Equilibrium Existence in Price-Quantity Games: A Sunk Cost Paradox*

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Abstract

Nonexistence of a pure-strategy Nash equilibrium is a notorious problem in price-quantity games. What drives this problem is the presence of spillover demand, *i.e.*, demand coming from competitors' unserved customers. We argue that such demand spillovers may stem from a strong implicit assumption that costs associated with obtaining a product are sunk and do not affect consumers' future payoffs. We relax this assumption by considering a more general class of cost functions. This is shown to admit a pure-strategy equilibrium that coincides with the Bertrand price equilibrium.

Keywords: Price-Quantity Competition; Spillover Demand; Sunk Cost.

JEL Classification: C72; D43; L13.

"*No problem is so formidable that you cannot walk away from it.*" (Charles Schulz)

1 Introduction

In the classical models of Cournot quantity competition and Bertrand price competition, a pure-strategy Nash equilibrium exists under mild conditions.[1] The same cannot be said for models where sellers compete in prices *and* quantities. Nonexistence of an equilibrium is indeed a persistent problem in price-quantity games. At the heart of this problem is the presence of spillover demand, *i.e.*, indirect demand coming from competitors' unserved customers. At a candidate equilibrium, this creates an incentive for firms to hike their price and act as a monopolist on their residual demand curve. In this paper, we argue that such demand spillovers may stem from

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[1] See, *e.g.*, Vives (1999) for a detailed discussion.

a strong implicit assumption that costs associated with obtaining a product are sunk and do not affect consumers' future payoffs.

To fix ideas, consider the story of a buyer named Brainy. Brainy lives in a house surrounded by one hundred bakeries and Brainy wants bread. At the set prices, he prefers bakery 1 to bakery 2 and bakery 2 to bakery 3, *et cetera*. Moreover, since Brainy is starving, he prefers any baker to staying home. After arriving at bakery 1, he learns that all bread is sold out. He then moves to bakery 2 where, again, there is no bread left. The story is no different at bakery 3: all shelves are empty. This continues until he finally arrives at bakery 100 to discover that all bread is gone. Tired and hungry he returns home empty-handed and falls asleep on his couch.

One may accuse Brainy of being naive or admire his persistence. Either way, within the context of the story, he arguably did not do that much wrong. Unlike what it may seem, he in particular did not fall prey to the so-called *sunk cost fallacy*, which states that individuals have a greater tendency to continue a behavior or endeavor once they have invested effort, money, or time.[2] Instead, once having learned that a bakery was out of bread, he updated his information and, considering all investments sunk, simply went for the next-best alternative. And yet, Brainy's journey seems neither likely, nor optimal. Indeed, most people would probably have stopped their quest for bread at a much earlier stage and settle for rice, stew or a snack with beer for that matter.

What is (implicitly) assumed in the preceding tale is that: (i) all costs of visiting a particular bakery are sunk once incurred, and (ii) such investments have no effect on the (expected) payoffs from the remaining alternatives. The first is very much conceivable. After all, transportation and related costs are often not recoverable. The second seems far less plausible, however. For instance, within a given time frame (*e.g.*, a day), one typically has to perform a variety of different tasks. The more time spent on one task, *ceteris paribus*, the less time and energy there is left to properly complete others. That is, bygones may be bygones, but nevertheless feed into future payoffs and, therefore, potentially also affect the preference ranking over remaining alternatives.[3] After having visited, say, two bakeries, one can imagine Brainy to reevaluate his options and prefer going home to trying a third.

[2]This *sunk cost fallacy* is frequently referred to as *sunk cost effect* as introduced by Thaler (1980). For an extensive analysis and discussion of the sunk cost effect and the related Concorde effect, see Arkes and Blumer (1985) and Arkes and Ayton (1999), as well as the references therein.

[3]Contrary to conventional wisdom, McAfee, Mialon and Mialon (2010) argue that it may be rational to react to sunk costs because of informational content, reputational concerns, or financial and time constraints.

With this in mind, we analyze price-quantity competition in a spatial duopoly model, allowing for the possibility that costs to obtain a product are sunk, but nevertheless affect future payoffs. Specifically, as the story of Brainy illustrates, this may significantly reduce spillover demand and, therefore, potentially provide a solution to the nonexistence problem. We derive conditions under which there exists a pure-strategy Nash equilibrium and show that it coincides with the equilibrium in a price-only model. Price-quantity competition may thus yield Bertrand outcomes when consumers' preference ranking over remaining alternatives is affected by preceding sunk investments.

The nonexistence problem was formulated for the first time in the pioneering work of Edgeworth (1922, 1925). Ever since, scholars have explored several directions to address this problem. One is to allow for randomized strategies. A mixed-strategy Nash equilibrium indeed exists under relatively weak assumptions.[4] Another is to consider different information and timing structures. For instance, pure-strategy equilibria commonly exist in sequential-move price-quantity games.[5] Finally, there is work that considers arguments for why spillover demand for higher-priced firms may be limited. Dixon (1990), for instance, argues that it may be costly to turn customers away. This provides an incentive for firms to meet additional demand in case a competitor would raise its price. Similarly, Dixon (1992) assumes that suppliers select a price and a quantity they are willing to sell at that price. This also effectively reduces residual demand for higher-priced firms. These and related works all identify supply side factors that may limit demand spillovers. By contrast, our focus is on the demand side; the (un)avoidable costs for rationed customers.[6]

The next section introduces the model. Sections 3 and 4, respectively, consider the price and price-quantity versions of the model. This serves as a benchmark for the main analysis in Section 5 and Section 6, which establish existence and offer a characterization of a price-quantity equilibrium without and with spillover demand, respectively. Section 7 concludes. All proofs are relegated to the Appendix.

[4] See Maskin (1986). In a variety of different price-quantity games, the presence of a mixed-strategy equilibrium is shown by Levitan and Shubik (1972), Dixon (1984), Gertner (1986), van den Berg and Bos (2017), Tasnádi (2004, 2020), Montez and Schutz (2021), amongst many others.

[5] See Boyer and Moreaux (1987, 1988, 1989) and, more recently, Yousefimanesh, Bos and Vermeulen (2023).

[6] A notable exception is Tasnádi (1999) who shows there exists a pure-strategy Nash equilibrium in a homogeneous-good Bertrand-Edgeworth model when demand is sufficiently elastic.

2 Model

We conduct our study in the context of a spatial duopoly model as developed by Hotelling (1929). Suppliers produce a homogeneous good at common marginal cost $c > 0$ and are located at the endpoints of a unit interval $[0, 1]$. Without loss of generality, we assume that firm 1 is located at 0 and firm 2 is located at 1. Demand comes from consumers who are uniformly distributed along the interval with a density that is normalized to one. Each consumer either purchases one unit of the good or does not buy. Gross utility from consumption is v and production potentially creates value, *i.e.*, $v > c$.

To facilitate the ensuing analysis, it is useful to distinguish between *direct* and *indirect* demand. Firm i's direct demand comprises all customers who have a strict preference for firm i's product at the set prices. Firm i's indirect demand comes from its rival's unserved customers who prefer consuming firm i's product to some outside option. We here formally introduce direct demand. A detailed specification of indirect demand is provided in Section 4.

Consider some given price pair, (p_1, p_2), and suppose that transportation costs are linear and given by $t > 0$ per unit of distance. As is well-known, t has a broad interpretation and captures the degree of (spatial) product differentiation. Someone located at $x \in [0, 1]$ then makes one of the following choices:

- H: Stay home. This gives utility $u_x(h) = 0$.
- F_1: Buy firm 1's product at a price of p_1. This gives utility $u_x(1) = v - tx - p_1$.
- F_2: Buy firm 2's product at a price of p_2. This gives utility $u_x(2) = v - t(1 - x) - p_2$.

These three options induce an equal number of 'indifference points'. In the following, let z_1 indicate the location of consumers who are indifferent between H and F_1, *i.e.*, $u_{z_1}(1) = 0$. Likewise, z_2 is the location of consumers who are indifferent between H and F_2, *i.e.*, $u_{z_2}(2) = 0$. Last, let z_3 be the location of consumers who are indifferent between F_1 and F_2, *i.e.*, $u_{z_3}(1) = u_{z_3}(2)$. It can be easily verified that:

$$z_1 = \frac{v - p_1}{t}, \quad z_2 = 1 - \frac{v - p_2}{t}, \quad \text{and} \quad z_3 = \frac{p_2 - p_1 + t}{2t}.$$

Note that $z_3 = \frac{1}{2}$ when $p_1 = p_2$, *i.e.*, at equal prices, buyers who are indifferent between F_1 and F_2 are located precisely in the middle.

Firm 1's direct demand, $D_1 = D_1(p_1, p_2)$, can now be defined as the length of an interval of locations x for which $u_x(1)$ is highest:

$$I_1 = \{x \in [0,1] \mid u_x(1) \geq u_x(h) \text{ and } u_x(1) \geq u_x(2)\}.$$

If $I_1 = [0,1]$, then $D_1 = 1$, and if $I_1 = \emptyset$, then $D_1 = 0$. For the interior cases, $D_1 = \min\{z_1, z_3\}$. Direct demand for firm 2 is defined in a similar fashion.

In what follows, it is assumed that $p_1 < v$ and $p_2 < v$ so that $u_0(1) > 0$ and $u_1(2) > 0$. Hence, a firm does not select too high a price in that all buyers prefer H (staying home) to purchasing from this seller. Using the properties of $z_3 = \frac{p_2 - p_1 + t}{2t}$, we can then distinguish four different market configurations:

- Predatory pricing
- Monopolistic pricing
- Market-sharing pricing
- Competitive pricing

Let us now discuss each of these in turn.

PREDATORY PRICING. A price pair (p_1, p_2) is *predatory* if either $z_3 < 0$, or $z_3 > 1$. If $z_3 < 0$, then firm 2 is the predator. In this case, for a given price p_1, p_2 is low enough to ensure that even consumers located at $x = 0$ prefer F_2. Thus, $D_1 = 0$ and $D_2 = 1$. Conversely, if $z_3 > 1$, then firm 1 is the predator. In this case, $D_1 = 1$ and $D_2 = 0$.

Figure 1 depicts a situation in which $z_3 < 0$, which is equivalent to $u_0(1) < u_0(2)$. The condition $z_3 > 1$ is equivalent to $u_1(2) < u_1(1)$.

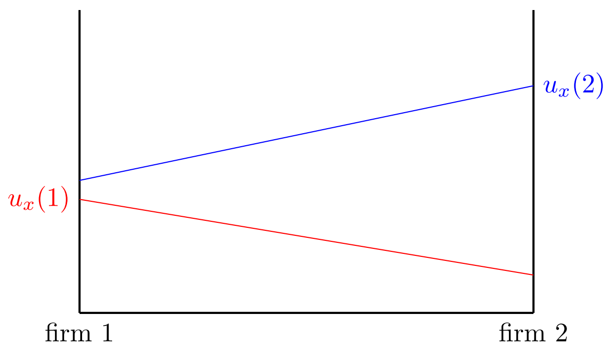

Figure 1: Predatory pricing illustrated.

MONOPOLISTIC PRICING. A price pair (p_1, p_2) is *monopolistic* if $u_{z_3}(1) < 0$. In this case, both firms set a price so high that some consumers prefer to stay home. In terms of direct demand, it holds that $D_1 = z_1$ and $D_2 = 1 - z_2$.

Figure 2 illustrates a monopolistic pricing regime.

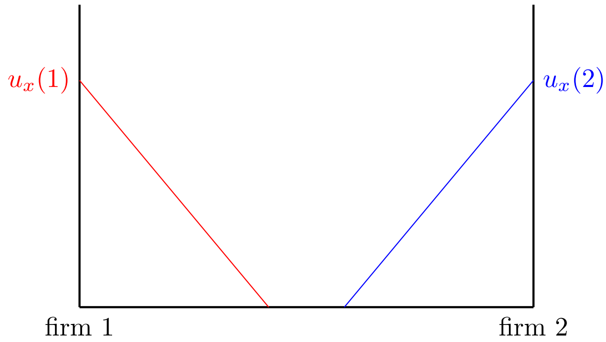

Figure 2: Monopolistic pricing illustrated.

MARKET-SHARING PRICING. A price pair (p_1, p_2) is *market-sharing* when $u_{z_3}(1) = 0$. In this case, all consumers buy and each seller has a strictly positive market share. In particular, customers located at z_3 are indifferent between the three options: H, F_1, and F_2. Hence, $z_1 = z_2 = z_3$ and, therefore, $D_1 = z_3$ and $D_2 = 1 - z_3$.

Market-sharing pricing is illustrated in Figure 3 below.

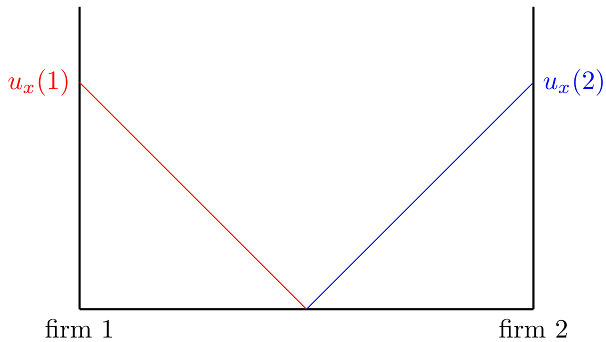

Figure 3: Market-sharing pricing illustrated.

COMPETITIVE PRICING. A price pair (p_1, p_2) is *competitive* when $0 \leq z_3 \leq 1$ and $u_{z_3}(1) > 0$. Comparable to market-sharing pricing, firms set their price in such a way that both face demand and all consumers purchase either from firm 1 or from firm 2. The difference is that customers located at z_3 obtain a strictly positive utility. That is, $u_{z_3}(1) = u_{z_3}(2) > 0$. Moreover, as with market-sharing pricing, $D_1 = z_3$ and $D_2 = 1 - z_3$.

A competitive pricing situation is shown in Figure 4.

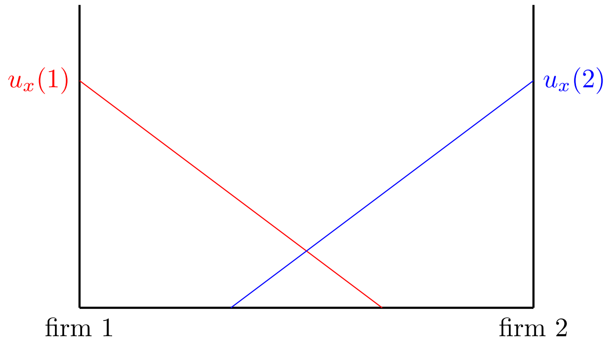

Figure 4: Competitive pricing illustrated.

Note that with predatory pricing, market-sharing pricing and competitive pricing it holds that $D_1 + D_2 = 1$ (*i.e.*, the market is covered). Yet, with predatory pricing, one firm captures the entire market, whereas both receive demand in the other two cases. The next result provides a definition of a predatory price pair.

Lemma 2.1 *A price pair* (p_1, p_2) *is predatory precisely when* $|p_1 - p_2| > t$.

In the following, our focus is on the remaining three market configurations, *i.e.*, it is assumed that $|p_1 - p_2| \leq t$. Lemma 2.2 specifies the partition.

Lemma 2.2 *Suppose that* $|p_1 - p_2| \leq t$. *A price pair* (p_1, p_2) *is:*

[A] *Monopolistic, precisely when* $p_1 + p_2 > 2v - t$.

[B] *Market-sharing, precisely when* $p_1 + p_2 = 2v - t$.

[C] *Competitive, precisely when* $p_1 + p_2 < 2v - t$.

Observe that B, even though it is a boundary case between monopolistic and competitive pricing, applies to a range of prices. Indeed, there are many price pairs for which the total price (*i.e.*, $p_1 + p_2$) equals $2v - t$.[7]

[7] A detailed analysis is provided in Bacchiega and Fedele (2022), which refers to the market-sharing case as *monopolistic duopoly*.

3 Price Competition: A Benchmark

Let us now use the model laid out in the preceding section to study price competition. This serves as a useful benchmark for the analysis of price-quantity competition in the ensuing sections.

In case of price competition, firms simultaneously select prices and meet the demand forthcoming to them at the set prices. Firm i's objective is then to maximize the following profit function:

$$\Pi_i(p_i, p_j) = (p_i - c) \cdot D_i(p_i, p_j), i \in \{1, 2\} \text{ and } i \neq j.$$

where $D_1 = \min\{z_1, z_3\}$ and $D_2 = 1 - \max\{z_2, z_3\}$ is the direct demand for firm 1 and firm 2, respectively.

The next definition specifies the applied solution concept.

Definition 3.1 *A price pair* (p_1, p_2) *is a* ***price equilibrium*** *if* p_1 *is a best-response of firm* 1 *to* p_2 *and* p_2 *is a best-response of firm* 2 *to* p_1.

In deriving the best-response functions, it is useful to distinguish between the following three regions. Define:

- $A = \{p_i \leq v \mid 3v < c + 2t + 2p_i\}$, and
- $B = \{p_i \leq v \mid 3v \geq c + 2t + 2p_i \text{ and } c + 3t + 3p_i \geq 4v\}$, and
- $C = \{p_i \leq v \mid c + 3t + 3p_i < 4v\}$.

It is clear that when firm i's price is in region A or C, then it is not in region B and *vice versa*. Moreover, it can be easily verified that its price cannot be in region A and C simultaneously.[8] The regions are thus mutually exclusive and completely partition a firm's strategy space $[0, v]$.

We now have all the ingredients to precisely specify the best-responses.

Proposition 3.2 *Consider a price pair* (p_i, p_j), $i = 1, 2$ *and* $i \neq j$. *For a given* p_j, *the best-*

[8] To see this, suppose that $p_i \in C$ so that $c + 3t + 3p_i < 4v$. It then holds that $2c + 6t + 6p_i < 8v$. Since $c \leq v$, it then also holds that $3c + 6t + 6p_i < 9v$ and, therefore, $c + 2t + 2p_i < 3v$. Thus, if $p_i \in C$, then $p_i \notin A$.

response $BR_i(p_j)$ of firm i to p_j is given by:

$$BR_i(p_j) = \begin{cases} \dfrac{v+c}{2} & \text{if } p_j \in A. \\ 2v - t - p_j & \text{if } p_j \in B. \\ \dfrac{c+t+p_j}{2} & \text{if } p_j \in C. \end{cases}$$

The next Lemma shows that if a price pair is a price equilibrium, then both prices must be part of the same region.

Lemma 3.3 *For $i \in \{1, 2\}$ and $i \neq j$:*

[1] *If $p_i \in A$ and $p_j \in C$, then (p_i, p_j) is not a price equilibrium.*

[2] *If (p_i, p_j) is a price equilibrium and $p_j \in B$, then $p_i \in B$.*

The following result identifies under what conditions a price equilibrium is in region A, B, or C.

Lemma 3.4 *Let (p_1, p_2) be a price equilibrium.*

[1] *If $p_1, p_2 \in A$, then $p_1 = p_2 = \frac{v+c}{2}$ and $v < c + t$.*

[2] *If $p_1, p_2 \in B$, then $p_1 + p_2 = 2v - t$ and $c + t \leq v \leq c + \frac{3}{2}t$.*

[3] *If $p_1, p_2 \in C$, then $p_1 = p_2 = c + t$ and $c + \frac{3}{2}t < v$.*

Finally, we conclude this section with three theorems showing existence of a price equilibrium for all feasible values of v, c, and t.

Theorem 3.5 *If $v < c + t$, then there is a unique price equilibrium given by:*

$$(p_1^*, p_2^*) = (\frac{v+c}{2}, \frac{v+c}{2}).$$

In this case, there is monopolistic pricing and $p_1^* = p_2^* \in A$.

Theorem 3.6 *If $c + t \leq v \leq c + \frac{3}{2}t$, then there is a line segment of price equilibria given by:*

$$(p_1^*, p_2^*) = (2v - t - \lambda, \lambda),$$

where λ is such that:

$$\max\{3v + 3c, 8v - 6t - 2c\} \leq 6\lambda \leq \min\{4v + 2c, 9v - 6t - 3c\}.$$

In this case, there is market-sharing pricing and $p_1^*, p_2^* \in B$.

Theorem 3.7 *If $c + \frac{3}{2}t < v$, then there is a unique price equilibrium given by:*

$$(p_1^*, p_2^*) = (c + t, c + t).$$

In this case, there is competitive pricing and $p_1^* = p_2^* \in C$.

In sum, for any constellation of v, c, and t, price equilibria can be of one type only: A, B, or C. Furthermore, for the cases A and C, the price equilibrium is unique, whereas in case of B there is a whole range of asymmetric price equilibria.

4 Price-Quantity Competition

Armed with the above price competition benchmark, we now direct our attention to competition in prices *and* quantities. In the following, firms are supposed to simultaneously select a price and a level of supply. A key difference with the price-only model is that sellers are not restricted to picking a price-production pair on their demand function. This, in particular, means that suppliers may not meet their demand in which case (part of) the unserved customers may switch to the rival. That is, firms potentially face indirect (or spillover) demand.[9]

Firm i's objective is then to maximize the following profit function:

$$\Pi_i(p_i, q_i, p_j, q_j) = p_i \cdot s_i(p_i, q_i, p_j, q_j) - c \cdot q_i,\ i \in \{1, 2\} \text{ and } i \neq j,$$

where sales, $s_i = \min\{q_i, d_i(p_i, p_j, q_j)\}$, is the minimum of its supply and demand. Specifically, firm i's total demand is given by $d_i(p_i, p_j, q_j) = D_i(p_i, p_j) + E_i(p_i, p_j, q_j)$, where E_i is indirect (or spillover) demand. It is assumed that information is imperfect in that consumers observe prices, but not the available supplies.[10] As with price competition, a buyer located at x chooses between H, F_1, and F_2 to maximize its utility. If (s)he is not served, (s)he returns home and chooses between the remaining two options. The spillover demand function E_i captures the part of rationed customers that prefers visiting a second seller.

Note that, for a given price combination (p_1, p_2), total demand equals direct demand when pricing is either *monopolistic* or *market-sharing.* To see this, suppose that firm j produces to meet its

[9] For a discussion of the role of spillover demand in price-quantity competition, see Friedman (1988) and Bos and Vermeulen (2021a, 2021b).

[10] With perfect information, consumers observe both prices and quantities. In that case there is no spillover demand since a customer can anticipate whether (s)he will be served and directly go to a seller that has a product available or stay home instead.

demand at the set prices, *i.e.*, $q_j = D_j(p_i, p_j)$. In this case, firm i can create a shortage at firm j's site by raising its price. Although such a shortage is a necessary condition for spillover demand, it is not sufficient. Indeed, when pricing is monopolistic or market-sharing, unserved customers prefer H to F_i and, therefore, will not visit firm i. In other words, the presence of demand spillovers ($E_i > 0$) requires pricing to be competitive, *i.e.*, $u_{z_3}(1) = u_{z_3}(2) > 0$.

Let us then suppose that (p_1, p_2) is competitive. To specify demand in case of rationing, it is assumed that the time of arrival is inversely related to distance. That is, customers located at 0 or 1 are served first and the ones located at z_3 latest. This can be interpreted literally, *i.e.*, it takes time to travel through space. Alternatively, the preference for a particular seller is stronger the closer a consumer is to that seller. That is, the most eager ones are served first. Taking firm 1's perspective, there are then basically two possibilities. If $q_2 \geq D_2(p_1, p_2) = 1 - z_3$, then all consumers in $[z_3, 1]$ are served by firm 2. By contrast, if $q_2 < D_2(p_1, p_2) = 1 - z_3$, then consumers in the (non-degenerate) interval $[z_3, 1 - q_2]$ are not served by firm 2.[11] These consumers return home and reconsider their options, stay home or visit firm 1. The mirror version provides firm 2's perspective.

In analyzing this model, we employ the following solution concept.

Definition 4.1 *A pair of price-quantity combinations* $((p_1, q_1), (p_2, q_2))$ *is a* ***price-quantity equilibrium*** *if* (p_1, q_1) *is a best-response by firm* 1 *to* (p_2, q_2) *and* (p_2, q_2) *is a best-response by firm* 2 *to* (p_1, q_1).

The next result specifies a best-response property, namely that firms produce to meet their demand.

Lemma 4.2 *Let* (p_j, q_j) *be a strategy of firm* j. *If* (p_i, q_i) *is a best-response of firm* $i \neq j$ *to* (p_j, q_j), *then* $q_i = d_i(p_i, p_j, q_j)$.

Hence, akin to the price-only model, any equilibrium solution is on the demand curve.[12] In fact, as the following result shows, any price-quantity equilibrium coincides with a price equilibrium.

Theorem 4.3 *If* $((p_1, q_1), (p_2, q_2))$ *is a price-quantity equilibrium, then* (p_1, p_2) *is a price equilibrium. Moreover,* $q_1 = D_1(p_1, p_2)$ *and* $q_2 = D_2(p_1, p_2)$.

[11] Recall that we disregard the option of predatory pricing so that $0 \leq z_3 \leq 1$.

[12] It is noteworthy that this is a very robust result in the literature on price-quantity competition with continuous demand. Indeed, it has been shown in a variety of different price-quantity models, including Alger (1979, Theorem 3.1), Friedman (1988, Lemma 3), Benassy (1989, Theorem 1), Canoy (1996, Lemma 1), and, more recently, Bos and Vermeulen (2021a, Lemma 2). Taking an evolutionary perspective, Khan and Peeters (2015) shows that this outcome may also emerge when sellers imitate the most profitable industry player.

We have provided a complete characterization of price equilibria in Section 3. To determine all price-quantity equilibria, it therefore suffices to evaluate which price equilibria remain an equilibrium in the price-quantity game.

The next result shows that the monopolistic and market-sharing price equilibria survive the extension of the strategy space.[13] That is, the monopolistic and market-sharing equilibrium outcomes in the price game are also equilibrium outcomes in the price-quantity game. In these cases, firms that compete in price and quantity effectively behave as if they compete in price alone.

Theorem 4.4 *If (p_1, p_2) is a monopolistic or market-sharing price equilibrium, then*

$$((p_1^*, q_1^*), (p_2^*, q_2^*)) = ((p_1, D_1(p_1, p_2)), (p_2, D_2(p_1, p_2)))$$

is a price-quantity equilibrium.

The same does not hold for the competitive price equilibrium, however.

Theorem 4.5 *If (p_1, p_2) is a competitive price equilibrium, then*

$$((p_1, q_1), (p_2, q_2)) = ((p_1, D_1(p_1, p_2)), (p_2, D_2(p_1, p_2)))$$

is not a price-quantity equilibrium.

5 A Competitive Price-Quantity Equilibrium

In the preceding section, we showed there is a price-quantity equilibrium with monopolistic or market-sharing pricing, but not when pricing is competitive. The logic underlying this nonexistence result is similar to the one underlying the Edgeworth paradox. At the candidate equilibrium, which is the equilibrium outcome in the price-only game (Theorem 4.3), firms produce to meet their demand. Since production precedes sales, this creates an incentive to increase prices. A firm, by raising its price, induces a shortage at the rival's site. More customers prefer the rival's product after the price hike, but this competitor is *de facto* capacity-constrained. Rationed customers then spill over to the higher-priced firm, which meets the residual demand at the higher price. In turn, this creates an incentive for the lower-priced firm to raise its own price, thereby starting a new Edgeworth price cycle.

[13] This closely resembles Somogyi (2020) who shows the existence of a pure-strategy equilibrium in a Bertrand-Edgeworth duopoly with sufficiently differentiated goods.

Although the argument for nonexistence of a pure-strategy equilibrium is in itself sound, there is something peculiar about it. In the above model, all costs incurred by a rationed customer are taken to be sunk and, in particular, do not affect future payoffs. Indeed, consider a buyer located at x and suppose that (s)he prefers F_2 to F_1 and F_1 to H. If this buyer is not served by firm 2, then (s)he spills over to firm 1. The fact that (s)he did not obtain the preferred product neither changed the preference ranking over the remaining alternatives (F_1 and H), nor their (expected) payoffs. While this is clearly a possibility, in many situations it seems more plausible to assume that future payoffs *are* affected and, therefore, potentially also their relative ranking. For example, as illustrated by the introductory example, a consumer may well get tired and hungry by moving from one shop to another. Also, there typically are opportunity costs and these can change over time. Unserved consumers may therefore not only update their information, but additionally reconsider and reevaluate their options.

To take account of this possibility, we assume in the following that rationed customers face different costs of transportation. Specifically, consider those who are not served by firm 2 at some given q_2 under competitive pricing, *i.e.*, all buyers located at $x \in [z_3, 1-q_2]$. For these customers, the cost of visiting firm 1 is then given by $C(x)$, with $C(0) = 0$, $C'(x) > 0$ and $C''(x) \geq 0$.[14] A rationed customer located at x thus obtains a utility of $U_x(1) = v - C(x) - p_1$ when buying from firm 1. Let Z be the unique solution to $U_x(1) = u_x(h)$, *i.e.*, Z is defined by the equation $v - C(Z) - p_1 = 0$. This allows to distinguish three cases: (1) If $Z < z_3$, then $E_1 = 0$, (2) If $Z > 1 - q_2$, then $E_1 = 1 - q_2 - z_3$, and (3) If $Z \in [z_3, 1-q_2]$, then $E_1 = Z - z_3$. As $D_1 = z_3$, firm 1's demand is then given by:

$$d_1(p_1, p_2, q_2) = \begin{cases} z_3 & \text{if } Z < z_3 \\ Z & \text{if } z_3 \leq Z \leq 1 - q_2 \\ 1 - q_2 & \text{if } Z > 1 - q_2, \end{cases}$$

or, equivalently, $d_1 = \min\{z, 1-q_2\}$, with $z = \max\{z_3, Z\}$.[15] Firm 2's demand, $d_2(p_1, p_2, q_1)$, has a similar structure.

To fix ideas, suppose that $C(x) = a \cdot x^2 + r \cdot x$. The utility function of a rationed customer located at x when buying from firm 1 is then given by $U_x(1) = v - p_1 - a \cdot x^2 - r \cdot x$. The next result provides conditions under which there exists a unique price-quantity equilibrium.

[14] Note that this includes the linear cost structure of Section 4 as a special case.

[15] Note that within the context of Section 4, $C(x) = t \cdot x$. In that case, $Z = z_1$. Since with competitive pricing it holds that $z_3 \leq z_1$, the formulation for total demand simplifies to $d_1 = \min\{z_1, 1 - q_2\}$.

Proposition 5.1 *Suppose that $c + \frac{3}{2}t \leq v \leq c + 2t$ and $C(x) = a \cdot x^2 + r \cdot x$. If $a + 4 \cdot r \geq 8 \cdot t$, then*

$$((p_1^*, q_1^*), (p_2^*, q_2^*)) = ((c+t, \frac{1}{2}), (c+t, \frac{1}{2}))$$

is the unique price-quantity equilibrium.

Next, let us offer sufficient conditions for the existence of a competitive price-quantity equilibrium with the generalized cost function $C(x)$. To that end, suppose that firm 1 raises its price by an amount $\Delta \geq 0$ from the competitive equilibrium price $p_1 = c + t$ to $p_1' = c + t + \Delta$ and let z_p be firm 1's market share at which its profit equals its price equilibrium profit $\frac{t}{2}$:

$$z_p = \frac{t}{2(t + \Delta)}.$$

Observe that firm 1's market share is less than 50 percent and that $z_p = \frac{1}{2}$ at $\Delta = 0$. Finally, we define two functions of Δ: $F(\Delta) = C(z_p)$ and $L(\Delta) = v - c - t - \Delta$.

The next result provides sufficient conditions for the existence of a competitive price-quantity equilibrium.

Theorem 5.2 *Suppose that $c + \frac{3}{2}t < v$. If $L(\Delta) \leq F(\Delta)$, then there exists a competitive price-quantity equilibrium.*

If $L(\Delta) \leq F(\Delta)$, then rationed customers find it too costly to visit a second seller. Hence, and unlike the situation described in Theorem 4.5 above, hiking the price from $c + t$ is no longer profitable since it does not create sufficient spillover demand.

Let us conclude this section with a simple, illustrative example.

EXAMPLE

Suppose that $c + \frac{3}{2}t < v$ and that $C(x) = r \cdot x$. Hence, $Z = \frac{v-c-t-\Delta}{r}$, $z_p = \frac{t}{2 \cdot (t+\Delta)}$, and $F(\Delta) = r \cdot \frac{t}{2 \cdot (t+\Delta)}$. We show there is a price-quantity equilibrium precisely when $(v-c)^2 \leq 2 \cdot r \cdot t$.

By Theorem 5.2, there is a competitive price-quantity equilibrium when $L(\Delta) \leq F(\Delta)$, or:

$$2 \cdot (t + \Delta) \cdot (v - c - t - \Delta) \leq r \cdot t.$$

Since the LHS is quadratic in Δ, and the RHS is constant, it is sufficient to check the inequality for the maximum of the LHS, $\Delta^* = \frac{v-c-2t}{2}$:

$$2 \cdot (t + \Delta^*) \cdot (v - c - t - \Delta^*) \leq r \cdot t,$$

which is equivalent to

$$2 \cdot (2t + 2 \cdot \Delta^*) \cdot (2v - 2c - 2t - 2 \cdot \Delta^*) \leq 2 \cdot r \cdot t,$$

or

$$(v - c) \cdot (v - c) \leq 2 \cdot r \cdot t.$$

Thus, there is a price-quantity equilibrium precisely when $(v - c)^2 \leq 2 \cdot r \cdot t$. ■

6 Price-Quantity Equilibrium with Spillover Demand

In the previous section, we have shown there exists a competitive price-quantity equilibrium when the prospective costs for rationed consumers to still obtain the product are sufficiently high. In this section, we explicitly allow for demand spillovers. Specifically, we consider spillover demand for any deviation from the candidate equilibrium and show that, also in these cases, there may be a competitive price-quantity equilibrium.

To begin, consider the candidate equilibrium prices (p_1, p_2) and suppose that firm 1 raises its price to $p_1' = p_1 + \Delta$. In that case, the new location of consumers who are indifferent between the two sellers, z_3', is given by:

$$z_3' = \frac{t - \Delta}{2t} = \frac{1}{2} - \frac{\Delta}{2t}.$$

Customers who are located in the interval $[z_3', \frac{1}{2}]$ then reconsider their first choice (firm 1) and now approach firm 2 first. Those located in the interval $[z_3', Z]$ visit firm 1 in case they are not served by firm 2. Figure 5 provides a graphical illustration.

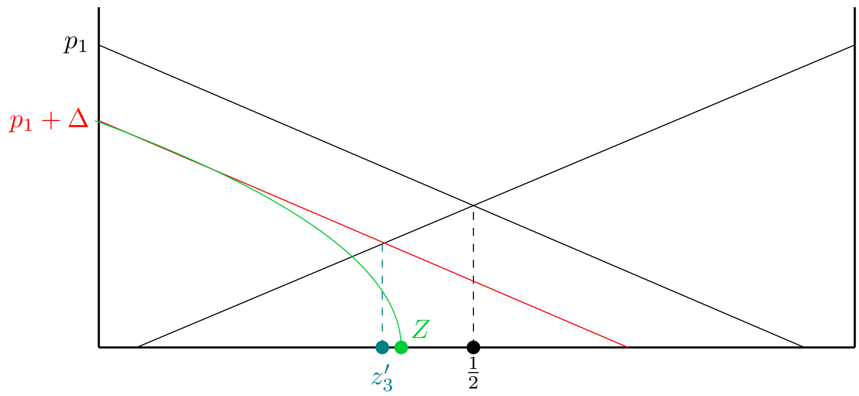

Figure 5: The effect of a price increase from p_1 to $p_1' = p_1 + \Delta$ illustrated.

Next, note that $z_3' \geq 0$ implies $0 \leq \Delta \leq t$. Moreover, both z_3' and z_p (as defined in Section 5) are decreasing functions of Δ. Specifically, both are equal to $\frac{1}{2}$ at $\Delta = 0$ and, respectively, equal to 0 and $\frac{1}{4}$ at $\Delta = t$. This is illustrated in Figure 6.

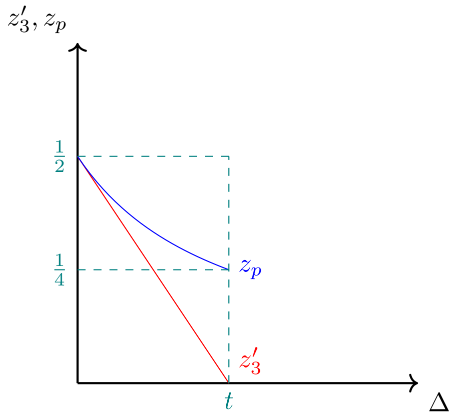

Figure 6: The functions $z_p(\Delta)$ and $z_3'(\Delta)$ illustrated.

Last, let $G(\Delta) = C(z_3')$ and recall that $F(\Delta) = C(z_p)$. It can be easily verified that $F(\Delta) \geq G(\Delta)$, for all $\Delta \in [0, t]$.[16]

Taking firm 1's perspective, there is spillover demand when $E_1 > 0$ for some $\Delta \in [0, t]$. That is, when $z_3' < Z$ for some $0 < \Delta < t$ so that there is a non-degenerate interval $[z_3', Z]$ of consumers who approach firm 1 after having visited firm 2 first. Now define:

$$f(\Delta) = \frac{E_1}{\frac{1}{2} - z_3'} = \frac{Z - z_3'}{\frac{1}{2} - z_3'}.$$

The function f measures the fraction of rationed customers who, after having visited firm 2 first, approach firm 1. The next result shows the existence of a price-quantity equilibrium in the presence of spillover demand.

Theorem 6.1 *Suppose that $Z \leq z_p$, for all $\Delta \in [0, t]$. Then, for all $\Delta \in [0, t]$, $f(\Delta) \leq \frac{1}{2}$. Moreover, $f(\Delta) \to 0$ when $\Delta \to 0$.*

This finding is illustrated in Figure 7. Since $L(\Delta) \leq F(\Delta)$, there is a competitive price-quantity equilibrium (Theorem 5.2). At relatively low and high values of Δ, there is no spillover demand.

[16] Since $\Delta \geq 0$, it holds that $t^2 \geq (t + \Delta) \cdot (t - \Delta)$. Rearranging gives $\frac{t}{2(t+\Delta)} \geq \frac{t-\Delta}{2t}$ so that $z_p \geq z_3'$, for all $\Delta \in [0, t]$. Since $C(x)$ is an increasing function, it follows that $C(z_p) \geq C(z_3')$, for all $\Delta \in [0, t]$.

In these cases, $G(\Delta) > L(\Delta)$ so that customers who are rationed out of firm 2's product find it too costly to visit firm 1. For intermediate values of Δ, however, $G(\Delta) < L(\Delta)$ so that unserved consumers prefer firm 1's product to the outside option.

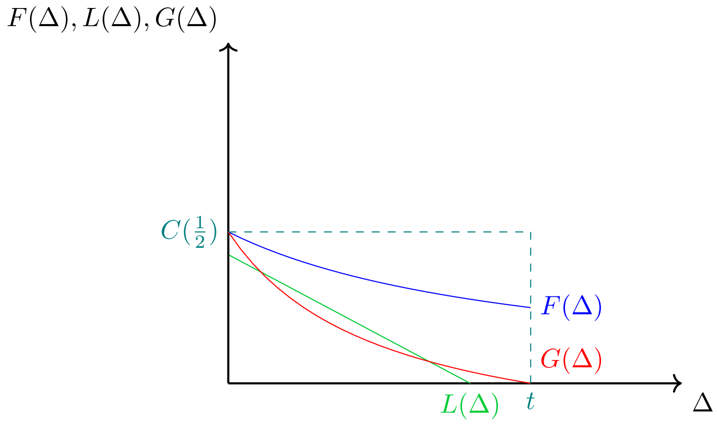

Figure 7: Price-quantity equilibrium with spillover demand.

Let us conclude this section with two illustrative cases: $C(x) = a \cdot x^2 + r \cdot x$ and $C(x) = r \cdot x$.

Proposition 6.2 *Assume $C(x) = a \cdot x^2 + r \cdot x$ and suppose that $c + \frac{3}{2}t \leq v \leq c + 2t$, $r < 2t$, and $a + 4 \cdot r \geq 8 \cdot t$. If $(2 \cdot t - r)^2 > 4 \cdot a \cdot (2t + c - v)$, then there is a $\Delta \in (0, t)$ for which there is spillover demand.*

It is worth noting that the conditions in Proposition 6.2 allow for a set of values of full dimension. To illustrate, all inequalities are strictly satisfied for $a = 225$, $v = 200$, $c = 5$, $t = 100$, and $r = 150$. Consequently, the conditions are still satisfied for sufficiently small changes in any of these numbers.

Finally, we consider the case where costs are linear: $C(x) = r \cdot x$.[17] The concluding example in Section 5 shows that equilibrium existence requires $(v - c)^2 \leq 2 \cdot r \cdot t$. The following result provides conditions for a price-quantity equilibrium in the presence of spillover demand.

Proposition 6.3 *Assume $C(x) = r \cdot x$ and suppose that $c + \frac{3}{2}t \leq v$ and $(v - c)^2 \leq 2 \cdot r \cdot t$. If $r \leq 2t$, then $f(\Delta) = 0$, for all $\Delta \in [0, t]$. If $r > 2t$, then $f(\Delta) > 0$ for at least one $\Delta \in [0, t]$ precisely when $v - c - 2t > 0$.*

For example, there is a price-quantity equilibrium with spillover demand when $t = 1$, $v = 5$, $c = 2$, and $r = 5$.

[17]Note that Proposition 6.2 does not apply, because the conditions are contradictory for the linear case $a = 0$.

7 Conclusion

Rational choice theory prescribes that a consumer's choice among available options should be guided only by incremental costs and benefits. All sunk costs should be ignored. Within the context of price-quantity games, such a rational approach may, paradoxically, lead consumers to behave as if they fall prey to the sunk cost fallacy. In principle, by considering costs associated with obtaining a product sunk, a rationed customer may prefer to continue shopping *ad infinitum.* We have argued that such costs may well be sunk, but nevertheless feed into future payoffs and, therefore, potentially affect the preference ranking over remaining alternatives. In this paper, we have taken account of this possibility by considering a more general class of cost functions. This is shown to admit a pure-strategy Nash equilibrium that coincides with the equilibrium in a price-only model.

Appendix: Proofs

Proof of Lemma 2.1

Suppose that firm 2 is the predator so that $z_3 < 0$. Notice that:

$$z_3 < 0 \quad \Leftrightarrow \quad \frac{p_2 - p_1 + t}{2t} < 0 \quad \Leftrightarrow \quad t < p_1 - p_2.$$

Hence, firm 2 is the predator when $t < p_1 - p_2$. By symmetry, firm 1 is the predator when $t < p_2 - p_1$. Taken together, we conclude that a price pair (p_1, p_2) is predatory precisely when $|p_1 - p_2| > t$. ∎

Proof of Lemma 2.2

Let (p_1, p_2) be a price pair with $|p_1 - p_2| \leq t$ so that $0 \leq z_3 \leq 1$.

A. A price pair (p_1, p_2) is monopolistic when $u_{z_3}(1) < 0$. Notice that:

$$u_{z_3}(1) < 0 \quad \Leftrightarrow \quad v - t \cdot z_3 - p_1 < 0 \quad \Leftrightarrow \quad v - t \cdot \frac{p_2 - p_1 + t}{2t} - p_1 < 0 \quad \Leftrightarrow \quad p_1 + p_2 > 2v - t.$$

We conclude that a price pair (p_1, p_2) is monopolistic precisely when $p_1 + p_2 > 2v - t$.

B. This is comparable to part A. The condition $u_{z_3}(1) = 0$ is equivalent to $p_1 + p_2 = 2v - t$.

C. This is comparable to part A. The condition $u_{z_3}(1) > 0$ is equivalent to $p_1 + p_2 < 2v - t$.

∎

Proof of Proposition 3.2

In proving this statement, we take firm 1's perspective. Firm 2's best-response can be derived in a similar fashion.

Consider some price $p_2 \leq v$. We compute the best-response of firm 1 to p_2. Let p_1 be a best-response to p_2. To begin, we argue that the resulting price pair (p_1, p_2) is non-predatory. By contradiction, suppose that (p_1, p_2) is predatory and that firm 1 is the predator. By Lemma 2.1, it then holds that $p_1 < p_2 - t$ and $z_3 > 1$ so that $D_1 = 1$. However, in this case firm 1 could raise its price to $p_1' = p_2 - t$ and sell the same amount of goods. Hence, p_1 is not a best-response to p_2; a contradiction. By symmetry, a similar argument excludes the possibility that firm 2 is the predator. We conclude that in any best-response $|p_1 - p_2| \leq t$ so that Lemma 2.2 applies.

1. Suppose that $p_2 \in A$. We argue that $BR_1(p_2) = \frac{v+c}{2}$.

1a. Consider any p_1 with $p_1 > 2v - t - p_2$. Then, by Lemma 2.2, pricing is monopolistic and firm 1's demand is $D_1 = z_1$. In this case, its profits are given by:

$$\Pi_1 = (p_1 - c) \cdot z_1 = (p_1 - c) \cdot \left(\frac{v - p_1}{t} \right),$$

which has its maximum at:

$$p_1^* = \frac{v + c}{2}.$$

We check whether it indeed holds that $p_1^* > 2v - t - p_2$. Since $p_2 \in A$, we know that $3v < c + 2t + 2p_2$ and, therefore,

$$2v - t - p_2 < \frac{v + c}{2} = p_1^*.$$

We conclude that $p_1^* > 2v - t - p_2$.

1b. Now consider any p_1 with $p_1 \leq 2v - t - p_2$. We show that $\Pi_1(p_1) < \Pi_1(p_1^*)$. Since $p_1 \leq 2v - t - p_2$, pricing is market-sharing or competitive (Lemma 2.2). In these cases, $D_1 = z_3$ and $z_3 \leq z_1$. Thus, since $p_1 \leq 2v - t - p_2 < p_1^*$, it holds that

$$\Pi_1(p_1) = (p_1 - c) \cdot z_3 < (p_1^* - c) \cdot z_1 = \Pi_1(p_1^*).$$

We conclude that $p_1^* = \frac{v+c}{2}$ is the unique best-response of firm 1 to p_2 when $p_2 \in A$.

2. Suppose that $p_2 \in C$. We argue that $BR_1(p_2) = \frac{c+t+p_2}{2}$.

2a. Consider any p_1 with $p_1 \leq 2v - t - p_2$. Then, by Lemma 2.2, pricing is market-sharing or competitive and $D_1 = z_3$. Therefore,

$$\Pi_1(p_1, p_2) = (p_1 - c) \cdot z_3 = (p_1 - c) \cdot \left(\frac{t + p_2 - p_1}{2t} \right),$$

which has its maximum at:

$$p_1^{**} = \frac{c + t + p_2}{2}.$$

We check whether it indeed holds that $p_1^{**} < 2v - t - p_2$. Since $p_2 \in B$, we know that $c + 3t + 3p_2 < 4v$ and, therefore,

$$p_1^{**} = \frac{c + t + p_2}{2} < 2v - t - p_2.$$

We conclude that $p_1^{**} < 2v - t - p_2$.

2b. Now consider any p_1 with $p_1 > 2v - t - p_2$. We show that $\Pi_1(p_1) < \Pi_1(p_1^{**})$. By Lemma 2.2, pricing is monopolistic in this case os that $D_1 = z_1$ and $z_1 < z_3$. Moreover, $p_1^{**} < 2v - t - p_2 < p_1$. So, $p_1^{**} \neq p_1$. It follows that:

$$(p_1 - c) \cdot z_1 \leq (p_1 - c) \cdot z_3 < \Pi_1(p_1^{**}, p_2).$$

We conclude that $p_1^{**} = \frac{c+t+p_2}{2}$ is the unique best-response of firm 1 to p_2 when $p_2 \in C$.

3. Finally, suppose that $p_2 \in B$. Notice that $c + 2t + 2p_2 \leq 3v$ implies $p_1^* = \frac{v+c}{2} \leq 2v - t - p_2$, and that $c + 3t + 3p_2 \geq 4v$ implies $p_1^{**} = \frac{c+t+p_2}{2} \geq 2v - t - p_2$. In this case, therefore, the best-response is $BR_1(p_2) = 2v - t - p_2$. ∎

Proof of Lemma 3.3

[1] Suppose that (p_i, p_j) is a price equilibrium with $p_i \in A$ and $p_j \in C$. It then holds that $3v < c + 2t + 2p_i$ and $c + 3t + 3p_j < 4v$. Moreover, by Proposition 3.2, we know that $p_i = \frac{c+t+p_j}{2}$ and $p_j = \frac{v+c}{2}$. Substituting $p_j = \frac{v+c}{2}$ into $c + 3t + 3p_j < 4v$ and rearranging yields $5c + 6t < 5v$. Alternatively, substituting $p_j = \frac{v+c}{2}$ into $p_i = \frac{c+t+p_j}{2}$ yields $4p_i = 3c + 2t + v$. Substituting this equation into $3v < c + 2t + 2p_i$ and rearranging yields $5v < 5c + 6t$, which contradicts the preceding conclusion that $5c + 6t < 5v$. We conclude that there is no price equilibrium with $p_i \in A$ and $p_j \in C$.

[2] Suppose that (p_i, p_j) is a price equilibrium and that $p_j \in B$. Then, by Proposition 3.2, $p_i = 2v - t - p_j$ and, therefore, $p_j = 2v - t - p_i$. Now suppose that $p_i \in A$ so that $p_j = \frac{v+c}{2}$ is a best-response. It then holds that:

$$2v - t - p_i = p_j = \frac{v+c}{2}.$$

This implies that $3v = c + 2t + 2p_i$, which contradicts $p_i \in A$.

Next, suppose that $p_i \in C$. Then, by Proposition 3.2, $p_j = \frac{c+t+p_i}{2}$. Therefore,

$$\frac{c+t+p_i}{2} = p_j = 2v - t - p_i.$$

This implies that $c + 3t + 3p_i = 4v$, which contradicts $p_i \in C$.

∎

Proof of Lemma 3.4

Let (p_1, p_2) be a price equilibrium.

[1] Suppose that $p_1, p_2 \in A$. Then, by Proposition 3.2, $p_i = \frac{v+c}{2}$, $i = 1, 2$. Moreover, $3v < c + 2t + 2p_i$. Substituting $p_i = \frac{v+c}{2}$ into $3v < c + 2t + 2p_i$ and rearranging yields $v < c + t$.

[2] Suppose that $p_1, p_2 \in B$. Then, by Proposition 3.2, $p_1 + p_2 = 2v - t$. Next, we know that (p_1, p_2) is a price equilibrium, and that $p_1, p_2 \in B$. Since B is convex,

$$p = \frac{p_1 + p_2}{2} = \frac{2v - t}{2}$$

is also an element of B and, therefore, $p = 2v - t - p$ by Proposition 3.2. Thus, (p, p) is a price equilibrium. Moreover, since $p \in B$, it holds that $3v \geq c+2t+2p$ and $c+3t+3p \geq 4v$. Substituting $p = \frac{2v-t}{2}$ into these inequalities and rearranging yields $c + t \leq v \leq c + \frac{3}{2}t$.

[3] Suppose that $p_1, p_2 \in C$. Then, by Proposition 3.2, $p_1 = \frac{c+t+p_2}{2}$ and $p_2 = \frac{c+t+p_1}{2}$. Solving for p_1 and p_2 yields $p_i = c+t$, $i = 1, 2$. Moreover, $c+3t+3p_i < 4v$. Substituting $p_i = c+t$ into $c + 3t + 3p_i < 4v$ and rearranging yields $c + \frac{3}{2}t < v$.

■

Proof of Theorem 3.5

Suppose that $v < c + t$. By Lemma 3.4, this leaves

$$(p_1, p_2) = (\frac{v + c}{2}, \frac{v + c}{2})$$

as the only equilibrium candidate. It, therefore, suffices to show that $(p_1, p_2) = (\frac{v+c}{2}, \frac{v+c}{2})$ is a price equilibrium. Let us show that $p_1 = \frac{v+c}{2} \in A$. Using $v < c + t$, it holds that:

$$3v < 2c + 2t + v = c + 2t + v + c = c + 2t + 2p_1.$$

Hence, $p_1 \in A$. Furthermore, following Proposition 3.2, $p_2 = \frac{v+c}{2}$ is a best-response to $p_1 = \frac{v+c}{2}$. By symmetry, we conclude that $(p_1^*, p_2^*) = (\frac{v+c}{2}, \frac{v+c}{2})$ is the unique price equilibrium. ■

Proof of Theorem 3.6

Let (p_1, p_2) be a price equilibrium and suppose that $c + t \leq v \leq c + \frac{3}{2}t$. Therefore, by Lemma 3.4, it holds that $p_1 + p_2 = 2v - t$ and $p_1, p_2 \in B$. Choosing $\lambda = p_2$, we can then write:

$$(p_1, p_2) = (2v - t - \lambda, \lambda).$$

It suffices to check for which values of λ it holds that $p_1 = 2v - t - \lambda \in B$ and $p_2 = \lambda \in B$. By the definition of B, we know that $p_2 = \lambda \in B$ precisely when:

$$c + 3t + 3\lambda \geq 4v \quad \text{and} \quad 3v \geq c + 2t + 2\lambda.$$

Rearranging gives:

$$8v - 6t - 2c \leq 6\lambda \leq 9v - 6t - 3c.$$

Likewise, $p_1 = 2v - t - \lambda \in B$ precisely when:

$$c + 3t + 3 \cdot (2v - t - \lambda) \geq 4v \quad \text{and} \quad 3v \geq c + 2t + 2 \cdot (2v - t - \lambda).$$

Rearranging gives:

$$3v + 3c \leq 6\lambda \leq 4v + 2c.$$

Taken together, this gives the range of λ values as specified in the statement. ■

Proof of Theorem 3.7

Suppose that $c + \frac{3}{2}t < v$. By Lemma 3.4, this leaves

$$(p_1, p_2) = (c + t, c + t)$$

as the only equilibrium candidate. It, therefore, suffices to show that $(p_1, p_2) = (c + t, c + t)$ is a price equilibrium. Let us show that $p_1 = c + t \in C$. Using $c + \frac{3}{2}t < v$, it holds that

$$c + 3t + 3p_1 = c + 3t + 3c + 3t = 4c + 6t < 4v.$$

Hence, $p_1 \in C$. Furthermore, following Proposition 3.2, $p_2 = \frac{c+t+p_1}{2}$ is a best-response. Substituting $p_1 = c + t$ gives $p_2 = c + t$. By symmetry, we conclude that $(p_1^*, p_2^*) = (c + t, c + t)$ is the unique price equilibrium. ■

Proof of Lemma 4.2

In proving this statement, we take firm 1's perpsective, which is without loss of generality. Consider a strategy (p_2, q_2) of firm 2, and let (p_1, q_1) be a best-response of firm 1. Suppose that, by contradiction, $q_1 < d_1$. Since a predatory price is not a best-response, firm 1's demand is given by $d_1 = \min\{z_1, z_3\}$. Both z_1 and z_3 depend continuously on p_1. Therefore, one can increase p_1 slightly to p_1' in such a way that still $q_1 < d_1'$. The pair (p_1', q_1) has the same cost, as q_1 did not change, and the same sales since $s_1' = \max\{d_1', q_1\} = q_1 = s_1$, but a higher price p_1'. Consequently,

$$\Pi_1'(p_1', q_1) = p_1' \cdot q_1 - C_1(q_1) > p_1 \cdot q_1 - C_1(q_1) = \Pi_1(p_1, q_1).$$

Thus, (p_1, q_1) is not a best-response.

Now suppose that $q_1 > d_1$. Firm 1's profit is then $\Pi_1 = p_1 \cdot d_1 - C_1(q_1)$. Now take $q_1' = d_1$. Since $C(q_i)$ is strictly increasing, it holds that

$$\Pi_1(p_1, q_1') = p_1 \cdot d_1 - C_1(q_1') > p_1 \cdot d_1 - C_1(q_1) = \Pi_1(p_1, q_1).$$

Thus, (p_1, q_1) is not a best-response. We conclude that in any best-response, a firm produces to meet its demand. ■

Proof of Theorem 4.3

Let $((p_1, q_1), (p_2, q_2))$ be a price-quantity equilibrium. By Lemma 4.2, we know that $q_1 = d_1$ and $q_2 = d_2$.

We show that (p_1, p_2) is a price equilibrium. Since $((p_1, q_1), (p_2, q_2))$ is a price-quantity equilibrium, we know that (p_i, q_i) is the best-response of firm i to (p_j, q_j). So, (p_i, q_i) solves the maximization problem

$$\max_{p_i, q_i} \Pi_i$$

By lemma 4.2, $q_i = d_i$. Hence, p_i solves the maximization problem

$$\max_{p_i, q_i} \Pi_i$$

$$\text{s.t. } q_i = d_i,$$

which means that p_i is a best-response in the price game. ■

Proof of Theorem 4.4

Let (p_1, p_2) be a monopolistic or market-sharing price equilibrium. We prove that the strategy $(p_1, D_1(p_1, p_2))$ is a best-response of firm 1 against the strategy $(p_2, D_2(p_1, p_2))$. One can take the same steps for firm 2.

Let $(\widetilde{p_1}, \widetilde{q_1})$ be a best-response of firm 1 against $(p_2, D_2(p_1, p_2))$. We prove that $(\widetilde{p_1}, \widetilde{q_1}) = (p_1, D_1(p_1, p_2))$.

Since $(\widetilde{p_1}, \widetilde{q_1})$ is a best-response of firm 1 against $(p_2, D_2(p_1, p_2))$, we know that $(\widetilde{p_1}, \widetilde{q_1})$ solves:

$$\max_{p_1', q_1'} \Pi_1 = p_1' \cdot s_1 - C(q_1').$$

For brevity, write $q_2 = D_2(p_1, p_2)$. By Lemma 4.2, we then know that $\widetilde{q_1} = d_1(\widetilde{p_1}, p_2, q_2)$. It then follows that $\widetilde{p_1}$ maximizes the profit function:

$$\Pi_1 = p_1' \cdot d_1(p_1', p_2, q_2) - C(d_1(p_1', p_2, q_2)).$$

However, since pricing is monopolistic or market-sharing, we know that $d_1(p_1', p_2, q_2) = D_1(p_1', p_2)$ for all p_1' and, therefore, that $\widetilde{p_1}$ maximizes:

$$\Pi_1 = p_1' \cdot D_1(p_1', p_2) - C(D_1(p_1', p_2)).$$

It follows that $\widetilde{p_1} = p_1$ so that $\widetilde{q_1} = d_1(\widetilde{p_1}, p_2, q_2) = D_1(p_1, p_2)$. ■

Proof of Theorem 4.5

Let (p_1, p_2) be a competitive price equilibrium. By Theorem 3.7, the competitive price equilibrium is $(p_1, p_2) = (c + t, c + t)$. At these prices, direct demand for firm 1 is $D_1 = \frac{1}{2}$.

Fix firm 2's strategy $(p_2, D_2(p_1, p_2)) = (c + t, \frac{1}{2})$. We show that $((p_1, D_1(p_1, p_2)) = (c + t, \frac{1}{2})$ is not a best-response of firm 1.

For brevity, write $D_i = D_i(p_1, p_2)$ and $u = u_{\frac{1}{2}}(1)$. Since the price equilibrium is competitive, we know that $u > 0$. Define $p'_1 = p_1 + u$. We first argue that $s'_1 = \frac{1}{2}$. Notice that

$$u'_{\frac{1}{2}}(1) = v - p'_1 - t \cdot \frac{1}{2} = v - p_1 - u - t \cdot \frac{1}{2} = u - u = 0.$$

So, $z'_1 = \frac{1}{2}$. Since $q_1 = q_2 = \frac{1}{2}$, it follows that $d'_1 \geq \min\{z'_1, 1 - q_2, q_1\} = \frac{1}{2} = d_1$. Thus, $s'_1 = \min\{d'_1, q_1\} = \frac{1}{2}$. Hence,

$$\begin{aligned}\Pi_1((p'_1, D_1)(p_2, D_2)) &= p'_1 \cdot s'_1 - t \cdot q_1 \\ &= (c + t + u) \cdot \frac{1}{2} - t \cdot q_1 \\ &> (c + t) \cdot \frac{1}{2} - t \cdot q_1 = \Pi_1((p_1, D_1)(p_2, D_2))\end{aligned}$$

Therefore, $(c + t + u, \frac{1}{2})$ is a better reply for firm 1 to $(c + t, \frac{1}{2})$ than $(c + t, \frac{1}{2})$. ■

Proof of Proposition 5.1

Suppose that $c + \frac{3}{2}t \leq v \leq c + 2t$ and that $a + 4 \cdot r \geq 8 \cdot t$. Let (p_1, q_1, p_2, q_2) be any price-quantity equilibrium. By Theorem 4.3, it then holds that (p_1, p_2) is a price equilibrium. As $c + \frac{3}{2}t \leq v$, it follows from Theorem 3.7 that

$$(p_1, p_2) = (c + t, c + t),$$

$D_1 = D_2 = \frac{1}{2}$, and therefore $q_1 = q_2 = \frac{1}{2}$ by Theorem 4.3. Thus, it suffices to show that

$$((p_1, q_1), (p_2, q_2)) = ((c + t, \frac{1}{2}), (c + t, \frac{1}{2}))$$

is a price-quantity equilibrium. Suppose that firm 2 chooses $(p_2, q_2) = (c + t, \frac{1}{2})$. We argue that $(p_1, q_1) = (c + t, \frac{1}{2})$ is the unique best-response for firm 1.

Suppose that firm 1 chooses $p'_1 = p_1 + \Delta$. If $\Delta < 0$, then $d'_1 = D_1(p'_1, p_2)$, which is not a profitable deviation. Suppose then that $\Delta > 0$.

Let z_p be the point where firm 1's profit at p'_1 equals its price equilibrium profit. More specifically, z_p is the smallest value of d for which:

$$\Pi'_1 = (c + t + \Delta) \cdot d - c \cdot d \geq \frac{t}{2} = \Pi^*_1.$$

Rewriting yields $(t+\Delta)\cdot d \geq \frac{t}{2}$ and, therefore:

$$z_p = \frac{t}{2(t+\Delta)}.$$

Consequently, the price equilibrium outcome is a price-quantity equilibrium when the consumers located at z_p have a negative utility when buying from firm 1. We argue that indeed $u_{z_{\mathrm{p}}}(1) \leq 0$.

A. We first argue that:

$$a\cdot t^2 + 2\cdot r\cdot t^2 + 2\cdot t\cdot r\cdot \Delta \geq 4\cdot t^3 + 4\cdot t^2\cdot \Delta,$$

for all $\Delta \leq t$. First note that this is true for any $\Delta \geq 0$ when $r \geq 2\cdot t$. Suppose, therefore, that $r \leq 2\cdot t$. By assumption, $a + 4\cdot r \geq 8\cdot t$ so that multiplying by t^2 and rearranging yields:

$$a\cdot t^2 + 2\cdot r\cdot t^2 - 4\cdot t^3 \geq 4\cdot t^3 - 2\cdot r\cdot t^2.$$

So, since $r \leq 2\cdot t$, for any $\Delta \leq t$, it holds that:

$$a\cdot t^2 + 2\cdot r\cdot t^2 - 4\cdot t^3 \geq t\cdot \left(4\cdot t^2 - 2\cdot r\cdot t\right) \geq \Delta\cdot \left(4\cdot t^2 - 2\cdot r\cdot t\right).$$

Rewriting gives the above inequality.

B. Next, notice that $|p_1 - p_2| = \Delta$. By Lemma 2.1 and Lemma 2.2, it holds that $\Delta \leq t$. By using the inequality under A, we then have that:

$$a\cdot t^2 + 2\cdot r\cdot t^2 + 2\cdot t\cdot r\cdot \Delta \geq 4\cdot t^3 + 4\cdot t^2\cdot \Delta = 4\cdot (t+\Delta)\cdot t^2 \geq 4\cdot (t+\Delta)\cdot (t^2 - \Delta^2).$$

It follows that:

$$a\cdot t^2 + 2\cdot r\cdot t\cdot (t+\Delta) = a\cdot t^2 + 2\cdot r\cdot t^2 + 2\cdot t\cdot r\cdot \Delta \geq 4\cdot (t+\Delta)\cdot (t^2 - \Delta^2) = 4\cdot (t+\Delta)^2\cdot (t-\Delta).$$

Since $v \leq c + 2t$, $v - c - t \leq t$ and, therefore:

$$a\cdot t^2 + 2\cdot r\cdot t\cdot (t+\Delta) \geq 4\cdot (t+\Delta)^2\cdot (t-\Delta) \geq 4\cdot (t+\Delta)^2\cdot (v - c - t - \Delta).$$

Dividing both sides by $4\cdot (t+\Delta)^2$ yields:

$$a\cdot \left(\frac{t}{2(t+\Delta)}\right)^2 + r\cdot \left(\frac{t}{2(t+\Delta)}\right) \geq v - c - t - \Delta.$$

Hence,

$$a\cdot (z_p)^2 + r\cdot (z_p) \geq v - c - t - \Delta,$$

which is equivalent to

$$v - p_1' - C(z_p) \le 0.$$

By symmetry, a similar analysis can be conducted for firm 2. ■

Proof of Theorem 5.2

Suppose that $L(\Delta) \le F(\Delta)$. Recall that Z is the location of consumers who are indifferent between staying home or buying at firm 1 at price p_1'. Equilibrium existence is then ensured when $Z \le z_p$ for any $\Delta \in [0, t]$. This implies a negative utility for buying from firm 1 for consumers at z_p. It, therefore, needs to be checked that $U_{z_p}(1) \le 0$. Substituting gives:

$$v - c - t - \Delta - C(z_p) \le 0,$$

which is equivalent to $L(\Delta) \le F(\Delta)$. ■

Proof of Theorem 6.1

We consider a deviation $p_1' = p_1 + \Delta$ by firm 1 from the candidate equilibrium price $p_1 = c + t$. First, notice that

$$z_p - z_3' = \frac{t}{2(t+\Delta)} - \frac{t-\Delta}{2t} = \frac{2t^2 - 2(t^2 - \Delta^2)}{2t(t+\Delta)} = \frac{\Delta^2}{2t(t+\Delta)}.$$

Moreover,

$$\frac{1}{2} - z_3' = \frac{1}{2} - \frac{p_2 - p_1' + t}{2t} = \frac{1}{2} - \frac{\frac{1}{2} - \frac{1}{2} - \Delta + t}{2t} = \frac{1}{2} - \frac{t-\Delta}{2t} = \frac{\Delta}{2t}.$$

By assumption, $Z < z_p$, for all $\Delta \in [0, t]$. Therefore,

$$f(\Delta) = \frac{Z - z_3'}{\frac{1}{2} - z_3'} = (Z - z_3') \cdot \frac{2t}{\Delta} \le (z_p - z_3') \cdot \frac{2t}{\Delta} = \frac{\Delta^2}{2t(t+\Delta)} \cdot \frac{2t}{\Delta} = \frac{\Delta}{t+\Delta} \le \frac{1}{2}$$

for all $\Delta \in [0, t]$. Furthermore, it follows that:

$$f(\Delta) = \frac{\Delta}{t+\Delta} = \frac{1}{\frac{t}{\Delta} + 1} \to 0$$

as $\Delta \to 0$. Thus, spillover is negligible for small enough price deviations. ■

Proof of Proposition 6.2

We need to show that $Z > z_3'$. To that end, it is sufficient to show that $U_{z_3'} > 0$. We can rewrite

$$U_{z_3'} > 0$$

as

$$v - c - t - \Delta > a \cdot \left(\frac{t-\Delta}{2t}\right)^2 + r \cdot \frac{t-\Delta}{2t},$$

which can be rewritten to:

$$4t^2 \cdot (v - c - t - \Delta) > a \cdot (t - \Delta)^2 + 2 \cdot r \cdot t \cdot (t - \Delta).$$

Define:

$$\Delta^* = \frac{t \cdot (a + r - 2t)}{a}.$$

Notice that $4 \cdot a + 4 \cdot r > a + 4 \cdot r \geq 8 \cdot t$. So, $a + r > 2 \cdot t$, which implies that $\Delta^* > 0$. Furthermore, since $r < 2 \cdot t$, it follows that $a + r - 2 \cdot t < a$. So, $\Delta^* < t$. We conclude that $0 < \Delta^* < t$.

It then suffices to show that:

$$4t^2 \cdot (v - c - t - \Delta^*) > a \cdot (t - \Delta^*)^2 + 2 \cdot r \cdot t \cdot (t - \Delta^*).$$

Note that:

$$t - \Delta^* = t - \frac{t \cdot (a + r - 2t)}{a} = \frac{a \cdot t - a \cdot t - rt + 2t^2}{a} = \frac{t \cdot (2 \cdot t - r)}{a}.$$

So, we need to prove that:

$$4t^2 \cdot (v - c - t - \Delta^*) > a \cdot \left(\frac{t \cdot (2 \cdot t - r)}{a}\right)^2 + 2 \cdot r \cdot t \cdot \frac{t \cdot (2 \cdot t - r)}{a}.$$

Multiplication by $\frac{a}{t^2}$ and rearranging yields:

$$4a \cdot (v - c - t - \Delta^*) > (2 \cdot t - r)^2 + 2 \cdot r \cdot (2 \cdot t - r),$$

which can be rewritten to:

$$4 \cdot (a \cdot (v - c - t) - t \cdot (a + r - 2t)) > (2 \cdot t - r) \cdot (2 \cdot t + r),$$

which, in turn, is equivalent to:

$$4 \cdot (a \cdot (v - c - 2t) + t \cdot (2t - r)) > (2 \cdot t - r) \cdot (2 \cdot t + r).$$

This inequality is equivalent to:

$$4 \cdot a \cdot (v - c - 2t) > (2 \cdot t - r) \cdot (r - 2 \cdot t),$$

and

$$(2 \cdot t - r)^2 > 4 \cdot a \cdot (2t + c - v),$$

which is the condition in the proposition. ∎

Proof of Proposition 6.3

Notice that $f(\Delta) > 0$, for at least one $\Delta \in [0,t]$, precisely when $U(z_3') > 0$, for at least one $\Delta \in [0,t]$. This inequality can be written as:

$$2t \cdot (v - c - t - \Delta) > r \cdot (t - \Delta),$$

for at least one $\Delta \in [0,t]$. This is equivalent to:

$$2t \cdot (v - c - t - \frac{1}{2} \cdot r) > (2t - r) \cdot \Delta,$$

for at least one $\Delta \in [0,t]$. Since the LHS does not depend on Δ, this is equivalent to

$$2t \cdot (v - c - t - \frac{1}{2} \cdot r) > 0,$$

for $r \leq 2t$. In this case, therefore, $f(\Delta) > 0$, for at least one $\Delta \in [0,1]$ precisely when $2v - 2c - 2t - r > 0$. However, it then holds that:

$$2 \cdot t \cdot r \geq (v-c)^2 > \left(t + \frac{1}{2} \cdot r\right)^2 = t^2 + t \cdot r + \frac{1}{4} \cdot r^2,$$

which implies:

$$\left(t - \frac{1}{2} \cdot r\right)^2 < 0.$$

Hence, in this case, $f(\Delta) = 0$, for all $\Delta \in [0,t]$.

If $r > 2t$, then the above inequality is equivalent to:

$$2t \cdot (v - c - t - \frac{1}{2} \cdot r) > (2t - r) \cdot t.$$

Dividing both sides by $2 \cdot t$ and rearranging yields $v - c - 2t > 0$. ■

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