Efficient Effort Equilibrium in Cooperation with Pairwise Cost Reduction

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Abstract

There are multiple situations in which bilateral interaction between agents results in considerable cost reductions. The cost reduction that an agent obtains depends on the effort made by other agents. We model this situation as a bi-form game with two states. In the first stage, agents decide how much effort to exert. We model this first stage as a non-cooperative game, in which these efforts will reduce the cost of their partners in the second stage. This second stage is modeled as a cooperative game in which agents reduce each other's costs as a result of cooperation, so that the total reduction in the cost of each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. The proposed cost allocation for the cooperative game in the second stage determines the payoff function of the non-cooperative game in the first stage. Based on this model, we explore the costs, benefits, and challenges associated with setting up a pairwise effort situation. We identify a family of cost allocations with weighted pairwise reductions which are always feasible in the cooperative game and contain the Shapley value. We also identify the cost allocation with weighted pairwise reductions that generate an efficient equilibrium effort level.

Keywords Allocation, Cost models, Efficiency, Game Theory, Mechanism Design.

1 Introduction

The search for greater efficiency, access to new markets and greater competitiveness are some of the main factors that result in inter-organization or inter-corporate cooperation structures. There are different forms of cooperation depending on the degree of integration or interdependence of partners

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and on the intended goals of agreements. These forms have been widely studied in economic literature (see e.g. Todeva and Knoke (2005) for a survey). There is one specific type of cooperation whose properties and characteristics differentiate it from the rest. It can occur between agents that share, for example, resources, knowledge or infrastructure. The common purpose is to obtain individual advantages such as reducing their respective individual costs. The particularity of this form of cooperation lies in the fact that the cost reduction is based on bilateral interactions.

We consider that form of cooperation here in which, given any pair of cooperating agents, one agent reduces the cost of the other by a certain amount which is independent of cooperation with other agents. This means that if there are more agents in the coalition the amount of the cost reduction does not change. This pairwise cost reduction is independent of the coalition to which the pair of agents may belong. Therefore, for any agent, the total cost reduction in any coalition can easily be calculated as the sum of the reductions obtained from each bilateral interaction with the other members of the coalition.

There are several situations where this kind of cooperation with pairwise cost reduction occurs and is profitable, e.g. strategic collaboration agreements between firms to reduce logistical operational costs. The need to increase market share requires logistics firms to expand their radius of action as far as possible. This means major investments in expensive infrastructures at new sites, which increase operational costs. Agreements are established between companies to reduce those costs while maintaining control of their respective markets and hindering access by new competitors. They offer the resources held by each firm in its respective area of influence under advantageous conditions. This enables them to expand their operating ranges with significant cost savings. Interactions occur bilaterally, with each company using the resources of the other. These cost reductions are independent of any cost reductions that can also be obtained by interacting with other agents in larger coalitions.

The second situation is that of bilateral free trade agreements between countries. In a globalized economy, free trade agreements are quite common. They facilitate trade in goods and services between countries, reducing trade barriers and consequently the cost of trade. These cost reductions are specific to each pair of countries, and are independent of any other agreements that either may decide to establish with other countries.

A third situation is the sharing of market data. Currently, information on customers and their purchase patterns is vitally important for firms. It enables them to maximize returns on advertising costs and focus on their ideal target markets. Cooperation between firms (usually from complementary sectors) consists of sharing information about their respective customers. This reduces the costs of each of the firms involved. The information that a particular firm provides is specific to it, so the value of the information that it receives from another specific firm is independent of information from other firms. Even if two firms provide information about the same customer, the information itself is different because it describes the purchase of a different good or service. This can increase the value of that particular customer as a target, which again boosts the value of this particular kind of cooperation.

The last situation presented here is that of inter-firm cooperation agreements to reduce costs by increasing the range of firms' respective telecommunication networks. In eminently competitive sectors such as mobile telephony and online services, cooperation between operators has become quite common. For example, they may share the locations of their respective antennas, which enables them to expand the reach of their networks. This means greater benefits thanks to the offering of a broader service, while avoiding the costs that would be entailed by each company installing its own structures. Here again, cost reduction is bilateral when two agents decide to share and use each other's antennas. These cost savings are independent of any collaboration agreements that each firm may have with other agents to share antennas in larger coalitions.

In this kind of cooperation, the cost reduction between agents may be highly asymmetric when they cooperate in pairs. For example, if two agents A and B decide to cooperate, agent A could provide a major reduction for agent B, while the reduction provided in the opposite direction could be more modest. These asymmetries can induce imbalances or discriminations that could jeopardize cooperation. A fair distribution mechanism for the costs generated by cooperation is undoubtedly needed to ensure the stability of any strategic partnership, as Thomson (2010) points out.

In addition, it is quite common for this kind of cooperation to require the agents involved to make a set level of effort. It is natural to think that the amount by which one agent can reduce the costs of the other (if they decide to cooperate) could depend on the effort that the agent exerts. For example, if one country can obtain information relevant to another (e.g. information on tax evasion and the flight of capital involving its citizens), the amount and quality of the specific information may depend on the effort exerted by the first country in gathering it. This extends the situation beyond a cooperative model. For this reason, we model the sequence of decisions as a bi-form game (Brandenburger and Stuart, 2007). In the first stage of the bi-form game, agents decide how much (costly) effort they are willing to exert. This has a direct impact on their pairwise cost reductions. This first stage is modeled as a non-cooperative game in which agents determine the level of pairwise effort to reduce the costs of their partners. In the second stage, agents engage in bilateral interactions with multiple independent partners where the cost reduction brought by each agent to another agent is independent of any possible coalition. We study this bilateral cooperation in the second stage as a cooperative game in which cooperation leads agents to reduce their respective costs, so that the total reduction in costs for each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. In the non-cooperative game of the first stage, the agents anticipate the cost allocation that will result from the cooperative game in the second stage by incorporating the effect of the effort made into their cost functions. Based on this model, we explore costs, benefits, and challenges associated with setting up a pairwise effort situation.

We investigate the impact of pairwise efforts on cost reductions and the resulting cost structure for this framework. In particular, we explore the design of a cost-allocation mechanism that efficiently allocates the gains from pairwise effort to all parties. To that end, we first compute the optimal level of cost reduction, taking into account the pairwise cost reductions collectively accrued by all agents. An ideal allocation scheme should encourage agents to participate in it and, at the same time, establish proper incentives to make efforts prior to cooperation. Specifically, we first show that it is profitable for all agents to participate in a pairwise effort situation. Then we study how the total reduction in costs should be allocated to the participants in such a situation. We do this by modeling the pairwise cost reduction between agents that takes place in the second stage as a cooperative game, which we refer to as the pairwise effort game or "PE-game".

We prove that the marginal contribution of an agent diminishes as a coalition grows in PE-games (i.e. they are concave games) and thus all-included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off or, at least, not worse off (i.e. PE-games are balanced, which means that the core is not empty). This all-included cooperation is also consistent (i.e. PE-games are totally balanced, which means the core of every subgame is non-empty). We identify various allocation mechanisms that enable all-included cooperation to be feasible (i.e. allocation mechanisms that belong to the core of PE-games). In particular, we discuss a family of cost allocations with weighted pairwise reduction which is always a subset of the core of PE-games. This is a broad family of core-allocations which includes the Shapley value, which is obtained when all the weights work out to a half. We provide a highly intuitive, simple expression for the Shapley value, which matches the Nucleolus in our model. To select one of these core-allocations in the second stage, we take into account the incentives that it generates in the efforts made by agents, and consequently in the aggregate cost of a coalition. We show that the Shapley value can induce inefficient effort strategies in equilibrium in the non-cooperative model. However, it is always possible to find core-allocations with weighted pairwise reductions that create appropriate incentives for agents to make optimal efforts that minimize aggregate costs, i.e. core-allocations that generate an efficient level of effort in equilibrium.

This paper contributes to the literature by presenting a doubly robust cost sharing mechanism. That mechanism not only has good properties regarding the cooperative game in the second stage but also creates appropriate incentives in the non-cooperative game in the first stage that enable efficiency to be achieved.

Cooperative game theory has developed a substantial mathematical framework for identifying and providing suitable cost sharing allocations (see, e.g., Fiestras-Janeiro et al. 2011; Guajardo and Rönnqvist 2016; Luo et al. 2022 for a survey). Multiple solutions have been proposed from a wide range of approaches (see, e.g., Moulin 1987; Slikker and Van den Nouweland 2012; Lozano et al. 2013; Yu et al. 2017; Omrani et al. 2018; Fardi et al. 2019; Ciardello et al 2019; Mitridati et al. 2021; Tajbakhsh and Hassini (2022); Meng et al 2023;). The Shapley value (Shapley 1953) is considered one of the most outstanding of them, and a suitable solution concept (see, e.g., Moretti and Patrone 2008; Serrano 2009 for a survey). As an allocation rule it has very good properties, such as efficiency, proportionality, and individual and coalitional rationality. However, it has the disadvantage of posing computational difficulties, which increase as the number of players increases. Nonetheless, there is a large body of literature in which the Shapley value is proposed as a simple, easy-to-apply solution in different economic scenarios (see, e.g., Littlechild and Owen 1973; Bilbao et al. 2008; Li and Zhang 2009; Kimms and Kozeletskyi 2016; Le et al. 2018; Meca et al. 2019). These papers give simplified solutions for different classes of games. They take the cost structure as given and do not consider the system externalities that arise when agents make efforts to give and receive cost reductions. Our paper here incorporates both the non-cooperative aspects of making efficient efforts (by modeling decisions related to pairwise cost reductions) and the cooperative nature of giving and receiving cost reductions in pairwise effort situations.

As in principal-agent literature, we refer to action by agents as "effort". In this setting, the concept of "effort" is widely used in analyzing different kinds of problem. One of the first was the moral hazard problems: See for example Holmstrom (1982). Other examples are Holmstrom (1999) and Dewatripont et al. (1999), who identify conditions under which more information can induce an agent to make less effort. The approach in our model is quite different, in that we do not consider any kind of principal. As far as we know, our model is novel in that it analyzes the incentive for agents to make efforts in a bi-form game: A non-cooperative stage where agents choose how much effort to make and a cooperative second stage. As mentioned, we show that the solution of the cooperative game determines the incentives of agents to make an effort in the first stage, and consequently the efficiency of the final outcome.

Bernstein et al. (2015) also use a bi-form model to analyze the role of process improvement in a decentralized assembly system in which an assembler lays in components from several suppliers. The assembler faces a deterministic demand and suppliers incur variable inventory costs and fixed production setup costs. In the first stage of the game suppliers invest in process improvement activities to reduce their fixed production costs. Upon establishing a relationship with suppliers, the assembler sets up a knowledge sharing network which is modeled as a cooperative game between suppliers in which all suppliers obtain reductions in their fixed costs. They compare two classes of allocation mechanism: Altruistic allocation enables non-efficient suppliers to keep the full benefits of the cost reductions achieved due to learning from the efficient supplier. The Tute allocation mechanism benefits a supplier by transferring the incremental benefit generated by including an efficient supplier in the network. They find that the system-optimal level of cost reduction is achieved under the Tute allocation rule. Our bi-form game is novel in terms of incentive for efforts by agents and is also richer in results: We find the allocation rule that generates the unique efficient effort in equilibrium in cooperation with pairwise cost reduction.

The paper is organized as follows. Section 2 presents the bi-form game and describes in detail the two stages in which the model is developed. Section 3 is devoted to analyzing the second stage which is a cooperative game. In this cooperative game, agents reduce each other's costs as a result of cooperation, so that the total reduction in the cost of each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. In Section 4 the first stage is studied, that is the non-cooperative game that precedes the cooperative game in the second stage. Here, the agents anticipate the cost allocation that results from the cooperative game in the second stage by incorporating the effect of the effort exerted into their cost functions. We consider a family of cost allocation rules (in the second state) with pairwise reductions weighted separately (WPR family) and obtain the corresponding effort equilibria in the first state. Then, we develop a general and complete analysis of the efficient effort equilibria. Finally, in this section, we found the core-allocation rule in this WPR family that generates the unique efficient effort equilibria. Section 5 focuses on a subfamily of the WPR family in which pairwise reductions are not weighted separately, but are weighted as aggregated reduction, this is the WPAR family. We find out that the level of efficiency is lower than that attained when the pairwise reductions are weighted separately for each agent. Then, we found the rule, within this WPAR family, that generates the equilibrium efforts closest to the efficient ones. Finally, Section 6 completes the study of our model by comparing the two families of core-allocation analyzed. We complete the paper with a section of conclusions and four appendices containing the proofs of the results and tables of summaries (notation and optimization problems).

2 Model

We consider a model with a finite set of agents $N = \{1, 2, ..n\}$, where each agent has a good (for example resources, knowledge or infrastructure) and has to perform a certain activity. The total cost of an agent's activity can be reduced if it cooperates with another agent, which means that the two agents share their goods. These cost reductions obtained by sharing goods in pairs depend on the effort made previously by each agent. Our model consists of two different stages. In the first stage, agents choose their effort levels as in a non-cooperative game. In the second stage, agents cooperate to reduce their costs, and allocate the minimum cost they achieve by pairwise cost reductions as in a cooperative game. The proposed cost allocation for the cooperative game in the second stage determines the payoff function of the non-cooperative game in the first stage. Therefore, we model the sequence of decisions as a bi-form game (Brandenburger and Stuart, 2007). The two stages of the model are described in detail below.

First Stage (non-cooperative game): Each agent $i \in N$ chooses in this state an effort level $e_i = (e_{i1}, ..., e_{i(i-1)}, e_{i(i+1)}, ...e_{in}) \in [0, 1]^{n-1}$, where $e_{ij} \in [0, 1]$ stands for the level of effort by agent i to reduce the cost of agent j if they cooperate in the second stage. These efforts have a cost $c_i(e_i) \in \mathbb{R}_+$ for any $i \in N$. We assume that $c_i(.) : [0, 1]^{n-1} \to \mathbb{R}_+$ is a scalar field of class $C^2([0, 1]^{n-1})$.[1] Moreover, for all $e_{ij} \in [0, 1]$ with $j \in N\backslash\{i\}$, it is assumed that $\frac{\partial c_i(e_i)}{\partial e_{ij}} > 0$, $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$, and $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}\partial e_{ih}} = 0$ for all $h \neq i, j$, which implies that the marginal cost $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is independent of the effort that i exerts with agents other than j.[2]

Second Stage (cooperative game): Given the effort made in the first stage, agents cooperate,

[1] A scalar field is said to be class C^2 at $[0, 1]^{n-1}$ if its 2-partial derivatives exist at all points of $[0, 1]^{n-1}$ and are continuous.

[2] This last assumption implies that the Hessian matrix is a diagonal matrix. In addition, note that, given our assumptions about c_i, w.l.o.g. we could consider that $c_i(e_i) = \sum_{j\in N\backslash\{i\}} c_{ij}(e_{ij})$ where $c_{ij}(.) : [0, 1] \to \mathbb{R}_+$. We omit it from the paper so as not to introduce more notation into the model.

so for any pair of cooperating agents $i, j \in N$ and a given effort e_{ij}, agent i reduces the total cost of agent j by an amount $r_{ji}(e_{ij}) \in R_+$, and vice versa. These particular reductions between agents $i, j \in N$ are independent of cooperation with other agents. We also assume for all $j \in N\backslash\{i\}$ that function $r_{ij}(.) : [0,1] \to R_+$ is class C^2, increasing and concave[3] at $[0,1]$. Thus, these agents participate in bilateral interactions with multiple independent partners whose cost reductions are coalitionally independent, i.e. the cost reduction given by each agent to another agent is independent of any possible coalition. This means that the total reduction in cost for each agent in a coalition $S \subset N$ is the sum of the pairwise cost reductions given to that agent by the rest of the members of the coalition, i.e. for agent i, it is $\sum_{j\in S\backslash\{i\}} r_{ij}(e_{ji})$. We assume perfect information regarding agents' costs and cost reductions depending on efforts.

Given an effort profile $e = (e_1, e_2, ..., e_n) \in [0,1]^{n(n-1)}$ in the first stage, the second stage can be seen as a cooperative game, more specifically a transferable utility cost game (N, e, c), where N is the finite set of players, and $c : 2^N \to R$ is the so-called characteristic function of the game, which assigns to each subset $S \subseteq N$ the cost $c(S)$ that is incurred if agents in S cooperate. By convention, $c(\emptyset) = 0$. The cost of agent i in coalition $S \subseteq N$ is given by $c^S(i) := c_i(e_i) - \sum_{j\in S\backslash\{i\}} r_{ij}(e_{ji})$. This cost can be interpreted as the reduced cost of agent i in coalition S. Note that the larger the coalition, the greater the cost reduction it achieves, i.e. for all $i \in S \subseteq T \subseteq N$, $c^T(\{i\}) \leq c^S(\{i\})$. The total reduced cost for coalition S is given by

$$c(S) := \sum_{i\in S} c^S(\{i\}) = \sum_{i\in S}[c_i(e_i) - \sum_{j\in S\backslash\{i\}} r_{ij}(e_{ji})]. \tag{1}$$

When all agents cooperate, they form what is called the grand coalition, which is denoted by N. Thus, $c(N)$ is the aggregate cost of the grand coalition. The allocation of the grand coalition cost achieved through cooperation, in the second stage, assigns a reduced final cost to each agent, that is, $\psi_i(e)$, for all $i \in N$, where $\psi_i : E \to R$ with $E := \prod_{i\in N} E_i$ and $E_i := [0,1]^{n-1}$. Then, we define the cost allocation rule $\psi : E \to R^n$ s.t. $\psi(e) = (\psi_i(e))_{i\in N}$ and $\sum_{i\in N} \psi_i(e) = c(N)$.

The non-cooperative cost game in the first stage is defined through that cost allocation rule ψ by $(N, \{E_i\}_{i\in N}, \{\psi_i\}_{i\in N})$, where E_i is the strategy space of agent $i \in N$ (its effort level space), and ψ_i is the payoff function of agent i, but in this case is a cost function. Hence, for an effort profile $e \in E$, the corresponding cost function is $\psi(e)$. That effort is made in anticipation of the result of the cooperative cost game that follows in the second stage. Therefore, we first analyze the second stage (see Section 3), and focus on different ways of allocating the grand coalition cost. We determine cost allocation rules with good computability properties and coalitional stability for this cooperative cost game. Notice that a given cost allocation rule will generate precise incentives in the first state and consequently particular equilibrium effort strategies [4]. In turn, these particular effort strategies will have an associate cost of the grand coalition. At this point, a question about efficiency arises. The

[3] $\partial r_{ji}(e_{ij})/\delta e_{ij} > 0$ (increasing) and $\partial^2 r_{ji}(e_{ij})/\delta e_{ij}^2 < 0$ (concave).

[4] An effort strategy profile is said to be in equilibrium when each agent has nothing to gain by changing only their own effort strategy given the strategies of all the other agents (Nash equilibrium).

lower the cost of the grand coalition generated in equilibrium is, the more efficient the equilibrium effort strategies and the allocation rule considered will be.

Therefore, there are two dimensions to be considered. First, the cost allocation rule for the cooperative game should have good properties (computability and coalitional stability). Second, the allocation rule must induce the right incentives in the non-cooperative game to obtain the lowest cost of the grand coalition. This interesting, relevant question is analyzed in Section 4 and 5.

Throughout the paper, we consider the following assumptions:

(CA) Cost assumptions: $c_i \in C^2$, and $\frac{\partial c_i(e_i)}{\partial e_{ij}} > 0$ (increasing), $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$ (convex), and $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}\partial e_{iK}} = 0$, if $k \neq j$ (additively separable).

(RA) Reduction assumptions: $r_{ji} \in C^2$, and $\partial r_{ji}(e_{ij})/\delta e_{ij} > 0$ (increasing), $\partial^2 r_{ji}(e_{ij})/\delta e_{ij}^2 < 0$ (concave).

A summary of the notation and the main optimization problems (Table 1 and 2) can be found in Appendix D.

3 Cooperation with Pairwise Cost Reduction

This section presents the analysis of cooperation with pairwise cost reduction in the second stage. Agents make their efforts in pairwise sharing in the first stage, and initiate cooperation with efforts $e = (e_1, ..., e_i, ..., e_n)$. We model the PE-game as a multiple-agent cooperative game where each agent i incurs an initial cost of $c_i(e_i)$. All agents in a pairwise effort group (coalition) give cost reductions to and receive such reductions from other agents. As a result, all agents in the coalition reduce their initial costs to levels that depend on the efforts made in the first stage by the others. No agent outside the pairwise effort situation benefits from this pairwise cost reduction opportunity. We introduce all the game-theoretic concepts used in this paper, but readers are referred to González-Díaz et al. (2010) for more details on cooperative and non-cooperative games.

We refer to the pairwise effort situation as a PE-situation and denote it by the tuple $(N, e, \{c_i(e_i), \{r_{ji}(e_{ij})\}_{j \in N \setminus \{i\}}\}_{i \in N})$. We associate a cost game (N, e, c) with each PE-situation as defined by (1).

The class of PE-games has some similarities with the class of linear cost games introduced in Meca and Sosic (2014). They define the concept of cost-coalitional vectors as a collection of certain a priori information, available in the cooperative model, represented by the costs of the agents in all possible coalitions to which they could belong. The cost of a coalition in their study is thus the sum of the costs of all agents in that coalition. However, the PE-games considered here are significantly different from their linear cost games. Linear cost games focus on the role played by benefactors (giving) and beneficiaries (receiving) as two groups of disjoint agents, but PE-games consider that all agents could be dual benefactors, in the sense that they be benefactors and beneficiaries at the same time. In addition, PE-games are based on a bilateral cooperation between agents that enables both to reduce their costs but is coalitionally independent.

We now consider a PE-situation $(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j\in N\setminus\{i\}}\}_{i\in N})$ and consider whether it is profitable for the agents in N to form the grand coalition to obtain a significant reduction in costs. We find that the answer is yes, and show that the associated PE-game (N, e, c) is concave, in the sense that for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subset N$ with $i \in S$, so $c(S) - c(S\setminus\{i\}) \geq c(T) - c(T\setminus\{i\})$. This concavity property provides additional information about the game: the marginal contribution of an agent diminishes as a coalition grows. This is well-known and is called the "snowball effect".

The first result in this section shows that PE-games are always concave. This means that the grand coalition can obtain a significant reduction in costs. In that case, the reduced total cost is given by $c(N) = \sum_{i\in N} c_i(e_i) - R(N)$, where $R(N) = \sum_{i\in N}\sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})$ is the total reduction produced by bilateral reductions between all agents in the situation, which turns out to be the total cost savings for all agents. The proof of Proposition 1, together with all our other proofs for this section, is shown in Appendix A.

Proposition 1 *Every PE-game is concave.*

An allocation rule for PE-games is a map ψ which assigns a vector $\psi(e) \in R^n$ to every (N, e, c), satisfying efficiency, that is, $\sum_{i\in N} \psi_i(e) = c(N)$. Each component $\psi_i(e)$ indicates the cost allocated to $i \in N$, so an allocation rule for PE-games is a procedure for allocating the reduced total to all the agents in N when they cooperate. It is a well-known result in cooperative game theory that concave games are totally balanced: The core of a concave game is non-empty, and since any subgame of a concave game is concave, the core of any subgame is also non-empty. That means that coalitionally stable allocation rules can always be found for PE-games. We interpret a non-empty core for PE-games as indicating a setting where all included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off (or, at least, not worse off). The totally balanced property suggests that this all-included cooperation is consistent, i.e. for every group of agents whole-group cooperation is also feasible.

A highly natural allocation rule for PE-games is $\varphi_i(e) = c^N(\{i\}) = c_i(e_i) - R_i(N)$, for all $i \in N$, with $R_i(N) = \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})$ being the total reduction received by agent $i \in N$ from the rest of the agents $j \in N\setminus\{i\}$. It has good properties at least with respect to computability and coalitional stability in the sense of the core. Formally, the core of a PE-game (N, c) is defined as follows

$$Core\,(N, c) = \{x \in \mathbb{R}^n / \sum_{i\in N} x_i = c(N), \sum_{i\in S} x_i \leq c(S) \forall S \subseteq N\}. \tag{2}$$

Notice that $\varphi(e) \in Core\,(N, c)$. Indeed, $\sum_{i\in N} \varphi_i(e) \leq c(N)$ and for every $S \subseteq N, \sum_{i\in S} \varphi_i(e) =$

$\sum_{i\in S} c^N(i) \leq \sum_{i\in S} c^S(i) = c(S)$. Nevertheless, the agents could argue that this allocation does not provide sufficient compensation for their dual role of giving and receiving. Note that the allocation only considers their role as receivers.

PE-games are concave, so cooperative game theory provides allocation rules for them with good properties, at least with respect to coalitional stability and acceptability of items. We highlight the Shapley value (see Shapley 1953), which assigns a unique allocation (among the agents) of a total surplus generated by the grand coalition. It measures how important each agent is to the overall cooperation, and what cost can it reasonably expect. The Shapley value of a concave game is the center of gravity of its core (see Shapley 1971). In general, this allocation becomes harder to compute as the number of agents increases. We present a simple expression here for the Shapley value of PE-games that takes into account all bilateral relations between agents and compensates them for their dual role of giving and receiving.

Given a general cost game (N, c), we denote the Shapley value by $\phi(c)$, where the corresponding cost allocation for each agent $i \in N$, is

$$\phi_i(c) = \sum_{i\in T\subseteq N} \frac{(n-t)!(t-1)!}{n!} \left[(c(T) - c(T\backslash\{i\})\right], \text{ with } \mid T \mid = t. \tag{3}$$

The Shapley value has many desirable properties, and it is also the only allocation rule that satisfies a certain subset of those properties (see Moulin, 2004). For example, it is the only allocation rule that satisfies the four properties of Efficiency, Equal treatment of equals, Linearity and Null player (Shapley, 1953).

Given a PE-game (N, e, c), we denote by $\phi(e)$ the Shapley value of the cost game. The following Theorem shows that the Shapley value provides an acceptable allocation for PE-games. Indeed, it reduces the individual cost of an agent by the average of the total reduction that it obtains from the others $(R_i(N))$ plus half of the total reduction that it provides to the rest of the agents, i.e. $G_i(N) = \sum_{j\in N\backslash\{i\}} r_{ji}(e_{ij})$.

Theorem 1 *Let (N, e, c) be a PE-game. For each agent $k \in N, \phi_k(e) = c_k(e_k) - \frac{1}{2}[R_k(N) + G_k(N)]$.*

From Theorem 1 it can be derived that the Shapley value, $\phi(e)$, considers the dual role of giving and receiving of all agents, and the final effect on those agents depends on which role is stronger. As mentioned above, if an allocation does not compensate them for their dual role of giving and receiving, and it only considers their role as receivers, as the individual cost in the grand coalition, $\varphi(e)$, does, the cooperation would not be desirable for those dual agents. This non-acceptability can be avoided by using the Shapley value, which also coincides with the Nucleolus (Schmeiler 1989) for PE-games.

The nucleolus selects the allocation in which the coalition with the smallest excess (the worst treated) has the highest possible excess. The nucleolus maximizes the "welfare" of the worst treated coalitions. Denote by $\nu(e) \in R^n$ the *Nucleolus* of the PE-game (N, e, c), associated with a PE-situation

$(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j\in N\setminus\{i\}}\}_{i\in N})$. First, we define the excess of coalition S in (N, e, c) with respect to allocation x as $d(S, x) = c(S) - \sum_{i\in S} x_i$. This excess can be considered as an index of the "welfare" of coalition S at x: The greater $d(S, x)$, the better coalition S is at x. Let $d^*(x)$ be the vector of the 2^n excesses arranged in (weakly) increasing order, i.e., $d_i^*(x) \le d_j^*(x)$ for all $i < j$. Second, we define the lexicographical order $\succ_l$. For any $x, y \in R^n$, $x \succ_l y$ if and only if there is an index k such that for any $i < k$, $x_i = y_i$ and $x_k > y_k$. The nucleolus of the PE-game (N, e, c) is the set

$$\nu(e) = \{x \in X : d^*(x) \succ_l d^*(y) \text{ for all } y \in X\} \tag{4}$$

with $X = \{x \in R^n : \sum_{i\in N} x_i = c(N),\ x_i \ge c(\{i\}) \text{ for all } i \in N\}$.

It is well known that the Nucleolus is a singleton for balanced games and that it is always a core-allocation.

The Proposition 2 proves that for PE-games the Shapley value matches the Nucleolus. This is a very good property that few cost games satisfy.

Proposition 2 *Let (N, e, c) be a PE-game. For each agent $k \in N$, $\nu_k(e) = \phi_k(e)$.*

Therefore, given an effort profile, the Shapley value is a very suitable way of allocating the reduced cost due to cooperation. Note that, the cost reduction as a result of cooperation between any pair of agents $i, j \in N$ is $r_{ij}(e_{ji}) + r_{ji}(e_{ij})$, and the Shapley value assigns one half of this amount to i and the other half to j. This seems a reasonable way to split this aggregate cost reduction. However, if agents knew before choosing their levels of efforts that the cost reductions resulting from their efforts were going to be allocated according to the Shapley value, the incentives created could generate inefficiencies. Some agents could find it optimal to exert too little effort and in some situations this could be inefficient.

For example, consider a PE-situation in which one agent has the ability to produce a substantial reduction in costs for other agents with a low effort cost and the rest of the agents have almost no ability to reduce costs for others even with a high effort cost. If the Shapley value is used as the allocation rule for this game, agents may not have incentives to make any level of effort. Note that in the first step agents have to decide how much effort to make. However, if the Shapley value is modified to give a greater portion of the pairwise cost reduction to the especially productive agent, it might make more effort and thus produce a greater reduction in cost for other agents. This change in the Shapley value generates new allocation rules, which can reduce the cost of the grand coalition regarding the Shapley allocation. The following example with three agents illustrates these ideas.

Example 1 . *Consider a pairwise inter-organizational situation with three firms, i.e. $N = \{1, 2, 3\}$. For any effort profile $e \in [0, 1]^6$, the PE-situation is given by the following initial costs,*

$c_1(e_{12}, e_{13}) = 100 + 100e_{12} + 4e_{12}^2 + 100e_{13} + 4e_{13}^2$
$c_2(e_{21}, e_{23}) = 100 + 100e_{21} + 4e_{21}^2 + 100e_{23} + 4e_{23}^2$
$c_3(e_{31}, e_{32}) = 100 + 100e_{31} + 4e_{31}^2 + 100e_{32} + 4e_{32}^2$

and the following pairwise reduced costs, all of them in thousands of Euros,

$r_{i1}(e_{1i}) = 2 + 200e_{1i} - 3e_{1i}^2$ *with* $i = 2, 3$
$r_{i2}(e_{2i}) = 2 + 3e_{2i} - e_{2i}^2$ *with* $i = 1, 3$
$r_{i3}(e_{3i}) = 2 + 3e_{3i} - e_{3i}^2$ *with* $i = 1, 2$

If the allocation rule in the second stage is the Shapley value, the firms choose their levels of effort according to this cost allocation function. It is straight forward to show that in this case the unique effort equilibrium e^**, is one in which the three firms make no effort, i.e.* $e^*_{ij} = 0$ *for* $i, j \in N$.[5] *Thus, the Shapley value distributes the cost of the grand coalition* $c^*(N) = 288$ *equally, i.e. for each firm* $i = 1, 2, 3$, $\phi_i(e^*) = c_i(e^*_i) - \frac{1}{2}\sum_{j\in N\setminus\{i\}}[r_{ij}(e^*_{ji}) + r_{ji}(e^*_{ij})] = 100 - \frac{1}{2}((2+2)+(2+2)) = 96$.

Note that, for example, in the relationship between firm 1 *and* 2*, the pairwise cost reduction is* $r_{12}(e_{21}) + r_{21}(e_{12})$*, and the Shapley value gives* $\frac{1}{2}$ *of this amount to firm* 1 *and the other* $\frac{1}{2}$ *to firm* 2*. However, if the proportion that firm* 1 *obtains is increased, e.g. from* $\frac{1}{2}$ *to* $\frac{3}{4}$*, and the part for firm* 2 *is thus reduced to* $\frac{1}{4}$*, the incentive of firm* 1 *to make an effort can be increased. The same goes for firms* 1 *and* 3 *so that the incentive of firm* 1 *to make an effort for firm 3 is also increased. These changes in the Shapley value lead to a new allocation rule which we denote by* $\Omega(e) = (\Omega_1(e), \Omega_2(e), \Omega_3(e))$ *for any effort profile* $e \in [0, 1]^6$*. With this new allocation rule, the equilibrium efforts are zero for firms* 2 *and* 3*, and one for firm* 1*. That is,* $e^{**}_{1j} = 1$*, for* $j = 2, 3$, $e^{**}_{2j} = 0$*, for* $j = 1, 3$*, and* $e^{**}_{3j} = 0$*, for* $j = 1, 2$*. In this case, the grand coalition cost* $c^{**}(N) = 102$ *is allocated equally between firms* 2 *and* 3*, and the rest to firm* 1*. That is,* $\Omega_i(e^{**}) = 100 - \frac{1}{4}[(2+200-3)+2] - \frac{1}{2}(2+2) = 47,75$ *for* $i = 2, 3$*, and* $\Omega_1(e^{**}) = 100 + 100 + 4 + 100 + 4 - \frac{3}{4}[(2 + (2 + 200 - 3)) + (2 + (2 + 200 - 3))] = 6,5$.

Hence, the new allocation rule $\Omega(e^{**})$ *greatly reduces the grand coalition cost (by* 136.000 *Euros) as well as the costs of each firm; i.e. a reduction of* 89.500 *Euros for firm* 1 *and* 23.250 *Euros for firms* 2 *and* 3*. In relative terms, with the Shapley value each company pays 33.33% of the total cost. However, with the modified Shapley value agent 1 only pays 4.4% of the total cost, while agents 2 and 3 pay 47.8% each. Therefore, the modified Shapley value generates a more efficient outcome in the sense that it creates more appropriate incentives for firms.*

To reach efficient effort strategies in equilibrium (henceforth EEE) in the first stage, we consider a new family of allocation rules, for PE-games (second stage), based on the Shapley value. This family consists of the rules $\Omega(e) \in R^n$, where for all $i \in N$,

$$\Omega_i(e) = c_i(e_i) - \sum_{j\in N\setminus\{i\}} [\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})],$$

[5] Theorem 3, in Section 4, shows the efforts of equilibrium in the non-cooperative game in the general case.

with $\omega_{ij}^i, \omega_{ji}^i \in [0,1]$, for all $j \in N\backslash\{i\}$, such that $\omega_{ij}^i = 1 - \omega_{ij}^j$ and $\omega_{ji}^i = 1 - \omega_{ji}^j$. The Shapley value is a particular case of this family of rules in which $\omega_{ij}^i = \omega_{ji}^i = \frac{1}{2}$, for all $i \in N$ and all $j \in N\backslash\{i\}$. This family of cost allocation for PE-games is referred to as *cost allocation with weighted pairwise reduction.*

The Theorem below shows that the family of cost allocations with weighted pairwise reduction is always a subset of the core of PE-games. This property identifies a wide subset of the large core of PE-games, including the Shapley value (and thus the Nucleolus).

Theorem 2 *Let (N, e, c) be a PE-game. For every family of weights $\omega_{ij}^i, \omega_{ji}^i \in [0,1]$, $i, j \in N, i \neq j$, such that $\omega_{ij}^i = 1 - \omega_{ij}^j$ and $\omega_{ji}^i = 1 - \omega_{ji}^j$, $\Omega(e)$ belongs to the core of (N, e, c).*

Now a complete analysis of the EEE for cooperation in pairwise cost reduction can be conducted.

4 Efficiency, Equilibrium Strategies, and Optimal Rule

We first define an *efficient effort profile* as the effort profile that minimizes the cost of the grand coalition, $c(N) = \sum_{i\in N}[c_i(e_i) - \sum_{j\in N\backslash\{i\}} r_{ij}(e_{ji})]$.

Definition 1 *An effort profile $\tilde{e} = (\tilde{e}_1, ..., \tilde{e}_i, ..., \tilde{e}_n)$ with $\tilde{e}_i = (\tilde{e}_{i1}, ..., \tilde{e}_{i(i-1)}, \tilde{e}_{i(i+1)}, ...\tilde{e}_{in}) \in [0,1]^{n-1}$ is efficient if $\tilde{e} = \arg \underset{e\in[0,1]^{n(n-1)}}{min} \sum_{i\in N}[c_i(e_i) - \sum_{j\in N\backslash\{i\}} r_{ij}(e_{ji})]$*

An efficient effort profile $\tilde{e}$ is well defined because $c(N)$ as a function of e is strictly convex in e_{ij} for all $i, j \in N, i \neq j$. [6]

The following proposition shows that the effort e_{ij} is efficient if the marginal cost of that effort equals the marginal reduction that this effort generates; otherwise, the effort is zero or one. The proof of Proposition 3 appears in Appendix B, together with those of all the other proofs in this section.

Proposition 3 *There exists a unique efficient effort profile $\tilde{e} = (\tilde{e}_1, ..., \tilde{e}_i, ..., \tilde{e}_n)$ with $\tilde{e}_i = (\tilde{e}_{i1}, ..., \tilde{e}_{i(i-1)}, \tilde{e}_{i(i+1)}, ...\tilde{e}_{in}) \in [0,1]^{n-1}$, such that*

- $\tilde{e}_{ij} = 0$ *if* $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ *for all* $e_{ij} \in [0,1]$,
- $\tilde{e}_{ij} = 1$ *if* $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ *for all* $e_{ij} \in [0,1]$,
- $\tilde{e}_{ij} \in (0,1)$ *is the unique solution of* $\frac{\partial c_i(e_i)}{\partial e_{ij}}\Big|_{e_{ij}=\tilde{e}_{ij}} = \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\Big|_{e_{ij}=\tilde{e}_{ij}}$, *otherwise.*

[6] Note that the second derivative in e_{ij} is equal to $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} - \frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2}$, which is always positive because $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} < 0$.

We now focus on the non-cooperative effort game that arises under the family of *cost allocation with weighted pairwise reduction* (henceforth, WPR family). Then we analyze efficiency in equilibrium.

Consider the WPR family, i.e., $\Omega_i(e) = c_i(e_i) - \sum_{j\in N\setminus\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})]$ for all $i \in N$ with $\omega^i_{ij}, \omega^i_{ji} \in [0,1]$, i, $j \in N, i \neq j$, such that $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$. For each specification of these weights, a particular allocation rule can be obtained that induces a certain equilibrium effort strategy in the first stage, which in turn generates the associated cost allocation in equilibrium. The aim of this section is twofold. First, we identify the efficient allocation rule within the WPR family, i.e., that which results in the lowest cost of the grand coalition. Second, we show that the effort profile induced in equilibrium by this allocation rule coincides with the efficient effort profile of Proposition 3.

The non-cooperative cost game associated with $\Omega = (\Omega_i)_{i\in N}$ in the first stage is defined by $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$, where for every agent $i \in N$, $E_i := [0,1]^{n-1}$ is the players' i strategy set, and for all effort profiles $e \in E := \prod_{i\in N} E_i$, and Ω_i is the cost function for agent $i \in N$. We call this an effort game.

In this game, we use the following definition of equilibrium.

Definition 2 *The effort profile $e^* = (e^*_1, ..., e^*_n) \in E$ is an equilibrium for the game $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$ if e^*_i is the optimal effort for agent $i \in N$ given the strategies of all the other agents $j \in N\setminus\{i\}$.*

First, note that the optimal effort for agent $i \in N$ given the strategies of all the other agents $j \in N\setminus\{i\}$ is the effort e_i that minimizes $\Omega_i(e_i, e_{-i})$. Note that the function $\Omega_i(e_i, e_{-i})$ is strictly convex in the effort e_{ij} that agent i exerts for any $j \in N\setminus\{i\}$.[7] This means that for agent i there is a unique optimal level of effort $\hat{e}_{ij}$ for each $j \in N\setminus\{i\}$. That optimal level $\hat{e}_{ij}$ depends on the parameter ω^i_{ji}, on the marginal cost of agent i in regard to effort $\hat{e}_{ij}$ (i.e. $\frac{\partial c_i(e_i)}{\partial e_{ij}}$), and on the marginal cost-reduction for agent j in regard to effort $\hat{e}_{ij}$, (i.e. $\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$). Consequently, although the cost function of agent i depends on other agents' efforts (e_{ji} for all $j \in N\setminus\{i\}$), the optimal effort does not.

To obtain the optimal effort, we analyze the derivative of the convex function $\Omega_i(e)$ with respect to e_{ij}, for any $j \in N\setminus\{i\}$. It must be noted that $\frac{\partial \Omega_{ii}(e)}{\partial e_{ij}} \geq 0 \iff \frac{\partial c_i(e_i)}{\partial e_{ij}} \geq \omega^i_{ji} \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0,1]$. The following result characterizes the optimal effort level for agent $i \in N$ in the first stage of the game.

Lemma 1 *Let $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$ be an effort game and $\hat{e}_{ij}$ be the optimal level of effort that agent i exerts to reduce the costs of agent j. Thus,*

- $\hat{e}_{ij} = 0$ *if and only if* $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \omega^i_{ji} \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, *for all* $e_{ij} \in [0,1]$,

[7] Note that $\frac{\partial \Omega_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \omega^i_{ji} \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $\frac{\partial^2_i \Omega(e)}{\partial e^2_{ij}} = \frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}} - \omega^i_{ji} \frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}} > 0$ because, as assumed above, $\frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}} < 0$

- $\hat{e}_{ij} = 1$ *if and only if* $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \omega_{ji}^i \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, *for all* $e_{ij} \in [0,1]$,
- $\hat{e}_{ij} \in (0,1)$ *that holds* $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=\hat{e}_{ij}} = \omega_{ji}^i \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=\hat{e}_{ij}}$, *otherwise.*

The following theorem shows the unique allocation rule of the WPR family that induces an efficient effort profile in equilibrium. This allocation rule gives all the reductions to the agent that generates them. Formally, let $H(e) := (H_i(e))_{i \in N}$ be the allocation rule in the WPR family with $\omega_{ji}^i = 1$ for i, $j \in N, i \neq j$, that is $H_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} r_{ji}(e_{ij})$ for $i \in N$. We consider an allocation rule as efficient if it induces an efficient effort profile in equilibrium.

Theorem 3 *Consider the effort game* $(N, \{E_i\}_{i \in N}, \{H_i\}_{i \in N})$*. Let* e_{ij}^* *be the level of effort that an agent* i *exerts to reduce the costs of agent* j *in the unique equilibrium with* i*,* $j \in N, i \neq j$*. Thus,*

- $e_{ij}^* = 0$ *if and only if* $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=0} > \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=0}$
- $e_{ij}^* = 1$ *if and only if* $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=1} < \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=1}$
- $e_{ij}^* \in (0,1)$ *that holds* $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=e_{ij}^*} = \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=e_{ij}^*}$, *otherwise.*

In addition, $e_{ij}^* = \tilde{e}_{ij}$ *for* i*,* $j \in N, i \neq j$ *and* $H_i(e)$ *is the only allocation rule of the WPR family that always induces an efficient effort profile in equilibrium.*

The next Corollary shows that the allocation rule H is not only the only efficient one within the WPR family, but that it induces the lowest possible grand coalition cost for any possible allocation rule.

Corollary 1 *Let* Θ *be the set of all allocation rules for PE-games. There is no* $\psi \in \Theta$ *such that the effort equilibrium profile induced in the non cooperative game* $(N, \{E_i\}_{i \in N}, \{\psi_i\}_{i \in N})$ *generates a lower cost of the grand coalition than allocation rule* H*.*

As mentioned, the effort e_{ij} is efficient when its marginal cost matches the marginal reduction that it generates; otherwise, the effort is zero or one. Allocation rule $H(e)$ aligns the incentives of agents in the first stage game with this idea. It gives all the reduction to the agent that generates it. In that case, the best response of any agent is to make its marginal cost equal to the marginal reduction that its effort generates; otherwise, this agent exerts the minimal or maximal effort depending on which is higher: the marginal cost or the marginal reduction.

We illustrate this analysis with the 3-firm case given in Example 2 in Section 6.

In this section we work out the allocation rule (in the second stage) within the WPR family that generates the unique efficient effort equilibrium (in the first stage). However, there are situations in which pairwise reductions cannot be weighted separately, i.e. it is not possible to assign different weights to what an agent gives and what the same agent receives in a pairwise interaction. For example, there may be situations in which there is a unique cost reduction for any pair of agents that depends on the effort exerted by both agents, i.e. an aggregate reduction. In that case they have to decide how to split the whole cost reduction. Such cases require a weight to be assigned to the pairwise aggregate reduction.

The question that arises in this new scenario is whether the level of efficiency maintained is the same as that attained when the pairwise reductions are weighted separately for each agent. Unfortunately, the answer is no: the level of efficiency decreases in this new scenario. The next section focuses on measuring the level of efficiency of efforts in equilibrium for a particular family of weighted pairwise aggregate reductions.

5 Measuring Efficiency for Pairwise Aggregate Reduction

Consider the family of cost allocation with weighted pairwise aggregate reduction $A(e) \in R^n$ defined as follows:

$$A_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} \alpha_{ij}[r_{ij}(e_{ji}) + r_{ji}(e_{ij})], \tag{5}$$

with $\alpha_{ij} \in [0, 1]$. The interaction between agents i and j generates an aggregate cost reduction which is $r_{ij}(e_{ji}) + r_{ji}(e_{ij})$. The parameter α_{ij} measures the proportions in which this reduction is shared between agents i and j, i.e. α_{ij} is the proportion for agent i and $\alpha_{ji} = 1 - \alpha_{ij}$ for agent j.

Note that $A(e)$ is a subfamily of the WPR family $\Omega(e)$, where now $\omega^i_{ij} = \omega^j_{ij} = \alpha_{ij}$, for all $i, j \in N$. From now on we refer to this subfamily as the WPAR family. It is important to note that the Shapley value and the Nucleolus belong to the WPAR family with $\alpha_{ij} = \frac{1}{2}$ for all i, $j \in N$, $i \neq j$. We consider whether the allocation rule $H(e)$, which generates the efficient effort in equilibrium, is applicable in this situation. Unfortunately, $H(e)$ does not fit the scheme of pairwise aggregate reduction.

This section analyzes the non-cooperative effort game that arises in the first stage when cost allocations in the WPAR family are considered.

Our goal is to find out, within the WPAR family, a core-allocation in the cooperative game of the second stage that induce the effort equilibrium level in the first stage closest to the efficient one. We consider that an effort profile $e' \in E$ is more efficient than a profile $e'' \in E$ if the aggregate cost generated in the second stage by e' is lower than that generated by e''.

We therefore first study the non-cooperative effort game that arises under this new cost allocation $A(e)$, that is $(N, \{E_i\}_{i \in N}, \{A_i\}_{i \in N})$.

To simplify notation and analysis, we consider that for all $i \in N$ and $j \in N \setminus \{i\}$, $c'_i(e_{ij}) := \frac{\partial c_i(e_i)}{\partial e_{ij}}$, $c''_i(e_{ij}) := \frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}}$, $r'_{ji}(e_{ij}) := \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $r''_{ji}(e_{ij}) := \frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}}$. Note that, as the WPAR family is a subfamily of WPR, the properties of the latter apply to the former.

Before analyzing the EEE of the above non-cooperative effort game, we define thresholds of alpha parameters that enable them to be reached.

Definition 3 *Given an effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$, we define the following lower and upper thresholds for each pair of agents i and j,*

$$\underline{\alpha}_{ij} := \frac{c_i'(0)}{r_{ji}'(0)},\ \bar{\alpha}_{ij} := \frac{c_i'(1)}{r_{ji}'(1)},\ \underline{\alpha}_{ji} := \frac{c_j'(0)}{r_{ij}'(0)},\ \text{and } \bar{\alpha}_{ji} := \frac{c_j'(1)}{r_{ij}'(1)}.$$

It is clear that $0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij}$ because c_i' is an increasing function and r_{ji}' decreasing one. Analogously, $0 < \underline{\alpha}_{ji} < \bar{\alpha}_{ji}$.

The first Theorem in this section characterizes all possible types of effort equilibrium according to the value of the parameter α_{ij}, for all $i, j \in N, i \neq j$. The proof of Theorem 4 appears in Appendix C, together with all the other proofs in this section.

Theorem 4 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be an effort game. The pairwise efforts in any unique equilibrium (e_{ij}^*, e_{ji}^*) are given by*

$$e_{ij}^* = \begin{cases} 0 \text{ if and only if } \alpha_{ij} \leq \underline{\alpha}_{ij} \\ e^I \text{ if and only if } \underline{\alpha}_{ij} < \alpha_{ij} < \bar{\alpha}_{ij} \\ 1 \text{ if and only if } \alpha_{ij} \geq \bar{\alpha}_{ij} \end{cases} \qquad e_{ji}^* = \begin{cases} 0 \text{ if and only if } \alpha_{ij} \geq 1 - \underline{\alpha}_{ji} \\ e^J \text{ if and only if } 1 - \bar{\alpha}_{ji} < \alpha_{ij} < 1 - \underline{\alpha}_{ji} \\ 1 \text{ if and only if } \alpha_{ij} \leq 1 - \underline{\alpha}_{ji} \end{cases}$$

where $e^I \in (0, 1)$ is the unique solution of $c_i'(e_i) - \alpha_{ij} r_{ji}'(e_{ij}) = 0$ and $e^J \in (0, 1)$ is the unique solution of $c_j'(e_j) - (1 - \alpha_{ij}) r_{ij}'(e_{ji}) = 0$.

It is demonstrated in Appendix C that e^I increases with α_{ij} while e^J decreases, see Corollary 2. The findings of Corollary 2 are valuable when the objective is to incentivize agents $i, j \in N$ to increase their pairwise effort e_{ij} by adjusting the parameter α_{ij}. However, our aim is to go beyond this and achieve optimal efficiency within the WPAR family. In other words, we seek to determine the optimal values of α_{ij}^*, for all $i, j \in N$, which minimizes the aggregate cost function $\sum_{i\in N} A_i(e^*)$ at equilibrium, where both A_i and the effort equilibrium e^* depend on α_{ij}.

The search for alpha parameters which will lead to the EEE can be simplified by taking into account the bilateral independent interactions of agents. Note first that any pair of agents have a particular α_{ij}, and second that the optimal effort made by any agent $i \in N$ in regard to any agent $j \in N\backslash\{i\}$ is independent of the optimal effort that agent i exerts in regard to any other agent $h \in N\backslash\{i, j\}$. Thus, minimizing $\sum_{i\in N} A_i(e^*)$ in terms of α_{ij} is equivalent to minimizing $A_i(e^*) + A_j(e^*)$, since each particular α_{ij} only appears in $A_i(e^*)$ and $A_j(e^*)$. Fortunately, the problem can be further simplified: Note that, $A_i(e^*)$ and $A_j(e^*)$ are the sums of different terms, but α_{ij} only appears in those terms related to the interaction between i and j (see (5)). These terms are $c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$ from $A_i(e^*)$, and $c_j(e_j^*) - (1 - \alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ from $A_j(e^*)$.

Thus, a new function $A_i^*(\alpha_{ij}) := c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$ can be considered, and analogously $A_j^*(1-\alpha_{ij})$. Note that $\frac{\partial^x(A_i(e^*))}{\partial\alpha_{ij}^x} = \frac{\partial^x(A_i^*(\alpha_{ij}))}{\partial\alpha_{ij}^x}$ and $\frac{\partial^x(A_j(e^*))}{\partial\alpha_{ij}^x} = \frac{\partial^x(A_j^*(1-\alpha_{ij}))}{\partial\alpha_{ij}^x}$for $x = 1, 2, ...$ Therefore, for each pair i and j, it is possible to define the function $L_{ij}^*(\alpha_{ij}) := A_i^*(\alpha_{ij}) + A_j^*(1-\alpha_{ij})$. Hence, minimizing $\sum_{i\in N} A_i(e^*)$ is equivalent to minimizing $L_{ij}^*(\alpha_{ij})$, with

$$\begin{aligned} L_{ij}^*(\alpha_{ij}) &= c_i(e_i^*) + c_j(e_j^*) - \left[\alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)) + (1-\alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))\right] \\ &= c_i(e_i^*) + c_j(e_j^*) - (r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)) \end{aligned} \quad (6)$$

The function $L_{ij}^*(\alpha_{ij})$ depends on α_{ij} through the equilibrium efforts e_{ij}^* and e_{ji}^* because they depend on α_{ij}. We now focus on finding the α_{ij} that minimizes function $L_{ij}^*(\alpha_{ij})$, and provide a procedure for finding the EEE for pairwise aggregate reduction.

We can summarize this reasoning as follows.[8] Let $\alpha = (\alpha_i)_{i\in N}$ and $\alpha_i = (\alpha_{ij})_{j\in N\setminus\{i\}}$, $\alpha^* = \arg \min_{\alpha\in[0,1]^{n(n-1)}} \sum_{i\in N} A_i(e^*) \iff \alpha_{ij}^* = \arg \min_{\alpha_{ij}\in[0,1]} A_i(e^*) + A_j(e^*)$ for all $i \in N \iff \alpha_{ij}^* = \arg \min_{\alpha_{ij}\in[0,1]} c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)) + c_j(e_j^*) - (1-\alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ for all $i, j \in N, i \neq j \iff \alpha_{ij}^* = \arg \min_{\alpha_{ij}\in[0,1]} c_i(e_i^*) + c_j(e_j^*) - (r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ for all $i,\, j \in N, i \neq j$. As $L_{ij}^*(\alpha_{ij}) = c_i(e_i^*) + c_j(e_j^*) - (r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$, then $\alpha_{ij}^* = \arg \min_{\alpha_{ij}\in[0,1]} L_{ij}^*(\alpha_{ij})$ for all $i,\, j \in N, i \neq j$.

For any effort game considered here, there are only six possible distributions of the lower and upper thresholds of the alpha parameter.[9] These cases are

$$\begin{array}{ll} \textbf{Case A} & \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} \\ \textbf{Case B} & \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} \\ \textbf{Case C} & \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} \\ \textbf{Case D} & 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} \\ \textbf{Case E} & 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} \\ \textbf{Case F} & 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} \end{array} \quad (7)$$

[8] In principle, this problem could be considered a bilevel optimization problem (Bracken and McGill ,1973). The main characteristic of a bilevel programing problem is a kind of hierarchy, because its constraints are defined, in part, by a second optimization problem. In our case, the second level (lower level or follower' level) will be the problem $\min_{e_i\in[0,1]^{(n-1)}} A_i(e)$ with solution $e^* = (e_i^*)_{i\in N}$ where e^* depends on α. The first level (upper level or leader's problem) will be $\min_{\alpha\in[0,1]^{n(n-1)}} \sum_{i\in N} A_i(e^*)$. Thus, we can rewrite the problem as follows:

$$\begin{array}{ll} \min_{\alpha,e} \sum_{i\in N} A_i(\alpha, e) & \\ s.t. & (\alpha, e) \in [0,1]^{n(n-1)} \times [0,1]^{n(n-1)} \\ & e_i \in G_i(\alpha) \text{ for all } i \in N \\ & \text{with } e = (e_i)_{i\in N} \\ \text{where} & G_i(\alpha) = \arg \min_{e_i} A_i(\alpha, e) \\ & s.t. \quad e_i \in [0,1]^{(n-1)},\ \alpha \in [0,1]^{n(n-1)} \end{array}$$

However, it is difficult to see this problem as a Stakelberg game, as described for example in Dempe (2002), because α is not a strategy profile but a parameter of the reduction cost functions. We believe that our setting better fits a bi-form game that was introduced by Brandenburger and Stuart (2007).

[9] Note that $\underline{\alpha}_{ji} < \bar{\alpha}_{ji}$ and $\underline{\alpha}_{ij} < \bar{\alpha}_{ij}$.

The last theorem characterizes the optimal α^*_{ij} in cases A-F. Thus, Theorem 5 provides the α^*_{ij} that incentivizes an efficient effort equilibrium for WPAR.[10] In Theorem 5 we use the following notation:

1. $\check{\alpha}^{[a,b]}_{ij} \in [a,b]$ with $0 \leq a < b \leq 1$ is:
$$\check{\alpha}^{[a,b]}_{ij} = \begin{cases} a & \text{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} > 0 \text{ for all } \alpha_{ij} \in [a,b] \\ b & \text{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} < 0 \text{ for all } \alpha_{ij} \in [a,b] \\ \text{Solution of } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} = 0 & \text{otherwise} \end{cases} .$$

2. $$\Lambda(\alpha) = \begin{cases} 0 \text{ if } \alpha < 0 \\ \alpha \text{ if } \alpha \in (0,1) \\ 1 \text{ if } \alpha > 1 \end{cases}$$

Theorem 5 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i \in N})$ be an effort game, and $L^*_{ij}(\alpha_{ij}) = c_i(e^*_i) + c_j(e^*_j) - (r_{ij}(e^*_{ji}) + r_{ji}(e^*_{ij}))$. The optimal solution $\alpha^*_{ij} = \arg \min_{\alpha_{ij} \in [0,1]} L^*_{ij}(\alpha_{ij})$ is in each case,*

Case A α^*_{ij} *is any element of* $[\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}]$.

Case B $\alpha^*_{ij} = \check{\alpha}^{[1-\bar{\alpha}_{ji}, \bar{\alpha}_{ij}]}_{ij}$

Case C $$\alpha^*_{ij} = \begin{cases} \textit{any element of } [\bar{\alpha}_{ij}, 1] \textit{ if } \alpha^C = \Lambda(\bar{\alpha}_{ij}) \textit{ and } \Lambda(\bar{\alpha}_{ij}) < 1 \\ \alpha^C \textit{ otherwise} \end{cases} ,$$

where $\alpha^C = \arg min\{L^*_{ij}(\check{\alpha}^{[1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}]}_{ij}), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$.

Case D $$\alpha^*_{ij} = \begin{cases} \textit{any element of } [0, 1-\bar{\alpha}_{ji}] \textit{ if } \alpha^D = \Lambda(1-\bar{\alpha}_{ji}) \textit{ and } \Lambda(1-\bar{\alpha}_{ji}) > 0 \\ \alpha^D \textit{ otherwise} \end{cases} ,$$

where $\alpha^D = \arg min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), L^*_{ij}(\check{\alpha}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]}_{ij})\}$.

Case E $$\alpha^*_{ij} = \begin{cases} \textit{any element of } [0, 1-\bar{\alpha}_{ji}] \textit{ if } \alpha^E = \Lambda(1-\bar{\alpha}_{ji}) \textit{ and } \Lambda(1-\bar{\alpha}_{ji}) > 0 \\ \textit{any element of } [\bar{\alpha}_{ij}, 1] \textit{ if } \alpha^E = \Lambda(\bar{\alpha}_{ij}) \textit{ and } \Lambda(\bar{\alpha}_{ij}) < 1 \\ \alpha^E \textit{ otherwise} \end{cases} ,$$

where $\alpha^E = \arg min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), \check{\alpha}^{[\underline{\alpha}_{ij}, 1-\underline{\alpha}_{ji}]}_{ij}, L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$.

Case F $$\alpha^*_{ij} = \begin{cases} \textit{any element of } [0, 1-\bar{\alpha}_{ji}] \textit{ if } \alpha^F = \Lambda(1-\bar{\alpha}_{ji}) \textit{ and } \Lambda(1-\bar{\alpha}_{ji}) > 0 \\ \textit{any element of } [\bar{\alpha}_{ij}, 1] \textit{ if } \alpha^F = \Lambda(\bar{\alpha}_{ij}) \textit{ and } \Lambda(\bar{\alpha}_{ij}) < 1 \\ \alpha^F \textit{ otherwise} \end{cases}$$

[10] The function L^*_{ij} is a piecewise function, and although it is continuous in $\alpha_{ij} \in [0,1]$, it is not differentiable at all points in its domain. Since it is defined over intervals, it is generally non-differentiable at the endpoints of these intervals. Therefore, to compute the minimum, it is also necessary to evaluate the function at the interval endpoints. In addition, due to its convexity, the minimum can also be an interior point within any of the intervals. However, each interval entails a distinct derivative function, thereby contributing to the complexity of the computation process. The introduction of Theorem 5 streamlines the evaluation procedure by reducing the number of points to be assessed, presenting them in a case-by-case framework.

where $\alpha^F = \arg min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$.

To conclude the section, we describe a procedure for finding an efficient effort in equilibrium induced by the WPAR family.

EEE Procedure

Given an effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$,

1. we first calculate the lower and upper thresholds of the bilateral interaction between any pair of agents by using Definition 3;
2. we then focus on the list (7) and determine which case (A-F) applies;
3. Theorem 5 provides an optimal α^*_{ij} for all $i, j \in N$, to minimize the centralized (aggregate) cost allocation $\sum_{i\in N} A_i(e^*)$;
4. with this α^*_{ij}, Theorem 4 gives the associated efficient effort equilibrium (e^*_{ij}, e^*_{ji}) for every pair of agents, and thus an efficient effort equilibrium e^* for the game;
5. at this point the optimal cost allocation that incentivizes agents $i, j \in N$ to make an efficient effort equilibrium e^*_{ij} and e^*_{ji} is known, i.e.

$$A^*_i(e^*) = c_i(e^*_i) - \sum_{j\in N\setminus\{i\}} \alpha^*_{ij}[r_{ij}(e^*_{ji}) + r_{ji}(e^*_{ij})];$$

We illustrate this procedure with the 3-firm case given in Example 2 in Section 6.

6 Comparison of WPR and WPAR families

We complete the study of our model of cooperation with pairwise cost reduction by comparing the two families of core-allocations analyzed. We find that there is a loss of efficiency when cooperation is restricted to a pairwise aggregate cost reduction. That loss of efficiency can be measured. In addition, we show that those agents who receive less than the total reduction generated and bear the total cost of this effort always exert less effort than the efficient agent.

As mentioned above, the allocation rule $H(e)$ induces an equilibrium effort e^{*H} that matches the efficient effort of Proposition 3, i.e. $e^{*H} = \tilde{e}$. This means that there is no rule that generates a lower cost of the grand coalition, see Corollary 1. However, as also mentioned above, WPAR is a subfamily of WPR, but $H(e)$ is not in WPAR, so e^{*A} is not always equal to e^{*H}.

Let $A^*(e)$ be the allocation rule in WPAR that induces the effort profile e^{*A^*} that minimizes the cost of the grand coalition, i.e. the efficient allocation in this subfamily. The difference, in terms of efficiency, between the cost of the grand coalition with e^{*A^*} and $\tilde{e}$ can be measured. Note that for any particular functions $c_i(e_i)$ and $r_{ij}(e_{ji})$ for $i,\, j \in N,\, i \neq j$, the associated e^{*A^*} and $\tilde{e}$ can be obtained. Let Δ be this difference or loss of efficiency, where

$$\Delta = \sum_{i \in N}[c_i(e_i^{*A^*}) - \sum_{j \in N\backslash\{i\}} r_{ij}(e_{ji}^{*A^*})] - \sum_{i \in N}[c_i(\tilde{e}_i) - \sum_{j \in N\backslash\{i\}} r_{ij}(\tilde{e}_{ji})]. \tag{8}$$

The following proposition shows the relation between efforts e^{*A^*} and $\tilde{e}$. The proof of Proposition 4 appears in Appendix B.

Proposition 4 *Let $e_{ij}^{*A^*}$ for $i,\, j \in N,\, i \neq j$ be the equilibrium efforts of $A^*(e)$, that minimize the cost of the grand coalition in the family WPAR. Thus, the efficient effort $\tilde{e}_{ij} \geq e_{ij}^{*A^*}$ for all $i,\, j \in N$, $i \neq j$.*

As mentioned above, when an agent receives less than the total reduction that it generates and bears the total cost of that effort, then that agent always exerts less effort than the efficient one

Finally, readers may think that the rationale behind the efficient rule, $H(e)$, in the WPR family, could also apply to the WPAR family. However, this is not the case. To reach an efficient effort equilibrium in the WPR family, for each pair of agents $i, j \in N, i \neq j$, the weight ω_{ji}^i must be 1, because $\frac{\partial \Omega_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \omega_{ji}^i \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, and ω_{ij}^j must also be 1, because $\frac{\partial \Omega_j(e)}{\partial e_{ji}} = \frac{\partial c_j(e_j)}{\partial e_{ji}} - \omega_{ij}^j \frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$. However, this is no longer true for the WPAR family.[11]

The following example with three agents illustrates the comparison of the two core allocation families and completes the paper.

Example 2 *Consider a pairwise inter-organizational situation with three firms, i.e. $N = \{1, 2, 3\}$. For any effort profile $e \in [0,1]^6$, the PE-situation is given by the following initial costs,*

$c_1(e_{12}, e_{13}) = 100 + 100e_{12} + 4e_{12}^2 + 100e_{13} + 4e_{13}^2$
$c_2(e_{21}, e_{23}) = 100 + 100e_{21} + 4e_{21}^2 + 100e_{23} + 4e_{23}^2$
$c_3(e_{31}, e_{32}) = 100 + 100e_{31} + 4e_{31}^2 + 100e_{32} + 4e_{32}^2$

and the following pairwise reduced costs, all of them in thousands of Euros,

$r_{i1}(e_{1i}) = 2 + 110e_{1i} - 2e_{1i}^2$ *with* $i = 2, 3$
$r_{i2}(e_{2i}) = 2 + 105e_{2i} - 3e_{2i}^2$ *with* $i = 1, 3$
$r_{i3}(e_{3i}) = 2 + 105e_{3i} - 3e_{3i}^2$ *with* $i = 1, 2$

[11] In WPAR, for each pair of agents $i, j \in N, i \neq j$, the weight α_{ij} is not always 1, because $\frac{\partial A_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \alpha_{ij} \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $\frac{\partial A_j(e)}{\partial e_{ji}} = \frac{\partial c_j(e_j)}{\partial e_{ji}} - \alpha_{ji} \frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$ but $\alpha_{ij} = 1 - \alpha_{ji}$. Note that if $\alpha_{ij} = 1$, then $\alpha_{ji} = 0$ and the derivative conditions for efficiency in Proposition 3 would be violated. Bear in mind that the weights ω_{ji}^i that appear in each derivative $\frac{\partial \Omega_i(e)}{\partial e_{ij}}$ for $i, j \in N, i \neq j$ are independent of one another. However, the weights α_{ij} that appear in the each derivative $\frac{\partial A_i(e)}{\partial e_{ij}}$ for $i, j \in N, i \neq j$ are not, because $\alpha_{ij} = 1 - \alpha_{ji}$. In addition, it is known that $\omega_{ij}^i = \omega_{ji}^i = \alpha_{ij}$ in WPAR for all $i, j \in N, i \neq j$, where $\omega_{ij}^i = 1 - \omega_{ij}^j$ and $\omega_{ji}^i = 1 - \omega_{ji}^j$. The fact that pairwise cost reduction is aggregated by α_{ij} in the subfamily WPAR means that it is not possible to apply the efficient argument used for the WPR family.

By Definition 3, the pair of firms $\{1,2\}$ *has the thresholds* $\underline{\alpha}_{12} = 0.91$, $\bar{\alpha}_{12} = 1.02$, $\underline{\alpha}_{21} = 0.95$, *and* $\bar{\alpha}_{21} = 1.09$, *which correspond to Case F in the Table 7. By using Theorem 5, it can easily be checked that* $\alpha^F = \Lambda(\bar{\alpha}_{12}) < 1$ *and* $\alpha^*_{12} = 1$. *Thus, by Theorem 4,* $e^*_{12} = 0.833$, $e^*_{21} = 0$. *As firms 2 and 3 are identical,* $\alpha^*_{13} = 1$, $e^*_{13} = 0.833$ *and* $e^*_{31} = 0$. *Finally, for the pair* $\{2,3\}$, $\underline{\alpha}_{23} = 0.95$, $\bar{\alpha}_{23} = 1.09$, $\underline{\alpha}_{32} = 0.95$, *and* $\bar{\alpha}_{32} = 1.09$. *This is again Case F. Note that in case F,* $\alpha^F = \arg min\{L^*_{23}(\Lambda(1-\bar{\alpha}_{32})), L^*_{23}(\Lambda(\bar{\alpha}_{23}))\}$, *where in this particular case* $L^*_{23}(\Lambda(1-\bar{\alpha}_{32})) = L^*_{23}(\Lambda(\bar{\alpha}_{23}))$ *with* $\Lambda(1-\bar{\alpha}_{32}) = 0$ *and* $\Lambda(\bar{\alpha}_{23}) = 1$ *Thus, two solutions emerge: (i)* $e^*_{23} = 0.357$, $e^*_{32} = 0$, *and* $\alpha^*_{23} = 1$, *and (ii)* $e^*_{23} = 0$, $e^*_{32} = 0.357$, *and* $\alpha^*_{23} = 0$. *Therefore, there are two EEE in WPAR.*

(i) $e^*_{12} = e^*_{13} = 0.833$, $e^*_{21} = 0$, $e^*_{23} = 0.357$, $e^*_{31} = e^*_{32} = 0$

(ii) $e^*_{12} = e^*_{13} = 0.833$, $e^*_{21} = e^*_{23} = 0, e^*_{31} = 0$, $e^*_{32} = 0.357$

We now calculate the efficient efforts in this example by Proposition 3. They are the solutions of $c'_i(e_{ij}) - r'_{ji}(e_{ij}) = 0$, *thus,* $\tilde{e}_{12} = \tilde{e}_{13} = 0.833$, *and* $\tilde{e}_{21} = \tilde{e}_{23} = \tilde{e}_{31} = \tilde{e}_{32} = 0.357$. *Note that by Theorem 3 these efforts are also the effort equilibrium obtained by the allocation rule* $H(e)$.

This example is a particular subcase of Case F. This implies that α^*_{ij} *is zero or one, which in turn implies that one of the agents makes no effort and the other makes the efficient value. However, they are never able to make the efficient effort simultaneously under WPAR. The loss of efficiency in WPAR with regard to WPR can be calculated with the help of (8).*

$\Delta = \sum_{i\in N}[c_i(e_i^{*A^*}) - \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji}^{*A^*})] - \sum_{i\in N}[c_i(\tilde{e}_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(\tilde{e}_{ji})] = 278.776 - 276.104 = 2.67$.

7 Conclusions and future research

This paper presents a model of cooperation with pairwise cost reduction. The direct impact of pairwise effort on cost reductions is investigated by means of a bi-form game. First, the agents determine the level of pairwise effort to be made to reduce the costs of their partners. Second, they participate in a bilateral interaction with multiple independent partners where the cost reduction that each agent gives to another agent is independent of any possible coalition. As a result of cooperation, agents reduce each other's costs. In the non-cooperative game that precedes cooperation, the agents anticipate the cost allocation that will result from the cooperative game by incorporating the effect of the effort made into their cost functions. We show that all-included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off (or, at least, not worse off), and consistent. We then identify a family of feasible cost allocations with weighted pairwise reduction. One of these cost allocations is selected by taking into account the incentives generated in the efforts that agents make, and consequently in the total cost of coalitions. Surprisingly, we find that the Shapley value, which coincides with the Nucleolus in this model, can induce inefficient effort strategies in equilibrium in the non-cooperative model. However, it is always possible to select a core-allocation with appropriate pairwise weights that can generate an efficient effort.

Future research could take any of several directions. First, this paper assumes that the individual effort cost function $c_i(e_i)$ is independent of the effort of other agents, and that the marginal cost $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is independent of the effort that i makes in regard to agents other than j, i.e. $\frac{\partial c_i^2(e_i)}{\partial e_{ij} \partial e_{ih}} = 0$. We make a similar assumption with the cost reduction function $r_{ij}(e_{ji}^*)$. There is some degree of independence between efforts. This is a reasonable assumption in many contexts, but in some settings different assumptions might be needed. For example, there are situations with strategic complementarity in which the efforts of agents reinforce each other. In such cases the cost function is supermodular. In other cases there is strategic substitutability, so that efforts offset each other and the function is submodular. Focusing on the effort cost function of one agent, if $\frac{\partial c_i^2(e_i)}{\partial e_{ij} \partial e_{ih}} > 0$ then there is complementarity between the efforts, and if $\frac{\partial c_i^2(e_i)}{\partial e_{ij} \partial e_{ih}} < 0$, then there is substitutability. This is a very interesting future extension. It could also be worth considering this complementarity/substitutability not only between the different efforts that one agent makes in regard to other agents but also between the efforts made by different agents. This assumption can be made on both the effort cost functions and the cost reduction function. Obviously, complementarity on the effort cost function has the opposite effect to that on the cost reduction function.

The second direction is close to the first. The pairwise total cost reduction could be considered as a general function which is increasing in the efforts e_{ij} and e_{ji}, that is $R_{ij}(e_{ij}, e_{ji})$. In our model, this function is additively separable, i.e. $R_{ij}(e_{ij}, e_{ji}) = r_{ij}(e_{ji}) + r_{ji}(e_{ij})$. However, as mentioned above, there could be situations with strategic complementarity or substitutability in which the efforts of agents reinforce or offset each other. In that case, the function $R_{ij}(e_{ij}, e_{ji})$ would not be separable. This is also an interesting question for analysis.

Another direction is related to the assumption of bilateral interaction between agents. This has the advantage of being analytically more tractable and is widely applied in practice (e.g., Fang and Wang 2019; Amin et al. 2020, Park et al. 2010), but overall interaction between agents, dependent on groups, is an important factor that we believe does not affect the success of cooperation. One possible future extension would be to investigate the cooperative model with multiple cost reduction and the impact of the efforts made on those cost reductions.

Finally, we identify a large family of core-allocations with weighted pairwise reduction which contains the Shapley value and the Nucleolus and always provides a level of efficient effort in equilibrium. This family is very rich in itself, as a set solution concept for our cooperative model. Research into this core-allocation family can be furthered through an in-depth analysis of its structure and its geometric relationship to the core.

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Appendix A

Proposition 1, in Section 3, shows that PE-games are always concave. To prove this, the class of unanimity games must be described. Shapley (1953) proves that the family of unanimity games $\{(N, u_T), T \subseteq N\}$ forms a basis of the vector space of all games with set of players N, where (N, u_T) is defined for each $S \subseteq N$ as follows:

$$u_T(S) = \begin{cases} 1, & T \subseteq S \\ 0, & otherwise \end{cases}$$

Hence, for each cost game (N, c) there are unique real coefficients $(\alpha_T)_{T \subseteq N}$ such that $c = \sum_{T \subseteq N} \alpha_T u_T$. Many different classes of games, including airport games (Littlechild and Owen, 1973) and sequencing games (Curiel et al., 1989), can be characterized through constraints on these coefficients.

Proof of Proposition 1

Proof. Let $(N, e, \{c_i(e_i), \{r_{ji}(e_{ij})\}_{j \in N \setminus \{i\}}\}_{i \in N})$ be a PE-situation and (N, e, c) the associated PE-game. First, we prove that this game can be rewritten as a weighted sum of unanimity games $u_{\{i\}}$ and $u_{\{i,j\}}$ for all $i, j \in N$ as follows:

$$c = \sum_{i \in N} c_i(e_i) u_{\{i\}} - \sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}. \tag{9}$$

Indeed, for all $S \subseteq N$,

$$\begin{aligned} c(S) &= \sum_{i \in N} c_i(e_i) u_{\{i\}}(S) - \sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}(S) = \\ &= \sum_{i \in S} c_i(e_i) - \sum_{i,j \in S; i \neq j} r_{ij}(e_{ji}) = \\ &= \sum_{i \in S} c_i(e_i) - \sum_{i \in S} \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji}). \end{aligned}$$

It is easily shown that the additive game $\sum_{i \in N} c_i(e_i) u_{\{i\}}$ is concave and that $u_{\{i,j\}}$ is convex. Thus, the game $-\sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}$ is concave because of $r_{ij}(e_{ji}) > 0$ for all $i, j \in N$. Finally, the concavity of (N, e, c) follows from the fact that game c is the sum of two concave games. ■

The Theorem 1, in Section 3, shows that the Shapley value reduces the individual cost of an agent by half the total reduction that it obtains from the others $(R_i(N))$ plus a half of the total reduction that it provides to the rest of the agents, which is $G_i(N) = \sum_{j\in N\setminus\{i\}} r_{ji}(e_{ij})$.

The Shapley value is the only allocation rule that satisfies the four properties of Efficiency, Equal treatment of equals, Linearity and Null player. Next, we describe all of these properties of the Shapley value, which are useful in demonstrating the Theorem 1.

(EFF) *Efficiency.* The sum of the Shapley values of all agents equals the value of the grand coalition, so all the gain is allocated to the agents:

$$\sum_{i\in N} \phi_i(c) = c(N). \tag{10}$$

(ETE) *Equal treatment of equals.* If i and j are two agents who are equivalent in the sense that $c(S\cup\{i\}) = c(S\cup\{j\})$ for every coalition S of N which contains neither i nor j, then $\phi_i(c) = \phi_j(c)$.

(LIN) *Linearity.* If two cost games c and c^* are combined, then the cost allocation should correspond to the costs derived from c and the costs derived from c^*:

$$\phi_i(c + c^*) = \phi_i(c) + \phi_i(c^*), \forall i \in N. \tag{11}$$

Also, for any real number a,

$$\phi_i(ac) = a\phi_i(c), \forall i \in N. \tag{12}$$

(NUP) *Null Player.* The Shapley value $\phi_i(c)$ of a null player i in a game c is zero. A player i is null in c if $c(S\cup\{i\}) = c(S)$ for all coalitions S that do not contain i.

Proof of the Theorem 1

Proof. Consider the PE-game (N, e, c) rewritten as a weighted sum of unanimity games given by (9), i.e.

$$c = \sum_{i\in N} c_i(e_i)u_{\{i\}} - \sum_{i,j\in N;i\neq j} r_{ij}(e_{ji})u_{\{i,j\}}.$$

Take an agent $k \in N$. By the (LIN) property of the Shapley value, $\phi_k(e)$, it follows that

$$\begin{aligned}\phi_k(e) &= \phi_k\left(\sum_{i\in N} c_i(e_i)u_{\{i\}}\right) - \phi_k\left(\sum_{i,j\in N;i\neq j} r_{ij}(e_{ji})\left(u_{\{i,j\}}\right)\right)\\ &= \sum_{i\in N} c_i(e_i)\phi_k\left(u_{\{i\}}\right) - \sum_{i\in N}\sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})\phi_k\left(u_{\{i,j\}}\right).\end{aligned} \tag{13}$$

In addition, it is known from the (NUP) property that

$$\phi_k\left(u_{\{i\}}\right) = \begin{cases} 1, & i=k \\ 0, & otherwise \end{cases} \tag{14}$$

and from (ETE) and (NUP), that

$$\phi_k\left(u_{\{i,j\}}\right) = \begin{cases} 1/2, & i=k, j=k, i \neq j \\ 0, & otherwise \end{cases} \tag{15}$$

Consequently, by substituting the values (14) and (15) in equation (13), the following is obtained:

$$\begin{aligned} \phi_k(e) &= c_k(e_k) - \sum_{j \in N\setminus\{k\}} r_{kj}(e_{jk})\phi_k\left(u_{\{k,j\}}\right) - \sum_{j \in N\setminus\{k\}} r_{jk}(e_{kj})\phi_k\left(u_{\{j,k\}}\right) \\ &= c_k(e_k) - \frac{1}{2}\sum_{j \in N\setminus\{k\}} [r_{kj}(e_{jk}) + r_{jk}(e_{kj})]. \end{aligned}$$

Finally, it can be concluded that, for each agent $k \in N$,

$$\phi_k(e) = c_k(e_k) - \frac{1}{2}[R_k(N) + G_k(N)].$$

■

Proof of Proposition 2

Proof. To prove that the Shapley value coincides with the Nucleolus for PE-games, it is first necessary to describe the class of PS-games introduced by Kar et al (2009).

Denote by $M_i c(T)$ the marginal contribution of player $i \in T$, that is $M_i c(T) = c(T) - c(T \setminus \{i\})$, for all $i \in T \subseteq N$. A cost game (N, c) satisfies the PS property if for all $i \in N$ there exists $k_i \in \mathbb{R}$ such that $M_i c(T \cup \{i\}) + M_i c(N \setminus T) = k_i$, for all $i \in N$ and all $T \subseteq N \setminus \{i\}$. Kar et al (2009) show that for PS games, the Shapley value coincides with the Nucleolus, i.e. $\phi_i(c) = \nu_i(c) = \frac{k_i}{2}$, for all $i \in N$.

Therefore, it only remains to show that (N, e, c) is a PS-game with $k_i = [c_i(e_i) - R_i(N)] + [c_i(e_i) - G_i(N)]$, for all $i \in N$.

First, it is straightforward to prove that $M_i c(T) = c_i(e_i) - \sum_{j \in T\setminus\{i\}} [r_{ji}(e_{ij}) + r_{ij}(e_{ji})]$ for all $i \in T \subseteq N$. Second, we show that $M_i c(T \cup \{i\}) + M_i c(N \setminus T) = [c_i(e_i) - R_i(N)] + [c_i(e_i) - G_i(N)]$ for all $i \in N$ and $T \subseteq N\setminus\{i\}$. Indeed, take a coalition $T \subseteq N$ and an agent $i \in T$. It is shown that $M_i c(T \cup \{i\}) = c_i(e_i) - \sum_{j \in T} \left(r_{ji}(e_{ij}) + r_{ij}(e_{ji})\right)$, and $M_i c(N \setminus T) = c_i(e_i) - \sum_{j \in N\setminus(T\cup\{i\})} \left(r_{ji}(e_{ij}) + r_{ij}(e_{ji})\right)$. Therefore,

$$M_i c(T \cup \{i\}) + M_i c(N \setminus T) = 2c_i(e_i) - \sum_{j \in N\setminus\{i\}} \left(r_{ji}(e_{ij}) + r_{ij}(e_{ji})\right) =$$

$$\left[c_i(e_i) - \sum_{j \in N\setminus\{i\}} r_{ij}(e_{ji})\right] + \left[c_i(e_i) - \sum_{j \in N\setminus\{i\}} r_{ji}(e_{ij})\right].$$

Hence, $M_i c(T \cup \{i\}) + M_i c(N \setminus T) = [c_i(e_i) - R_i(N)] + [c_i(e_i) - G_i(N)] = k_i)$, and so (N, e, c) is a PS game. ■

Proof of Theorem 2

Proof. Consider the PE-game (N, e, c) associated with the PE-situation $(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j \in N \setminus \{i\}}\}_{i \in N})$. Take a family of weights $\omega_{ij}^i, \omega_{ji}^i \in [0,1]$, for all $j \in N \setminus \{i\}$, such that $\omega_{ij}^i = 1 - \omega_{ij}^j$ and $\omega_{ji}^i = 1 - \omega_{ji}^j$, and $\Omega(e)$ the corresponding cost allocation with weighted pairwise reduction with $\Omega_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})]$, for all $i \in N$. To prove that $\Omega(e)$ belongs to the core of (N, e, c) it must be checked that (1) $\sum_{i \in N} \Omega_i(e) = c(N)$, (2) $\sum_{i \in S} \Omega_i(e) \leq c(S)$, for all $S \subset N$.

We start by checking (1). Notice that $\sum_{i \in N} \Omega_i(e) = c(N)$ is equivalent to

$\sum_{i \in N} \sum_{j \in N \setminus \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})] = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji})$.

Indeed,

$\sum_{i \in N} \sum_{j \in N \setminus \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})] = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} (\omega_{ij}^i + \omega_{ij}^j) r_{ij}(e_{ji}) = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji})$,

where the last equality is due to $\omega_{ij}^i + \omega_{ij}^j = 1$ for all $i, j \in N$.

Next we check (2). Take $S \subset N$. Notice now that $\sum_{i \in S} \Omega_i(e) \leq c(S)$ is equivalent to

$\sum_{i \in S} \sum_{j \in N \setminus \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})] - \sum_{i \in S} \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji}) \geq 0$.

Indeed, an argument similar to that used in (1) leads to

$\sum_{i \in S} \sum_{j \in N \setminus \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})] - \sum_{i \in S} \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji}) =$

$\sum_{i \in S} \sum_{j \in S \setminus \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})] + \sum_{i \in S} \sum_{j \in N \setminus S \cup \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})] - \sum_{i \in S} \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji}) =$

$\sum_{i \in S} \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji}) + \sum_{i \in S} \sum_{j \in N \setminus S \cup \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})] - \sum_{i \in S} \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji}) =$

$\sum_{i \in S} \sum_{j \in N \setminus S \cup \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})] \geq 0$. ■

Appendix B

Proof of Proposition 3.

To prove this result it is necessary to analyze $c(N)$ as a function of e. First, It is easy to prove that $c(N)$ is strictly convex in e_{ij} for all i, $j \in N, i \neq j$. Indeed, $\frac{\partial^2 c(N)}{\partial e_{ij}^2} = \frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} - \frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} > 0$, because $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} < 0$. Thus, there is a unique effort profile $\tilde{e}$ that minimizes $c(N)$.

Second, we focus on finding this efficient effort profile $\tilde{e}$. Note that the derivative $\frac{\partial c(N)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ only depends on e_{ij} because $\frac{\partial c_i^2(e_i)}{\partial e_{ij} \partial e_{ih}} = 0$ for all $h \neq i, j$. Therefore, if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0,1]$, then the function $c(N)$ is increasing in e_{ij}, which implies that $\tilde{e}_{ij} = 0$. Analogously, if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0,1]$, then $\tilde{e}_{ij} = 1$. Finally, if there is a solution of $\frac{\partial c_i(e_i)}{\partial e_{ij}} = \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, that solution is $\tilde{e}_{ij}$. ■

Proof of Lemma 1.

Consider the non-cooperative game $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$. To learn the optimal level of effort $\hat{e}_{ij}$ that agent i must exert to reduce the costs of agent j in this game, it is necessary to analyze the function $\Omega_i(e) = c_i(e_i) - \sum_{j\in N\setminus\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})]$ for all $i \in N$ with $\omega^i_{ij}, \omega^i_{ji} \in [0,1]$, i, $j \in N, i \neq j$, such that $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$.

As above, we also prove that the function $\Omega_i(e)$ is strictly convex in e_{ij}. Indeed, $\frac{\partial^2_i \Omega(e)}{\partial e^2_{ij}} = \frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}} - \omega^i_{ji}\frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}} > 0$ because $\frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}} < 0$. Hence, there is a unique optimal level of effort $\hat{e}$.

Again, we focus on finding this optimal level of effort $\hat{e}$. We know that $\frac{\partial \Omega_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, but $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ only depends on e_{ij}, because $\frac{\partial c^2_i(e_i)}{\partial e_{ij}\partial e_{ih}} = 0$ for all $h \neq i, j$. Moreover, for all $e_{ij} \in [0,1]$, $\frac{\partial \Omega_{ii}(e)}{\partial e_{ij}} \geq 0 \iff \frac{\partial c_i(e_i)}{\partial e_{ij}} \geq \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$.

Therefore, if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0,1]$, then $\hat{e}_{ij} = 0$. If $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0,1]$, then $\hat{e}_{ij} = 1$. Finally, if there is a solution of $\frac{\partial c_i(e_i)}{\partial e_{ij}} = \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, that solution is $\hat{e}_{ij}$ and is unique. Hence, there is a unique optimal level of effort. ■

Proof of Theorem 3.

Now consider the non-cooperative game $(N, \{E_i\}_{i\in N}, \{H_i\}_{i\in N})$. Note that, both derivative functions $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ and $\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ only depend on e_{ij}. Thus, by Lemma 1, the optimal level of effort of a particular agent $i \in N$ with another particular agent $j \in N\setminus\{i\}$, i.e. $\hat{e}_{ij}$, is independent of any other effort made by i or by any other agent. Thus, the equilibrium is also characterized by Lemma **1** with $\omega^i_{ji} = 1$ for i, $j \in N, i \neq j$. Comparing Lemma 1 with Proposition 3, it follows directly that the equilibrium must also be efficient. ■

Proof of Corollary 1.

This is straightforward from the proof of **Theorem 3**. ■

Proof of Proposition 4.

Take $A^*(e)$ the allocation rule in WPAR with α^*_{ij} for all $i, j \in N$ which induces the effort profile e^{*A^*} that minimizes the cost of the grand coalition. Since WPAR is a subfamily of WPR in which $\omega^i_{ij} = \omega^j_{ij} = \alpha_{ij} \in [0,1]$ for all $i, j \in N$, by Lemma 1 the optimal level of effort for $A^*(e)$ can be also characterized.

Thus, the efforts are optimal in equilibrium and so e^{*A^*} must hold that

$e^{*A^*}_{ij} = 0$ if and only if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \alpha^*_{ij}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, for all $e_{ij} \in [0,1]$,

$e^{*A^*}_{ij} = 1$ if and only if $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \alpha^*_{ij}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, for all $e_{ij} \in [0,1]$,

Otherwise, $e^{*A^*}_{ij} \in (0,1)$ so $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=e^{*A^*}_{ij}} = \alpha^*_{ij}\left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=e^{*A^*}_{ij}}$ holds.

Comparing the above expressions with Proposition 3 and taking into account that $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is a positive increasing function , $\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ a positive decreasing function, and $\alpha^*_{ij} \in [0,1]$, it can be concluded that $\tilde{e}_{ij} \geq e^{*A^*}_{ij}$ for all i, $j \in N$. ■

Appendix C

Theorem 4, in Section 5, characterizes all possible types of effort equilibrium according to the value of the parameter α_{ij}, for all $i, j \in N, i \neq j$. Before proving this theorem, we consider a previous Lemma that is very useful for latter results. It characterizes the optimal effort level for agent $i \in N$ in the first stage non-cooperative game.

Lemma 2 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game, with $\hat{e}_{ij}$ being the optimal level of effort that agent i exerts to reduce the costs of agent j. Thus,*

1. *$\hat{e}_{ij} = 0$ if and only if $\alpha_{ij} \leq \underline{\alpha}_{ij}$*
2. *There is a unique $\hat{e}_{ij} \in (0,1)$ that holds $c'_i(\hat{e}_{ij}) - \alpha_{ij} r'_{ji}(\hat{e}_{ij}) = 0$ if and only if $\underline{\alpha}_{ij} < \alpha_{ij} < \bar{\alpha}_{ij}$.*
3. *$\hat{e}_{ij} = 1$ if and only if $\alpha_{ij} \geq \bar{\alpha}_{ij}$.*

Proof. First, remember that the cost function $A_i(e)$ is convex for all $i \in N$. To obtain the optimal effort, the derivative of this function can be analyzed with respect to e_{ij} for any $j \in N\backslash\{i\}$. It must be noted that $\frac{\partial A_i(e)}{\partial e_{ij}} > 0 \iff c'_i(e_{ij}) > \alpha_{ij} r'_{ji}(e_{ij})$ for all $e_{ij} \in [0,1]$, which is a necessary and sufficient condition for $\hat{e}_{ij} = 0$ to be the optimal effort. [12]

We begin by proving point 1. Note that $\underline{\alpha}_{ij} = \frac{c'_i(0)}{r'_{ji}(0)} < \frac{c'_i(e_{ij})}{r'_{ji}(e_{ij})}$ because $c'_i > 0$, $r'_{ji} > 0$, $c''_i > 0$, and $r''_{ji} < 0$. Thus, $c'_i(e_{ij})$ is a positive and increasing function, and $r'_{ji}(e_{ij})$ a positive and decreasing function, so for any $e_{ij} > 0$, $c'_i(0) < c'_i(e_{ij})$ and $r'_{ji}(0) > r'_{ji}(e_{ij})$. Therefore, $\alpha_{ij} \leq \underline{\alpha}_{ij} \iff c'_i(e_{ij}) > \alpha_{ij} r'_{ji}(e_{ij})$ for all $e_{ij} > 0 \iff \hat{e}_{ij} = 0$.

The demonstration in point 3 is similar to that of point 1. The above arguments are the same and only the signs of the inequalities change.

To end the proof, we prove point 2. First, we show that there is a unique $\hat{e}_{ij} \in (0,1)$ such that $c'_i(\hat{e}_{ij}) = \alpha_{ij} r'_{ji}(\hat{e}_{ij})$, which is the unique optimal effort because $\left.\frac{\partial A_i(e)}{\partial e_{ij}}\right|_{e_i=\hat{e}_{ij}} = 0$ and $A_i(e)$ is a convex function. In addition, $c'_i(e_{ij})$ is a positive increasing function and $r'_{ji}(e_{ij})$ a positive decreasing function, in $e_{ij} \in [0,1]$. This means that equation $\frac{\partial A_i(e)}{\partial e_{ij}} = c'_i(e_{ij}) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ has a unique root, which belongs to $(0,1)$ if and only if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$. Note that if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$ then $c'_i(0) < \alpha_{ij} r'_{ji}(0)$ and $c'_i(1) > \alpha_{ij} r'_{ji}(1)$, and so there is a unique point $\hat{e}_{ij}$ where $c'_i(\hat{e}_{ij}) = \alpha_{ij} r'_{ji}(\hat{e}_{ij})$. ■

Proof of Theorem 4

[12] This occurs because $A_i(e)$ is an increasing function in e_{ij} and the minimum value is obtained for $\hat{e}_{ij} = 0$, which is the optimal effort for agent i.

Proof. As we already mention, the optimum $\hat{e}_{ij}$ is independent of other efforts. Therefore, the equilibrium effort is determined by Lemma 2. In addition, we want to characterize the effort equilibrium according to the value of the parameter α_{ij}. Thus, in the case of agent j, $\underline{\alpha}_{ji} < \alpha_{ji} < \bar{\alpha}_{ji} \Leftrightarrow \underline{\alpha}_{ji} < 1-\alpha_{ij} < \bar{\alpha}_{ji} \Leftrightarrow 1-\bar{\alpha}_{ji} < \alpha_{ij} < 1-\underline{\alpha}_{ji}$. ■

The next corollary shows how the pairwise equilibrium efforts e^*_{ij} depend on α_{ij}, for all $i,j \in N, i \neq j$. As expected, as the proportion of aggregate cost reduction obtained by an agent increases, the effort that agent exerts also increases (or at least stays the same).

Corollary 2 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game and (e^*_{ij}, e^*_{ji}) the pairwise efforts equilibrium. Thus,*

- $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} > 0$, *if* $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$; $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$, *otherwise.*
- $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} < 0$, *if* $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$; $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$, *otherwise.*

Proof of Corollary 2

Proof. By the implicit function theorem, $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = -\frac{\frac{\partial (c'_i(e^*_{ij}) - \alpha_{ij} r'_{ji}(e^*_{ij}))}{\partial \alpha_{ij}}}{\frac{\partial (c'_i(e^*_{ij}) - \alpha_{ij} r'_{ji}(e^*_{ij}))}{\partial e^*_{ij}}} = \frac{r'_{ji}(e^*_{ij})}{c''_i(e^*_{ij}) - \alpha_{ij} r''_{ji}(e^*_{ij})} > 0$, because $r'_{ji}(e^*_{ij}) > 0$, $c''_i(e^*_{ij}) > 0$, and $r''_{ji}(e^*_{ij}) < 0$. Thus, for any $\alpha_{ij} \leq \underline{\alpha}_{ij}$, Lemma 2 implies that $e^*_{ij} = 0$, thus, $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$. However, if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$, then $e^*_{ij} \in (0,1)$ and $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} > 0$. Finally, if $\alpha_{ij} \geq \bar{\alpha}_{ij}$, then $e^*_{ij} = 1$ and $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$. Analogously, if $\alpha_{ji} \leq \underline{\alpha}_{ji} \iff \alpha_{ij} \geq 1 - \underline{\alpha}_{ji}$, then $e^*_{ji} = 0$ and $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$, if $\alpha_{ji} \in (\underline{\alpha}_{ji}, \bar{\alpha}_{ji}) \iff \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, then $e^*_{ji} \in (0,1)$ and $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} < 0$. Finally, if $\alpha_{ji} \geq \bar{\alpha}_{ji} \iff \alpha_{ij} \leq 1 - \bar{\alpha}_{ji}$, then $e^*_{ij} = 1$ and $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$. ■

Theorem 5, in Section 5, provides the weights α_{ij} that minimizes function $L^*_{ij}(\alpha_{ij})$, and the efficient effort equilibrium. To solve the above optimization problem it is necessary to know the function $L^*_{ij}(\alpha_{ij})$ very accurately.

To demonstrate Theorem 5, three technical lemmas are needed first. Lemmas 3, 4, and 5 characterize the derivatives $\frac{\partial (A^*_i(\alpha_{ij}))}{\partial \alpha_{ij}}$, $\frac{\partial (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}}$, and $\frac{\partial^2 (L^*_{ij}(\alpha_{ij}))}{\partial \alpha^2_{ij}}$ respectively.

The first lemma shows how the optimal cost function of agent $i \in N$ depends on α_{ij}. Henceforth, to simplify notation, we consider that for any $i,j \in N$, $\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}$ and $\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}}$ stand for derivatives $\frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$ and $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ evaluated in the unique effort equilibrium.

Lemma 3 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game and e^* the effort equilibrium. Thus,*

1. $$\frac{\partial (A_i(e^*))}{\partial \alpha_{ij}} = \frac{\partial (A^*_i(\alpha_{ij}))}{\partial \alpha_{ij}} = \begin{cases} -r_{ij}(e^*_{ji}) - \alpha_{ij} \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} \frac{\partial e^*_{ji}}{\partial \alpha_{ij}} - r_{ji}(e^*_{ij}), & \text{if} \quad \alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \\ -r_{ij}(e^*_{ji}) - r_{ji}(e^*_{ij}) < 0, & \text{otherwise} \end{cases}$$

2. $$\frac{\partial (A_j(e^*))}{\partial \alpha_{ij}} = \frac{\partial (A^*_j(1-\alpha_{ij}))}{\partial \alpha_{ij}} = \begin{cases} r_{ji}(e^*_{ij}) - (1-\alpha_{ij}) \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}} \frac{\partial e^*_{ij}}{\partial \alpha_{ij}} + r_{ij}(e^*_{ji}), & \text{if} \quad \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \\ r_{ji}(e^*_{ij}) + r_{ij}(e^*_{ji}) > 0, & \text{otherwise.} \end{cases}$$

Proof. It is known that $A_i(e^*) = c_i(e_i^*) - \sum_{z\in N\setminus\{i\}} \alpha_{iz}(r_{iz}(e_{zi}^*) + r_{zi}(e_{iz}^*))$, and $A_i^*(\alpha_{ij}) = c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$, thus

$$\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = \frac{\partial(A_i^*(\alpha_{ij}))}{\partial\alpha_{ij}} = \frac{\partial c_i(e_i^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}} - r_{ij}(e_{ji}^*) - \alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} - r_{ji}(e_{ij}^*) - \alpha_{ij}\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}},$$

$$= \left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \alpha_{ij}\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right)\frac{\partial e_{ij}^*}{\partial\alpha_{ij}} - r_{ij}(e_{ji}^*) - \alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} - r_{ji}(e_{ij}^*).$$

The first term of the above expression is always zero, i.e. $\left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \alpha_{ij}\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right)\frac{\partial e_{ij}^*}{\partial\alpha_{ij}} = 0$. To see this, note that if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$, then $e_{ij}^* \in (0,1)$ by Lemma 2, so $\left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \alpha_{ij}\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right) = 0$ because it is evaluated in equilibrium. In the other case, where $\alpha_{ij} \le \underline{\alpha}_{ij}$ or $\alpha_{ij} \ge \bar{\alpha}_{ij}$, $e_{ij}^* = 0$ by Proposition 2, so $\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} = 0$. Therefore, $\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = -r_{ij}(e_{ji}^*) - \alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} - r_{ji}(e_{ij}^*)$.

It is known by assumption that $r_{ij}(e_{ji}^*) \ge 0$, $\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} > 0$. If $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, then by Proposition 2, $\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} < 0$. However, if $\alpha_{ij} \notin (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$ then, by Proposition 2, $\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} = 0$, so $\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = -r_{ij}(e_{ji}^*) - r_{ji}(e_{ij}^*)$.

The proof is analogous for $\frac{\partial(A_j(e^*))}{\partial\alpha_{ij}}$. ■

Notice that the effect of α_{ij} on the cost function of agent i could be positive or negative because of two simultaneous effects. First effect: As expected, if α_{ij} increases so does the proportion of cost reduction that agent i can obtain, and thus the cost function, $A_i(e^*)$, decreases. This decrease is measured by the term $-r_{ij}(e_{ji}^*) - r_{ji}(e_{ij}^*) < 0$ in the derivative. Second effect: When α_{ij} increases, the effort of agent j decreases in equilibrium, so the cost function of agent i increases. The term $-\alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{e_{ji}^*}{\partial\alpha_{ij}} > 0$ measures this second effect. The sum of these two effects determines the sign of the derivative. Therefore, an increase in the proportion of the aggregate cost reduction that an agent obtains could increase the cost of that agent if the second effect dominates the first. This is an interesting result: Giving too much to a particular agent could be not only worse for the aggregate cost but also for that particular agent.

The second lemma calculates the derivative of the aggregate cost function $L_{ij}^*(\alpha_{ij})$ in the effort equilibrium for any $i, j \in N$.

Lemma 4 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game, and e^* the effort equilibrium. Thus,*

$$\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e_j^*)}{\partial e_{ji}^*} - \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\right)\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} I_j + \left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right)\frac{\partial e_{ij}^*}{\partial\alpha_{ij}} I_i,$$

$$\text{where } I_i = \begin{cases} 1 & \text{if} \quad \alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \\ 0 & \text{otherwise} \end{cases} \text{ and } I_j = \begin{cases} 1 & \text{if} \quad \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, there are four possible cases:

- $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}}$ *can be positive and/or negative if* $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$
- $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}} = 0$*, if* $\alpha_{ij} \notin (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cup (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$
- $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}} > 0$ *if* $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \cap \big((0, \underline{\alpha}_{ij}) \cup (\bar{\alpha}_{ij}, 1)\big)$

- $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} < 0$ *if* $\alpha_{ij} \in \big((0, 1-\bar{\alpha}_{ji}) \cup (1-\underline{\alpha}_{ji}, 1)\big) \cap \big(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\big)\big)$

Proof. From (6), we calculate that $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$. Simplifying for the different subsets of α_{ij}, the following emerges:

1. if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$ then, by Theorem 4, $e^*_{ji} \in (0,1)$ and $e^*_{ij} \in (0,1)$, thus, by Corollary 2, $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} < 0$ and $\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} > 0$. In addition, since $\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - (1-\alpha_{ij})\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} = 0$ and $\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \alpha_{ij}\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}} = 0$, it follows that $\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} < 0$ and $\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}} < 0$. Therefore, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$, which can be positive or negative in this case.

2. if $\alpha_{ij} \notin (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cup (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$ then, by Theorem 4, $e^*_{ji} \in \{0,1\}$ and $e^*_{ij} \in \{0,1\}$, and by Corollary 2, $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} = \frac{\partial e^*_{ij}}{\partial\alpha_{ij}} = 0$. Therefore, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0$.

3. if $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \cap \big((0, \underline{\alpha}_{ij}) \cup (\bar{\alpha}_{ij}, 1)\big)$, then, as above, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} > 0$.

4. if $\alpha_{ij} \in \big((0, 1-\bar{\alpha}_{ji}) \cup (1-\underline{\alpha}_{ji}, 1)\big) \cap \big(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\big)\big)$ then $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} < 0$.

■

The derivative is a piecewise function and there are intervals where its sign is independent of the particular form of the functions of the game. For those cases, it is straightforward to find the optimal α_{ij} that minimizes the function $L^*_{ij}(\alpha_{ij})$. In those intervals, the derivative is either positive, negative or zero throughout the interval. These cases are respectively $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} > 0$, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} < 0$, and $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0$. However, there is an interval where the sign of the derivative depends on the particular form of functions of the game. In this particular case $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$. This occurs when $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, which implies that in equilibrium simultaneously $0 < e^*_{ij} < 1$ and $0 < e^*_{ji} < 1$. Therefore, in this case only, the derivative may be zero for some α_{ij} within this interval. In that case, the second derivative is needed to solve the optimization problem.

The third Lemma shows that the aggregate cost function $L^*_{ij}(\alpha_{ij})$ is convex in α_{ij}. Two additional assumptions about third derivatives need to be introduced.

Lemma 5 *Let* $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ *be the effort game,* e^* *the effort equilibrium, and* $\frac{\partial^3 c_i(e^*_i)}{\partial e^{*3}_{ij}} > 0$ *and* $\frac{\partial^3 r_{ji}(e^*_{ij})}{\partial e^{*3}_{ij}} < 0$, *for any* $i, j \in N$. *Thus* $\frac{\partial^2 L^*_{ij}(\alpha_{ij}))}{\partial\alpha^{*2}_{ij}} > 0$ *for all* $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$.

Proof. Take $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$. Thus,

$$\frac{\partial^2(L^*_{ij}(\alpha_{ij}))}{\partial\alpha^2_{ij}} = \frac{\partial^2\left[\left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}\right]}{\partial\alpha^2_{ij}}$$

$$\left(\frac{\partial^2 c_j(e^*_j)}{\partial e^*_{ji}\partial\alpha_{ij}} - \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^*_{ji}\partial\alpha_{ij}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}}$$

$$+\left(\frac{\partial^2 c_i(e_i^*)}{\partial e_{ij}^*\partial\alpha_{ij}}-\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^*\partial\alpha_{ij}}\right)\frac{\partial e_{ij}^*}{\partial\alpha_{ij}}+\left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*}-\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right)\frac{\partial^2 e_{ij}^*}{\partial\alpha_{ij}^2}$$

$$=\left(\frac{\partial^2 c_j(e_j^*)}{\partial^2 e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}-\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial^2 e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}\right)\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}+\left(\frac{\partial c_j(e_j^*)}{\partial e_{ji}^*}-\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\right)\frac{\partial^2 e_{ji}^*}{\partial\alpha_{ij}^2}$$

$$+\left(\frac{\partial^2 c_i(e_i^*)}{\partial^2 e_{ij}^*}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}}-\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial^2 e_{ij}^*}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}}\right)\frac{\partial e_{ij}^*}{\partial\alpha_{ij}}+\left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*}-\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right)\frac{\partial^2 e_{ij}^*}{\partial\alpha_{ij}^2}$$

$$=\left(\frac{\partial^2 c_j(e_j^*)}{\partial^2 e_{ji}^*}-\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial^2 e_{ji}^*}\right)\left(\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}\right)^2+\left(\frac{\partial c_j(e_j^*)}{\partial e_{ji}^*}-\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\right)\frac{\partial^2 e_{ji}^*}{\partial\alpha_{ij}^2}$$

$$+\left(\frac{\partial^2 c_i(e_i^*)}{\partial^2 e_{ij}^*}-\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial^2 e_{ij}^*}\right)\left(\frac{\partial e_{ij}^*}{\partial\alpha_{ij}}\right)^2+\left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*}-\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right)\frac{\partial^2 e_{ij}^*}{\partial\alpha_{ij}^2}>0$$

Now we prove that $\frac{\partial^2 e_{ji}^*}{\partial\alpha_{ij}^2}<0$ and $\frac{\partial^2 e_{ij}^*}{\partial\alpha_{ij}^2}<0$, so $\frac{\partial^2(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}^2}>0$.

We first prove that $\frac{\partial^2 e_{ji}^*}{\partial\alpha_{ij}^2}<0$. It is known that

$$\frac{\partial A_j(e^*)}{\partial e_{ji}}=\frac{\partial c_j(e_j^*)}{\partial e_{ji}^*}-(1-\alpha_{ij})\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}=0$$

We now derive the second term regarding α_{ij}.

$$\frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}+\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}-(1-\alpha_{ij})\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}=0$$

We now do the same for α_{ij}.

$$\left(\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}}\left(\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}\right)^2+\frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}}\frac{\partial^2 e_{ji}^*}{\partial\alpha_{ji}^2}\right)+\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}$$

$$-(1-\alpha_{ij})\left(\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}}\left(\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}\right)^2+\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial^2 e_{ji}^*}{\partial\alpha_{ji}^2}\right)=0$$

$$\left(\frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}}-(1-\alpha_{ij})\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\right)\frac{\partial^2 e_{ji}^*}{\partial\alpha_{ji}^2}+\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}$$

$$+\left(\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}}-(1-\alpha_{ij})\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}}\right)\left(\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}\right)^2=0$$

$$\frac{\partial^2 e_{ji}^*}{\partial\alpha_{ij}^2}=\frac{-\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}-\left(\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}}-(1-\alpha_{ij})\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}}\right)\left(\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}\right)^2}{\frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}}-(1-\alpha_{ij})\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}}$$

Clearly, this expression is lower than zero if $\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}}>0$ and $\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}}<0$; note that $\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}<0$ by Proposition 2.

Analogously, we obtain

$$\frac{\partial^2 e_{ij}^*}{\partial\alpha_{ij}^2}=\frac{\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*2}}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}}-\left(\frac{\partial^3 c_i(e_i^*)}{\partial e_{ij}^{*3}}-\alpha_{ij}\frac{\partial^3 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*3}}\right)\left(\frac{\partial e_{ij}^*}{\partial\alpha_{ij}}\right)^2}{\frac{\partial^2 c_i(e_i^*)}{\partial e_{ij}^{*2}}-\alpha_{ij}\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*2}}}<0.\ \blacksquare$$

Lemma 5 enables us to state that in any interval where the piecewise derivative function takes the value $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}}=-\alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}}-(1-\alpha_{ij})\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}}$, the function $L_{ij}^*(\alpha_{ij})$ is convex (see also Lemma 4).

The following proposition shows that, according to the value of the effort equilibrium, the cost function $L_{ij}^*(\alpha_{ij})$ is a continuous piecewise function with four types of piece. This result characterizes all of those pieces, showing the shape of $L_{ij}^*(\alpha_{ij})$ and the optimal α_{ij} in each type of piece.

Proposition 5 *Consider the effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ and e^* as the effort equilibrium. Let $\alpha_{ij} \in [a,b]$ be a piece of $L^*_{ij}(\alpha_{ij})$ with $0 \le a < b \le 1$, $L^*_{ij}(\alpha_{ij})$ can have only four types of piece:*

1. **Constant:** (e^*_{ij}, e^*_{ji}) *is either* $(0,0)$, $(1,0)$, $(0,1)$ *or* $(1,1)$. *Thus* $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0$ *and* $L^*_{ij}(\alpha_{ij})$ *is always constant. Therefore, any* $\alpha_{ij} \in [a,b]$ *minimizes* $L^*_{ij}(\alpha_{ij})$.

2. **Increasing:** e^*_{ij} *is either* 0 *or* 1, *and* $0 < e^*_{ji} < 1$. *Thus* $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} > 0$ *and* $L^*_{ij}(\alpha_{ij})$ *is always increasing. Therefore,* $\alpha_{ij} = a$ *minimizes* $L^*_{ij}(\alpha_{ij})$.

3. **Decreasing:** $0 < e^*_{ij} < 1$, *and* e^*_{ji} *is either* 0 *or* 1. *Thus* $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} < 0$ *and* $L^*_{ij}(\alpha_{ij})$ *is always decreasing. Therefore,* $\alpha_{ij} = b$ *minimizes* $L^*_{ij}(\alpha_{ij})$.

4. **Depending on cost function shape:** $0 < e^*_{ij} < 1$ *and* $0 < e^*_{ji} < 1$. *Thus,*
$\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$.
In this case, there is always a unique $\breve{\alpha}^{[a,b]}_{ij} \in [a,b]$ *that minimizes* $L^*_{ij}(\alpha_{ij})$, *which is:*
$$\breve{\alpha}^{[a,b]}_{ij} = \begin{cases} a & \textit{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} > 0 \textit{ for all } \alpha_{ij} \in [a,b] \\ b & \textit{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} < 0 \textit{ for all } \alpha_{ij} \in [a,b] \\ \textit{Solution of } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0 & \textit{otherwise} \end{cases}$$

Proof. The proof of Lemma 4 shows four possible cases for $L^*_{ij}(\alpha_{ij})$. The point 2. of the proof of Lemma 4 proves the point 1. (Constant). The point 3. proves the point 2. (Increasing), and point 4. proves point 3 (decreasing). Finally, to prove the point 4. (Depending on cost function shape) we need the point 1. of Lemma 4 and Lemma 5 which proves that $L^*_{ij}(\alpha_{ij})$ is convex in this case. Therefore, in this last case, it is also straightforward to show that $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}}$ is continuous, so there is always a unique α_{ij} that minimizes $L^*_{ij}(\alpha_{ij})$ in such pieces. The procedure for calculating $\breve{\alpha}^{[a,b]}_{ij}$ is the following: First, by Theorem 4, we calculate e^*_{ij} and e^*_{ji} as a function of α_{ij} from $c'_i(e_{ij}) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0$. Second, we build the function $L^*_{ij}(\alpha_{ij})$ with the $e^*_{ij}(\alpha_{ij})$ and $e^*_{ji}(\alpha_{ij})$ previously calculated. Finally, we calculate $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}}$ and obtain $\breve{\alpha}^{[a,b]}_{ij}$. ■

Finally, Theorem 5 characterizes the optimal α^*_{ij}, for all $i,j \in N$ with $i \neq j$, which incentivizes an efficient effort equilibrium, which is also provided.

Proof of Theorem 5

Proof. As $L^*_{ij}(\alpha_{ij})$ is a continuous piecewise function, we analyze the five pieces that define it in each case. Lemma 4, 5 and Proposition 5 enable the type of piece to be determined, thus giving the value of α_{ij} that minimizes $L^*_{ij}(\alpha_{ij})$ in each piece. Comparing the pieces gives the α^*_{ij} that minimizes the aggregate cost for each of the six cases. This value need not be unique. Note, in addition, that $\underline{\alpha}_{ij}$, $\bar{\alpha}_{ij}$, $\bar{\alpha}_{ji}$ and $\underline{\alpha}_{ji}$ are always greater than zero, but any of them may be greater than one, which implies that some pieces of certain cases may not exist. We prove the theorem case by case:

Case A $(\underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji})$

Note that those thresholds are always greater than zero, so $0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < 1$.

By Lemma 4,

if $\alpha_{ij} \in \big(0, \underline{\alpha}_{ij}\big)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \big(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\big)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji})$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \big(1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji}\big)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \big(1 - \underline{\alpha}_{ji}, 1\big)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

Therefore, α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}]$.

Case B $(\underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji})$

Analogously, $0 < \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1$, and by Lemma 4, 5 and Proposition 5,

if $\alpha_{ij} \in \big(0, \underline{\alpha}_{ij}\big)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \big(\underline{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}\big)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in (1 - \bar{\alpha}_{ji}, \bar{\alpha}_{ij})$, then $\check{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$, where $\check{\alpha}_{ij}$ is define in Proposition 5.

If $\alpha_{ij} \in \big(\bar{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}\big)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \big(1 - \underline{\alpha}_{ji}, 1\big)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

Therefore, $\alpha^*_{ij} = \check{\alpha}_{ij}^{[1-\bar{\alpha}_{ji}, \bar{\alpha}_{ij}]}$.

Case C $(\underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij})$

It may happen here that either $\bar{\alpha}_{ij} < 1$ or $\bar{\alpha}_{ij} \geq 1$. Thus there are two subcases:

$0 < \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} < 1$

$0 < \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < 1 < \bar{\alpha}_{ij}$

Starting with the first subcase, by Lemma 4, 5 and Proposition 5

if $\alpha_{ij} \in \big(0, \underline{\alpha}_{ij}\big)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \big(\underline{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}\big)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \big(1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji}\big)$, then $\check{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \big(1 - \underline{\alpha}_{ji}, \bar{\alpha}_{ij}\big)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $L^*_{ij}(\alpha_{ij})$ is constant, in this interval.

However, in the second subcase $\bar{\alpha}_{ij} > 1$, which implies that the last interval described above does not exist. The rest of the analysis is similar to the first subcase.

Therefore, $\alpha^*_{ij} = \arg min\{L^*_{ij}(\check{\alpha}_{ij}^{[1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}]}), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that, if $\alpha^*_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$.

Case D $(1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji})$

It may happen here that either $1 - \bar{\alpha}_{ji} > 0$ or $1 - \bar{\alpha}_{ji} \leq 0$. Thus there are two subcases:

$0 < 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1$

$1 - \bar{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1$

Starting with the first subcase, by Lemma 4, 5 and Proposition 5

if $\alpha_{ij} \in (0, 1 - \bar{\alpha}_{ji})$, then $e^*_{ij} = 0$, $e^*_{ji} = 1$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \big(1 - \bar{\alpha}_{ji}, \underline{\alpha}_{ij}\big)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \big(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\big)$, then $\check{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \big(\bar{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}\big)$, then $e^*_{ij} = 1$, $0 < e^*_{ji} < 1$, and $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

However, if $1 - \bar{\alpha}_{ji} < 0$ the first interval above does not exist. Again, the rest of the analysis is similar to the first subcase.

Therefore, $\alpha^*_{ij} = \arg min\{L^*_{ij}(\Lambda(1 - \bar{\alpha}_{ji})), L^*_{ij}(\check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]})\}$. Note that if $\alpha^*_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1 - \bar{\alpha}_{ji}]$.

Case E $(1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij})$

In this case, it may happen that either $1 - \bar{\alpha}_{ji} > 0$ or $1 - \bar{\alpha}_{ji} \leq 0$, and either $\bar{\alpha}_{ij} < 1$ or $\bar{\alpha}_{ij} \geq 1$. Thus there are four subcases:

$0 < 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} < 1$

$1 - \bar{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} < 1$

$0 < 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1 < \bar{\alpha}_{ij}$

$1 - \bar{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1 < \bar{\alpha}_{ij}$

Focusing on the first subcase, by Lemma 4, 5 and Proposition 5.

if $\alpha_{ij} \in (0, 1 - \bar{\alpha}_{ji})$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \big(1 - \bar{\alpha}_{ji}, \underline{\alpha}_{ij}\big)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, 1-\underline{\alpha}_{ji}\right)$, then $\check{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \left(1-\underline{\alpha}_{ji}, \bar{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

In the other three subcases, the first and/or last interval may not exist. Once again, the rest of the analysis for those subcases is similar to the first one.

Therefore, $\alpha^*_{ij} = \arg min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), \check{\alpha}_{ij}^{\left[\underline{\alpha}_{ij}, 1-\underline{\alpha}_{ji}\right]}, L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that if $\alpha^*_{ij} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} > 0$ then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1-\bar{\alpha}_{ji}]$, and if $\alpha^E_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$.

Case F $(1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij})$

This is the most general case and anything could happen with thresholds greater than one. Thus there are nine subcases. First consider the case $0 < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1:$

If $\alpha_{ij} \in (0, 1-\bar{\alpha}_{ji})$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \left(1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \left(1-\underline{\alpha}_{ji}, \underline{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = \bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

In any other subcase, the first, second, to last, and last intervals considered above, may not exist. The rest of the analysis for those subcases is similar to the first one.

Therefore, $\alpha^*_{ij} = \arg Min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that, if $\alpha^*_{ij} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1-\bar{\alpha}_{ji}]$, but if $\alpha^*_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$. Additionally, if $1-\underline{\alpha}_{ji} < 0$ and $\bar{\alpha}_{ij} > 1$, then $L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})) = L^*_{ij}(\Lambda(\bar{\alpha}_{ij})$, so α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1]$.

■

Appendix D

Table 1: Notation summary	
$N = \{1, 2, ..n\}$	Agents
$E_i = [0,1]^{n-1}$	Strategy space of agent i of the non-cooperative game
$E = \prod_{i\in N} E_i = [0,1]^{n(n-1)}$	Strategy profile space of the non-cooperative game
$e_{ij} \in [0,1]$	Effort exerted by agent i to reduce the cost of agent j
$e_i = (e_{ij})_{j\neq i} \in E_i$	Efforts exerted by agent i
$e \in E$	Effort profile
$c_i : E_i \to \mathbb{R}_+$	Cost function for agent i with $c_i(e_i)$ the cost of effort e_i
$r_{ij} : [0,1] \to \mathbb{R}_+$	Cost reduction function of agent i given by agent j
$r_{ij}(e_{ji})$	Cost reduction for agent i due to effort e_{ji}
$c : 2^N \to \mathbb{R}$	Characteristic function of the cooperative cost game
$S \subseteq N$	Coalition of agents
$c^S(\{i\}) = c_i(e_i) - \sum_{j\in S\setminus\{i\}} r_{ij}(e_{ji})$	The reduced cost of agent i in coalition S
$c(S) = \sum_{i\in S} c^S(\{i\})$	The reduced cost for coalition S
$\psi_i : E \to \mathbb{R}$	Allocation to agent i
$\psi(e) = (\psi_i(e))_{i\in N}$	Allocation rule, with $\sum_{i\in N} \psi_i(e) = c(N)$
$\Omega_i(e) = c_i(e_i) - \sum_{j\in N\setminus\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})]$	WPR allocation for agent i, where $\omega^i_{ij} \in [0,1]$, and $\omega^i_{ji} = 1 - \omega^j_{ji}$ with i, $j \in N, i \neq j$
$A_i(e) = c_i(e_i) - \sum_{j\in N\setminus\{i\}} \alpha_{ij}[r_{ij}(e_{ji}) + r_{ji}(e_{ij})]$	WPAR allocation for agent i, where $\alpha_{ij} \in [0,1]$ and $\alpha_{ji} = 1 - \alpha_{ij}$.
$\alpha = (\alpha_i)_{i\in N}$ with $\alpha_i = (\alpha_{ij})_{j\in N\setminus\{i\}}$	Weights of WPAR allocation
$\phi(c)$	Shapley value
$\nu(e)$	Nucleolus

Table 2: Summary of optimization problems		
$\tilde{e}$	Efficient effort profile	$\tilde{e} = \arg \min_{e\in[0,1]^{n(n-1)}} c(N)$
$\hat{e}_i$	Optimal efforts of agent i given efforts of other agents	$\hat{e}_i = \arg \min_{e_i\in[0,1]^{(n-1)}} A_i(e)$
e^*_i	Equilibrium strategy of agent i	$e^*_i = \hat{e}_i$
α^*	Optimal weights of WPAR allocation	$\alpha^* = \arg \min_{\alpha\in[0,1]^{n(n-1)}} \sum_{i\in N} A_i(e^*)$ $\Updownarrow$ $\alpha^*_{ij} = \arg \min_{\alpha_{ij}\in[0,1]} L^*_{ij}(\alpha_{ij})$ for $i \neq j \in N$ with $L^*_{ij}(\alpha_{ij}) = c_i(e^*_i) + c_j(e^*_j)$

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