end{aligned}$$

Thus we know that under the conditions $\alpha > \frac{-5+\sqrt{97}}{12}$ and $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(4\alpha+5)}{6\alpha^2+5\alpha-3}$, $f(\lambda_1)$ is positive when λ_1 is small and close to 1. Therefore there exist parameter configurations where the overconfident player's equilibrium winning probability P_1 is higher than that of the rational player in the other semi-final P_3.

2. When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(1+2\alpha)}{\alpha(1+3\alpha)}$ or $\lambda_1 \geqslant \hat{\lambda}$

(1) P_1

Since player 3 and player 4 are identical, the equilibrium winning probability of player 1 is

$$\begin{aligned}
P_1 &= p_{13}^f p_{12}^s \\
&< \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}
\end{aligned}$$

(2) P_2

$$\begin{aligned}
P_2 &= p_{23}^f p_{21}^s \\
&= \frac{1}{2} p_{21}^s \\
&> \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}
\end{aligned}$$

(3) P_3 and P_4

$P_3 = P_4 > \frac{1}{4}$ still holds.

Proof of Proposition 6

The perceived winning probabilities of the players are:

$$\widetilde{p}_{13}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha} & \text{if } \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha} & \text{if } \lambda_1 e_1^\alpha \leqslant e_3^\alpha \end{cases}$$

$$\widetilde{p}_{31}^f = \begin{cases} 1 - \frac{1}{2}\frac{e_1^\alpha}{\lambda_3 e_3^\alpha} & \text{if } \lambda_3 e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{\lambda_3 e_3^\alpha}{e_1^\alpha} & \text{if } \lambda_3 e_3^\alpha \leqslant e_1^\alpha \end{cases}$$

Overconfident player 1 $\quad max \quad \widetilde{E}^f(U_{13}) = \widetilde{p}_{13}^f[u(w_1) - u(w_2)] + u(w_2) - ce_1$

$$=\begin{cases}\left(1-\frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)[u(w_1)-u(w_2)]+u(w_2)-ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_3^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}[u(w_1)-u(w_2)]+u(w_2)-ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_3^\alpha\end{cases}$$

Overconfident player 3 $\quad max \quad \widetilde{E}^f(U_{31})=\widetilde{p}_{31}^f[u(w_1)-u(w_2)]+u(w_2)-ce_3$

$$=\begin{cases}\left(1-\frac{1}{2}\frac{e_1^\alpha}{\lambda_3 e_3^\alpha}\right)[u(w_1)-u(w_2)]+u(w_2)-ce_3 & \text{if} \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{\lambda_3 e_3^\alpha}{e_1^\alpha}[u(w_1)-u(w_2)]+u(w_2)-ce_3 & \text{if} \quad \lambda_3 e_3^\alpha \leqslant e_1^\alpha\end{cases}$$

There are 4 cases.

$$\begin{cases}\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \leqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \leqslant e_1^\alpha\end{cases}$$

Since $\lambda_1 > \lambda_3 > 1$, the fourth case is impossible.

1. equilibrium efforts

   (1) case 1: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha$

   Player 1 $\quad max \quad \left(1-\frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)[u(w_1)-u(w_2)]+u(w_2)-ce_1$

   Player 3 $\quad max \quad \left(1-\frac{1}{2}\frac{e_1^\alpha}{\lambda_3 e_3^\alpha}\right)[u(w_1)-u(w_2)]+u(w_2)-ce_3$

   F.o.c

   $[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}[u(w_1)-u(w_2)]-c=0$

   $[e_3] \quad \frac{\alpha}{2\lambda_3}\frac{e_1^\alpha}{e_3^{\alpha+1}}[u(w_1)-u(w_2)]-c=0$

   S.o.c

   $[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_3^\alpha}{e_1^{\alpha+2}}[u(w_1)-u(w_2)]<0$

   $[e_3] \quad \frac{\alpha}{2\lambda_3}(-\alpha-1)\frac{e_1^\alpha}{e_3^{\alpha+2}}[u(w_1)-u(w_2)]<0$

   Solve F.O.C , we get

$$e_1=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}[u(w_1)-u(w_2)]$$

$$e_3=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1)-u(w_2)]$$

Check the condition $\lambda_3 e_3^\alpha \geqslant e_1^\alpha$:

$$\lambda_3 e_3^\alpha \geqslant e_1^\alpha \iff \frac{\lambda_3 e_3^\alpha}{e_1^\alpha} \geqslant 1$$
$$\iff \lambda_1^{\frac{\alpha}{2\alpha+1}} \lambda_3^{\frac{\alpha+1}{2\alpha+1}} \geqslant 1$$

Check the condition $\lambda_1 e_1^\alpha \geqslant e_3^\alpha$:

$$\lambda_1 e_1^\alpha \geqslant e_3^\alpha \iff \frac{\lambda_1 e_1^\alpha}{e_3^\alpha} \geqslant 1$$
$$\iff \lambda_1^{\frac{\alpha+1}{2\alpha+1}} \lambda_3^{\frac{\alpha}{2\alpha+1}} \geqslant 1$$

which is always satisfied.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \leqslant e_1^\alpha$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_3^\alpha}{\lambda_1 e_1^\alpha}\right)[u(w_1) - u(w_2)] + u(w_2) - ce_1$

Player 3 $\quad max \quad \frac{1}{2}\frac{\lambda_3 e_3^\alpha}{e_1^\alpha}[u(w_1) - u(w_2)] + u(w_2) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_3^\alpha}{e_1^{\alpha+1}}[u(w_1) - u(w_2)] - c = 0$

$[e_3] \quad \frac{\alpha}{2}\lambda_3\frac{e_3^{\alpha-1}}{e_1^\alpha}[u(w_1) - u(w_2)] - c = 0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \lambda_1\lambda_3 > 1$$

which contradicts the condition that $\lambda_3 e_3^\alpha \leqslant e_1^\alpha$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_3^\alpha \quad and \quad \lambda_3 e_3^\alpha \geqslant e_1^\alpha$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_3^\alpha}[u(w_1) - u(w_2)] + u(w_2) - ce_1$

Player 3 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_1^\alpha}{\lambda_3 e_3^\alpha}\right)[u(w_1) - u(w_2)] + u(w_2) - ce_3$

F.o.c

$[e_1] \quad \frac{\alpha}{2}\lambda_1\frac{e_1^{\alpha-1}}{e_3^\alpha}[u(w_1) - u(w_2)] - c = 0$

$[e_3] \quad \frac{\alpha}{2\lambda_3}\frac{e_1^\alpha}{e_3^{\alpha+1}}[u(w_1) - u(w_2)] - c = 0$

divide the two F.O.C , we get

$$\frac{e_3}{e_1} = \frac{1}{\lambda_1\lambda_3} < 1$$

which contradicts the condition that $\lambda_1 e_1^\alpha \leqslant e_3^\alpha$.

Thus the unique equilibrium is

$$e_1^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)]$$

$$e_3^f = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]$$

Since $\lambda_1 > \lambda_3 > 1$, we can get $e_1^f < e_3^f < \bar{e}^f$.

2. winning probabilities

The true winning probabilities are

$$\begin{aligned} p_{13}^f &= \frac{1}{2}\left(\frac{e_1^f}{e_3^f}\right)^\alpha \\ &= \frac{1}{2}\left(\lambda_1^{-\frac{1}{2\alpha+1}}\lambda_3^{\frac{1}{2\alpha+1}}\right)^\alpha \\ &= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{\frac{\alpha}{2\alpha+1}} \\ &< \frac{1}{2} \end{aligned}$$

$$p_{31}^f = 1 - p_{13}^f = 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{\frac{\alpha}{2\alpha+1}} > \frac{1}{2}$$

The perceived winning probabilities are

$$\begin{aligned} \widetilde{p}_{13}^f &= 1 - \frac{1}{2}\frac{(e_3^f)^\alpha}{\lambda_1(e_1^f)^\alpha} \\ &= 1 - \frac{1}{2\lambda_1}\left(\lambda_1^{\frac{1}{2\alpha+1}}\lambda_3^{-\frac{1}{2\alpha+1}}\right)^\alpha \\ &= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}} \end{aligned}$$

$$\begin{aligned} \widetilde{p}_{31}^f &= 1 - \frac{1}{2}\frac{(e_1^f)^\alpha}{\lambda_3(e_3^f)^\alpha} \\ &= 1 - \frac{1}{2\lambda_3}\left(\lambda_1^{-\frac{1}{2\alpha+1}}\lambda_3^{\frac{1}{2\alpha+1}}\right)^\alpha \\ &= 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}} \end{aligned}$$

Thus we have

$$\widetilde{p}_{13}^f > \widetilde{p}_{31}^f > p_{31}^f > \frac{1}{2} > p_{13}^f$$

3. expected utilities of final

$$\begin{aligned} \widetilde{E}^f(U_{13}) &= \widetilde{p}_{13}^f u(w_1) + (1 - \widetilde{p}_{13}^f)u(w_2) - ce_1^f \\ &= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)u(w_1) + \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}u(w_2) \\ &\quad - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)] \\ &= u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)] \end{aligned}$$

$$
\begin{aligned}
\widetilde{E}^f(U_{31}) &= \widetilde{p}^f_{31}u(w_1) + (1-\widetilde{p}^f_{31})u(w_2) - ce^f_3 \\
&= \left(1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)u(w_1) + \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}u(w_2) \\
&\quad - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)] \\
&= u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]
\end{aligned}
$$

Since $\alpha < 1$ and $\lambda_1 > \lambda_3 > 1$, $\widetilde{E}^f(U_{13}) > \widetilde{E}^f(U_{31}) > \overline{E}^f(U)$. The participation constraints are also satisfied.

Proof of Proposition 7

1. Continuation values

   Using Proposition 2, we can get the continuation values of each player.
   Overconfident player 1:

$$\widetilde{v}_1 = p^s_{34}\widetilde{E}^f(U_{13}) + p^s_{43}\widetilde{E}^f(U_{14})$$

   Since player 3 and 4 are identical, $\widetilde{E}^f(U_{13}) = \widetilde{E}^f(U_{14})$,

$$\widetilde{v}_1 = u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)]$$

   We show that $\widetilde{v}_1 > \overline{v}$:

$$
\begin{aligned}
\widetilde{v}_1 > \overline{v} &\iff u(w_1) - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)] > u(w_1) - \frac{1+\alpha}{2}[u(w_1) - u(w_2)] \\
&\iff \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} < 1
\end{aligned}
$$

   Overconfident player 2:

$$\widetilde{v}_2 = p^s_{34}\widetilde{E}^f(U_{23}) + p^s_{43}\widetilde{E}^f(U_{24})$$

   Since player 3 and 4 are identical, $\widetilde{E}^f(U_{23}) = \widetilde{E}^f(U_{24})$,

$$\widetilde{v}_2 = u(w_1) - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)] > \overline{v}$$

   We can easily get $\widetilde{v}_1 > \widetilde{v}_2$:

$$\widetilde{v}_1 > \widetilde{v}_2 \iff \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} < \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$$

   Thus we have $\widetilde{v}_1 > \widetilde{v}_2 > \overline{v}$.

2. The equilibrium

   Player 1 $\quad max \quad \widetilde{E}^s(U_{12}) = \widetilde{p}^s_{12}\widetilde{v}_1 - ce_1$

$$
= \begin{cases}
\left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\
\frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha
\end{cases}
$$

Player 2 $\quad max \quad \widetilde{E}^s(U_{21}) = \widetilde{p}^s_{21}\widetilde{v}_2 - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{\lambda_2 e_2^\alpha}\right)\widetilde{v}_2 - ce_2 & \text{if } \lambda_2 e_2^\alpha \geqslant e_1^\alpha \\ \frac{1}{2}\frac{\lambda_2 e_2^\alpha}{e_1^\alpha}\widetilde{v}_2 - ce_2 & \text{if } \lambda_2 e_2^\alpha \leqslant e_1^\alpha \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \leqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \geqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \geqslant e_1^\alpha \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \leqslant e_1^\alpha \end{cases}$$

Since $\lambda_1 > \lambda_2 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \leqslant e_1^\alpha$, which corresponds to (i) in proposition 7.

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}\frac{\lambda_2 e_2^\alpha}{e_1^\alpha}\widetilde{v}_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha\lambda_2}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha}\widetilde{v}_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha - 1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha\lambda_2}{2}(\alpha - 1)\frac{e_2^{\alpha-2}}{e_1^\alpha}\widetilde{v}_2 < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\lambda_2^\alpha(\widetilde{v}_1)^{1-\alpha}(\widetilde{v}_2)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^\alpha\lambda_2^{\alpha+1}(\widetilde{v}_1)^{-\alpha}(\widetilde{v}_2)^{\alpha+1}$$

$$\frac{e_2}{e_1} = \frac{\lambda_2\widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $\lambda_2 e_2^\alpha \leqslant e_1^\alpha$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

When $\lambda_2 e_2^\alpha \leqslant e_1^\alpha$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied.

② $\lambda_2 e_2^\alpha \leqslant e_1^\alpha$

$$\begin{aligned}
&\lambda_2 e_2^\alpha \leqslant e_1^\alpha \\
&\Longleftrightarrow \lambda_2\left(\frac{e_2}{e_1}\right)^\alpha \leqslant 1 \\
&\Longleftrightarrow \lambda_2\left(\frac{\lambda_1\lambda_2\widetilde{v}_2}{\widetilde{v}_1}\right)^\alpha \leqslant 1 \\
&\Longleftrightarrow \lambda_1^\alpha\lambda_2^{\alpha+1}(\widetilde{v}_1)^{-\alpha}(\widetilde{v}_2)^\alpha \leqslant 1 \\
&\Longleftrightarrow \lambda_2^{\alpha+1}(\widetilde{v}_2)^\alpha \leqslant \lambda_1^{-\alpha}(\widetilde{v}_1)^\alpha \\
&\Longleftrightarrow \lambda_1^\alpha\lambda_2^{\alpha+1}(\widetilde{v}_2)^\alpha \leqslant (\widetilde{v}_1)^\alpha \\
&\Longleftrightarrow \lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}}\widetilde{v}_2 \leqslant \widetilde{v}_1 \\
&\Longleftrightarrow \lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right) \leqslant \left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\
&\Longleftrightarrow \left(\lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}}-1\right)\frac{u(w_1)}{u(w_1)-u(w_2)} \leqslant \frac{1+\alpha}{2}\left(\lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}-\frac{\alpha+1}{2\alpha+1}}-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \\
&\Longleftrightarrow \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}
\end{aligned}$$

Therefore the solution

$$e_1^s = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\lambda_2^\alpha(\widetilde{v}_1)^{1-\alpha}(\widetilde{v}_2)^\alpha$$

$$e_2^s = \frac{\alpha}{2c}\lambda_1^\alpha\lambda_2^{\alpha+1}(\widetilde{v}_1)^{-\alpha}(\widetilde{v}_2)^{\alpha+1}$$

only applies when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$.

Therefore, in proposition 7 (i), $e_1^s > e_2^s$ is always satisfied since $\lambda_2 e_2^\alpha \leqslant e_1^\alpha$ and $\lambda_2 > 1$.

(2) case 2: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ *and* $\lambda_2 e_2^\alpha \geqslant e_1^\alpha$, which corresponds to (ii) and (iii) in proposition 7.

Player 1 $\quad max \quad \left(1-\frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \left(1-\frac{1}{2}\frac{e_1^\alpha}{\lambda_2 e_2^\alpha}\right)\widetilde{v}_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2\lambda_2}\frac{e_1^\alpha}{e_2^{\alpha+1}}\widetilde{v}_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2]\quad \frac{\alpha}{2\lambda_2}(-\alpha-1)\frac{e_1^{\alpha}}{e_2^{\alpha+2}}\widetilde{v}_2<0$

Solve F.O.C , we get

$$e_1=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}}$$
$$e_2=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha+1}{2\alpha+1}}$$
$$\frac{e_2}{e_1}=\left(\frac{\lambda_2^{-1}\widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1e_1^{\alpha}\geqslant e_2^{\alpha}$ and $\lambda_2e_2^{\alpha}\geqslant e_1^{\alpha}$:

① $\lambda_1e_1^{\alpha}\geqslant e_2^{\alpha}$

$$\begin{aligned}\lambda_1e_1^{\alpha}\geqslant e_2^{\alpha}&\Longleftrightarrow\frac{\lambda_1e_1^{\alpha}}{e_2^{\alpha}}\geqslant 1\\&\Longleftrightarrow\lambda_1^{\frac{\alpha+1}{2\alpha+1}}\lambda_2^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{-\frac{\alpha}{2\alpha+1}}\geqslant 1\\&\Longleftrightarrow\lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}\geqslant\lambda_2^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}}\end{aligned}$$

this is always satisfied. Therefore $\lambda_1e_1^{\alpha}\geqslant e_2^{\alpha}$ always holds.

② $\lambda_2e_2^{\alpha}\geqslant e_1^{\alpha}$

$$\begin{aligned}&\lambda_2e_2^{\alpha}\geqslant e_1^{\alpha}\\&\Longleftrightarrow\lambda_2\left(\frac{e_2}{e_1}\right)^{\alpha}\geqslant 1\\&\Longleftrightarrow\lambda_2\left(\frac{\lambda_2^{-1}\widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^{\frac{\alpha}{2\alpha+1}}\geqslant 1\\&\Longleftrightarrow\lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}}\geqslant 1\\&\Longleftrightarrow\lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}}\geqslant(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}\\&\Longleftrightarrow\lambda_1^{\alpha}\lambda_2^{\alpha+1}\widetilde{v}_2^{\alpha}\geqslant\widetilde{v}_1^{\alpha}\\&\Longleftrightarrow\lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}}\widetilde{v}_2\geqslant\widetilde{v}_1\\&\Longleftrightarrow\lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\\&\Longleftrightarrow\left(\lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}}-1\right)\frac{u(w_1)}{u(w_1)-u(w_2)}\geqslant\frac{1+\alpha}{2}\left(\lambda_1\lambda_2^{\frac{\alpha+1}{\alpha}-\frac{\alpha+1}{2\alpha+1}}-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\\&\Longleftrightarrow\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}\geqslant\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}\end{aligned}$$

Thus $\lambda_2e_2^{\alpha}\geqslant e_1^{\alpha}$ is satisfied when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}\geqslant\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$.

Therefore the solution

$$e_1^s=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(\widetilde{v}_2)^{\frac{\alpha+1}{2\alpha+1}}$$

only applies when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$.

When the above condition is satisfied,

$$e_1^s - e_2^s \begin{cases} > 0 & \text{when } \lambda_1^{-1}\widetilde{v}_1 > \lambda_2^{-1}\widetilde{v}_2 \\ = 0 & \text{when } \lambda_1^{-1}\widetilde{v}_1 = \lambda_2^{-1}\widetilde{v}_2 \\ < 0 & \text{when } \lambda_1^{-1}\widetilde{v}_1 < \lambda_2^{-1}\widetilde{v}_2 \end{cases}$$

$$\begin{aligned}
&\lambda_1^{-1}\widetilde{v}_1 \gtreqless \lambda_2^{-1}\widetilde{v}_2 \\
&\Longleftrightarrow \lambda_1^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) \gtreqless \lambda_2^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right) \\
&\Longleftrightarrow \frac{1+\alpha}{2}\left(\lambda_2^{-\frac{\alpha+1}{2\alpha+1}-1} - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right) \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\left(\lambda_2^{-1}-\lambda_1^{-1}\right) \\
&\Longleftrightarrow \frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}-1} - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}}{\lambda_2^{-1}-\lambda_1^{-1}} \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \\
&\Longleftrightarrow \frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}-1} - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}}{\frac{\lambda_1-\lambda_2}{\lambda_1\lambda_2}} \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \\
&\Longleftrightarrow \frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\lambda_1 - \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2}{\lambda_1-\lambda_2} \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \\
&\Longleftrightarrow \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \gtreqless \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}
\end{aligned}$$

Therefore,

$$e_1^s - e_2^s \begin{cases} > 0 & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \\ = 0 & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} = \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \\ < 0 & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} > \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \end{cases}$$

Combining the condition $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$ we can get

$$e_1^s - e_2^s \begin{cases} > 0 & \text{when } \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right) \\ \leqslant 0 & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \max\left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}\right] \end{cases}$$

where the first line corresponds to proposition 7 (ii) and the second line corresponds to proposition 7 (iii).

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad \lambda_2 e_2^\alpha \geqslant e_1^\alpha$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{e_1^\alpha}{\lambda_2 e_2^\alpha}\right)\widetilde{v}_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{\lambda_2 e_2^{\alpha+1}}\widetilde{v}_2 - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{\widetilde{v}_2}{\lambda_1\lambda_2\widetilde{v}_1} < 1$$

which contradicts the condition that $\lambda_1 e_1^\alpha \leqslant e_2^\alpha$

Therefore, the equilibrium in this semi-final is given by:

(1) Proposition 7 (i): when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$

$$\begin{aligned} e_1^s = \frac{\alpha}{2c}\lambda_1^{\alpha-1}\lambda_2^\alpha & \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha} \\ & \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha} [u(w_1) - u(w_2)] \end{aligned}$$

$$\begin{aligned} e_2^s = \frac{\alpha}{2c}\lambda_1^{\alpha}\lambda_2^{\alpha+1} & \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha} \\ & \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1+\alpha} [u(w_1) - u(w_2)] \end{aligned}$$

We show that $e_1^s > \overline{e}^s$:

$$\begin{aligned} e_1^s &= \frac{\alpha}{2c}\left(\lambda_1^{-1}\widetilde{v}_1\right)^{1-\alpha}\lambda_2^\alpha(\widetilde{v}_2)^\alpha \\ &> \frac{\alpha}{2c}\left(\lambda_2\widetilde{v}_2\right)^{1-\alpha}\lambda_2^\alpha(\widetilde{v}_2)^\alpha \\ &= \frac{\alpha}{2c}\lambda_2\widetilde{v}_2 \\ &> \frac{\alpha}{2c}\overline{v} = \overline{e}^s \end{aligned}$$

The equilibrium winning probabilities are

$$
\begin{aligned}
p_{21}^{s} &= \frac{1}{2}\left(\frac{e_2^s}{e_1^s}\right)^{\alpha} \\
&= \frac{1}{2}\left(\frac{\lambda_2 \widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^{\alpha} \\
&= \frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
p_{12}^{s} &= 1-p_{21}^{s} \\
&= 1-\frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{p}_{12}^{s} &= 1-\frac{1}{2}\frac{(e_2^s)^{\alpha}}{\lambda_1 (e_1^s)^{\alpha}} \\
&= 1-\frac{1}{2}\lambda_1^{\alpha-1}\lambda_2^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{p}_{21}^{s} &= \frac{1}{2}\frac{\lambda_2 (e_2^s)^{\alpha}}{(e_1^s)^{\alpha}} \\
&= \frac{1}{2}\lambda_2\left(\frac{\lambda_2 \widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^{\alpha} \\
&= \frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha+1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha}
\end{aligned}
$$

Since $e_1^s > e_2^s$, we can get $p_{12}^s > \frac{1}{2} > p_{21}^s$. Since $\lambda_2 (e_2^s)^{\alpha} \leqslant (e_1^s)^{\alpha}$, we can get $\widetilde{p}_{21}^s \leqslant \frac{1}{2}$.

Thus in Proposition 7 (i) we have

$$
\widetilde{p}_{12}^{s} > p_{12}^{s} > \frac{1}{2} \geqslant \widetilde{p}_{21}^{s} > p_{21}^{s}
$$

(2) Proposition 7 (ii) and (iii): when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$

The equilibrium efforts are

$$
\begin{aligned}
e_1^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}[u(w_1)-u(w_2)]
\end{aligned}
$$

$$
\begin{aligned}
e_2^s = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha+1}{2\alpha+1}}[u(w_1)-u(w_2)]
\end{aligned}
$$

and the efforts satisfy

$$e_1^s \begin{cases} > e_2^s > \overline{e}^s & \text{when } \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2} \left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}} \right) \\ \leqslant e_2^s & \text{when } \frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} \geqslant \max \left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \right] \end{cases}$$

① Proposition 7 (ii): when $\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2} \left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}} \right)$

We first show that $e_1^s > e_2^s > \overline{e}^s$ is satisfied under ①.
Since $e_2^s = \frac{\alpha}{2c} \left(\lambda_1^{-1}\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(\lambda_2^{-1}\widetilde{v}_2\right)^{\frac{\alpha+1}{2\alpha+1}}$ and $e_1^s > e_2^s$, if both $\lambda_1^{-1}\widetilde{v}_1 > \overline{v}$ and $\lambda_2^{-1}\widetilde{v}_2 > \overline{v}$ are satisfied then we can get $e_1^s > e_2^s > \overline{e}^s$.

We show that under the condition of $\frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2} \left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}} \right)$, both $\lambda_1^{-1}\widetilde{v}_1 > \overline{v}$ and $\lambda_2^{-1}\widetilde{v}_2 > \overline{v}$ are satisfied:

Let

$$f(\lambda_1) = \lambda_1^{-1} \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right)$$

$$\begin{aligned} f'(\lambda_1) &= -\lambda_1^{-2} \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right) + \lambda_1^{-1} \frac{1+\alpha}{2} \frac{\alpha+1}{2\alpha+1} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1} \\ &= \lambda_1^{-2} \left[-\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right) + \frac{1+\alpha}{2} \frac{\alpha+1}{2\alpha+1} \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} \right] \\ &= \lambda_1^{-2} \left[\frac{1+\alpha}{2} \left(\frac{\alpha+1}{2\alpha+1} + 1 \right) \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} - \frac{u(w_1)}{u(w_1)-u(w_2)} \right] \end{aligned}$$

Let $g(\lambda_1) = \frac{1+\alpha}{2} \left(\frac{\alpha+1}{2\alpha+1} + 1 \right) \lambda_1^{-\frac{\alpha+1}{2\alpha+1}} - \frac{u(w_1)}{u(w_1)-u(w_2)}$, we can easily get $g'(\lambda_1) < 0$.

$$g(\lambda_1 = 1) = \frac{1+\alpha}{2} \left(\frac{\alpha+1}{2\alpha+1} + 1 \right) - \frac{u(w_1)}{u(w_1)-u(w_2)}$$

$$g(\lambda_1 \to \infty) = -\frac{u(w_1)}{u(w_1)-u(w_2)} < 0$$

(a) When $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$

If $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, then $g(\lambda_1 = 1) < 0$, $g(\lambda_1) \leqslant 0$ is always satisfied and thus $f'(\lambda_1) < 0$ always holds. Therefore $f(\lambda_1) < f(\lambda_2) < f(1) = \overline{v}$. This contradicts the condition $\frac{u(w_1)}{u(w_1)-u(w_2)} \frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2} \left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}} \right)$ since we showed earlier that this condition is equivalent to $f(\lambda_1) > f(\lambda_2)$ and $e_1^s > e_2^s$.

(b) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$

If $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$, then $g(\lambda_1 = 1) > 0$ and thus

$$g(\lambda_1) \quad \text{and} \quad f'(\lambda_1) \begin{cases} > 0 & \text{when } \lambda_1 < \left(\frac{\frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1}+1\right)}{\frac{u(w_1)}{u(w_1)-u(w_2)}}\right)^{\frac{2\alpha+1}{\alpha+1}} \\ < 0 & \text{when } \lambda_1 > \left(\frac{\frac{1+\alpha}{2}\left(\frac{\alpha+1}{2\alpha+1}+1\right)}{\frac{u(w_1)}{u(w_1)-u(w_2)}}\right)^{\frac{2\alpha+1}{\alpha+1}} \end{cases}$$

$$f(\lambda_1) \begin{cases} > \overline{v} & \text{when } \lambda_1 < \hat{\lambda} \\ = \overline{v} & \text{when } \lambda_1 = \hat{\lambda} \\ < \overline{v} & \text{when } \lambda_1 > \hat{\lambda} \end{cases}$$

where $\hat{\lambda}$ is as derived in the proposition 3.

When $f(\lambda_1) > f(\lambda_2)$ is satisfied, $f(\lambda_1) > f(\lambda_2) > f(1) = f(\hat{\lambda}) = \overline{v}$ must be true.
If $f(\lambda_2) < \overline{v}$, then $\lambda_2 > \hat{\lambda}$. And since $f'(\lambda_1) < 0$ for $\forall\ \lambda_1 > \hat{\lambda}$, $f(\lambda_1) < f(\lambda_2)$ when $f(\lambda_2) < \overline{v}$. This contradicts $f(\lambda_1) > f(\lambda_2)$. Thus when the condition $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}\right)$ is satisfied, $e_1 > e_2 > \overline{e}^s$ is always true.

The true equilibrium winning probabilities under ① are

$$\begin{aligned} p^s_{21} &= \frac{1}{2}\left(\frac{e^s_2}{e^s_1}\right)^{\alpha} \\ &= \frac{1}{2}\left(\frac{\lambda_2^{-1}\widetilde{v}_2}{\lambda_1^{-1}\widetilde{v}_1}\right)^{\frac{\alpha}{2\alpha+1}} \\ &= \frac{1}{2}\lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\ &\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \end{aligned}$$

$$\begin{aligned} p^s_{12} &= 1 - p^s_{21} \\ &= 1 - \frac{1}{2}\lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\ &\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \end{aligned}$$

Since $e^s_1 > e^s_2$, we can get $p^s_{12} > \frac{1}{2} > p^s_{21}$.
Thus we have $\widetilde{p}^s_{12} > p^s_{12} > \frac{1}{2} > p^s_{21}$.

② Proposition 7 (iii): when $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \max\left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}\right]$

The true equilibrium winning probabilities are

$$
\begin{aligned}
p_{12}^s &= \frac{1}{2}\left(\frac{e_1^s}{e_2^s}\right)^{\alpha} \\
&= \frac{1}{2}\left(\frac{\lambda_1^{-1}\widetilde{v}_1}{\lambda_2^{-1}\widetilde{v}_2}\right)^{\frac{\alpha}{2\alpha+1}} \\
&= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

$$
\begin{aligned}
p_{21}^s &= 1-p_{12}^s \\
&= 1-\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

Since $e_1^s \leqslant e_2^s$, we can get $p_{21}^s \geqslant \frac{1}{2} \geqslant p_{12}^s$.

The perceived equilibrium winning probabilities under (2) are:

$$
\begin{aligned}
\widetilde{p}_{12}^s &= 1-\frac{1}{2}\frac{(e_2^s)^{\alpha}}{\lambda_1\,(e_1^s)^{\alpha}} \\
&= 1-\frac{1}{2}\lambda_1^{-1}\Bigg(\lambda_1^{\frac{1}{2\alpha+1}}\lambda_2^{-\frac{1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{1}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{1}{2\alpha+1}}\Bigg)^{\alpha} \\
&= 1-\frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{p}_{21}^s &= 1-\frac{1}{2}\frac{(e_1^s)^{\alpha}}{\lambda_2\,(e_2^s)^{\alpha}} \\
&= 1-\frac{1}{2}\lambda_2^{-1}\Bigg(\lambda_1^{-\frac{1}{2\alpha+1}}\lambda_2^{\frac{1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{1}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{1}{2\alpha+1}}\Bigg)^{\alpha} \\
&= 1-\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\quad \left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}
\end{aligned}
$$

We show that $\widetilde{p}^s_{12} > \widetilde{p}^s_{21}$:

$$
\begin{aligned}
&\widetilde{p}^s_{12} > \widetilde{p}^s_{21} \\
\iff & 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}} > 1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{-\frac{\alpha}{2\alpha+1}} \\
\iff & \lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{-\frac{\alpha}{2\alpha+1}} > \lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}} \\
\iff & \lambda_1^{\frac{1}{2\alpha+1}}\lambda_2^{-\frac{1}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{2\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{-\frac{2\alpha}{2\alpha+1}} > 1 \\
\iff & \frac{\lambda_1^{\frac{1}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{2\alpha}{2\alpha+1}}}{\lambda_2^{\frac{1}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{2\alpha}{2\alpha+1}}} > 1
\end{aligned}
$$

Thus in Proposition 7 (ii) we have $\widetilde{p}^s_{12} > p^s_{12} > \frac{1}{2} > p^s_{21}$ and in Proposition 7 (iii) we have $\widetilde{p}^s_{12} > \widetilde{p}^s_{21} > p^s_{21} \geqslant \frac{1}{2} \geqslant p^s_{12}$.

3. Participation constraint

(1) When $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$

$$
\begin{aligned}
\widetilde{E}^s(U_{12}) &= \widetilde{p}^s_{12}\widetilde{v}_1 - ce^s_1 \\
&> p^s_{12}\widetilde{v}_1 - ce^s_1 \\
&= \left(1 - \frac{1}{2}\lambda_1^\alpha\lambda_2^\alpha\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^\alpha\right)\widetilde{v}_1 - c\frac{\alpha}{2c}\lambda_1^{\alpha-1}\lambda_2^\alpha\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^\alpha \\
&= \widetilde{v}_1 - \frac{1}{2}\lambda_1^\alpha\lambda_2^\alpha\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^\alpha - \frac{\alpha}{2}\lambda_1^{\alpha-1}\lambda_2^\alpha\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^\alpha \\
&> \widetilde{v}_1 - \frac{1}{2}\lambda_1^\alpha\lambda_2^\alpha\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^\alpha - \frac{\alpha}{2}\lambda_1^{\alpha}\lambda_2^\alpha\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^\alpha \\
&= \widetilde{v}_1 - \frac{1+\alpha}{2}\lambda_1^\alpha\lambda_2^\alpha\left(\widetilde{v}_1\right)^{1-\alpha}\left(\widetilde{v}_2\right)^\alpha \\
&= \widetilde{v}_1\left[1 - \frac{1+\alpha}{2}\lambda_1^\alpha\lambda_2^\alpha\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^\alpha\right] \\
&= \widetilde{v}_1\left[1 - \frac{1+\alpha}{2}\left(\frac{e^s_2}{e^s_1}\right)^\alpha\right] \\
&> 0
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{E}^s(U_{21}) &= \widetilde{p}^s_{21}\widetilde{v}_2 - ce^s_2 \\
&= \frac{1}{2}\lambda_1^\alpha\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^\alpha\widetilde{v}_2 - c\frac{\alpha}{2c}\lambda_1^\alpha\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{\alpha+1} \\
&= \frac{1}{2}\lambda_1^\alpha\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{1+\alpha} - \frac{\alpha}{2}\lambda_1^\alpha\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{\alpha+1} \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha\lambda_2^{\alpha+1}\left(\widetilde{v}_1\right)^{-\alpha}\left(\widetilde{v}_2\right)^{1+\alpha} \\
&\geqslant 0
\end{aligned}
$$

(2) When $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}}$

$$\begin{aligned}
\widetilde{E}^s(U_{12}) &= \widetilde{p}^s_{12}\widetilde{v}_1 - ce^s_1 \\
&= \left(1-\frac{1}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}}\right)\widetilde{v}_1 - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}} \\
&= \widetilde{v}_1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}} \\
&= \left(\widetilde{v}_1\right)^{\frac{\alpha+1}{2\alpha+1}}\left[\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha}{2\alpha+1}}\right] \\
&> 0
\end{aligned}$$

$$\begin{aligned}
\widetilde{E}^s(U_{21}) &= \widetilde{p}^s_{21}\widetilde{v}_1 - ce^s_2 \\
&= \left(1-\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{-\frac{\alpha}{2\alpha+1}}\right)\widetilde{v}_2 - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \widetilde{v}_2 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \widetilde{v}_2\left[1 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}}\left(\widetilde{v}_2\right)^{-\frac{\alpha}{2\alpha+1}}\right] \\
&= \widetilde{v}_2\left[1 - \frac{1+\alpha}{2}\frac{(e^s_1)^\alpha}{\lambda_2(e^s_2)^\alpha}\right] \\
&> 0
\end{aligned}$$

Lemma 4

$$\lim_{\lambda_1\to\infty}\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}} = \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$$

$$\lim_{\lambda_1\to\lambda_2}\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}-\lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}-\lambda_1^{-1}} = \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$$

$$\lim_{\lambda_1\to\infty}\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}}-\frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2} = \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$$

$$\lim_{\lambda_1\to\lambda_2}\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}}-\frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2} = \frac{3\alpha+2}{2\alpha+1}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$$

Proof of Lemma 4

$$\lim_{\lambda_1\to\infty} \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} = \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}}}{\lambda_2^{\frac{\alpha+1}{\alpha}}} = \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}$$

$$\begin{aligned}
&\lim_{\lambda_1\to\lambda_2} \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \\
&= \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_2^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_2^{-1}} \\
&= \frac{\lambda_2^{-\frac{3\alpha+2}{2\alpha+1}}\left(\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}+\frac{3\alpha+2}{2\alpha+1}} - 1\right)}{\lambda_2^{-1}\left(\lambda_2^{\frac{\alpha+1}{\alpha}+1} - 1\right)} \\
&= \lambda_2^{-\frac{3\alpha+2}{2\alpha+1}+1}\frac{\lambda_2^{\frac{2\alpha+1}{\alpha}} - 1}{\lambda_2^{\frac{2\alpha+1}{\alpha}} - 1} \\
&= \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}
\end{aligned}$$

$$\begin{aligned}
\lim_{\lambda_1\to\infty} \frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2} &= \lim_{\lambda_1\to\infty} \frac{\frac{1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}+1}}}{1-\frac{\lambda_2}{\lambda_1}} \\
&= \frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}} - 0}{1-0} \\
&= \lambda_2^{-\frac{\alpha+1}{2\alpha+1}}
\end{aligned}$$

Let $t = \lambda_1 - \lambda_2$,

$$\begin{aligned}
\lim_{\lambda_1\to\lambda_2} \frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2} &= \lim_{t\to 0} \frac{\frac{\lambda_2+t}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{(\lambda_2+t)^{\frac{\alpha+1}{2\alpha+1}}}}{t} \\
&= \lim_{t\to 0} \frac{\frac{\partial\left(\frac{\lambda_2+t}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{(\lambda_2+t)^{\frac{\alpha+1}{2\alpha+1}}}\right)}{\partial t}}{\frac{\partial t}{\partial t}} \\
&= \lim_{t\to 0} \frac{\lambda_2^{-\frac{\alpha+1}{2\alpha+1}} - \lambda_2\left(-\frac{\alpha+1}{2\alpha+1}\right)(\lambda_2+t)^{-\frac{\alpha+1}{2\alpha+1}-1}}{1} \\
&= \left(\frac{3\alpha+2}{2\alpha+1}\right)\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}
\end{aligned}$$

Lemma 5

Lemma 5 *In the semifinal between two rational players of a two-stage elimination contest where the overconfident players 1 and 2 are seeded in one semifinal, the rational players*

3 and 4 are seeded in the other semifinal, and $\lambda_1 > \lambda_2 > 1 = \lambda_3 = \lambda_4$, the equilibrium efforts and winning probabilities satisfy $e_3^s = e_4^s > \overline{e}^s$ and $p_{34}^s = p_{43}^s = 1/2$.

Proof of Lemma 5

1. Continuation values

   Rational player 3:

$$
\begin{aligned}
v_3 &= p_{12}^s E^f(U_{31}) + p_{21}^s E^f(U_{32}) \\
&= \left[p_{12}^s\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right) + p_{21}^s\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\right] \\
&\quad \times [u(w_1)-u(w_2)] \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(\lambda_2^{-\frac{\alpha}{2\alpha+1}} + p_{12}^s\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\right)\right][u(w_1)-u(w_2)] \\
&> \overline{v}
\end{aligned}
$$

   where p_{12}^s is as derived in the proof of proposition 7.

   Rational player 4:

   Since player 3 and player 4 are identical,

$$v_4 = v_3 > \overline{v}$$

2. The equilibrium

   (1) When $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \leqslant \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}$

$$
\begin{aligned}
e_3^s = e_4^s &= \frac{\alpha}{2c}v_3 \\
&= \frac{\alpha}{2c}\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(\lambda_2^{-\frac{\alpha}{2\alpha+1}}\right.\right. \\
&\quad + \left(1 - \frac{1}{2}\lambda_1^{\alpha}\lambda_2^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\right. \\
&\quad \left.\left.\left.\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\alpha}\right)\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\right)\right][u(w_1)-u(w_2)]
\end{aligned}
$$

$$p_{34}^s = p_{43}^s = \frac{1}{2}$$

   (2) When $\frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} < \frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1 - \lambda_2}$

$$
\begin{aligned}
e_3^s = e_4^s &= \frac{\alpha}{2c} v_3 \\
&= \frac{\alpha}{2c}\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(\lambda_2^{-\frac{\alpha}{2\alpha+1}}\right.\right. \\
&+ \left(1-\frac{1}{2}\lambda_1^{\frac{\alpha}{2\alpha+1}}\lambda_2^{-\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\right. \\
&\left.\left.\left.\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}}\right)\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\right)\right][u(w_1)-u(w_2)]
\end{aligned}
$$

$$p_{34}^s = p_{43}^s = \frac{1}{2}$$

(3) When $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha} \geqslant \max\left[\frac{\frac{\lambda_1}{\lambda_2^{\frac{\alpha+1}{2\alpha+1}}} - \frac{\lambda_2}{\lambda_1^{\frac{\alpha+1}{2\alpha+1}}}}{\lambda_1-\lambda_2}, \frac{\lambda_2^{\frac{(\alpha+1)^2}{\alpha(2\alpha+1)}} - \lambda_1^{-\frac{3\alpha+2}{2\alpha+1}}}{\lambda_2^{\frac{\alpha+1}{\alpha}} - \lambda_1^{-1}}\right]$

$$
\begin{aligned}
e_3^s = e_4^s &= \frac{\alpha}{2c} v_3 \\
&= \frac{\alpha}{2c}\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(\lambda_2^{-\frac{\alpha}{2\alpha+1}}\right.\right. \\
&+ \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_2^{\frac{\alpha}{2\alpha+1}}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{\frac{\alpha}{2\alpha+1}} \\
&\left.\left.\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_2^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \lambda_2^{-\frac{\alpha}{2\alpha+1}}\right)\right)\right][u(w_1)-u(w_2)]
\end{aligned}
$$

$$p_{34}^s = p_{43}^s = \frac{1}{2}$$

Since $v_3 = v_4 > \overline{v}$, $e_3^s = e_4^s > \overline{e}^s$ is always satisfied.

3. Participation constraint

$$
\begin{aligned}
E^s(U_{34}) &= p_{34}^s v_3 - c e_3 \\
&= \frac{1-\alpha}{2} v_3 \\
&\geqslant 0
\end{aligned}
$$

$$E^s(U_{43}) = E^s(U_{34}) \geqslant 0$$

Proof of Proposition 8

1. Continuation values of each player

Overconfident player 1:

$$\begin{aligned}
\widetilde{v}_1 &= p^s_{34}\widetilde{E}^f(U_{13}) + p^s_{43}\widetilde{E}^f(U_{14}) \\
&= p^s_{34}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)[u(w_1)-u(w_2)] \\
&\quad + p^s_{43}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)[u(w_1)-u(w_2)] \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1-p^s_{34}+p^s_{34}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right][u(w_1)-u(w_2)]
\end{aligned}$$

Rational player 2:

$$\begin{aligned}
v_2 &= p^s_{34}E^f(U_{23}) + p^s_{43}E^f(U_{24}) \\
&= p^s_{34}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)[u(w_1)-u(w_2)] \\
&\quad + p^s_{43}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)[u(w_1)-u(w_2)] \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1-p^s_{34}+p^s_{34}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\right][u(w_1)-u(w_2)]
\end{aligned}$$

Overconfident player 3:

$$\begin{aligned}
\widetilde{v}_3 &= p^s_{12}\widetilde{E}^f(U_{31}) + p^s_{21}\widetilde{E}^f(U_{32}) \\
&= p^s_{12}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)[u(w_1)-u(w_2)] \\
&\quad + p^s_{21}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)[u(w_1)-u(w_2)] \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1-p^s_{12}+p^s_{12}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right][u(w_1)-u(w_2)]
\end{aligned}$$

Rational player 4:

$$\begin{aligned}
v_4 &= p^s_{12}E^f(U_{41}) + p^s_{21}E^f(U_{42}) \\
&= p^s_{12}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)[u(w_1)-u(w_2)] \\
&\quad + p^s_{21}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)[u(w_1)-u(w_2)] \\
&= \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1-p^s_{12}+p^s_{12}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\right][u(w_1)-u(w_2)]
\end{aligned}$$

2. The equilibrium of the semifinal between player 1 and player 2

Player 1 $\quad max \quad \widetilde{E}^s(U_{12}) = \widetilde{p}^s_{12}\widetilde{v}_1 - ce_1$

$$= \begin{cases} \left(1-\frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \geqslant e_2^\alpha \\ \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1 & \text{if} \quad \lambda_1 e_1^\alpha \leqslant e_2^\alpha \end{cases}$$

Player 2 $\quad max \quad E^s(U_{21}) = p^s_{21} v_2 - ce_2$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right) v_2 - ce_2 & \text{if}\, e_2 \geqslant e_1 \\ \frac{1}{2}\frac{e_2^\alpha}{e_1^\alpha} v_2 - ce_2 & \text{if}\, e_2 \leqslant e_1 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \\ \lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1 \\ \lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1 \end{cases}$$

Since $\lambda_1 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_1 e_1^\alpha \geqslant e_2^\alpha \quad and \quad e_2 \leqslant e_1$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right) \widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \frac{1}{2}(\frac{e_2}{e_1})^\alpha v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_2^{\alpha-1}}{e_1^\alpha} v_2 - c = 0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha - 1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1 < 0$

$[e_2] \quad \frac{\alpha}{2}(\alpha - 1)\frac{e_2^{\alpha-2}}{e_1^\alpha} v_2 < 0$

Solve the two F.O.C , we get

$$e_1 = \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(v_2)^\alpha$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^\alpha(\widetilde{v}_1)^{-\alpha}(v_2)^{\alpha+1}$$

$$\frac{e_2}{e_1} = \lambda_1\frac{v_2}{\widetilde{v}_1}$$

$$\begin{aligned} p^s_{21} &= \frac{1}{2}\left(\frac{e_2}{e_1}\right)^\alpha \\ &= \frac{1}{2}\left(\frac{\lambda_1 v_2}{\widetilde{v}_1}\right)^\alpha \end{aligned}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_1 \geqslant e_2$:

As long as $e_1 \geqslant e_2$ is satisfied, $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ is satisfied. So we only have to check $e_1 \geqslant e_2$.

$$\begin{aligned}
e_2 \leqslant e_1 &\iff \frac{e_2}{e_1} \leqslant 1 \\
&\iff \frac{1}{2}\left(\frac{e_2}{e_1}\right)^\alpha \leqslant \frac{1}{2} \\
&\iff \frac{e_2}{e_1} \leqslant 1 \\
&\iff \frac{v_2}{\lambda_1^{-1}\widetilde{v}_1} \leqslant 1 \\
&\iff \frac{\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)}{\lambda_1^{-1}\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right]} \leqslant 1
\end{aligned}$$

Let

$$\begin{aligned}
f(p_{34}^s) = \lambda_1^{-1}&\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right] \\
&- \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{34}^s + p_{34}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)\right]
\end{aligned}$$

Rearrange the terms we can get

$$\begin{aligned}
f(p_{34}^s) = \lambda_1^{-1}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&+ \frac{1+\alpha}{2}\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(1 - \lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right) p_{34}^s
\end{aligned}$$

Since $\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(1 - \lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right) < 0$ and $p_{34}^s \in [0,1]$, $f(p_{34}^s)$ reaches minimum at $p_{34}^s = 1$. Thus, $e_2 \leqslant e_1$ is always satisfied as long as $f(p_{34}^s = 1) \geqslant 0$.

$$\begin{aligned}
f(p_{34}^s = 1) = \lambda_1^{-1}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&+ \frac{1+\alpha}{2}\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(1 - \lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)
\end{aligned}$$

$$\lambda_1^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)-\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)$$
$$+\frac{1+\alpha}{2}\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant 0$$
$$\Longleftrightarrow\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)-\lambda_1\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)$$
$$+\lambda_1\frac{1+\alpha}{2}\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant 0$$
$$\Longleftrightarrow\frac{1+\alpha}{2}\left[\lambda_1\left(1+\left(\lambda_3^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\right)-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right]\geqslant(\lambda_1-1)\frac{u(w_1)}{u(w_1)-u(w_2)}$$
$$\Longleftrightarrow\frac{1}{\lambda_1-1}\left[\lambda_1\left(\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}+\lambda_3^{-\frac{\alpha}{2\alpha+1}}-\lambda_3^{-\frac{\alpha}{2\alpha+1}}\lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right)-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right]\geqslant\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}$$
$$\Longleftrightarrow\frac{1}{\lambda_1-1}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}$$

To ensure $e_1>e_2$ and $p^s_{12}>p^s_{21}$, we need $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}<\frac{1}{\lambda_1-1}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)$.

Since $\frac{\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}}{\lambda_1-1}$ decreasing in λ_1 for $\lambda_1>1$ and its limit when $\lambda_1\to 1$ is $\frac{3\alpha+2}{2\alpha+1}$, we can get $\frac{1}{\lambda_1-1}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)<\frac{3\alpha+2}{2\alpha+1}$. Hence, to satisfy the inequality $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}<\frac{1}{\lambda_1-1}\lambda_3^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_1-\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)$, we also need $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}<\frac{3\alpha+2}{2\alpha+1}$, which is equivalent to $\frac{u(w_1)-u(w_2)}{u(w_2)}>\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$.

(2) case 2: $\lambda_1 e_1^\alpha\geqslant e_2^\alpha \quad and \quad e_2\geqslant e_1$

Player 1 $\quad max \quad \left(1-\frac{1}{2}\frac{e_2^\alpha}{\lambda_1 e_1^\alpha}\right)\widetilde{v}_1-ce_1$

Player 2 $\quad max \quad \left(1-\frac{1}{2}\frac{e_1^\alpha}{e_2^\alpha}\right)v_2-ce_2$

F.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}\frac{e_2^\alpha}{e_1^{\alpha+1}}\widetilde{v}_1-c=0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2-c=0$

S.o.c

$[e_1] \quad \frac{\alpha}{2\lambda_1}(-\alpha-1)\frac{e_2^\alpha}{e_1^{\alpha+2}}\widetilde{v}_1<0$

$[e_2] \quad \frac{\alpha}{2}(-\alpha-1)\frac{e_1^\alpha}{e_2^{\alpha+2}}v_2<0$

Solve F.O.C , we get

$$e_1=\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha+1}{2\alpha+1}}(v_2)^{\frac{\alpha}{2\alpha+1}}$$

$$e_2 = \frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{\frac{\alpha+1}{2\alpha+1}}$$

$$\frac{e_2}{e_1} = \lambda_1^{\frac{1}{2\alpha+1}}(\widetilde{v}_1)^{-\frac{1}{2\alpha+1}}(v_2)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$ and $e_2 \geqslant e_1$:

① $\lambda_1 e_1^\alpha \geqslant e_2^\alpha$

$$\begin{aligned}\lambda_1 e_1^\alpha \geqslant e_2^\alpha &\Longleftrightarrow \frac{\lambda_1 e_1^\alpha}{e_2^\alpha} \geqslant 1 \\ &\Longleftrightarrow \lambda_1^{\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_1)^{\frac{\alpha}{2\alpha+1}}(v_2)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1\end{aligned}$$

Since $\lambda_1 > 1$ and $\widetilde{v}_1 > v_2$, $\lambda_1 e_1^\alpha > e_2^\alpha$ is always satisfied.

② $e_2 \geqslant e_1$

$e_2 \geqslant e_1$ is always satisfied as long as $f(p_{34}^s = 0) \leqslant 0$.

$$f(p_{34}^s = 0) = \lambda_1^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)$$

$$\begin{aligned}f(p_{34}^s = 0) \leqslant 0 &\Longleftrightarrow \frac{u(w_1)}{u(w_1)-u(w_2)}\left(\lambda_1^{-1} - 1\right) + \frac{1+\alpha}{2}\left(1 - \lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right) \leqslant 0 \\ &\Longleftrightarrow \frac{1+\alpha}{2}\left(1 - \lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}\right) \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\left(1 - \lambda_1^{-1}\right) \\ &\Longleftrightarrow \frac{1 - \lambda_1^{-1-\frac{\alpha+1}{2\alpha+1}}}{1 - \lambda_1^{-1}} \leqslant \frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}\end{aligned}$$

(3) case 3: $\lambda_1 e_1^\alpha \leqslant e_2^\alpha \quad and \quad e_2 \geqslant e_1$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_1 e_1^\alpha}{e_2^\alpha}\widetilde{v}_1 - ce_1$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}(\frac{e_1}{e_2})^\alpha\right)v_2 - ce_2$

F.o.c

$[e_1] \quad \frac{\alpha\lambda_1}{2}\frac{e_1^{\alpha-1}}{e_2^\alpha}\widetilde{v}_1 - c = 0$

$[e_2] \quad \frac{\alpha}{2}\frac{e_1^\alpha}{e_2^{\alpha+1}}v_2 - c = 0$

divide the two F.O.C , we get

$$\frac{e_2}{e_1} = \frac{v_2}{\lambda_1\widetilde{v}_1} < 1$$

which contradicts the condition that $e_2 \geqslant e_1$

3. The equilibrium between player 3 and player 4

Player 3 $\quad max \quad \widetilde{E}^s(U_{34}) = \widetilde{p}_{34}^s\widetilde{v}_3 - ce_3$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_4^\alpha}{\lambda_3 e_3^\alpha}\right)\widetilde{v}_3 - ce_3 & \text{if} \quad \lambda_3 e_3^\alpha \geqslant e_4^\alpha \\ \frac{1}{2}\frac{\lambda_3 e_3^\alpha}{e_4^\alpha}\widetilde{v}_3 - ce_3 & \text{if} \quad \lambda_3 e_3^\alpha \leqslant e_4^\alpha \end{cases}$$

Player 4 $\quad max \quad E^s(U_{43}) = p_{43}^s v_4 - ce_4$

$$= \begin{cases} \left(1 - \frac{1}{2}\frac{e_3^\alpha}{e_4^\alpha}\right) v_4 - ce_4 & \text{if}\, e_4 \geqslant e_3 \\ \frac{1}{2}\frac{e_4^\alpha}{e_3^\alpha} v_4 - ce_4 & \text{if}\, e_4 \leqslant e_3 \end{cases}$$

There are 4 cases.

$$\begin{cases} \lambda_3 e_3^\alpha \geqslant e_4^\alpha \quad and \quad e_4 \leqslant e_3 \\ \lambda_3 e_3^\alpha \geqslant e_4^\alpha \quad and \quad e_4 \geqslant e_3 \\ \lambda_3 e_3^\alpha \leqslant e_4^\alpha \quad and \quad e_4 \geqslant e_3 \\ \lambda_3 e_3^\alpha \leqslant e_4^\alpha \quad and \quad e_4 \leqslant e_3 \end{cases}$$

Since $\lambda_3 > 1$, the fourth case is impossible.

(1) case 1: $\lambda_3 e_3^\alpha \geqslant e_4^\alpha \quad and \quad e_4 \leqslant e_3$

Player 1 $\quad max \quad \left(1 - \frac{1}{2}\frac{e_4^\alpha}{\lambda_3 e_3^\alpha}\right)\widetilde{v}_3 - ce_3$

Player 2 $\quad max \quad \frac{1}{2}(\frac{e_4}{e_3})^\alpha v_4 - ce_4$

F.o.c

$[e_3] \quad \frac{\alpha}{2\lambda_3}\frac{e_4^\alpha}{e_3^{\alpha+1}}\widetilde{v}_3 - c = 0$

$[e_4] \quad \frac{\alpha}{2}\frac{e_4^{\alpha-1}}{e_3^\alpha} v_4 - c = 0$

S.o.c

$[e_3] \quad \frac{\alpha}{2\lambda_3}(-\alpha - 1)\frac{e_4^\alpha}{e_3^{\alpha+2}}\widetilde{v}_3 < 0$

$[e_4] \quad \frac{\alpha}{2}(\alpha - 1)\frac{e_4^{\alpha-2}}{e_3^\alpha} v_4 < 0$

Solve the two F.O.C , we get

$$e_3 = \frac{\alpha}{2c}\lambda_3^{\alpha-1}(\widetilde{v}_3)^{1-\alpha}(v_4)^\alpha$$

$$e_4 = \frac{\alpha}{2c}\lambda_3^{\alpha}(\widetilde{v}_3)^{-\alpha}(v_4)^{\alpha+1}$$

$$\frac{e_4}{e_3} = \lambda_3\frac{v_4}{\widetilde{v}_3}$$

$$
\begin{aligned}
p_{43}^s &= \frac{1}{2}\left(\frac{e_4}{e_3}\right)^{\alpha} \\
&= \frac{1}{2}\left(\frac{\lambda_3 v_4}{\widetilde{v}_3}\right)^{\alpha}
\end{aligned}
$$

Check the conditions $\lambda_3 e_3^\alpha \geqslant e_4^\alpha$ and $e_3 \geqslant e_4$:
As long as $e_3 \geqslant e_4$ is satisfied, $\lambda_3 e_3^\alpha \geqslant e_4^\alpha$ is satisfied. So we only have to check $e_3 \geqslant e_4$.

$$
\begin{aligned}
e_4 \leqslant e_3 &\iff \frac{e_4}{e_3} \leqslant 1 \\
&\iff \frac{1}{2}\left(\frac{e_4}{e_3}\right)^{\alpha} \leqslant \frac{1}{2} \\
&\iff p_{43}^s \leqslant \frac{1}{2} \\
&\iff \frac{v_4}{\lambda_3^{-1}\widetilde{v}_3} \leqslant 1 \\
&\iff \frac{\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{12}^s + p_{12}^s \lambda_3^{-\frac{\alpha}{2\alpha+1}}\right)}{\lambda_3^{-1}\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{12}^s + p_{12}^s \lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right]} \leqslant 1
\end{aligned}
$$

Let

$$
\begin{aligned}
f(p_{12}^s) = \lambda_3^{-1}&\left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{12}^s + p_{12}^s \lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right] \\
&- \left[\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\left(1 - p_{12}^s + p_{12}^s \lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\right]
\end{aligned}
$$

Rearrange the terms we can get

$$
\begin{aligned}
f(p_{12}^s) = \lambda_3^{-1}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&+ \frac{1+\alpha}{2}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(1 - \lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)p_{12}^s
\end{aligned}
$$

Since $\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(1 - \lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right) < 0$ and $p_{12}^s \in [0,1]$, $f(p_{12}^s)$ reaches minimum at $p_{12}^s = 1$. Thus, $e_4 \leqslant e_3$ is always satisfied as long as $f(p_{12}^s = 1) \geqslant 0$.

$$
\begin{aligned}
f(p_{12}^s = 1) = \lambda_3^{-1}&\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right) - \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right) \\
&+ \frac{1+\alpha}{2}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}} - 1\right)\left(1 - \lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)
\end{aligned}
$$

$$\lambda_3^{-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)-\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)$$
$$+\frac{1+\alpha}{2}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant 0$$
$$\Longleftrightarrow\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)-\lambda_3\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)$$
$$+\lambda_3\frac{1+\alpha}{2}\left(\lambda_1^{-\frac{\alpha}{2\alpha+1}}-1\right)\left(1-\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant 0$$
$$\Longleftrightarrow\frac{1+\alpha}{2}\left[\lambda_3\left(1-\left(1-\lambda_1^{-\frac{\alpha}{2\alpha+1}}\right)\left(1-\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)\right)-\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right]\geqslant(\lambda_3-1)\frac{u(w_1)}{u(w_1)-u(w_2)}$$
$$\Longleftrightarrow\frac{1}{\lambda_3-1}\left[\lambda_3\left(\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}+\lambda_1^{-\frac{\alpha}{2\alpha+1}}-\lambda_1^{-\frac{\alpha}{2\alpha+1}}\lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right)-\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right]\geqslant\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}$$
$$\Longleftrightarrow\frac{1}{\lambda_3-1}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_3-\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)\geqslant\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}$$

To ensure $e_3>e_4$ and $p_{34}^s>p_{43}^s$, we need $\frac{u(w_1)}{u(w_1)-u(w_2)}\frac{2}{1+\alpha}<\frac{1}{\lambda_3-1}\lambda_1^{-\frac{\alpha}{2\alpha+1}}\left(\lambda_3-\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}\right)$. Similar to the equilibrium in the semifinal between player 1 and player 2, we also need $\frac{u(w_1)-u(w_2)}{u(w_2)}>\frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$.

(2) case 2: $\lambda_3 e_3^\alpha\geqslant e_4^\alpha$ *and* $e_4\geqslant e_3$

Player 1 *max* $\left(1-\frac{1}{2}\frac{e_4^\alpha}{\lambda_3 e_3^\alpha}\right)\widetilde{v}_3-ce_3$

Player 2 *max* $\left(1-\frac{1}{2}\frac{e_3^\alpha}{e_4^\alpha}\right)v_4-ce_4$

F.o.c

$[e_3]$ $\frac{\alpha}{2\lambda_3}\frac{e_4^\alpha}{e_3^{\alpha+1}}\widetilde{v}_3-c=0$

$[e_4]$ $\frac{\alpha}{2}\frac{e_3^\alpha}{e_4^{\alpha+1}}v_4-c=0$

S.o.c

$[e_3]$ $\frac{\alpha}{2\lambda_3}(-\alpha-1)\frac{e_4^\alpha}{e_3^{\alpha+2}}\widetilde{v}_3<0$

$[e_4]$ $\frac{\alpha}{2}(-\alpha-1)\frac{e_3^\alpha}{e_4^{\alpha+2}}v_4<0$

Solve F.O.C , we get

$$e_3=\frac{\alpha}{2c}\lambda_3^{-\frac{\alpha+1}{2\alpha+1}}(\widetilde{v}_3)^{\frac{\alpha+1}{2\alpha+1}}(v_4)^{\frac{\alpha}{2\alpha+1}}$$
$$e_4=\frac{\alpha}{2c}\lambda_3^{-\frac{\alpha}{2\alpha+1}}(\widetilde{v}_3)^{\frac{\alpha}{2\alpha+1}}(v_4)^{\frac{\alpha+1}{2\alpha+1}}$$
$$\frac{e_4}{e_3}=\lambda_3^{\frac{1}{2\alpha+1}}(\widetilde{v}_3)^{-\frac{1}{2\alpha+1}}(v_4)^{\frac{1}{2\alpha+1}}$$

Check the conditions $\lambda_3 e_3^\alpha\geqslant e_4^\alpha$ and $e_4\geqslant e_3$:

① $\lambda_3 e_3^\alpha \geqslant e_4^\alpha$

$$\lambda_3 e_3^\alpha \geqslant e_4^\alpha \iff \frac{\lambda_3 e_3^\alpha}{e_4^\alpha} \geqslant 1$$
$$\iff \lambda_3^{\frac{\alpha+1}{2\alpha+1}} (\widetilde{v}_3)^{\frac{\alpha}{2\alpha+1}} (v_4)^{-\frac{\alpha}{2\alpha+1}} \geqslant 1$$

Since $\lambda_3 > 1$ and $\widetilde{v}_3 > v_4$, $\lambda_3 e_3^\alpha > e_4^\alpha$ is always satisfied.

② $e_4 \geqslant e_3$

$e_2 \geqslant e_1$ is always satisfied as long as $f(p_{12}^s = 0) \leqslant 0$.

$$f(p_{12}^s = 0) = \lambda_3^{-1} \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \lambda_3^{-\frac{\alpha+1}{2\alpha+1}} \right) - \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)$$

$$f(p_{12}^s = 0) \leqslant 0 \iff \frac{u(w_1)}{u(w_1) - u(w_2)} \left(\lambda_3^{-1} - 1\right) + \frac{1+\alpha}{2} \left(1 - \lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right) \leqslant 0$$
$$\iff \frac{1+\alpha}{2} \left(1 - \lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}\right) \leqslant \frac{u(w_1)}{u(w_1) - u(w_2)} \left(1 - \lambda_3^{-1}\right)$$
$$\iff \frac{1 - \lambda_3^{-1-\frac{\alpha+1}{2\alpha+1}}}{1 - \lambda_3^{-1}} \leqslant \frac{u(w_1)}{u(w_1) - u(w_2)} \frac{2}{1+\alpha}$$

(3) case 3: $\lambda_3 e_3^\alpha \leqslant e_4^\alpha$ *and* $e_4 \geqslant e_3$

Player 1 $\quad max \quad \frac{1}{2}\frac{\lambda_3 e_3^\alpha}{e_4^\alpha}\widetilde{v}_3 - ce_3$

Player 2 $\quad max \quad \left(1 - \frac{1}{2}(\frac{e_3}{e_4})^\alpha\right) v_4 - ce_4$

F.o.c

$[e_3] \quad \frac{\alpha\lambda_3}{2}\frac{e_3^{\alpha-1}}{e_4^\alpha}\widetilde{v}_3 - c = 0$

$[e_4] \quad \frac{\alpha}{2}\frac{e_3^\alpha}{e_4^{\alpha+1}} v_4 - c = 0$

divide the two F.O.C , we get

$$\frac{e_4}{e_3} = \frac{v_4}{\lambda_3 \widetilde{v}_3} < 1$$

which contradicts the condition that $e_4 \geqslant e_3$.

Proof of Proposition 9

1. Equilibrium expected utility of overconfident player 1 in the semifinal with rational player 2.

The equilibrium continuation value of overconfident player 1 is

$$\begin{aligned}
v_1 &= p_{34}^f E^f(U_{13}) + p_{43}^f E^f(U_{14}) \\
&= E^f(U_{13}) \\
&= \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}}[u(w_1) - u(w_2)] - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}[u(w_1) - u(w_2)] + u(w_2) \\
&= \left(\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)}{u(w_1) - u(w_2)}\right)[u(w_1) - u(w_2)]
\end{aligned}$$

and his equilibrium expected utility in the semifinal with rational player 2 when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda < \hat{\lambda}$ is

$$\begin{aligned}
E^s(U_{12}) &= p_{12}^s v_1 - ce_1^s \\
&= \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right] \\
&\quad \left(\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)}{u(w_1) - u(w_2)}\right)[u(w_1) - u(w_2)] \\
&\quad - \frac{\alpha}{2}\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha} \\
&\quad \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}[u(w_1) - u(w_2)]
\end{aligned}$$

The benchmark equilibrium expected utility of the semifinal is

$$\overline{E}^s(U) = \frac{1-\alpha}{2}\overline{v} = \frac{1-\alpha}{2}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)[u(w_1) - u(w_2)]$$

Let

$$\begin{aligned}
f(\lambda_1) &= \frac{E^s(U_{12}) - \overline{E}^s(U)}{[u(w_1) - u(w_2)]} \\
&= \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right] \\
&\quad \left(\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)}{u(w_1) - u(w_2)}\right) \\
&\quad - \frac{\alpha}{2}\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \\
&\quad - \frac{1-\alpha}{2}\left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2}\right)
\end{aligned}$$

We can easily get that $f(\lambda_1 = 1) = 0$.

$$f'(\lambda_1) = \left[1 - \frac{1}{2}\lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\right]$$
$$\left(-\frac{1}{2}\frac{\alpha}{2\alpha+1}\lambda_1^{-\frac{\alpha}{2\alpha+1}-1} + \frac{\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right)$$
$$+ \left[-\frac{1}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\left(\alpha\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\right.\right.$$
$$\left.\left. + \lambda_1^{\alpha}(-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right)\right]$$
$$\left(\frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}} + \frac{u(w_2)}{u(w_1)-u(w_2)}\right)$$
$$- \frac{\alpha}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\left[(\alpha-1)\lambda_1^{\alpha-2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\right.$$
$$\left. + \lambda_1^{\alpha-1}(1-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\right]$$

$$
\begin{aligned}
f'(\lambda_1 = 1) = &\frac{1}{2}\frac{\alpha}{2}\frac{\alpha}{2\alpha+1}+ \\
&\left[-\frac{\alpha}{2}+\frac{\alpha}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)^{-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right]\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right) \\
&-\frac{\alpha}{2}(\alpha-1)\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)-\frac{\alpha}{2}(1-\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
=&\frac{1}{2}\frac{\alpha}{2}\frac{\alpha}{2\alpha+1}-\frac{\alpha}{2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)+\frac{\alpha}{2}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
&-\frac{\alpha}{2}(\alpha-1)\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)-\frac{\alpha}{2}(1-\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1} \\
=&\frac{\alpha}{2}\left[\frac{1}{2}\frac{\alpha}{2\alpha+1}-\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)+\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right. \\
&\left.-(\alpha-1)\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)-(1-\alpha)\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
=&\frac{\alpha}{2}\left[\frac{1}{2}\frac{\alpha}{2\alpha+1}-\alpha\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)+\alpha\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
=&\frac{\alpha^2}{2}\left[\frac{1}{2}\frac{1}{2\alpha+1}-\left(\frac{u(w_1)}{u(w_1)-u(w_2)}-\frac{1+\alpha}{2}\right)+\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\right] \\
=&\frac{\alpha^2}{2}\left[\frac{1}{2}\frac{1}{2\alpha+1}+\frac{1+\alpha}{2}\left(1+\frac{\alpha+1}{2\alpha+1}\right)-1-\frac{u(w_2)}{u(w_1)-u(w_2)}\right] \\
=&\frac{\alpha^2}{2}\left[\frac{\alpha(3\alpha+1)+1}{2(2\alpha+1)}-\frac{u(w_2)}{u(w_1)-u(w_2)}\right]
\end{aligned}
$$

$\frac{\alpha(3\alpha+1)+1}{2(2\alpha+1)} - \frac{u(w_2)}{u(w_1)-u(w_2)} > 0$ is satisfied when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$.
Therefore there exist $\lambda_1 \in (1, \hat{\lambda})$ for which $E^s(U_{12}) > \overline{E}^s(U)$.

2. Equilibrium expected utility of rational player 2 in the semifinal with overconfident player 1.

(1) When $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$

$$\begin{aligned}
E^s(U_{21}) &= p_{21}^s v_2 - c e_2^s \\
&= \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^\alpha v_2 - c\frac{\alpha}{2c}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{\alpha+1} \\
&= \frac{1}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{1+\alpha} - \frac{\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{\alpha+1} \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(v_2\right)^{1+\alpha} \\
&= \frac{1-\alpha}{2}\lambda_1^\alpha \left(\widetilde{v}_1\right)^{-\alpha} \left(\overline{v}\right)^{1+\alpha} \\
&= \frac{1-\alpha}{2}\left(\frac{\overline{v}}{\lambda_1^{-1}\widetilde{v}_1}\right)^\alpha \overline{v} \\
&< \frac{1-\alpha}{2}\overline{v} = \overline{E}^s(U)
\end{aligned}$$

(2) When either $\frac{u(w_1)-u(w_2)}{u(w_2)} \leqslant \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ or $\lambda_1 \geqslant \hat{\lambda}$

$$\begin{aligned}
E^s(U_{21}) &= p_{21}^s v_2 - c e_2^s \\
&= \left[1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{-\frac{\alpha}{2\alpha+1}}\right] v_2 - c\frac{\alpha}{2c}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left[1 - \frac{1}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{-\frac{\alpha}{2\alpha+1}}\right] v_2 - \frac{\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= v_2 - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(v_2\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \overline{v} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha}{2\alpha+1}} \left(\widetilde{v}_1\right)^{\frac{\alpha}{2\alpha+1}} \left(\overline{v}\right)^{\frac{\alpha+1}{2\alpha+1}} \\
&= \left[1 - \frac{1+\alpha}{2}\left(\frac{\lambda_1^{-1}\widetilde{v}_1}{\overline{v}}\right)^{\frac{\alpha}{2\alpha+1}}\right]\overline{v} \\
&> \frac{1-\alpha}{2}\overline{v} = \overline{E}^s(U)
\end{aligned}$$

3. Equilibrium expected utility of rational player 3 (4) in the semifinal with rational player 4 (3).

$$E^s(U_{34}) = E^s(U_{43}) = \frac{1-\alpha}{2}v_3 = \frac{1-\alpha}{2}v_4 > \frac{1-\alpha}{2}\overline{v} = \overline{E}^s(U).$$

Proof of Proposition 10

Part (i) follows directly from Propositions 1 and 2. Let's then prove part (ii). We know that the equilibrium efforts in the semifinal between the two rational players are higher than the benchmark, thus the equilibrium aggregate effort in the semifinals stage is higher than that of the benchmark if the equilibrium total effort of the semifinal with an overconfident and a rational player is higher than that of the benchmark. We also know from Proposition 3 that if $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\lambda_1 < \hat{\lambda}$, then total effort in the semifinal with an overconfident and a rational player is given by

$$\begin{aligned} e_1^s + e_2^s &= \frac{\alpha}{2c}\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(v_2)^{\alpha} + \frac{\alpha}{2c}\lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(v_2)^{\alpha+1} \\ &= \frac{\alpha}{2c}\overline{v}\left[\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(\overline{v})^{\alpha-1} + \lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(\overline{v})^{\alpha}\right] \end{aligned}$$

Hence, we have

$$\begin{aligned} \frac{e_1^s + e_2^s}{2\overline{e}^s} &= \frac{\frac{\alpha}{2c}\overline{v}\left[\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(\overline{v})^{\alpha-1} + \lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(\overline{v})^{\alpha}\right]}{2\frac{\alpha}{2c}\overline{v}} \\ &= \frac{1}{2}\left[\lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(\overline{v})^{\alpha-1} + \lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(\overline{v})^{\alpha}\right] \end{aligned}$$

Let $f(\lambda) = \lambda_1^{\alpha-1}(\widetilde{v}_1)^{1-\alpha}(\overline{v})^{\alpha-1} + \lambda_1^{\alpha}(\widetilde{v}_1)^{-\alpha}(\overline{v})^{\alpha}$,

$$\begin{aligned} f(\lambda) = \lambda_1^{\alpha-1} &\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha-1} \\ &+ \lambda_1^{\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha} \end{aligned}$$

We can easily get that $f(\lambda_1 = 1) = 2$.

$$\begin{aligned} f'(\lambda_1) = &\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha-1}\Bigg[(\alpha-1)\lambda_1^{\alpha-2}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{1-\alpha} \\ &+ \lambda_1^{\alpha-1}(1-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\Bigg] \\ &+ \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\right)^{\alpha}\Bigg[\alpha\lambda_1^{\alpha-1}\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha} \\ &+ \lambda_1^{\alpha}(-\alpha)\left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}}\right)^{-\alpha-1}\frac{1+\alpha}{2}\frac{\alpha+1}{2\alpha+1}\lambda_1^{-\frac{\alpha+1}{2\alpha+1}-1}\Bigg] \end{aligned}$$

$$
\begin{aligned}
f'(\lambda_1 = 1) = & \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{\alpha - 1} \left[(\alpha - 1) \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{1-\alpha} \right. \\
& \left. + (1-\alpha) \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{-\alpha} \frac{1+\alpha}{2} \frac{\alpha+1}{2\alpha+1} \right] \\
& + \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{\alpha} \left[\alpha \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{-\alpha} \right. \\
& \left. + (-\alpha) \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{-\alpha-1} \frac{1+\alpha}{2} \frac{\alpha+1}{2\alpha+1} \right] \\
= & (\alpha - 1) + (1-\alpha) \frac{1+\alpha}{2} \frac{\alpha+1}{2\alpha+1} \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{-1} + \alpha \\
& - \alpha \frac{1+\alpha}{2} \frac{\alpha+1}{2\alpha+1} \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{-1} \\
= & (2\alpha - 1) + (1-2\alpha) \frac{1+\alpha}{2} \frac{\alpha+1}{2\alpha+1} \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{-1} \\
= & (1-2\alpha) \left[-1 + \frac{1+\alpha}{2} \frac{\alpha+1}{2\alpha+1} \left(\frac{u(w_1)}{u(w_1) - u(w_2)} - \frac{1+\alpha}{2} \right)^{-1} \right]
\end{aligned}
$$

$-1 + \frac{1+\alpha}{2} \frac{\alpha+1}{2\alpha+1} \left(\frac{u(w_1)}{u(w_1)-u(w_2)} - \frac{1+\alpha}{2} \right)^{-1} > 0$ is equivalent to $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$.
Thus we have
If $\alpha < \frac{1}{2}$, then $f'(\lambda_1 = 1) > 0$.
If $\alpha > \frac{1}{2}$, then $f'(\lambda_1 = 1) < 0$.
Therefore, when $\frac{u(w_1)-u(w_2)}{u(w_2)} > \frac{2(2\alpha+1)}{\alpha(3\alpha+1)}$ and $\alpha < \frac{1}{2}$, there exist $\lambda_1 \in (1, \hat{\lambda})$ such that $e_1^s + e_2^s > 2\overline{e}^s$ which implies

$$
\sum_{i=1}^{4} e_i^s > 4\overline{e}^s
$$

Proof of Proposition 11

We know from Proposition 6 that the equilibrium effort in a final between two equally overconfident players is equal to

$$
e^f = \frac{\alpha}{2c\lambda} \left[u(w_1) - u(w_2) \right].
$$

The perceived expected utility of the final is then given by

$$
\begin{aligned}
\widetilde{E}(U^f) &= \frac{1}{2} u(w_1) + \frac{1}{2} u(w_2) - c \frac{\alpha}{2c\lambda} \left[u(w_1) - u(w_2) \right] \\
&= \frac{\lambda - \alpha}{2\lambda} u(w_1) + \frac{\lambda + \alpha}{2\lambda} u(w_2).
\end{aligned}
$$

The equilibrium effort in a semifinal between two equally overconfident players is equal to

$$
\begin{aligned}
e^s &= \frac{\alpha}{2c\lambda}\widetilde{E}(U^f) \\
&= \frac{\alpha}{2c\lambda}\left[\frac{\lambda-\alpha}{2\lambda}u(w_1) + \frac{\lambda+\alpha}{2\lambda}u(w_2)\right].
\end{aligned}
$$

Hence, total effort amounts to

$$
\begin{aligned}
\epsilon &= 2\frac{\alpha}{2c\lambda}\left[u(w_1) - u(w_2)\right] + 4\frac{\alpha}{2c\lambda}\left[\frac{\lambda-\alpha}{2\lambda}u(w_1) + \frac{\lambda+\alpha}{2\lambda}u(w_2)\right] \\
&= \frac{\alpha}{c\lambda}\left[\left(2-\frac{\alpha}{\lambda}\right)u(w_1) + \frac{\alpha}{\lambda}u(w_2)\right].
\end{aligned}
$$

The problem of the contest designer is to maximize total effort subject to $w_1 + w_2 = W$. Since $\alpha \in (0,1]$ and $\lambda \geqslant 1$ the optimal solution to this problem when players are risk neutral is to set $w_1 = W$ and $w_2 = 0$, that is, to allocate all the prize money to the winner of the final. When players are risk averse this is no longer the case. Substituting the constraint $w_2 = W - w_1$ into the objective function we have the unconstrained problem

$$
max \quad \frac{\alpha}{c\lambda}\left[\left(2-\frac{\alpha}{\lambda}\right)u(w_1) + \frac{\alpha}{\lambda}u(W-w_1)\right]
$$

The first-order condition is

$$
\left(2-\frac{\alpha}{\lambda}\right)u'(w_1) - \frac{\alpha}{\lambda}u'(W-w_1) = 0.
$$

Rearranging the first-order condition we have

$$
\frac{u'(w_1)}{u'(W-w_1)} = \frac{\frac{\alpha}{\lambda}}{2-\frac{\alpha}{\lambda}}. \tag{18}
$$

Hence, when players are risk averse, the optimal prize structure involves multiple prizes with the winner of the final receiving most of the prize money and a smaller part being assigned to the runner-up. Since the right-hand side of (18) is decreasing in λ, it follows that an increase in overconfidence raises the share of the prize money allocated to the winner of the final.