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# Asymmetric Platform Oligopoly

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# Asymmetric Platform Oligopoly*

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**Abstract:** We propose a tractable model of asymmetric platform oligopoly in which users from two distinct groups are subject to within-group and cross-group network effects and decide which platform to join. We characterize the equilibrium when platforms manage user access by setting participation fees. We explore the effects of platform entry, change of incumbent platforms' quality under free entry, and partial compatibility on market outcomes. We show how the analysis can be extended to partial user participation and zero fees for one of the user groups.

**Keywords:** oligopoly theory; aggregative games; network effects; two-sided markets; two-sided single-homing; free entry; compatibility

**JEL Codes:** L13; L41; D43

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# 1 Introduction

Recent decades have seen the emergence of large digital platforms, such as Alphabet, Amazon, Apple, Meta, and Microsoft, that cater to two or more user groups. Some of their activities have been increasingly scrutinized by legislators, competition watchdogs, and regulators. The assessment of competition policy and regulatory interventions requires a framework of oligopolistic platform competition that accommodates platforms of different sizes. What is more, asymmetries are also a common feature in platform markets in which Big Tech is not present. Yet, as Jullien, Pavan and Rysman (2021, p. 522) note, "the literature still lacks a tractable model of platform competition in asymmetric [...] markets."[1] This paper aims to filling this gap by proposing a tractable yet flexible model of asymmetric oligopolistic platform competition.

We model platforms as firms that bring together users from two groups. Each user cares about the participation of other users in their own group and/or in the other group; for example, some firms provide competing software services that are made available to business and private users and each user benefits from improved functionality as the number of other users of the service increases. Every user in the same group obtains an average maximal utility (when network effects play out fully) that is adjusted by the realized network size plus an utility realization of their idiosyncratic taste. Then, each user makes a discrete choice between the different (asymmetric) platforms; in other words, each user single-homes.

More specifically, we consider a multinomial logit demand model augmented by network effects. While, for tractability reasons, most of the theoretical literature assumes linear network effects, we assume that user benefits depend on the logarithm of the sizes of the two user groups; this is a specification widely adopted in the empirical analysis of network effects and platforms (e.g. Ohashi, 2003; Rysman, 2004, 2007; Zhu and Iansiti, 2012). In line with our modelling choice, according to practitioners, the incremental benefit of additional users typically declines with the user level; for instance, Chen (2021, p. 256) writes: "... network effects become less incrementally powerful. In eBay's case, when you search something like 'Rolex vintage daytona,' the product experience (and associated conversion rate) improve dramatically as you add the first few listings. It might even continue with a first few dozen. But you don't need the search to return 1,000 or 5,000 listings ..."

Platform competition with single-homing by users of each group is of high theoretical interest because platforms directly compete for users in each group. It formalizes real-world markets when heterogeneous users make a discrete choice between different systems, standards, or applications, and the providers of such offers price discriminate between user groups. An example is competing software packages with offers for business and private users that are subject to

[1] We removed the words "and/or partially covered" from this quote. In Section 5.2 we address partial coverage.

network effects. Another is competing cloud storage services that are offered to business and private users where network effects arise due to file-sharing possibilities. Yet another is enterprise resource planning softwares (e.g. by Oracle or SAP) that cater to large and small enterprises.

Platforms are heterogenous with respect to their costs and the average value they offer to users (after controlling for network effects). They simultaneously set participation fees for both user groups to maximize own profit. A platform's profit function depends on the vector of all the platforms prices for both goup; in our setting it can be rewritten as one that depends on two choice variables and their aggregates, which are the sum of the respective choice variables over all platforms. We show that there exists an equilibrium in the pricing game and provide several characterization results; in case of multiple equilibria, these equilibria are ordered. In line with earlier work (Armstrong, 2006; Tan and Zhou, 2021), the fees set by each platform in each group feature a "discount" to attract users in the same or the other group, triggered by within- and cross-group network effects. New to the literature, we also establish conditions under which the higher-quality platform sets higher fee for both user groups than a lower-quality platform and conditions under which it does not.

Exogenous platform entry necessarily increases consumer surplus if there are no cross-group network effects. In the presence of cross-group network effects, in our setting, one or both of the user groups benefits from entry; however, it is possible that one of the groups suffers.

Under endogenous entry, the number of fringe platforms depend on market conditions and the strategic choices of incumbent platforms. For example, a subset of incumbent platforms change the quality of their offers for at least one group of consumers. Under free entry such that some fringe platforms are active, we show that, under a weak condition, after a change of quality offered to one or both user groups by one or several incumbent platforms, one of the two user groups is better off, while the other group is worse off – this present a strong and novel see-saw property.

Turning to the analysis of partial compatibility, we show that better compatibility in some situations increases and in others decreases consumer surplus if there are no cross-group network effects; this result already holds under symmetry and extends to asymmetric networks. We also discuss how better compatibility tends to affect the two user groups when they are connected through cross-group network effects.

**Related literature** This paper contributes to the literature on (two-sided) platform competition. The literature on platform competition has examined the importance of network effects in platform competition (see Jullien, Pavan and Rysman, 2021, for a review of the literature). Prominent early works with two-sided single-homing include Armstrong (2006), Tan and Zhou

(2021), and Jullien and Pavan (2019). Armstrong (2006, section 4) proposes a model with linear cross-group network effects and two symmetric platforms within a Hotelling setting on each side and examines the pricing implications of cross-group network effects;[2] Tan and Zhou (2021) examine the welfare property of free entry equilibria in a model with general network effects and symmetric platforms; Jullien and Pavan (2019) examine the pricing implications in duopoly with linear cross-group network effects when platforms and users face uncertainty about the distribution of users' tastes and derive insights regarding the platforms' information management policies. As a methodological contribution, we analyze a oligopoly model with asymmetric platforms and two-sided single-homing as an aggregative game.

Earlier literature focused on platforms catering to a single user group characterized by direct network effects. Contributions within the multinomial logit setting include Anderson, De Palma and Thisse (1992, chapter 7.8) and Starkweather (2003), both of which assumed linear direct network effects. In these settings, there is no explicit solution for the participation game with asymmetric platforms.[3] As a special case of our framework, we characterize the unique price equilibrium under asymmetric platform competition and direct network effects.

In our analysis we make use of the aggregative game property of our model. Platform competition with two-sided single-homing implies that we cannot resort to a single aggregate in contrast to the oligopoly models analyzed by Anderson, Erkal and Piccinin (2020) and Nocke and Schutz (2018) as well as the platform models in Anderson and Peitz (2020, 2023). In our construction, profits can be written as a function of a platform's actions (such that there is a one-to-one relationship between actions and platform fees) and the corresponding aggregates as the sums of the actions over all platforms; thus, we work with a two-dimensional aggregate. We provide more specific references below.

The paper is organized as follows. In Section 2 we present the model. In Section 3.1, we characterize participation equilibria for any given platform fees and show that there is a unique interior equilibrium; we identify this as the unique asymptotically stable participation equilibrium and use this in the subsequent analysis. We then express profit functions as functions of two choice variables and their aggregates and express user welfare as a function of the aggregates (Section 3.2). In Section 3.3, we show that there exists an equilibrium of the platform

[2]For an empirical application to the German magazine market, see Kaiser and Wright (2006). The model with asymmetric platforms is used to analyze platform taxation (Belleflamme and Toulemonde, 2018) and the relationship between profits and market shares (Belleflamme, Peitz and Toulemonde, 2022). Sato (2021b) also looked at the relationship between profits and market shares, albeit in the oligopoly model with logit demand that we develop here.

[3]The operations research literature has looked at monopoly pricing and assortment problems in the presence of direct network effects and multinomial logit demand; see e.g. Du, Cooper and Wang (2016) and Wang and Wang (2017). Wang and Wang (2017) include an explicit solution of the participation game when network effects are logarithmic.

pricing game and provide several characterization results. In Section 4, we provide comparative statics results with respect to platform entry, incumbent platforms' "quality" under free entry, and partial compatibility. In the main analysis we postulate full market coverage and that each platform sets a price for each group; in Section 5.1 we show how our analysis can be applied when platforms charge only one user group; and in Section 5.2, we show how our analysis extends to an environment with partial coverage in which some users in each group choose an outside option. Section 6 concludes. Proofs are relegated to the Appendix.

# 2 The platform oligopoly model

Consider $M > 1$ platforms competing for users from two groups, $A$ and $B$. Each platform $i \in \{1, ..., M\}$ charges a membership or subscription fee $p_i^k \in \mathbb{R}$ to users from group $k \in \{A, B\}$. We consider the game in which, first, platforms simultaneously set participation fees $p_i^A, p_i^B$ and then a unit mass of users from both groups simultaneously decide which platform to join. We solve for subgame perfect Nash equilibria (applying the selection criterion detailed below). In the following, we describe the platforms' problem and the user demand model.

## 2.1 Platforms

Each platform $i$ incurs a constant marginal cost $c_i^k \geq 0$ for serving group-$k$ users. We denote platform $i$'s number of group-$k$ users by $n_i^k$ and the vector of prices for group $k$ by $p^k = (p_1^k, \ldots, p_M^k)$. Then, we can write platform $i$'s profit as $\pi_i(p^A, p^B) = (p_i^A - c_i^A)n_i^A(p^A, p^B) + (p_i^B - c_i^B)n_i^B(p^A, p^B)$, where $n_i^A$ and $n_i^B$ depend on the fees set by *all* platforms for *both* groups.

## 2.2 Users

A unit mass of users from each group decide which platform to join. Each user's utility from joining a platform consists of a maximal value of the platform, network effects, and an idiosyncratic preference for the platform. Formally, the utility of a group-$k$ consumer from joining platform $i$ is given by

$$u_i^k = a_i^k - p_i^k + \alpha^k \log n_i^k + \beta^k \log n_i^l + \varepsilon_i^k. \tag{1}$$

The first term $a_i^k - p_i^k$ is the expected value of platform $i$ for group-$k$ users if all users from both groups joined this platform, where $a_i^k$ represents the "quality" of platform $i$ for group $k$. The second and third terms, $\alpha^k \log n_i^k$ and $\beta^k \log n_i^l$, capture within-group and cross-group network effects, where $\alpha^k \in [0, 1)$ and $\beta^k \in [0, 1)$ are the parameters that represent the importance of platform-specific within-group and cross-group network effects, and $n_i^k$ and $n_i^l$ are the number

| Notation | Meaning |
|---|---|
| $k, l$ | indices for the two user groups |
| $a_i^k$ | group-$k$ quality of platform $i$ |
| $c_i^k$ | marginal cost for group-$k$ participation on platform $i$ |
| $p_i^k$ | group-$k$ fee of platform $i$ |
| $n_i^k$ | group-$k$ network size of platform $i$ |
| $\alpha^k$ | parameter for within-group network effect of group $k$ |
| $\beta^k$ | parameter for cross-group network effect enjoyed by group $k$ |

Table 1: Notation

of group-$k$ and group-$l(\neq k)$ consumers who join platform $i$. We call $n_i^k$ group $k$'s *network size* of platform $i$. We note that the chosen logarithmic specification of network effects is broadly adopted in the empirical literature (e.g., Ohashi, 2003; Rysman, 2004, 2007; Zhu and Iansiti, 2012).[4]

The last term, $\varepsilon_i^k$, is an idiosyncratic taste shock from an i.i.d. type-I extreme value distribution. We assume that network effects are not too strong, that is, $\alpha^k + \beta^l < 1$ hold for any $k, l \in \{A, B\}$. Thus, $\max\{\alpha^A, \alpha^B\} + \max\{\beta^A, \beta^B\} < 1$. Table 1 summarizes the notation.

In e-commerce marketplaces, sellers and buyers constitute the two user groups and parameters $\beta^A$ and $\beta^B$ are positive, while, in the simplest version, $\alpha^A = \alpha^B = 0$. Here, there are mutual cross-group network effects since buyers are attracted to platforms with many sellers and sellers to platforms with many buyers. On an ad-funded social network, advertisers and members constitute the two user groups $A$ and $B$, respectively, and, in its simplest version, $\beta^A$ and $\alpha^B$ are positive and $\alpha^A = \beta^B = 0$. In words, members benefit from interacting with each other, but do not care about advertising, while advertisers seek network members' attention, but do not care about fellow advertisers. For a discussion, see Belleflamme and Peitz (2021).

For given network sizes $(\bar{n}_i^A, \bar{n}_i^B)$, group-$k$ consumer demand of platform $i$ can be written as

$$n_i^k = \Pr\left(u_i^k \geq u_j^k \text{ for all } j \neq i\right) = \frac{\exp(a_i^k - p_i^k)\left(\bar{n}_i^k\right)^{\alpha^k}\left(\bar{n}_i^l\right)^{\beta^k}}{\sum_{j=1}^M \exp(a_j^k - p_j^k)\left(\bar{n}_j^k\right)^{\alpha^k}\left(\bar{n}_j^l\right)^{\beta^k}}.$$

This is the multinomial demand structure with network sizes endogenously determining platform quality.

[4] Most of the existing theoretical literature postulates linear network effects (e.g., Armstrong, 2006). However, in many real-world applications a strictly concave function looks more plausible.

# 3 Equilibrium analysis

We first characterize the participation equilibrium at stage 2 for given platform fees. We then analyze subgame perfect Nash equilibria of the price-then-participation game.

## 3.1 Participation equilibrium

In a participation equilibrium, network sizes $n_i^k$ on the left-hand side are equal to $\bar{n}_i^k$ on the right-hand side of equation (2.2) for all $k \in \{A, B\}$ and $i \in \{1, ..., M\}$.

Due to complementarity in platform choices, there may be multiple participation equilibria, an issue pointed out by Anderson et al. (1992, chapter 7.8) and Tan and Zhou (2021), among others. In the present setting, equation (2.2) indicates that whenever consumers expect $\bar{n}_i^k = 0$, such an expectation will be self-fulfilling. Therefore, there are several equilibria in which some platforms are chosen with probability zero.

We will first characterize the equilibrium in which all platforms have strictly positive demand for both groups and then show that this is the unique equilibrium under a selection criterion that we will formulate below. We call such an equilibrium an *interior participation equilibrium.*

**Proposition 1.** *For any prices $(p_1^A, ..., p_M^A, p_1^B, ..., p_M^B)$, there exists a unique interior participation equilibrium. Equilibrium participation levels are given by*

$$n_i^k(p) = \frac{\exp[\Gamma^{kk}(a_i^k - p_i^k) + \Gamma^{kl}(a_i^l - p_i^l)]}{\sum_{j=1}^M \exp[\Gamma^{kk}(a_j^k - p_j^k) + \Gamma^{kl}(a_j^l - p_j^l)]} \tag{2}$$

*for all $i \in \{1, \dots, M\}$ and $k, l \in \{A, B\}$ with $l \neq k$, where $\Gamma^{kk}$ and $\Gamma^{kl}$ are given by*

$$\Gamma^{kk} = \frac{1 - \alpha^l}{(1 - \alpha^k)(1 - \alpha^l) - \beta^k \beta^l} \geq 1, \ \ \textit{and } \Gamma^{kl} = \frac{\beta^k}{(1 - \alpha^k)(1 - \alpha^l) - \beta^l \beta^k} \geq 0.$$

The demand system given by equation (2) is a logit demand system augmented by within-group and cross-group network effects. To see this, consider first the case that $\alpha^k = \beta^k = 0$. Then, equation (2.2) gives the standard logit choice probability

$$n_i^k = \frac{\exp(a_i^k - p_i^k)}{\sum_{j=1}^M \exp(a_j^k - p_j^k)}.$$

Second, consider the case of within-group network effects but no cross-group network effects ($\alpha^k > 0,\ \beta^k = 0$ for $k \in \{A, B\}$). Logit choice probabilities are then adjusted by those within-group network effects:

$$n_i^k = \frac{\exp\left(\frac{a_i^k - p_i^k}{1-\alpha^k}\right)}{\sum_{j=1}^M \exp\left(\frac{a_j^k - p_j^k}{1-\alpha^k}\right)}.$$

Third, consider the case of cross-group network effects but no within-group network effects ($\alpha^k = 0$, $\beta^k > 0$ for $k \in \{A, B\}$). Logit choice probabilities are then:

$$n_i^k = \frac{\exp\left(\frac{a_i^k - p_i^k + \beta^k(a_i^l - p_i^l)}{1-\beta^k\beta^l}\right)}{\sum_{j=1}^M \exp\left(\frac{a_j^k - p_j^k + \beta^k(a_j^l - p_j^l)}{1-\beta^k\beta^l}\right)}.$$

Finally, consider the case that $\alpha^k$ and $\beta^k$ are positive. In an interior participation equilibrium, each platform's maximal average value in group $k$, $a_i^k - p_i^k$, is amplified by within-group and cross-group network effects represented by $\Gamma^{kk}$ and $\Gamma^{kl}$, respectively. These amplifiers translate the base values of platform $i$ in the two groups into the externality-adjusted group-$k$ values of platform $i$, which is given by $\Gamma^{kk}(a_i^k - p_i^k) + \Gamma^{kl}(a_i^l - p_i^l)$. In the interior consumption equilibrium, it turns out that consumers make a choice based on this externality-adjusted value rather than the original values, leading to expression (2).

Hence, we obtain a tractable closed-form expression of user participation with network effects because network effects are logarithmic in network size and demand takes the logit form.[5]

We impose *asymptotic stability* of best-response dynamics as our selection criterion and show that the only equilibrium that meets the selection criterion is the interior participation equilibrium. The notion of best-response dynamics corresponds to that used in the literature of population games (Sandholm, 2010, Chapter 6.2), and the notion of asymptotic stability is used to capture the stability of dynamic systems (Luenberger, 1979, Chapter 5.9).

**Definition 1.** *Define the best-response dynamics and asymptotic stability of network sizes as follows:*

1. *A best-response dynamics $\{n_t\}_{t=0}^\infty$ from the initial network sizes $n_0 = \left(n_{i,0}^A, n_{i,0}^B\right)_{i\in\{1,\dots,M\}}$ is defined by a sequence of network sizes $n_t = \left(n_{i,t}^A, n_{i,t}^B\right)_{i\in\{1,\dots,M\}}$ such that $n_{i,t}^k = T_i^k(n_{t-1})$ according to the best-response functions $T_i^k$ for all $t \in \{1, 2, \dots\}$, $i \in \{1, \dots, M\}$ and $k \in \{A, B\}$.*

2. *A network size vector $n = \left(n_i^A, n_i^B\right)_{i\in\{1,\dots,M\}}$ is the limit of the best-response dynamics $\{n_t\}_{t=0}^\infty$ from the initial network size $n_0$ if $n = \lim_{t\to\infty} n_t$.*

[5] Linear demand models with linear network effects also give rise to closed-form demand functions (e.g., Armstrong, 2006). In a linear demand model with linear network effects, the choice probability can be written as a linear function of expected network sizes, which makes it possible to use linear algebra to obtain the closed-form solution for network sizes.

3. *A participation equilibrium with the equilibrium network sizes $n$ is asymptotically stable if for any strictly positive $n_0$, $n$ is the limit of the best-response dynamics from the initial network sizes $n_0$.*

Definition 1 requires that the equilibrium network sizes are the result of best-response dynamics starting from any interior starting point.[6] We call a participation equilibrium with asymptotically stable network sizes an *asymptotically stable participation equilibrium.*

In our model, the best response functions are given by (compare with equation (2.2))

$$T_i^k(n_{t-1}) = \frac{\exp(a_i^k - p_i^k)\left(n_{i,t-1}^k\right)^{\alpha^k}\left(n_{i,t-1}^l\right)^{\beta^k}}{\sum_{j=1}^M \exp(a_j^k - p_j^k)\left(n_{j,t-1}^k\right)^{\alpha^k}\left(n_{j,t-1}^l\right)^{\beta^k}}.$$

The following proposition establishes that the interior consumption equilibrium is the only equilibrium that is asymptotically stable.

**Proposition 2.** *The interior participation equilibrium characterized by equations* (2) *is the unique asymptotically stable participation equilibrium.*

## 3.2 Aggregates, profit functions, and consumer surplus

We will write platform profits as functions of own actions and corresponding aggregates and user surplus of the two groups as functions of these aggregates. To do so, we define a platform's own actions as

$$\begin{aligned} h_i^A &:= \exp\left[\Gamma^{AA}(a_i^A - p_i^A) + \Gamma^{AB}(a_i^B - p_i^B)\right], \\ h_i^B &:= \exp\left[\Gamma^{BB}(a_i^B - p_i^B) + \Gamma^{BA}(a_i^A - p_i^A)\right]. \end{aligned} \tag{3}$$

and the corresponding aggregates $H^A := \sum_{j=1}^M h_i^A$ and $H^B := \sum_{j=1}^M h_i^B$. Thus, group-$k$ demand on platform $i$ is $n_i^k = h_i^k / H^k$.

There is a one-to-one mapping between $(p_i^A, p_i^B)$ and $(h_i^A, h_i^B)$. As we show in the following lemma, any $(h_i^A, h_i^B)$ induce prices $(p_i^A(h_i^A, h_i^B), p_i^B(h_i^A, h_i^B))$.

**Lemma 1.** *Platform fees can be written as functions of* $(h_i^A, h_i^B)$*:*

$$p_i^A(h_i^A, h_i^B) = a_i^A - (1 - \alpha^A)\log h_i^A + \beta^A \log h_i^B, \tag{4}$$

$$p_i^B(h_i^A, h_i^B) = a_i^B - (1 - \alpha^B)\log h_i^B + \beta^B \log h_i^A. \tag{5}$$

[6]Other selection criteria used in the literature on network effects in industrial organization include: Pareto dominance (Katz and Shapiro, 1986; Fudenberg and Tirole, 2000), coalitional rationalizability or coalition proofness (Ambrus and Argenziano, 2009; Karle, Peitz and Reisinger, 2020), focality advantage (Caillaud and Jullien, 2003; Halaburda, Jullien and Yehezkel, 2020), and potential maximization (Chan, 2021).

Recall that platform $i$'s profit as a function of platform fees is $(p_i^A - c_i^A)n_i^A + (p_i^B - c_i^B)n_i^B$. Since $n_i^k = h_i^k/H^k$ and there is a one-to-one mapping between $(p_i^A, p_i^B)$ and $(h_i^A, h_i^B)$, the profit of platform $i$ can be written as the function of the two action variables $h_i^A$ and $h_i^B$ and their aggregates $H^A$ and $H^B$:

$$\begin{aligned}\Pi_i(h_i^A, h_i^B, H^A, H^B) &= \Pi_i^A(h_i^A, h_i^B, H^A) + \Pi_i^B(h_i^A, h_i^B, H^B) \\ &= \frac{h_i^A}{H^A}[p_i^A(h_i^A, h_i^B) - c_i^A] + \frac{h_i^B}{H^B}[p_i^B(h_i^A, h_i^B) - c_i^B],\end{aligned}$$

where we defined $\Pi_i^k = \frac{h_i^k}{H^k}[p_i^k(h_i^k, h_i^l) - c_i^k]$, $k, l \in \{A, B\}$, $l \neq k$.

User surplus $CS^k$ of group $k$ is given by the expected indirect utility of consumers, and the aggregate consumer surplus $CS$ is given by the sum of the consumer surplus in both groups:

$$\begin{aligned}CS^k &:= \log\left[\sum_{i=1}^{M} \exp(a_i^k - p_i^k)(n_i^k)^{\alpha^k}(n_i^l)^{\beta^k}\right] \\ &= (1-\alpha^k)\log H^k - \beta^k \log H^l, \\ CS &:= CS^A + CS^B \\ &= (1-\alpha^A-\beta^B)\log H^A + (1-\alpha^B-\beta^A)\log H^B.\end{aligned}$$

Note that $CS^k$ is increasing in $H^k$ but decreasing in $H^l$. Because the group-$k$ aggregate $H^k$ measures the intensity of competition in group $k$, it is natural for $H^k$ to have a positive impact on $CS^k$. On the contrary, a large value of group-$l$ aggregate $H^l$ reduces the network sizes of platforms on side $l$. This works in a way that reduces group-$k$ users' benefit from cross-group network effects, thereby negatively affecting $CS^k$. Nonetheless overall user surplus $CS = CS^A + CS^B$ increases in each of the two aggregates $H^A$ and $H^B$.

## 3.3 Price equilibrium

Using the demand system obtained from the participation equilibrium, we analyze price competition between platforms using the continuation profits in the unique participation equilibrium at stage 2.

We establish the following lemma that guarantees that we can restrict attention to the first-order conditions of profit maximization when analyzing platform pricing.

**Lemma 2.** *For any given $H_{-i}^A = \sum_{j\neq i} h_j^A$ and $H_{-i}^B = \sum_{j\neq i} h_j^B$, there is a unique solution to the first-order conditions of profit maximization of $\Pi_i(h_i^A, h_i^B, h_i^A + H_{-i}^A, h_i^B + H_{-i}^B)$ with respect to $h_i^A$, $h_i^B$, and this solution is a global maximizer of platform $i$'s pricing problem.*

The derivative of $\Pi_i$ with respect to $h_i^A$ is

$$\begin{aligned}\frac{\partial \Pi_i}{\partial h_i^A} &= \left(\frac{1}{H^A} - \frac{\partial H^A}{\partial h_i^A}\frac{h_i^A}{(H^A)^2}\right)[p_i^A(h_i^A, h_i^B) - c_i^A] + \frac{h_i^A}{H_i^A}\frac{\partial p_i^A}{\partial h_i^A} + \frac{h_i^B}{H^B}\frac{\partial p_i^B}{\partial h_i^A} \\ &= \frac{1}{h_i^A}\left[\frac{h_i^A}{H^A}\left(1 - \frac{h_i^A}{H^A}\right)[p_i^A(h_i^A, h_i^B) - c_i^A] - (1-\alpha^A)\frac{h_i^A}{H^A} + \beta^B\frac{h_i^B}{H^B}\right].\end{aligned}$$

Therefore, from $\partial \Pi_i/\partial h_i^A = 0$, we have the characterization of the markup level:

$$p_i^A(h_i^A, h_i^B) - c_i^A = \frac{1}{1 - \frac{h_i^A}{H^A}}\left(1 - \alpha^A - \beta^B\frac{h_i^B}{H^B}\frac{H^A}{h_i^A}\right) = \frac{1}{1 - n_i^A}\left(1 - \alpha^A - \beta^B\frac{n_i^B}{n_i^A}\right).$$

In the standard multinomial logit model without network effects ($\alpha^k = \beta^k = 0$, for all $k \in \{A, B\}$), the markup is equal to $1/(1 - n_i^k)$. In the presence of within-group network effects $\alpha^k > 0$, the markup is reduced by $\alpha^k$. The lower markup is due to the larger price elasticity of demand arising from within-group network effects. In the presence of cross-group network effect $\beta^l > 0$, the markup for group $k$ is reduced by the amount $\beta^l n_i^l/n_i^k$. Here, the lower markup is due to the cross-subsidization incentive of the platform: it expands participation of group $k$ to attract users in group $l$; this is in line with the markup formulas reported in Armstrong (2006) and Tan and Zhou (2021).

Before considering the general case that includes cross-group network effects, it is insightful to consider the special case of only within-group network effects (i.e., $\beta^A = \beta^B = 0$). Users in one group do not care about user participation in the other group and it is sufficient to consider group $A$. The pricing equation for platform $i$ becomes $p_i^A - c_i^A = a_i^A - c_i^A - (1-\alpha^A)\log h_i^A$. Thus, the first-order condition of profit maximization for group $A$ can be written as

$$(1-\alpha^A)\frac{H^A}{H^A - h_i^A} = (a_i^A - c_i^A) - (1-\alpha^A)\log h_i^A. \tag{6}$$

Note that the right-hand side is decreasing in $h_i^A$, while the left-hand side is increasing in $h_i^A$. Thus, for any $H^A$ there is a unique $h_i^A(H^A)$. Note also that the right-hand side does not depend on $H^A$, while the left-hand side is shifted downward after an increase in $H^A$. Hence, $h_i^A(\cdot)$ is increasing in $H^A$.

**Proposition 3.** *There exists a unique price equilibrium when $\beta^A = \beta^B = 0$.*

In the general case, the system of equations

$$\begin{aligned} a_i^A - c_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B &= \frac{1}{1-\frac{h_i^A}{H^A}}\left(1-\alpha^A-\beta^B\frac{h_i^B}{H^B}\frac{H^A}{h_i^A}\right) \\ a_i^B - c_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A &= \frac{1}{1-\frac{h_i^B}{H^B}}\left(1-\alpha^B-\beta^A\frac{h_i^A}{H^A}\frac{H^B}{h_i^B}\right) \end{aligned}$$

must be satisfied for all $i \in \{1, 2, \dots, M\}$. As shown in the following lemma, for each $i$, this defines implicit best replies $(h_i^A(H^A, H^B), h_i^B(H^A, H^B))$.

**Lemma 3.** *For any $(H^A, H^B)$, the system of first-order conditions defines implicit best replies $(h_i^A(H^A, H^B), h_i^B(H^A, H^B))$ for each platform $i \in \{1, \dots, M\}$.*

Summing over all $i$, an equilibrium satisfies

$$\sum_{j=1}^{M} h_j^A(H^A, H^B) = H^A, \tag{7}$$

$$\sum_{j=1}^{M} h_j^B(H^A, H^B) = H^B. \tag{8}$$

With the following proposition, we establish that there exists a price equilibrium and that, whenever multiple equilibria exist, these are ordered in terms of surplus of one of the two user groups: if one equilibrium features higher surplus for one group then the other equilibrium features a higher surplus for the other group.

**Proposition 4.** *There exists a price equilibrium given by $(H^{A*}, H^{B*})$. When there are multiple price equilibria we obtain the ranking for any pair of equilibrium given by $(H_1^{A*}, H_1^{B*})$ and $(H_2^{A*}, H_2^{B*})$ with associated user surpluses $(CS_1^{A*}, CS_1^{B*})$ and $(CS_2^{A*}, CS_2^{B*})$: $CS_1^{A*} > CS_2^{A*}$ holds if and only if $CS_1^{B*} < CS_2^{B*}$. Furthermore, the extremal equilibria (i.e., the equilibria that maximize $CS^A$ or $CS^B$) are stable.*

A price equilibrium is characterized by the pair of aggregates $(H^{A*}, H^{B*})$ that satisfy the system of equations (7) and (8). Furthermore, since the surplus of group-$k$ users, $CS^k = (1-\alpha^k)\log H^k - \beta^k \log H^l$, depends only on aggregates $(H^A, H^B)$, the characterization of equilibrium aggregates directly characterizes user surplus in equilibrium. These properties have been obtained in the aggregative-games frameworks of price competition in standard oligopoly (Anderson, Erkal and Piccinin, 2020) and platform competition with competitive bottlenecks (Anderson and Peitz, 2020). Our result covers two-sided single-homing working with a two-dimensional aggregate.

As we will see next, the relative position of a platform with respect to the size of its user groups is determined by "type" $\tau_i^k := \exp\{a_i^k - c_i^k\}$, where $a_i^k - c_i^k$ is the net surplus (or net quality) if the platform caters to all users in both groups at marginal costs. Thus, $\tau_i^k$ is the platform's ability to provide value to group-$k$ users. Proposition 4 allows us to conduct an equilibrium analysis of platform oligopoly with arbitrarily heterogeneous platforms. Indeed, for any market structure, we can find a profile of net qualities that decentralizes any chosen outcome as an equilibrium outcome, as we formally establish in the following proposition.

**Proposition 5.** *Pick any profile of network sizes $(n_i^A, n_i^B)_{i\in\{1,\ldots,M\}}$ such that $\sum_{j\in\{1,\ldots,M\}} n_j^k = 1$ for $k \in \{A, B\}$.*

1. *For any given aggregates $(H^A, H^B) \in \mathbb{R}^2_{++}$, there exists a unique type profile $(\tau_i^A, \tau_i^B)_{i\in\{1,\ldots,M\}}$ such that the equilibrium network sizes and aggregates in the pricing equilibrium are $(n_i^A, n_i^B)_{i\in\{1,\ldots,M\}}$ and $(H^A, H^B)$, respectively; and*
2. *for any aggregate type $(\bar\tau^A, \bar\tau^B) \in \mathbb{R}^2_{++}$, there exists a unique type profile $(\tau_i^A, \tau_i^B)_{i\in\{1,\ldots,M\}}$ such that $\sum_{i\in\{1,\ldots,M\}} \tau_i^k = \bar\tau^k$ for $k \in \{A, B\}$, which, in equilibrium, leads to network sizes $(n_i^A, n_i^B)_{i\in\{1,\ldots,M\}}$ with appropriate aggregates $(H^A, H^B) \in \mathbb{R}^2_{++}$.*

How do market shares and price-cost margins differ across different platforms when they are asymmetric with respect to what they offer to users in *one* group? We now use notation $v_i^k := a_i^k - c_i^k = \log \tau_i^k$ as the cost-adjusted quality that platform $i$ offers to group-$k$ users. As the following result shows, the platform with higher cost-adjusted quality for one user group, has weakly larger market shares for this user group and, if at least one of the cross-group network effects is positive ($\beta^A > 0$ or $\beta^B > 0$) also a larger market share for the other group.

**Proposition 6.** *Take any two platforms $i$ and $j$ with $v_i^A > v_j^A$ and $v_i^B = v_j^B$. Then, in equilibrium, $n_i^A > n_j^A$ and $n_i^B \geq n_j^B$. Furthermore, $n_i^B > n_j^B$ if and only if $\beta^A > 0$ or $\beta^B > 0$.*

We next look at the pricing implications for users in one user group (group $B$ in the proposition below) if platforms are asymmetric with respect to the other group. To do so, we consider two polar cases: (i) only the other group benefits from cross-group network effects and (ii) the reverse; that is, the group for which platforms are symmetric with respect to the cost-adjusted quality they offer to that group benefits from cross-group network effects.

**Proposition 7.** *Take any two platforms $i$ and $j$ with $v_i^A > v_j^A$ and $v_i^B = v_j^B$. (i) Suppose that $\beta^A > 0$ and $\beta^B = 0$. Then, the price-cost margin for group-$B$ users is smaller on platform $i$ than on $j$. (ii) Suppose that $\beta^A = 0$ and $\beta^B > 0$. Then, the price-cost margin for group-$B$ users is larger on platform $i$ than on $j$.*

Thus, it depends on the direction of cross-group network effects whether the user group that considers two platforms to be symmetric in their cost-adjusted quality (say group $B$) faces a higher or lower price-cost margin on the platform with higher cost-adjusted quality for the other user group (say group $A$). If only group $A$ benefits from cross-group network effects ($\beta^A > 0, \beta^B = 0$), the platform with the higher cost-adjusted quality for group $A$ has a lower price-cost margin for group $B$ than the competing platform. This lower price-cost margin for group-$B$ users fosters the participation of those users. Since $\beta^A > 0$, this gives an extra push to group-$A$ users to join this platform. The platform with the higher (ex ante) cost-adjusted quality benefits more from this. This implies that the asymmetry between platforms for group $A$ is amplified by $\beta^A(\log n_i^B - \log n_j^B)$.

In the opposite case, in which only group $B$ benefits from cross-group network effects ($\beta^A = 0, \beta^B > 0$), the platform with the higher cost-adjusted quality for group $A$ will have more group-$A$ participation, which translates into an endogenous quality advantage for group $B$, $\beta^B(\log n_i^A - \log n_j^A)$. In equilibrium, this results a higher price-cost margin to group-$B$ users than the one charged by the competing platform.

Our next result takes a look at the user group that experiences different cost-adjusted qualities across platforms. One might expect that the platform that offers the higher cost-adjusted quality always has a higher price-cost margin for the same group. While this is correct under a number of conditions (parameter conditions or outcome variables), we show by example that this is not always the case.

**Proposition 8.** *Take any two platforms $i$ and $j$ with $v_i^B = v_j^B$. Then, the price-cost margin for group-$A$ users is larger on platform $i$ with the higher cost-adjusted quality $v_i^A > v_j^A$ if cross-group network effects are not mutual, that is, (1) $\beta^A = 0$ or (2) $\beta^B = 0$, (3) platforms $i$ and $j$ attract weakly more users from group $A$ than $B$, or (4) platforms set fees above costs for both user groups. However, there are environments in which the platform with the lower quality has a higher price-cost margin for group-$A$ users; this can only happen if $\beta^B, \beta^A > 0$ and, in equilibrium, $n_i^B > n_i^A$.*

We conclude that net quality differences between platforms for one user group (when net quality for the other user group is the same across platforms) gives rise to non-trivial differences in user participation across platform in the presence of cross-group network effects.

# 4 Entry and partial compatibility

In this section, we investigate comparative statics properties of three shocks or interventions: platform entry, changes to the incumbent platforms' characteristics under free entry, and partial compatibility. We analyze two scenarios with platform entry: first, we consider the effect of exogenous entry of a new platform and, second, we consider endogenous entry of "fringe" platforms and analyze what happens if the "quality" of some of the inframarginal platforms changes. Then, taking the number of platforms as given, we evaluate the effects of changing the degree of compatibility by varying the parameters that measure the strength of platform-specific network effects.

## 4.1 Exogenous entry

Under exogenous entry we investigate what happens when an additional platform enters the industry. Entry implies that the number of platforms increases from $M \geq 2$ to $M+1$ platforms.

It is instructive to first consider the case in which all the existing and new platforms are symmetric. In the symmetric setting with $M$ platforms, we must have $n_i^A = n_i^B = 1/M$ for all $i \in \{1, \ldots, M\}$ in any equilibrium. There can be no asymmetric equilibria. Profit maximization requires that $\partial \Pi_i / \partial h_i^k = 0$ for all $i \in \{1, \ldots, M\}$ and $k \in \{A, B\}$. Using symmetry $n_i^A = n_i^B = 1/M$ first-order conditions for group $A$ become

$$\left(1 - \frac{1}{M}\right) \left[ v^A - (1 - \alpha^A) \log \left( \frac{1}{M} \right) + \beta^A \log \left( \frac{1}{M} \right) - CS^A \right] - 1 + \alpha^A + \beta^B = 0.$$

This leads to the equilibrium consumer surplus

$$CS^{A*} = v^A + (1 - \alpha^A - \beta^A) \log M - (1 - \alpha^A - \beta^B) \frac{M}{M-1}.$$

By taking a derivative of $CS^{A*}$ with respect to $M$, we obtain

$$\frac{\partial CS^{A*}}{\partial M} = \frac{1 - \alpha^A - \beta^A}{M} + \frac{1 - \alpha^A - \beta^B}{(M-1)^2} > 0,$$

implying that platform entry always benefits users in symmetric environments. Correspondingly, this also holds for group $B$. The result that entry is consumer welfare increasing in symmetric platform oligopoly obtains under our functional-form assumptions on the distribution of taste shocks and the network effects. With more general functional forms, Tan and Zhou (2021) show that entry may hurt users through fragmenting the network benefits.

We turn to the case in which platforms are asymmetric. We establish below that in that case, one user group may be worse off after a new platform enters (while the other group is better off).

**Proposition 9.** *Consider the effect of entry of a new platform $E$ on user surplus.*

1. *For any given entry of a platform with* $(a_E^A, c_E^A, a_E^B, c_E^B)$, *there exists a value* $\underline{\beta}$ *such that entry increases user surplus for both groups if* $\beta^A < \underline{\beta}$ *and* $\beta^B < \underline{\beta}$.

2. *For any given* $\beta^A > 0$, *there exists a type of platform with* $(a_E^A, c_E^A, a_E^B, c_E^B)$ *such that the minimal user surplus of group* $A$ *or group* $B$ *decreases after entry.*

3. *Entry increases the minimal or maximal user surplus of at least one user group.*

Proposition 9-1 shows that in the absence of cross-group network effects ($\beta^A = \beta^B = 0$), $H^k$ increases with entry and, thus, consumer surplus must go up. While this property is satisfied in standard oligopoly models without network effects, it is a priori not obvious that this result carries over to a model with network effects. The reason is that, under full participation, the entering platforms attract consumers from the incumbent platforms reducing the network benefits of the consumers active on incumbent platforms due to reduced participation on those platforms. Nonetheless, in our setting, entry of a new platform always benefits users if cross-group network effects are sufficiently weak. Proposition 9-1 establishes this result.

In the presence of cross-group network effects, entry of a platform may hurt one of the user groups, as established in Proposition 9-2. The proof of Proposition 9-2 indicates that a typical example of entry that lowers the user surplus for one group (group $A$) is the entry of a platform that primarily caters to the needs of the other user group ($a_E^B - c_E^B$ large, and $a_E^A - c_E^A$ small and possibly negative); one may call such a platform "highly specialized". In such a case, entry will not add surplus to group $A$ users, but reduces the market shares of the incumbent platforms. This reduces the network benefits that group-$A$ users enjoy from joining existing platforms or the incumbent platforms' incentive to attract group-$A$ users. Then, such entry lowers group-$A$ user surplus.

Although entry may harm one user group, Proposition 9-3 establishes that at least one user group benefits from entry. Our results indicate that the welfare effects of entry of a two-sided platform crucially depends on the characteristics of the entrant. Entry of a highly specialized platform may hurt the users in the group that the entrant is not specialized in.

As a remark on the previous literature, the symmetric, but otherwise more flexible model by Tan and Zhou (2021) has the property that platform entry can lead to a lower consumer surplus of both groups. Other works address welfare effects of platform entry in different market environments. Correia-da-Silva, Jullien, Lefouili and Pinho (2019) consider homogeneous-product Cournot platform models and examine the welfare effects of exogenous entry. They find that platform entry may reduce consumer surplus of all groups due to the negative effect of entry on network benefits; Gama, Lahmandi-Ayed and Pereira (2020) find such a result when the platform caters to a single user group and this group experiences network effects.[7]

[7] Anderson and Peitz (2020) consider an asymmetric platform oligopoly in which one user group multi-homes

## 4.2 Shocks to incumbent platforms under free entry

To study long-run competition, we consider platform competition under free entry of "fringe" platforms. To this end, we extend the baseline framework by incorporating symmetric entrants as in Anderson, Erkal and Piccinin (2013).

Suppose that, along with $M_I \geq 1$ incumbents $\{1, \ldots, M_I\}$, $\bar{M}_E \geq 1$ (potential) entrants $\mathcal{E} := \{M_I+1, \ldots, M_I+\bar{M}_E\}$ choose whether to enter. Entrants $e \in \mathcal{E}$ all have the same characteristics $\left(a_E^A, a_E^B, c_E^A, c_E^B\right)$ and incurred entry cost $K > 0$. Incumbent platform $i \in \{1, \ldots, M_I\}$ has characteristics $(a_i^A, a_i^B, c_i^A, c_i^B)$ that may differ from those of other platforms. We assume that $v_i^k \geq v_E^k$ for all $i = 1, \ldots, M_I$. The number of potential entrants $\bar{M}_E$ is sufficiently large so that the number of actual entrants $M_E$ is endogenously determined by free entry. In our analysis we ignore integer constraints.

Let $\pi_E\left(H^A, H^B\right)$ be the post-entry profit of an entrant when the values of the aggregates are given by $H^A$ and $H^B$. Specifically, the post-entry profit with aggregates $(H^A, H^B)$, $\pi_E\left(H^A, H^B\right)$, is given by

$$\pi_E(H^A, H^B) := \Pi_E(h_E^A(H^A, H^B), h_E^B(H^A, H^B), H^A, H^B) \tag{9}$$

Using this notation, we define the free-entry equilibrium as follows.

**Definition 2.** *The number of active entrants $M_E$ constitutes a free-entry equilibrium if the triple $\left(H^A, H^B, M_E\right)$ satisfies the following conditions:*

$$\begin{aligned} \pi_E\left(H^A, H^B\right) - K &= 0, \qquad (10) \\ \sum_{i=1}^{M_I} n_i^A\left(H^A, H^B\right) + M_E n_E^A\left(H^A, H^B\right) &= 1, \\ \sum_{i=1}^{M_I} n_i^B\left(H^B, H^A\right) + M_E n_E^B\left(H^B, H^A\right) &= 1. \end{aligned}$$

The definition of free-entry equilibrium endogenizes the number of active entrants $M_E$ through the zero-profit condition (10). Entrants sequentially enter as long as the post-entry profit exceeds the entry cost, and the entry stops once additional entry becomes unprofitable. Using Definition 2, we examine the welfare effects of a shock to the incumbent platforms' characteristics, which is captured by a change in $(a_i^A, a_i^B, c_i^A, c_i^B)$ for $i \in \{1, \ldots, M_I\}$.

In the aggregative game analysis of standard oligopoly, the zero-profit condition of entrants uniquely pins down the value of the aggregate (e.g., Davidson and Mukherjee, 2007; Ino and

and the other single-homes (competitive bottleneck) and study the welfare effect of platform entry.

Matsumura, 2012; Anderson et al., 2013, 2020). Because consumer surplus is determined solely by the value of the aggregate, any change in the competitive environment, such as incumbents' investment and platform mergers does not affect consumer surplus, as long as there is at least some entry. By contrast, with two-sided platforms, the zero profit condition (10) only pins down the relation between the two aggregates $(H^A, H^B)$. Therefore, the competitive environments are no longer necessarily neutral to the consumer surplus in each user group and the aggregate consumer surplus. In a particular setting, under some conditions, we establish a *strong see-saw property*: any change in the competitive environment that increases consumer surplus of one group reduces consumer surplus of the other group.

For instance, suppose that an incumbent invests in group-$A$ benefit $a_i^A$ so that entrants' network size on group $A$ decreases. In standard oligopoly, competition for group-$A$ users becomes more intense due to the incumbent's investment. As an equilibrium response, fewer entrants will join, so the competition for group-$A$ users becomes weaker. In two-sided markets, a more subtle strategic interaction may exist due to network effects and implied changes in the two-sided pricing structure.

**Proposition 10.** *Consider a free-entry equilibrium with a non-empty set of entrants and positive cross-group network effects $\beta^A, \beta^B > 0$. Suppose that $1 - \bar{\alpha} \geq \bar{\beta}\zeta(M_I + M_E)$ holds in equilibrium, where $\bar{\alpha} := \max\{\alpha^A, \alpha^B\}$, $\bar{\beta} := \max\{\beta^A, \beta^B\}$, and $\zeta(M) := \frac{M^2+M-1}{M(M-1)}$, which is a decreasing function taking values in $(1, 2.5]$. Then, any change in competitive environments that increases the surplus of one user group decreases the surplus of the other user group. Formally, holding the parameters $(\alpha^A, \alpha^B, \beta^A, \beta^B, a_E^A, a_E^B, c_E^A, c_E^B, K)$ fixed, compare two long-run equilibria that differ in other parameters. Denoting the equilibrium surplus of the two user groups under the two settings by $(CS^{A*}, CS^{B*})$ and $(CS^{A**}, CS^{B**})$, we have*

$$\left(CS^{A*} - CS^{A**}\right)\left(CS^{B*} - CS^{B**}\right) < 0.$$

The strong see-saw property poses a challenge to competition authorities evaluating business practices of large incumbent platforms in an environment with fringe platforms. Because an incumbent platform's practice generically benefits users in one group at the expense of those in the other group, the competition authority must decide which group to protect (or which weights to give them in an overall consumer welfare ranking). In the context of e-commerce, some authorities focus on private consumers, which is in line with a narrow interpretation of the consumer welfare standard. For instance, Khan (2017) argues that such an approach fails to recognize other harms of incumbent platforms' practices, including the harm to third-party sellers, which can be included under a broader interpretation of the consumer welfare standard. Proposition 10 states that there is a conflict between what benefits users of one group and what benefits the other. This conflict is inevitable in two-sided platform competition with free entry of the type studied in this paper (under the weak condition stated in the proposition).

## 4.3 Partial compatibility and multi-homing

In this section, we consider the effect of the degree of compatibility on market outcomes and focus on settings in which there are only within-group network effects. Thus the two groups operate independently and we can focus our attention on group $A$. Partial compatibility implies that a fraction $\lambda$ of network effects are industry-wide. Partial compatibility is gained if some of the functionalities are available to all users, not only those on the same platform, but also those on competing platforms. An example of a regulatory intervention with that goal is Article 7 in the Digital Markets Act (DMA) in Europe. According to this regulation, a gatekeeper of a number-independent interpersonal communications service must "make the basic functionalities of its number-independent interpersonal communications services interoperable with the number-independent interpersonal communications services of another provider."[8]

Users from group $A$ have utility

$$\begin{aligned} u_i^A &= a_i^A - p_i^A + \lambda\alpha^A \log \sum_{j=1}^{M} n_j^A + (1-\lambda)\alpha^A \log n_i^A + \varepsilon_i^k \\ &= a_i^A - p_i^A + (1-\lambda)\alpha^A \log n_i^A + \varepsilon_i^k. \end{aligned}$$

By Proposition 3 there exists a unique price equilibrium for any value of $\lambda \in [0,1]$. How does the equilibrium depend on the degree of compatibility $\lambda$? Using the first-order condition given by equation (6) adjusted by $\lambda$, we obtain

$$\frac{H^A}{H^A - h_i^A} = \frac{a_i^A - c_i^A}{1-(1-\lambda)\alpha^A} - \log h_i^A, \tag{11}$$

which implicitly defines a solution $h_i(H^A;\lambda)$. We note that, as $\lambda$ increases, the right-hand side decreases. This implies that an increase in compatibility pushes the function $h_i(\cdot;\lambda)$ downward. Since this holds for all functions $h_i$, $i \in \{1,\ldots,M\}$, it must be that the equilibrium aggregate $H^A$ decreases in $\lambda$.

We know that if $a_i^A - c_i^A > a_j^A - c_j^A$ platform $i$ has a larger market share than platform $j$. How does the relative market size $n_i^A/n_j^A$ change as compatibility increases? From equation (11) we see that $h_i$ receives a stronger downward push than $h_j$ as compatibility increases. This

[8]The provision applies only to gatekeeper platforms and interoperability has to be offered upon the request of another provider. As a caveat, our model does not accommodate the situation that some but not all of the competing providers ask for interoperability.

tends to reduce the market size asymmetry between firms. Also the equilibrium value of the aggregate changes in compatibility: because of the downward shift, the equilibrium value of $H^A$ must decrease.

We now take a closer look at the model to answer the question of how a change in the degree of compatibility affects market shares. Denoting $\tilde{\alpha}^A := \alpha^A(1-\lambda)$, the first-order condition (11) can be rewritten as

$$\frac{H^A - h_i^A}{H^A}\left(a_i^A - c_i^A + (1-\tilde{\alpha}^A)\log h_i^A\right) - (1-\tilde{\alpha}^A) = 0$$

or, equivalently,

$$(1-n_i^A)\left(\frac{v_i^A}{1-\tilde{\alpha}^A} - \log n_i^A - \log H^A\right) - 1 = 0$$

This defines platform $i$'s market share as a function of the aggregate $\widetilde{n}_i^A(H^A)$, which has slope

$$\frac{d\widetilde{n}_i^A}{dH^A} = \frac{-\frac{1-n_i^A}{H^A}}{\frac{1}{1-n_i^A} + \frac{1-n_i^A}{n_i^A}} < 0.$$

The equilibrium condition for $H^A$ is $\sum_{i=1}^{M} \widetilde{n}_i^A(H^A) = 1$. We obtain the result that lower-quality platforms gain market share when the degree of compatibility is increased, while higher-quality platforms lose. In other words, industry concentration (e.g., measured by the HHI) goes down. This is formally stated in the following proposition.

**Proposition 11.** *Supppose that $\beta^A = \beta^B = 0$ and order platforms such that $v_j^A \leq v_{j+1}^A$ for all $j \in \{1, \ldots, M-1\}$. Then an increase in the degree of compatibility $\lambda$ affects market shares as follows: there exists a critical platform $\hat{j} \in \{1, \ldots, M-1\}$ such that for all $j > \hat{j}$ market share decreases ($dn_i^{A*}/d\lambda < 0$), and for all $j \leq \hat{j}$ market share (weakly) increases ($dn_i^{A*}/d\lambda \geq 0$, where the inequality must be strict for $j = 1$ and for all $j < \hat{j}$ with $v_j^A < v_{\hat{j}}^A$).*

Prices are determined according to Lemma 1 through $p_i^A = a_i^A - [1-(1-\lambda)\alpha^A]\log h_i^A$. Increased compatibility leads to a downward shift of $h_i^A$ and the equilibrium value of the aggregate decreases. More compatibility reduces the equilibrium value of $h_i^A$. However, a larger $\lambda$ leads to an increase of $[1-(1-\lambda)\alpha^A]$, which points in the opposite direction to $h_i^A$.

Consumer surplus is $[1-(1-\lambda)\alpha^A]\log H^A$. The term in square brackets increases in the degree of compatibility $\lambda$, which captures the direct effect of increased compatibility on consumer surplus. By contrast, as just shown, $H^A$ decreases. The decrease in $H^A$ captures the strategic effect that an increase in partial compatibility causes the platforms to compete less intensely for users.

Consider the special case of symmetric platforms, implying that $h_i^A/H^A = 1/M$. The first-order condition can then be rewritten as

$$\log H^A = \frac{a^A - c^A}{1 - (1-\lambda)\alpha^A} + \log M - \frac{M}{M-1}.$$

Thus, consumer surplus can be expressed as

$$a^A - c^A + [1 - (1-\lambda)\alpha^A] \left( \log M - \frac{M}{M-1} \right).$$

Under symmetry, welfare increases in the degree of compatibility if and only if $\log M > \frac{M}{M-1}$. This implies that welfare decreases with compatibility if and only if $M = 2$ or $M = 3$, while it increases for $M \geq 4$. With a sufficiently large number of platforms, the strategic effect is less pronounced, and thus the direct effect dominates.[9]

When platforms are asymmetric, compatibility mitigates the asymmetry of market outcomes, as observed in Proposition 11. This gives rise to an additional effect pushing down the price of large platforms. Naturally, this effect is strong when the asymmetry is large. To illustrate the role of asymmetry, consider a duopoly. As shown above, in this case, the strategic effect dominates under symmetry. In the following proposition, we establish that even under duopoly an increase in the degree of compatibility lowers the price set by a larger platform and increases user surplus if the asymmetry between platforms is sufficiently large.

**Proposition 12.** *Suppose that $\beta^A = \beta^B = 0$, $M = 2$, and $v_1^A \leq v_2^A$. (1) There exists a critical value $\Delta_p v^A \in (0, \infty)$ such that the equilibrium price set by platform 1, $p_1^{A*}$ decreases with $\lambda$ if and only if $v_1^A - v_2^A \geq \Delta_p v^A$. (2) There exists a critical value $\Delta_{CS} v^A \in (0, \infty)$ such that the equilibrium user surplus $CS^{A*}$ increases with $\lambda$ if and only if $v_1^A - v_2^A \geq \Delta_{CS} v^A$.*

Our analysis has focused on the case with zero cross-group network effects. We take a quick look at cross-group network effects when platforms are symmetric. When cross-group network effects are positive, a group-$k$ consumer's utility from joining platform $i$ is given by $a_i^k - p_i^k + (1-\lambda)\alpha^k \log n_i^k + (1-\lambda)\beta^k \log n_i^l + \varepsilon_i^k$. Let $\tilde{\alpha}^k := (1-\lambda)\alpha^k$ and $\tilde{\beta}^k := (1-\lambda)\beta^k$. Then, from the first-order condition and the fact that $h_i^k / H^k = 1/M$, the symmetric equilibrium price-cost margin for group-$k$ consumers is given by $p^k - c^k = (1 - \tilde{\alpha}^k - \tilde{\beta}^l)\frac{M}{M-1}$, which is increasing in $\lambda$, as increased compatibility relaxes price competition between platforms.

The symmetric model allows us to obtain insights into which side benefits from increased

[9] Starting with Katz and Shapiro (1985), earlier literature has looked at the welfare effect of (no versus full) compatibility under Cournot competition; Amir, Evstigneev and Gama (2021) provides conditions under which full compatibility leads to a larger consumer surplus than no compatibility. For an extension to two-sided platforms, see Shekhar, Petropoulos, Van Alstyne and Parker (2022). Grilo, Shy and Thisse (2001) provide an early analysis of price competition with direct network effects and product differentiation but with a different focus.

compatibility. With partial compatibility, the expression for consumer surplus can be written as

$$CS^{k*} = v^k + (1 - \tilde{\alpha}^k - \tilde{\beta}^k)\log M - (1 - \tilde{\alpha}^k - \tilde{\beta}^l)\frac{M}{M-1},$$

for $k, l \in \{A, B\}$, $l \neq k$. The derivative with respect to the degree of partial compatibility is

$$\frac{\partial CS^{k*}}{\partial \lambda} = (\alpha^k + \beta^k)\log M - (\alpha^k + \beta^l)\frac{M}{M-1}.$$

Given a large number of platforms, $M$, partial compatibility tends to be beneficial for consumers in either group because an increase in compatibility has a strong direct effect on consumers by expanding interaction possibilities within and across groups for the service features that become compatible. The associated consumer benefit then dominates the loss from reduced price competition. Considering group-$k$ consumer surplus, we observe that increased compatibility tends to be beneficial if $\beta^k$ is large relative to $\beta^l$, $l \neq k$. The group that experiences rather small benefits from cross-group network effects tends to be harmed by increased compatibility.

To address the effect of compatibility on industry concentration under cross-group network effects, we restrict attention to the duopoly case. In line with Proposition 11, we establish that compatibility mitigates industry concentration even in the presence of cross-group network effects.

**Proposition 13.** *Suppose that $M = 2$ and that $v_1^k := a_1^k - c_1^k > a_2^k - c_2^k =: v_2^k$ for $k \in \{A, B\}$, that is, platform 1 is more efficient than platform 2. Then, the equilibrium market share of platform 1, $n_1^{k*}$ decreases with the degree of compatibility $\lambda$.*

Our results of partial compatibility have a different interpretation if an exogenous fraction of users multi-home because they have installed a multi-homing device (for example, users may use a meta search engine that allows them to access all platforms). Our model then tells us what happens if there is an exogenous change of the fraction of multi-homers. If there are only within-group network effects (e.g. because consumers leave valuable feedback), such an increase of the fraction of multi-homers corresponds to an increase in $\lambda$; our consumer surplus results then refer to the expected consumer surplus effect of a single-homing consumer.

# 5 Extensions

We sketch two extensions of our setting; one in which platforms monetize on one side only and another in which some but not all users participate.

## 5.1 Fee setting for one user group only

Participation may be free for one user group. For example, shopping malls and flea markets typically charge retailers but often not end users. This may be because platforms would charge negative fees (or fees below costs) and such fees are not feasible. Alternatively, platforms would like to charge end user fees but such positive fees would go hand-in-hand with high transaction costs or are simply not possible (as in traditional free-to-air radio or television broadcasting). Suppose that group $B$ is the zero-fee group. Using the equations from Lemma 1, we then must have

$$\begin{aligned} p_i^A(h_i^A, h_i^B) &= a_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B, \\ 0 &= a_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A. \end{aligned}$$

We rewrite the second equation as $\log h_i^B = a_i^B/(1-\alpha^B) + (\beta^B/(1-\alpha^B))\log h_i^A$ and substitute into the first equation to obtain (with an abuse of notation, we write $p_i^A$ as a function of $h_i^A$)

$$\begin{aligned} p_i^A(h_i^A) &= a_i^A - (1-\alpha^A)\log h_i^A + \beta^A \left[\frac{a_i^B}{1-\alpha^B} + \frac{\beta^B}{1-\alpha^B}\log h_i^A\right] \\ &= a_i^A + \frac{\beta^A}{1-\alpha^B} a_i^B - \left[(1-\alpha^A) - \frac{\beta^A\beta^B}{1-\alpha^B}\right]\log h_i^A \\ &= \tilde{a}_i^A - (1-\tilde{\alpha}^A)\log h_i^A \end{aligned}$$

where $\tilde{a}_i^A := a_i^A + \frac{\beta^A}{1-\alpha^B}a_i^B$ and $\tilde{\alpha}^A := \alpha^A + \frac{\beta^A\beta^B}{1-\alpha^B}$. Platform $i$'s profit as a function of $h_i^A$ and its aggregate is

$$\Pi_i(h_i^A, H^A) = \frac{h_i^A}{H^A}[p_i^A(h_i^A) - c_i^A] = \frac{h_i^A}{H^A}[\tilde{a}_i^A - (1-\tilde{\alpha}^A)\log h_i^A].$$

Then the analysis for platforms with only direct network effects for group $A$ in the text of Section 3.3 applies after a change of variables from $(a_i^A, \alpha^A)$ to $(\tilde{a}_i^A, \tilde{\alpha}^A)$. The first-order condition of profit maximization can thus be written as

$$(1-\tilde{\alpha}^A)\frac{H^A}{H^A - h_i^A} = (\tilde{a}_i^A - c_i^A) - (1-\tilde{\alpha}^A)\log h_i^A$$

A sufficient condition for a unique solution of this equation is that $\tilde{\alpha}^A < 1$, which is equivalent to

$$\beta^A\beta^B < (1-\alpha^A)(1-\alpha^B).$$

This is implied by our assumption $\alpha^k + \beta^l < 1$, $k, l \in \{A, B\}, l \neq k$. Thus, with the change of variables, Proposition 3 applies and a unique price equilibrium exists.

## 5.2 Partially covered markets

The main analysis also assumes that there is no outside option. This assumption can be relaxed in the following manner. Suppose that each group-$k$ consumer has an outside option with value $\log y_0^k + \varepsilon_0^k$, where $\varepsilon_0^k$ is drawn from an i.i.d. type-I extreme-value distribution, and chooses whether to join one of the networks or to pick the outside option.

Let $\phi^A(H^A, H^B)$ and $\phi^B(H^A, H^B)$ be the solution to the system of equations

$$\begin{aligned}\log \phi^A - \alpha_A \log(y_0^A \phi^A + H^A) - \beta_A \log(y_0^B \phi^B + H^B) &= 0,\\ \log \phi^B - \alpha_B \log(y_0^B \phi^B + H^B) - \beta_B \log(y_0^A \phi^A + H^A) &= 0.\end{aligned}$$

Then, the demand for platforms in a partially covered market can be characterized in the following way.

**Proposition 14.** *In a unique interior participation equilibrium, the demand for platform $i$ is given by the function*

$$n_i^A(h_i^A, H^A, H^B) = \frac{h_i^A}{y_i^A \phi^A(H^A, H^B) + H^A}, \tag{12}$$

$$n_i^B(h_i^B, H^A, H^B) = \frac{h_i^B}{y_i^B \phi^B(H^A, H^B) + H^B}. \tag{13}$$

Note that this extension nests as special cases (i) standard logit demand with an outside option but without network effects and (ii) our base specification without an outside option but with network effects. When $\alpha^A = \alpha^B = \beta^A = \beta^B = 0$, we must have $\phi^A = \phi^B = 1$. Then, the demand is given by

$$n_i^A = \frac{\exp(a_i^A - p_i^A)}{y_i^A + \sum_{j=1}^M \exp(a_j^A - p_j^A)}.$$

When $y_0^A = y_0^B = 0$, the demand for platform $i$ is given by equation (2).

The consumer surplus in the interior consumption equilibrium is given by

$$\begin{aligned}CS^A &= \log\left(y_0^A + \sum_j \exp(a_j^A - p_j^A)(n_j^A)^{\alpha_k}(n_j^B)^{\beta_k}\right)\\ &= \log(y_0^A \phi^A + H^A) - \log \phi^A\\ &= (1-\alpha_A)\log(y_0^A \phi^A + H^A) - \beta_A \log(y_0^B \phi^B + H^B).\end{aligned}$$

Therefore, consumer surplus on each side can be characterized by $(H^A, H^B)$ and $(y_0^A, y_0^B)$.

As seen above, although the form of the demand becomes complicated, it is still possible to model platform competition as an aggregative game. A more-tractable setting with variable total participation would be to postulate that users have heterogeneous opportunity costs to become active and make the following sequential decisions: first, after learning their opportunity cost of joining but before learning their idiosyncratic taste realization for the different platforms, they decide whether to become active (e.g. by buying the necessary hardware that enables them to install a software package) and, second, after learning their taste realization they decide which platform to join (e.g. by buying one of the competing software packages).[10]

A straightforward way to introduce partial coverage is to assume that the outside option is also subject to the same network effects and idiosyncratic taste shocks as the for-profit platforms. This applies if choosing the outside option does not mean abstaining from the market but choosing a non-commercial offer. In the case of software, this could be open-source software that is provided free of charge. Our model in Section 2 can easily accommodate such a free platform by adding platform 0 that offers quality $a_0^k$ to side $k \in \{A, B\}$ at zero price, $p_0^k = 0$. Following our change of variables, platform 0 then offers $(h_0^A, h_0^B)$, which is independent of the choices offered by the for-profit platforms, and we write $H^k = \sum_{i=0}^{M} h_i^k$. Our equilibrium characterization of the participation game (Proposition 2) and the existence of an ordered set of price equilibria (Proposition 4) generalize to the introduction of such an outside option.

# 6 Discussion and conclusion

We propose a two-sided single-homing model of platform competition that features differences between platforms with respect to (i) marginal costs incurred for users of the two groups and (ii) the utility that platforms offer to their users (for given participation rates by both groups). Incorporating platform asymmetries provides a rich setting that allows us to explore the relative outcomes of platforms in equilibrium and the impact of exogenous shocks on the performance of different platforms. After establishing the existence and uniqueness of the participation equilibrium, we demonstrate the equilibrium outcome under price competition and obtain insights with respect to exogenous platform entry, incumbent platform investments under free entry, and mandated partial compatibility. Our analysis makes use of the IIA structure of the demand systems of both groups. Platform profits can be written as functions of two action variables and their aggregates (as the sum of action variables across platforms).

We follow the seminal work on platform competition and focus on the platform's pricing decisions. Our analysis can be extended to cover other design decisions if these decisions are taken concurrently with the pricing decision.[11] It is also interesting to extend the analysis

[10]There are two versions of this setting, one in which prices are observed at the first stage of the consumer decision process and the other in which prices are observed at the second stage (and users correctly infer equilibrium prices from the parameters of the model).

[11]In this case, platforms compete in utilities $v_i^k = a_i^k - p_i^k$ for users and platforms may increase value $a_i^k$.

to environments in which platforms do not charge any fees to one user group, but can use non-price strategies that directly affect the attractiveness of the platform for that group. For example, social media platforms typically charge advertisers but do not charge end users and devise non-price strategies to attract end users. We leave extensions in this direction for future work, as they are outside the canonical platform competition model.

We make the functional form assumption that network effects enter as logarithmic functions of participation numbers of each group into user utility and that users experience taste shocks that lead to a logit structure. This specification can be seen as a special case of the model of Tan and Zhou (2021). While such a logarithmic specification of network effects is popular in empirical work, most previous theoretical work assumed linear network effects and few theoretical studies allow for more general forms of network effects (Hagiu, 2009; Weyl, 2010; Belleflamme and Peitz, 2019; Tan and Zhou, 2021). Within the logit demand setting, any generalization beyond logarithmic network effects would make it impossible to obtain closed-form solutions for the participation equilibrium and to subsequently write the platforms' profit functions as a function of their action variables and the aggregates thereof.

Arguably, the canonical model of platform competition features two-sided single-homing. This specification is widely adopted by the literature, including by Armstrong (2006), Jullien and Pavan (2019), and Tan and Zhou (2021). In various real-world environments, however, some users in one or both groups can multihome (see e.g. Armstrong, 2006, section 5, and Anderson and Peitz, 2020, section 6, for the former and Bakos and Halaburda, 2020, Adachi, Sato and Tremblay, forthcoming, and Teh, Liu, Wright and Zhou, forthcoming, for the latter).[12] As pointed out in Section 4.3, we can accommodate environments in which some users single-home and the others of that group join all platforms. Unfortunately, we do not see a way to provide a model with asymmetric platforms competing in prices that encompasses more-complex homing environments and maintains the aggregative game property.

In particular, suppose that there is a one-to-one relationship between value $a_i^k$ and per-user cost $c_i^k$ that depends on the user group and the identity of the platform. Thus, we can write $c_i^k(a_i^k)$, and platforms set $a_i^k$ such that $c_i^k(a_i^k)' = 1$.

[12] Work on ad-funded media platforms has also looked at the effects of viewer multi-homing; see, e.g., Ambrus, Calvano and Reisinger (2016) and Anderson, Foros and Kind (2019).

# 7 Appendix: Relegated proofs

*Proof of Proposition 1.* Denote $y_i^k = \exp(a_i^k - p_i^k)$. Since, in equibrium, $n_i^k = \bar{n}_i^k$, equations (2.2) can be written as

$$n_i^k = \frac{y_i^k \left(n_i^k\right)^{\alpha^k} \left(n_i^l\right)^{\beta^k}}{\sum_{j=1}^{M} y_j^k \left(n_j^k\right)^{\alpha^k} \left(n_j^l\right)^{\beta^k}}, \tag{14}$$

$$n_i^l = \frac{y_i^l \left(n_i^l\right)^{\alpha^l} \left(n_i^k\right)^{\beta^l}}{\sum_{j=1}^{M} y_j^l \left(n_j^l\right)^{\alpha^l} \left(n_j^k\right)^{\beta^l}}. \tag{15}$$

Using the above conditions, for each $j$ and $i$, we have

$$\begin{aligned}
\frac{n_i^k}{n_j^k} &= \left(\frac{y_i^k}{y_j^k}\right)\left(\frac{n_i^k}{n_j^k}\right)^{\alpha^k}\left(\frac{n_i^l}{n_j^l}\right)^{\beta^k} \\
\Longleftrightarrow \frac{n_i^k}{n_j^k} &= \left(\frac{y_i^k}{y_j^k}\right)^{\frac{1}{1-\alpha^k}}\left(\frac{n_i^l}{n_j^l}\right)^{\frac{\beta^k}{1-\alpha^k}} \\
&= \left(\frac{y_i^k}{y_j^k}\right)^{\frac{1}{1-\alpha^k}}\left[\left(\frac{y_i^l}{y_j^l}\right)^{\frac{1}{1-\alpha^l}}\left(\frac{n_i^k}{n_j^k}\right)^{\frac{\beta^l}{1-\alpha^l}}\right]^{\frac{\beta^k}{1-\alpha^k}} \\
\Longleftrightarrow \left(\frac{n_i^k}{n_j^k}\right)^{\frac{(1-\alpha^k)(1-\alpha^l)-\beta^k\beta^l}{(1-\alpha^k)(1-\alpha^l)}} &= \left(\frac{y_i^k}{y_j^k}\right)^{\frac{1}{1-\alpha^k}}\left(\frac{y_i^l}{y_j^l}\right)^{\frac{\beta^k}{(1-\alpha^k)(1-\alpha^l)}} \\
\Longleftrightarrow \frac{n_i^k}{n_j^k} &= \left(\frac{y_i^k}{y_j^k}\right)^{\Gamma^{kk}}\left(\frac{y_i^l}{y_j^l}\right)^{\Gamma^{kl}}.
\end{aligned}$$

By substituting the last equation into equation (14), we obtain the equation

$$n_i^k = \frac{\left(y_i^k\right)^{1+\alpha^k\Gamma^{kk}+\beta^k\Gamma^{lk}}\left(y_i^l\right)^{\alpha^k\Gamma^{kl}+\beta^k\Gamma^{ll}}}{\sum_j \left(y_j^k\right)^{1+\alpha^k\Gamma^{kk}+\beta^k\Gamma^{lk}}\left(y_j^l\right)^{\alpha^k\Gamma^{kl}+\beta^k\Gamma^{ll}}}. \tag{16}$$

Noting that

$$\begin{aligned}
1+\alpha^k\Gamma^{kk}+\beta^k\Gamma^{lk} &= \frac{(1-\alpha^k)(1-\alpha^l)-\beta^k\beta^l+\alpha^k(1-\alpha^l)+\beta^k\beta^l}{(1-\alpha^k)(1-\alpha^l)-\beta^k\beta^l} = \Gamma^{kk}, \\
\alpha^k\Gamma^{kl}+\beta^k\Gamma^{ll} &= \frac{\alpha^k\beta^k+\beta^k(1-\alpha^k)}{(1-\alpha^k)(1-\alpha^l)-\beta^k\beta^l} = \Gamma^{kl},
\end{aligned}$$

equation (16) can be written as

$$n_i^k = \frac{\left(y_i^k\right)^{\Gamma^{kk}} \left(y_i^l\right)^{\Gamma^{kl}}}{\sum_j \left(y_j^k\right)^{\Gamma^{kk}} \left(y_j^l\right)^{\Gamma^{kl}}}$$

Finally, noting that

$$\left(y_i^k\right)^{\Gamma^{kk}} \left(y_i^l\right)^{\Gamma^{kl}} = \exp\left(\Gamma^{kk}(a_i^k - p_i^k) + \Gamma^{kl}(a_i^l - p_i^l)\right),$$

we obtain equation (2). □

*Proof of Proposition 2.* Start with an initial value of the vector of network sizes $(n_{i,0}^A, n_{i,0}^B)_{i=1,\dots,M}$ such that $n_{i,0}^k > 0$ for all $i \in \{1, \dots, M\}$ and $k \in \{A, B\}$. For each $t > 0$, update the network sizes based on the value of network sizes in the previous iteration $t-1$. Then, the sequence of network sizes $\{(n_i^t)_{i=1,\dots,M}\}_{t=0,\dots}$ is obtained. Here, for any $t > 0$, we have

$$\frac{n_{i,t}^k}{n_{j,t}^k} = \frac{y_i^k}{y_j^k} \left(\frac{n_{i,t-1}^k}{n_{j,t-1}^k}\right)^{\alpha_k} \left(\frac{n_{i,t-1}^l}{n_{j,t-1}^l}\right)^{\beta_k}$$

By taking the logarithm and letting $x_t^k := \log(n_{i,t}^k / n_{j,t}^k)$ and $\psi^k := \log(y_i^k / y_j^k)$, we have

$$\begin{pmatrix} x_t^A \\ x_t^B \end{pmatrix} = J \begin{pmatrix} x_{t-1}^A \\ x_{t-1}^B \end{pmatrix} + \begin{pmatrix} \psi^A \\ \psi^B \end{pmatrix},$$

where

$$J = \begin{bmatrix} \alpha^A & \beta^A \\ \beta^B & \alpha^B \end{bmatrix}.$$

If any eigenvalue of $J$ has an absolute value less than 1, $(x_t^A, x_t^B)$ converges to a unique value $(x^A, x^B)$ regardless of the initial value $(x_0^A, x_0^B)$ (see Luenberger, 1979, Chapter 5.9). At such value, we must satisfy $x_t^k = x_{t-1}^k = x^k$. Solving for $x^k$, we have

$$x^k = \frac{(1-\alpha_l)\psi^k + \beta_k \psi^l}{(1-\alpha_k)(1-\alpha_l) - \beta_k \beta_l}.$$

Then, using the relation $\lim_{t\to\infty}(n_{i,t}^k / n_{j,t}^k) = n_i^k / n_j^k = \exp(x^k)$, we obtain the relation (14). Therefore, from any starting value of positive network sizes, the best-response dynamics converges to the interior participation equilibrium.

Lastly, we show that any eigenvalue of $J$ has an absolute value less than 1. A scalar $b$ is an eigenvalue of $J$ if and only if it is the solution to the quadratic equation

$$\xi(b) = b^2 - (\alpha^A + \alpha^B)b + (\alpha^A\alpha^B - \beta^A\beta^B) = 0.$$

Because $\xi(b)$ is quadratic, $\xi(b) = 0$ has at most two solutions. Furthermore, because

$$\begin{aligned} \xi(-1) &= 1 + \alpha^A + \alpha^B + \alpha^A\alpha^B - \beta^A\beta^B > 0, \\ \xi\left(\frac{\alpha^A + \alpha^B}{2}\right) &= -\frac{(\alpha^A)^2 + (\alpha^B)^2 + 2\beta^A\beta^B}{2} < 0, \\ \xi(1) &= (1-\alpha^A)(1-\alpha^B) - \beta^A\beta^B > 0, \end{aligned}$$

There are two solutions to $\xi(b) = 0$ that lie in $(-1, 1)$, which completes the proof.

Thus, the demand for platform in group $k$ is the group-$k$ network size of platform $i$ given by equation (2). □

*Proof of Lemma 1.* The expressions for $h_i^A$ and $h_i^B$ can be rewritten as

$$\begin{aligned} \log h_i^A &= \Gamma^{AA}(a_i^A - p_i^A) + \Gamma^{AB}(a_i^B - p_i^B), \\ \log h_i^B &= \Gamma^{BB}(a_i^B - p_i^B) + \Gamma^{BA}(a_i^A - p_i^A). \end{aligned}$$

Rewriting the second equation as

$$a_i^B - p_i^B = \frac{1}{\Gamma^{BB}} \log h_i^B - \frac{\Gamma^{BA}}{\Gamma^{BB}}(a_i^A - p_i^A),$$

the first equation can be rewritten as

$$\begin{aligned} \log h_i^A &= \left[\Gamma^{AA} - \frac{\Gamma^{AB}\Gamma^{BA}}{\Gamma^{BB}}\right](a_i^A - p_i^A) + \frac{\Gamma^{AB}}{\Gamma^{BB}} \log h_i^B \\ &= \frac{\Gamma^{AA}\Gamma^{BB} - \Gamma^{AB}\Gamma^{BA}}{\Gamma^{BB}}(a_i^A - p_i^A) + \frac{\Gamma^{AB}}{\Gamma^{BB}} \log h_i^B \\ &= \frac{1}{(1-\alpha^A)(1-\alpha^B) - \beta^A\beta^B}\frac{1}{\Gamma^{BB}}(a_i^A - p_i^A) + \frac{\beta^A}{1-\alpha^A} \log h_i^B. \\ &= \frac{1}{1-\alpha^A}(a_i^A - p_i^A) + \frac{\beta^A}{1-\alpha^A} \log h_i^B. \end{aligned}$$

Therefore, we obtain the values of $(p_i^A, p_i^B)$ as a function of $(h_i^A, h_i^B)$, given by equations (4) and (5). □

*Proof of Lemma 2.* In the first part of the proof, we show that any solution to the first-order conditions of profit maximization is a global maximizer. Define

$$\begin{aligned} f_i^A(h_i^A, h_i^B) &:= \left(1 - \frac{h_i^A}{H^A}\right)\left[p_i^A(h_i^A, h_i^B) - c_i^A\right] - 1 + \alpha^A + \beta^B \frac{h_i^B}{H^B}\frac{H^A}{h_i^A}, \\ f_i^B(h_i^A, h_i^B) &:= \left(1 - \frac{h_i^B}{H^B}\right)\left[p_i^B(h_i^A, h_i^B) - c_i^B\right] - 1 + \alpha^B + \beta^A \frac{h_i^A}{H^A}\frac{H^B}{h_i^B} \end{aligned}$$

and, thus, $\partial\Pi_i/\partial h_i^k = f_i^k(h_i^A, h_i^B)/H^k$, for $k \in \{A, B\}$. Hence, $\partial^2\Pi_i/(\partial h_i^k)^2 = -\frac{f_i^k(h_i^A,h_i^B)}{(H^k)^2} + \frac{1}{H^k}\frac{\partial f_i^k}{\partial h_i^k}$. When the first-order conditions of profit maximization hold, the first term on the right-hand side is zero. Then, $\Pi_i(h_i^A, h_i^B, h_i^A + H_{-i}^A, h_i^B + H_{-i}^B)$ is a local maximizer in $(h_i^A, h_i^B)$ at any point at which the first-order conditions of profit maximization hold if $\partial f_i^A/\partial h_i^A < 0$, $\partial f_i^B/\partial h_i^B < 0$, and $(\partial f_i^A/\partial h_i^A)(\partial f_i^B/\partial h_i^B) - (\partial f_i^A/\partial h_i^B)(\partial f_i^B/\partial h_i^A) > 0$. Furthermore, this establishes that the Jacobian of $(f_i^A, f_i^B))$ is a $P$-matrix. This implies that $(f_i^A, f_i^B)$ is injective on $(0, \infty)^2$ (Gale and Nikaido, 1965) and, therefore, a solution to the first-order conditions of profit maximization is a global maximizer, provided that such a solution exists.

To see that the three inequalities hold, first note that

$$\begin{aligned} \frac{\partial f_i^A}{\partial h_i^A} &= \frac{1}{h_i^A}\left[-n_i^A(1-n_i^A)[p_i^A - c_i^A] - (1-\alpha^A)(1-n_i^A) - \beta^B(1-n_i^A)\frac{n_i^B}{n_i^A}\right] \\ &= \frac{1}{h_i^A}\left[-n_i^A\left(1-\alpha^A - \beta^B\frac{n_i^B}{n_i^A}\right) - (1-n_i^A)\left(1-\alpha^A + \beta^B\frac{n_i^B}{n_i^A}\right)\right] \\ &= -\frac{1}{h_i^A}\left[1-\alpha^A - \beta^B n_i^B + \beta^B(1-n_i^A)\frac{n_i^B}{n_i^A}\right] < 0, \end{aligned}$$

which establishes the first inequality above. Correspondingly, the second inequality holds. Third, we establish $(\partial f_i^A/\partial h_i^A)(\partial f_i^B/\partial h_i^B) - (\partial f_i^A/\partial h_i^B)(\partial f_i^B/\partial h_i^A) > 0$. To do so, note that

$$\frac{\partial f_i^A}{\partial h_i^B} = \frac{1}{h_i^B}\left[\beta^A(1-n_i^A) + \beta^B(1-n_i^B)\frac{n_i^B}{n_i^A}\right] > 0.$$

Without loss of generality, assume that $\beta^A \geq \beta^B$. Recall that $1 - \max\{\alpha^A, \alpha^B\} > \max\{\beta^A, \beta^B\}$.

Therefore, $1-\alpha^A > \beta^A$ and $1-\alpha^B > \beta^A$. Then, we have

$$\begin{aligned}
& h_i^A h_i^B \left(\frac{\partial f_i^A}{\partial h_i^A}\frac{\partial f_i^B}{\partial h_i^B} - \frac{\partial f_i^A}{\partial h_i^B}\frac{\partial f_i^B}{\partial h_i^A}\right) \\
= \; & \left(1-\alpha^A-\beta^B n_i^B\right)\left(1-\alpha^B-\beta^A n_i^A\right) \\
& + \left(1-\alpha^A-\beta^B n_i^B\right)\beta^A(1-n_i^B)\frac{n_i^A}{n_i^B} + \left(1-\alpha^B-\beta^A n_i^A\right)\beta^B(1-n_i^A)\frac{n_i^B}{n_i^A} \\
& -(\beta^A)^2(1-n_i^A)^2\frac{n_i^A}{n_i^B} - (\beta^B)^2(1-n_i^B)^2\frac{n_i^B}{n_i^A} - \beta^A\beta^B(1-n_i^A)(1-n_i^B) \\
> \; & \underbrace{(\beta^A-\beta^B n_i^B)\beta^A(1-n_i^A)}_{\text{(i)}} + \underbrace{(\beta^A-\beta^B n_i^B)\beta^A(1-n_i^B)\frac{n_i^A}{n_i^B}}_{\text{(ii)}} + \underbrace{\beta^A\beta^B(1-n_i^A)^2\frac{n_i^B}{n_i^A}}_{\text{(iii)}} \\
& - \underbrace{(\beta^A)^2(1-n_i^A)^2\frac{n_i^A}{n_i^B}}_{\text{(iv)}} - \underbrace{(\beta^B)^2(1-n_i^B)^2\frac{n_i^B}{n_i^A}}_{\text{(v)}} - \underbrace{\beta^A\beta^B(1-n_i^A)(1-n_i^B)}_{\text{(vi)}}.
\end{aligned}$$

Every pair $(n_i^A, n_i^B)$ belongs to one of three cases, and we show that the above expression is positive in each case.

1. First, consider the case with $n_i^A \geq n_i^B$. In this case, (ii) > (iv), (iii) > (v), and (i) > (vi), so the expression under consideration is positive.

2. Next, consider the case with $n_i^B \in (n_i^A, n_i^A\beta^A/\beta^B]$. In this case, we have $\beta^B n_i^B < \beta^A n_i^A$, so (i) > (iv), (iii) > (vi), and (ii) > (v), so the expression under consideration is positive.

3. Finally, consider the case with $n_i^B > n_i^A\beta^A/\beta^B$. Because $\beta^A \geq \beta^B$, we have (i) $\geq$ (vi). Next we show that (ii) + (iii) > (iv) + (v). Noting that (ii) − (iv) $\geq -(\beta^A)^2(n_i^A/n_i^B)[(1-n_i^A)^2-(1-n_i^B)^2]$ and (iii) − (v) $\geq (\beta^A)^2[(1-n_i^A)^2-(1-n_i^B)^2]$ when $n_i^B > n_i^A\beta^A/\beta^B$, we have

   $$\text{(ii)} + \text{(iii)} - [\text{(iv)} + \text{(v)}] \geq (\beta^A)^2[(1-n_i^A)^2-(1-n_i^B)^2]\left(1-\frac{n_i^A}{n_i^B}\right) > 0,$$

   which shows that the expression under consideration is positive. This completes the first part of the proof.

In the second part of the proof, we show that there always exists a solution to the system of equations

$$\begin{pmatrix} f_i^A(h_i^A, h_i^B) \\ f_i^B(h_i^A, h_i^B) \end{pmatrix} = \begin{pmatrix} 0, \\ 0. \end{pmatrix} \tag{17}$$

Step 1: existence of a solution $\tilde{h}_i^B(h_i^A)$ to $f_i^B(h_i^A, h_i^B) = 0$ given $h_i^A$. Fix $h_i^A$ and consider the solution to the equation $f_i^B(h_i^A, h_i^B) = 0$ given $h_i^A$, denoted by $\tilde{h}_i^B(h_i^A)$. We show that $\tilde{h}_i^B(h_i^A)$ exists in $(0, \infty)$, for any given $h_i^A \in (0, \infty)$. To see this, note that we have

$$
\begin{aligned}
&\lim_{h_i^B \to 0} f_i^B(h_i^A, h_i^B) \\
&= \lim_{h_i^B \to 0} \left[a_i^B - c_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A\right] - 1 + \alpha^B + \lim_{h_i^B \to 0}\left(\beta^A \frac{h_i^A}{h_i^A + H_{-i}^A}\frac{h_i^B + H_{-i}^B}{h_i^B}\right) \\
&= \infty > 0, \\
&\lim_{h_i^B \to \infty} f_i^B(h_i^A, h_i^B) \\
&= \lim_{h_i^B \to \infty}\left[\frac{H_{-i}^B}{h_i^B + H_{-i}^B}\left[a_i^B - c_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A\right]\right] - 1 + \alpha^A + \beta^B \frac{h_i^A}{H^A} \\
&= -1 + \alpha^A + \beta^B \frac{h_i^A}{h_i^A + H_{-i}^A} < 0.
\end{aligned}
$$

Hence, by the intermediate value theorem, the solution $\tilde{h}_i^B(h_i^A) \in (0, \infty)$ exists for any given $h_i^A \in [0, \infty]$. Note that such a solution is unique and continuous in $h_i^A$. To see this, note that we already established that we have $\partial f_i^B / \partial h_i^B < 0$ whenever $f_i^B = 0$ holds. Hence, $\tilde{h}_i^B(h_i^A)$ is unique and, from the implicit function theorem, continuous.

Step 2: preliminaries on the existence of solution to equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$. We show that there exists a solution to the equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$. As preliminaries, we show the following four limit results: $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^B) = 0$, $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \infty$, $\lim_{h_i^A \to 0} \frac{h_i^A}{\tilde{h}_i^B(h_i^A)} = 0$, $\lim_{h_i^A \to \infty} \frac{h_i^A}{\tilde{h}_i^B(h_i^A)} = \infty$.

First, we show that $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^A) = 0$. Suppose to the contrary that $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^A) > 0$. Then, there exists $\underline{h}_i^B > 0$ such that $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \underline{h}_i^B$, and we would have

$$
\begin{aligned}
&\lim_{h_i^A \to 0} f_i^B(h_i^A, \tilde{h}_i^B(h_i^A)) \\
=\;& \frac{H_{-i}^B}{\underline{h}_i^B + H_{-i}^B}\left[a_i^B - c_i^B - (1-\alpha^B)\log \underline{h}_i^B + \beta^B \lim_{h_i^A \to 0}\left(\log h_i^A\right)\right] - 1 + \alpha^B \\
=\;& -\infty \\
<\;& 0,
\end{aligned}
$$

contradicting the definition of $\tilde{h}_i^B(h_i^A)$. Hence, $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^A) = 0$.

Second, we show that $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \infty$. Suppose to the contrary that $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) <$

$\infty$. Then, there exists $\overline{h}^B < \infty$ such that $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \overline{h}_i^B$, and we would have

$$\begin{aligned}
& \lim_{h_i^A \to \infty} f_i^B(h_i^A, \tilde{h}_i^B(h_i^A)) \\
= & \frac{H_{-i}^B}{\overline{h}_i^B + H_{-i}^B} \left[ a_i^B - c_i^B - (1 - \alpha^B) \log \overline{h}_i^B + \beta^B \lim_{h_i^A \to \infty} \log h_i^A \right] - 1 + \alpha^B + \beta^A \frac{H^B}{\overline{h}_i^B} \\
= & \infty > 0,
\end{aligned}$$

contradicting the definition of $\tilde{h}_i^B(h_i^A)$. Hence, $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \infty$.

Third, we show that $\lim_{h_i^A \to 0}[h_i^A / \tilde{h}_i^B(h_i^A)] = 0$. Otherwise, there exists a constant $\underline{\kappa} > 0$ such that $\lim_{h_i^A \to 0}[h_i^A / \tilde{h}_i^B(h_i^A)] = \underline{\kappa}$, and

$$\begin{aligned}
& \lim_{h_i^A \to 0} f_i^B(h_i^A, \tilde{h}_i^B(h_i^A)) \\
= & a_i^B - c_i^B - (1 - \alpha^B - \beta^B) \lim_{h_i^A \to 0} \log \tilde{h}_i^B(h_i^A) + \beta^B \log \underline{\kappa} - 1 + \alpha^B + \beta^A \underline{\kappa} \frac{H_{-i}^B}{H_{-i}^A} \\
= & \infty
\end{aligned}$$

Hence, we have $\lim_{h_i^A \to 0}[h_i^A / \tilde{h}_i^B(h_i^A)] = 0$.

Fourth, we show that $\lim_{h_i^A \to \infty}[h_i^A / \tilde{h}_i^B(h_i^A)] = \infty$. Otherwise, there exists $\overline{\kappa} < \infty$ such that $\lim_{h_i^A \to \infty}[h_i^A / \tilde{h}_i^B(h_i^A)] = \overline{\kappa}$. Then, we would have

$$\begin{aligned}
& \lim_{h_i^A \to \infty} f_i^B(h_i^A, \tilde{h}_i^B(h_i^A)) \\
= & \lim_{h_i^A \to \infty} \left[ \frac{H_{-i}^B}{\tilde{h}_i^B(h_i^A) + H_{-i}^B} \left( a_i^B - c_i^B - (1 - \alpha^B - \beta^B) \log \tilde{h}_i^B(h_i^A) + \beta^B \log \overline{\kappa} \right) \right] - 1 + \alpha^B + \beta^A \\
= & -1 + \alpha^B + \beta^A < 0,
\end{aligned}$$

contradicting the definition of $\tilde{h}_i^B(h_i^A)$. Hence, we have $\lim_{h_i^A \to \infty}[h_i^A / \tilde{h}_i^B(h_i^A)] = \infty$.

Step 3: proof of the existence of a solution to equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$. To show the existence of the solution to the equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$, we show that

$$\begin{aligned}
\lim_{h_i^A \to 0} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) &> 0, \\
\lim_{h_i^A \to \infty} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) &< 0.
\end{aligned}$$

Then, the intermediate value theorem implies that there exists a solution to the equation.

We first show that $\lim_{h_i^A \to 0} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) > 0$. To see this, note that we can write $f_i^A$ as

$$
\begin{aligned}
f_i^A(h_i^A, h_i^B) &= \frac{H_{-i}^A}{h_i^A + H_{-i}^A}\left[a_i^A - c_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B\right] - 1 + \alpha^A + \beta^B \frac{h_i^B}{h_i^A}\frac{h_i^A + H_{-i}^A}{h_i^B + H_{-i}^B} \\
&= \frac{H_{-i}^A}{h_i^A + H_{-i}^A}\left[a_i^A - c_i^A - (1-\alpha^A-\beta^A)\log h_i^A - \beta^A \log\left(\frac{h_i^A}{h_i^B}\right)\right] \\
&\quad -1 + \alpha^A + \beta^B \frac{h_i^B}{h_i^A}\frac{h_i^A + H_{-i}^A}{h_i^B + H_{-i}^B}.
\end{aligned}
$$

Hence, because $\lim_{h_i^A \to 0} \tilde{h}_i^B(h_i^A) = 0$ and $\lim_{h_i^A \to 0}[h_i^A/\tilde{h}_i^B(h_i^A)] = 0$, we have

$$
\begin{aligned}
& \lim_{h_i^A \to 0} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) \\
= \;& a_i^A - c_i^A + \lim_{h_i^A \to 0}\left[-(1-\alpha^A-\beta^A)\log h_i^A - \beta^A \log\left(\frac{h_i^A}{\tilde{h}_i^B(h_i^A)}\right) - 1 + \alpha^A + \beta^B \frac{\tilde{h}_i^B(h_i^A)}{h_i^A}\right] \\
= \;& \infty.
\end{aligned}
$$

Next, we show that $\lim_{h_i^A \to \infty} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) < 0$. To see this, note that

$$
\begin{aligned}
& f_i^A(h_i^A, h_i^B) \\
= \;& \frac{h_i^A}{h_i^A + H_{-i}^A}\frac{H_{-i}^A}{h_i^A}\left[a_i^A - a_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B\right] - 1 + \alpha^A + \beta^B \frac{h_i^B}{h_i^B + H_{-i}^B}\frac{h_i^A + H_{-i}^A}{h_i^A}.
\end{aligned}
$$

Hence, because $\lim_{h_i^A \to \infty} \tilde{h}_i^B(h_i^A) = \infty$ and $\lim_{h_i^A \to \infty}[h_i^A/\tilde{h}_i^B(h_i^A)] = \infty$, we have

$$
\begin{aligned}
& \lim_{h_i^A \to \infty} f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) \\
= \;& \lim_{h_i^A \to \infty}\left(\frac{\log h_i^A}{h_i^A}\right) + \lim_{h_i^A \to \infty}\left(\frac{\log \tilde{h}_i^B(h_i^A)}{\tilde{h}_i^B(h_i^A)}\frac{\tilde{h}_i^B(h_i^A)}{h_i^A}\right) - 1 + \alpha^A + \beta^B \\
= \;& -1 + \alpha^A + \beta^B < 0.
\end{aligned}
$$

Put together, there exists a solution to the equation $f_i^A(h_i^A, \tilde{h}_i^B(h_i^A)) = 0$. Letting $h_i^{A*}$ be a solution and $h_i^{B*} := \tilde{h}_i^B(h_i^{A*})$, the pair $(h_i^{A*}, h_i^{B*})$ is a solution to the system of equations (17). □

*Proof of Proposition 3.* We rewrite the first-order condition as

$$
F(h_i^A, H^A) := (1-\alpha^A)\frac{H^A}{H^A - h_i^A} - (a_i^A - c_i^A) + (1-\alpha^A)\log h_i^A = 0.
$$

By the implicit function theorem,

$$\frac{dh_i^A}{dH^A} = -\frac{\frac{\partial F(h_i^A, H^A)}{\partial H^A}}{\frac{\partial F(h_i^A, H^A)}{\partial h_i^A}}.$$

Since $\frac{\partial F(h_i^A,H^A)}{\partial h_i^A} = (1-\alpha^A)\frac{H^A}{(H^A-h_i^A)^2} + (1-\alpha^A)\frac{1}{h_i^A} = (1-\alpha^A)\frac{h_i^A H^A + (H^A-h_i^A)^2}{h_i^A(H^A-h_i^A)^2}$ and $\frac{\partial F(h_i^A,H^A)}{\partial H^A} = -(1-\alpha^A)\frac{h_i^A}{(H^A-h_i^A)^2}$, we have that

$$\frac{dh_i^A}{dH^A} = \frac{\frac{h_i^A}{(H^A-h_i^A)^2}}{\frac{h_i^A H^A + (H^A-h_i^A)^2}{h_i^A(H^A-h_i^A)^2}} = \frac{(h_i^A)^2}{h_i^A H^A + (H^A-h_i^A)^2} > 0$$

The equilibrium is unique if $\sum_i \frac{dh_i^A}{dH^A} < 1$. Hence, uniqueness is implied by inequalities

$$\frac{(h_i^A)^2}{h_i^A H^A + (H^A-h_i^A)^2} < \frac{h_i^A}{H^A}$$

for all $i \in \{1, \ldots, M\}$, which is always satisfied. To see this, we rewrite inequalities as $h_i^A H^A < h_i^A H^A + (H^A - h_i^A)^2$, which is equivalent to $0 < (H^A - h_i^A)^2$. $\square$

*Proof of Lemma 3.* Let

$$\widetilde{f}_i^A(h_i^A, h_i^B, H^A, H^B) = \left(1 - \frac{h_i^A}{H^A}\right)\left[p_i^A(h_i^A, h_i^B) - c_i^A\right] - 1 + \alpha^A + \beta^B \frac{h_i^B}{H^B}\frac{H^A}{h_i^A},$$
$$\widetilde{f}_i^B(h_i^A, h_i^B, H^A, H^B) = \left(1 - \frac{h_i^B}{H^B}\right)\left[p_i^B(h_i^A, h_i^B) - c_i^B\right] - 1 + \alpha^B + \beta^A \frac{h_i^A}{H^A}\frac{H^B}{h_i^B}$$

By the implicit function theorem, implicit best replies are well-defined, if the matrix

$$\begin{pmatrix} \sum \frac{\partial \widetilde{f}_i^A}{\partial h_i^A} & \sum \frac{\partial \widetilde{f}_i^A}{\partial h_i^B} \\ \sum \frac{\partial \widetilde{f}_i^B}{\partial h_i^A} & \sum \frac{\partial \widetilde{f}_i^B}{\partial h_i^B} \end{pmatrix}$$

has a determinant different from zero. Then, taking $(H^A, H^B)$ as given, we have

$$\begin{aligned}\frac{\partial \widetilde{f}_i^A}{\partial h_i^A} &= \frac{1}{h_i^A}\left[-n_i^A(p_i^A - c_i^A) - (1-\alpha^A)(1-n_i^A) - \beta^B\frac{n_i^B}{n_i^A}\right] \\ &= \frac{1}{h_i^A}\left[-\frac{n_i^A}{1-n_i^A}\left(1-\alpha^A-\beta^B\frac{n_i^B}{n_i^A}\right) - (1-\alpha^A)(1-n_i^A) - \beta^B\frac{n_i^B}{n_i^A}\right] \\ &= -\frac{1}{h_i^A}\frac{1}{1-n_i^A}\left\{\left[(1-n_i^A)^2 + n_i^A\right](1-\alpha^A) - \beta^B n^B\left(1 - \frac{1-n_i^A}{n_i^A}\right)\right\} < 0,\end{aligned}$$

which can be shown as follows: $(1-n_i^A)^2+n_i^A$ takes positive value, is minimized at $n_i^A=1/2$, and increasing in $n_i^A>1/2$, while $1-(1-n_i^A)/n_i^A$ is increasing, takes value zero at $n_i^A=1$ and is maximized at $n_i^A=1$. Thus, for $n_i^A\leq 1/2$ the derivative must be negative. For $n_i^A>1/2$, since $1-\alpha^A>\beta^B$ by assumption, it is sufficient to show that $(1-n_i^A)^2+n_i^A\geq 1-\frac{1-n_i^A}{n_i^A}$ which is equivalent to $(1-(n_i^A)^2)(1-n_i^A)\geq 0$ and, thus, always holds. Note that we can write

$$\frac{\partial \widetilde{f}_i^A}{\partial h_i^A}=-\frac{1}{h_i^A}\left\{\frac{(1-n_i^A)^2+n_i^A}{1-n_i^A}(1-\alpha^A)-\beta^B\frac{n_i^B}{1-n_i^A}+\beta^B\frac{n_i^B}{n_i^A}\right\}.$$

Also, we have

$$\frac{\partial \widetilde{f}_i^A}{\partial h_i^B}=\frac{1}{h_i^B}\left[\beta^A(1-n_i^A)+\beta^B\frac{n_i^B}{n_i^A}\right].$$

Therefore,

$$\begin{aligned}
&h_i^A h_i^B\left(\frac{\partial \widetilde{f}_i^A}{\partial h_i^A}\frac{\partial \widetilde{f}_i^B}{\partial h_i^B}-\frac{\partial \widetilde{f}_i^A}{\partial h_i^B}\frac{\partial \widetilde{f}_i^B}{\partial h_i^A}\right)\\
&=\left\{\left[1-n_i^A+\frac{n_i^A}{1-n_i^A}\right](1-\alpha^A)+\frac{n_i^B}{n_i^A}\beta^B-\frac{n_i^B}{1-n_i^A}\beta^B\right\}\\
&\quad\times\left\{\left[1-n_i^B+\frac{n_i^B}{1-n_i^B}\right](1-\alpha^B)+\frac{n_i^A}{n_i^B}\beta^A-\frac{n_i^A}{1-n_i^B}\beta^A\right\}\\
&\quad-\left(\beta^A(1-n_i^A)+\beta^B\frac{n_i^B}{n_i^A}\right)\left(\beta^B(1-n_i^B)+\beta^A\frac{n_i^A}{n_i^B}\right)
\end{aligned}$$

Suppose without loss of generality that $\beta^A\geq\beta^B$. Then, because $\min\{1-\alpha^A,1-\alpha^B\}\geq\beta^A$, the last expression is greater than

$$\begin{aligned}
&\left\{\left[1-n_i^A+\frac{n_i^A}{1-n_i^A}\right]\beta^A+\left(\frac{n_i^B}{n_i^A}-\frac{n^B}{1-n_i^A}\right)\beta^B\right\}\beta^A\left[1-n_i^B+\frac{n_i^B}{1-n_i^B}-\frac{n_i^A}{1-n_i^B}+\frac{n_i^A}{n_i^B}\right]\\
&-\left(\beta^A(1-n_i^A)+\beta^B\frac{n_i^B}{n_i^A}\right)\left(\beta^B(1-n_i^B)+\beta^A\frac{n_i^A}{n_i^B}\right),
\end{aligned}$$

which, by dividing by $(\beta^A)^2$, has the same sign as

$$
\begin{aligned}
&\left\{\left[1-n_i^A+\frac{n_i^A}{1-n_i^A}\right]+\left(\frac{n_i^B}{n_i^A}-\frac{n^B}{1-n_i^A}\right)\frac{\beta^B}{\beta^A}\right\}\left[1-n_i^B+\frac{n_i^B}{1-n_i^B}-\frac{n_i^A}{1-n_i^B}+\frac{n_i^A}{n_i^B}\right]\\
&-\left((1-n_i^A)+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{\beta^B}{\beta^A}(1-n_i^B)+\frac{n_i^A}{n_i^B}\right)\\
&=\left(1-n_i^A+\frac{n_i^B}{n_i^A}\frac{\beta^A}{\beta^B}\right)\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)+\frac{n_i^A}{1-n_i^A}\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)\\
&\quad+\frac{n_i^B}{1-n_i^B}\left(1-\frac{n_i^A}{n_i^B}\right)\left(1-n_i^A+\frac{n_i^B}{n_i^A}\frac{\beta^B}{\beta^A}\right)+\frac{(n_i^A)^2}{(1-n_i^A)(1-n_i^B)}\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{n_i^B}{n_i^A}-1\right)\\
&\quad-\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left[\frac{\beta^B}{\beta^A}(1-n_i^B)+\frac{n_i^A}{n_i^B}\right]\\
&=\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(1-\frac{\beta^B}{\beta^A}\right)(1-n_i^B)\\
&\quad+\frac{n_i^A}{1-n_i^B}\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{n_i^B}{n_i^A}-1\right)\\
&\quad+\frac{n_i^A}{1-n_i^A}\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\\
&\quad+\frac{(n_i^A)^2}{(1-n_i^A)(1-n_i^B)}\left(\frac{n_i^B}{n_i^A}-1\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\\
&=\frac{(1-n_i^A)(1-n_i^B)^2}{(1-n_i^A)(1-n_i^B)}\left[1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right]\left(1-\frac{\beta^B}{\beta^A}\right)\\
&\quad+\frac{n_i^A}{(1-n_i^A)(1-n_i^B)}(1-n_i^A)\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{n_i^B}{n_i^A}-1\right)\\
&\quad+\frac{n_i^A}{(1-n_i^A)(1-n_i^B)}(1-n_i^B)\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\\
&\quad+\frac{n_i^A}{(1-n_i^A)(1-n_i^B)}n_i^A\left(\frac{n_i^B}{n_i^A}-1\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right),
\end{aligned}
$$

which is positive if

$$
\begin{aligned}
&\underbrace{(1-n_i^A)\left(1-n_i^A+\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)\left(\frac{n_i^B}{n_i^A}-1\right)}_{\text{(i)}}\\
&+\underbrace{(1-n_i^B)\left(1-n_i^B+\frac{n_i^A}{n_i^B}\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)}_{\text{(ii)}}\\
&+\underbrace{n_i^A\left(\frac{n_i^B}{n_i^A}-1\right)\left(1-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)}_{\text{(iii)}}
\end{aligned}
$$

is positive. Any value of $n_i^A$ belongs to one of three cases, and we show that $(\mathrm{i})+(\mathrm{ii})+(\mathrm{iii})>0$ for each case.

1. The first case we consider is $n_i^A \geq n_i^B$. In this case, we have $n_i^B/n_i^A - 1 \leq 0$ and

$$
\begin{aligned}
&(\mathrm{i})+(\mathrm{ii})+(\mathrm{iii})\\
=&\left(1-\frac{n_i^B}{n_i^A}\right)\left[(1-n_i^B)^2-(1-n_i^A)^2+(1-n_i^B)\frac{n_i^A}{n_i^B}-(1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}-\left(n_i^A-\frac{\beta^B}{\beta^A}n_i^B\right)\right]\\
&+\frac{n_i^B}{n_i^A}\left(1-\frac{\beta^B}{\beta^A}\right)\left[(1-n_i^B)^2+(1-n_i^B)\frac{n_i^A}{n_i^B}\right]\\
=&\left(1-\frac{n_i^B}{n_i^A}\right)\left[(n_i^A-n_i^B)(2-n_i^A-n_i^B)+(1-n_i^B)\frac{n_i^A}{n_i^B}-(1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}-\left(n_i^A-\frac{\beta^B}{\beta^A}n_i^B\right)\right]\\
&+\frac{n_i^B}{n_i^A}\left(1-\frac{\beta^B}{\beta^A}\right)\left[(1-n_i^B)^2+(1-n_i^B)\frac{n_i^A}{n_i^B}\right]\\
=&\left(1-\frac{n_i^B}{n_i^A}\right)\left[(n_i^A-n_i^B)(2-n_i^A-n_i^B)+(1-n_i^B)\frac{n_i^A}{n_i^B}-n_i^A+\frac{\beta^B}{\beta^A}n_i^B\right]\\
&+\frac{n_i^B}{n_i^A}(1-n_i^B)^2\left(1-\frac{\beta^B}{\beta^A}\right)+(1-n_i^B)\left(1-\frac{\beta^B}{\beta^A}\right)\\
\geq&\left(1-\frac{n_i^B}{n_i^A}\right)\left[(n_i^A-n_i^B)(2-n_i^A-n_i^B)+(1-n_i^B)\frac{n_i^A}{n_i^B}-\frac{\beta^B}{\beta^A}(1-n_i^B)\right]\\
>&0
\end{aligned}
$$

where, for the third equation, we used

$$
\begin{aligned}
(1-n_i^B)\frac{n_i^A}{n_i^B}&=(1-n_i^A)\frac{n_i^A}{n_i^B}+(n_i^A-n_i^B)\frac{n_i^A}{n_i^B}\\
&=(1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}+(1-n_i^A)\left(\frac{n_i^A}{n_i^B}-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}\right)+(n_i^A-n_i^B)\frac{n_i^A}{n_i^B}\\
&=(1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}+\frac{n_i^A}{n_i^B}n_i^A-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}+(1-n_i^A)\frac{n_i^A}{n_i^B}-n_i^A+\frac{\beta^B}{\beta^A}n_i^B\\
&=(1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}+n_i^A-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}+\left(\frac{n_i^A}{n_i^B}-1\right)n_i^A+(1-n_i^A)\frac{n_i^A}{n_i^B}-n_i^A+\frac{\beta^B}{\beta^A}n_i^B\\
&=(1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}+n_i^A-\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A}+(1-n_i^B)\frac{n_i^A}{n_i^B}-n_i^A+\frac{\beta^B}{\beta^A}n_i^B
\end{aligned}
$$

and, for the inequality

$$
\begin{aligned}
1-n_i^B&\geq 1-\frac{n_i^B}{n_i^A},\\
1-\frac{\beta^B}{\beta^A}-n_i^A&\geq -\frac{\beta^B}{\beta^A}.
\end{aligned}
$$

2. The second case is $n_i^A \in [n_i^B \beta^B/\beta^A, n_i^B)$. In this case, (i) $> 0$, (ii) $\geq 0$, and (iii) $\geq 0$.

3. The third case is $n_i^A < n_i^B \beta^B/\beta^A$. In this case, we have $n_i^B/n_i^A - 1 > (\beta^B n_i^B)/(\beta^A n_i^A) - 1 > 0$. Therefore,

$$\begin{aligned}&\text{(i)} + \text{(ii)} + \text{(iii)}\\ =&\left(\frac{n_i^B}{n_i^A} - 1\right)\left\{(1-n_i^A)^2 - (1-n_i^B)^2 + (1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - (1-n_i^B)\frac{n_i^A}{n_i^B} - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right)\right\},\end{aligned}$$

which is positive if and only if

$$\begin{aligned}&(1-n_i^A)^2 - (1-n_i^B)^2 + (1-n_i^A)\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - (1-n_i^B)\frac{n_i^A}{n_i^B} - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right)\\ =&(n_i^B - n_i^A)\left(2 - n_i^B - n_i^A + \frac{n_i^A}{n_i^B}\right) + (1-n_i^A)\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B}\right) - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right)\end{aligned}$$

is positive. Note that we have

$$\begin{aligned}&(n_i^B - n_i^A)\left(2 - n_i^B - n_i^A + \frac{n_i^A}{n_i^B}\right) + (1-n_i^A)\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B}\right) - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right)\\ \geq&(n_i^B - n_i^A)\frac{n_i^A}{n_i^B} + (1-n_i^A)\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B}\right) - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right)\\ =&\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B} - \frac{\beta^B}{\beta^A}n_i^B + n_i^A - n_i^A\left(\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - 1\right)\\ =&\frac{\beta^B}{\beta^A}\frac{n_i^B}{n_i^A} - \frac{n_i^A}{n_i^B} + 2n_i^A - 2\frac{\beta^B}{\beta^A}n_i^B.\end{aligned}$$

At $n_i^A = \beta^B n_i^B/\beta^A$, the last expression above is

$$1 - \frac{\beta^B}{\beta^A} \geq 0.$$

Furthermore, for any region where $n_i^A \leq \beta^B n_i^B/\beta^A$, the expression under consideration has the following derivative with respect to $n_i^A$:

$$-\frac{1}{n_i^A}\frac{n_i^B}{n_i^A}\frac{\beta^B}{\beta^A} - \frac{1}{n_i^B} + 2 < 0.$$

Therefore, for any given $n_i^B$ and any $n_i^A < \beta^B n_i^B/\beta^A$, the expression under consideration is positive.

$\square$

*Proof of Proposition 4.* To show the existence of the equilibrium, recall from Section 3.2 that

$CS^A = (1-\alpha^A)\log H^A - \beta^A \log H^B$ and $CS^B = (1-\alpha^B)\log H^B - \beta^B \log H^A$.

Denote market share for group $k$ as a function of the aggregates by

$$n_i^k(H^A, H^B) = \frac{h_i^k(H^A, H^B)}{H^k}.$$

Noting that

$$\log h_i^A = \log n_i^A + \log H^A,$$

we can rewrite the first-order condition for $(h_i^A, h_i^B)$ as the condition for $(n_i^A, n_i^B)$ in the following way:

$$g_i^A = (1-n_i^A)[a_i^A - c_i^A - (1-\alpha^A)\log n_i^A + \beta^A \log n_i^B - CS^A] - 1 + \alpha^A + \beta^B \frac{n_i^B}{n_i^A} = 0, \quad (18)$$

$$g_i^B = (1-n_i^B)[a_i^B - c_i^B - (1-\alpha^B)\log n_i^B + \beta^B \log n_i^A - CS^B] - 1 + \alpha^B + \beta^A \frac{n_i^A}{n_i^B} = 0. \quad (19)$$

The solution can be written as the functions

$$n_i^A = \tilde{n}_i^A(CS^A, CS^B),$$
$$n_i^B = \tilde{n}_i^B(CS^A, CS^B).$$

By the implicit function theorem, we have

$$\text{Sign}\left[\frac{\tilde{n}_i^k}{\partial CS^l}(CS^A, CS^B)\right]_{k,l\in\{A,B\}} = -\text{Sign}\begin{pmatrix} \frac{\partial g_i^B}{\partial n_i^B} & -\frac{\partial g_i^A}{\partial n_i^B} \\ -\frac{\partial g_i^B}{\partial n_i^A} & \frac{\partial g_i^A}{\partial n_i^A} \end{pmatrix} \begin{pmatrix} \frac{\partial g_i^A}{\partial CS^A} & 0 \\ 0 & \frac{\partial g_i^B}{\partial CS^B} \end{pmatrix}$$

Thus,

$$\text{Sign}\left(\frac{\partial \tilde{n}_i^A}{\partial CS^A}\right) = \text{Sign}\left(-\frac{\partial g_i^A}{\partial CS^A}\frac{\partial g_i^B}{\partial n_i^B}\right) < 0,$$

and

$$\text{Sign}\left(\frac{\partial \tilde{n}_i^B}{\partial CS^A}\right) = \text{Sign}\left(\frac{\partial g_i^A}{\partial CS^A}\frac{\partial g_i^B}{\partial n_i^A}\right) \leq 0,$$

where

$$\frac{\partial g_i^A}{\partial n_i^A} = -\frac{1}{n_i^A}\left[n_i^A(p_i^A - c_i^A) + (1-\alpha^A)(1-n_i^A) + \beta^B \frac{n_i^B}{n_i^A}\right] < 0,$$
$$\frac{\partial g_i^A}{\partial n_i^B} = \frac{1}{n_i^B}\left[\beta^A(1-n_i^A) + \beta^B \frac{n_i^B}{n_i^A}\right] \geq 0$$

and

$$\frac{\partial g_i^A}{\partial CS^A} = -(1-n_i^A) < 0,$$
$$\frac{\partial g_i^A}{\partial CS^B} = 0,$$
$$\frac{\partial g_i^B}{\partial CS^A} = 0,$$
$$\frac{\partial g_i^B}{\partial CS^B} = -(1-n_i^B) < 0.$$

Fix $CS^A$ and let $\widehat{CS}^B(CS^A)$ be the solution to the equation

$$\sum_{i=1,\ldots,M} \widetilde{n}_i^B(CS^A, CS^B) = 1. \tag{20}$$

Because $\widetilde{n}_i^B$ is decreasing in $CS^B$, there exists a unique solution to the above equation, provided that it exists. To show existence, we establish that (1) $\lim_{CS^B\to\infty} \widetilde{n}_i^B(CS^A, CS^B) = 0$ and (2) $\lim_{CS^B\to-\infty} \widetilde{n}_i^B(CS^A, CS^B) = 1$.

On (1): to satisfy $g_i^B = 0$ while letting $CS^B \to \infty$, we must have $(1-\alpha^B)\log n_i^B - \beta^B \log n_i^A \to \infty$ or $n_i^A/n_i^B \to \infty$. In the former case, we must have $(1-\alpha^A)\log n_i^A - \beta^A \log n_i^B \to -\infty$ and thus $g_i^A \to \infty$, violating the requirement that $g_i^B = 0$. Hence, we must have $n_i^A/n_i^B \to \infty$, implying that $\lim_{CS^B\to\infty} \widetilde{n}_i^B(CS^A, CS^B) = 0$.

On (2): suppose that $CS^B \to -\infty$. In this case, we must have $\widetilde{n}_i^B(CS^A, CS^B) \to 1$, because $g_i^B \to \infty$ otherwise.

Thus, we have shown that

$$\lim_{CS^B\to\infty} \sum_{i=1}^{M} \widetilde{n}_i^B(CS^A, CS^B) = 0 < 1,$$
$$\lim_{CS^B\to-\infty} \sum_{i=1}^{M} \widetilde{n}_i^B(CS^A, CS^B) = M > 1.$$

By the intermediate value theorem and the monotonicity of $\widetilde{n}_i^B$ in $CS^B$, there exists a unique value $\widehat{CS}^B(H^A)$ that satisfies the equation $\sum_{i=1}^{M} \widetilde{n}_i^B(CS^A, CS^B) = 1$ given any $CS^A$.

Next, let $CS^A$ vary while requiring that $CS^B = \widehat{CS}^B(CS^A)$. Let $CS^A \to \infty$. We must have

$$(1-\alpha^A)\log n_i^A - \beta^A \log n_i^B \to -\infty$$

or

$$\frac{n_i^B}{n_i^A} \to \infty.$$

Both of these conditions require that $n_i^A$ converges to 0. As $CS^A \to -\infty$, we must have that for each $i$,

$$(1-n_i^A)\left[(1-\alpha^A)\log n_i^A - \beta^A \log n_i^B + CS^A\right]$$

is finite, which requires that either $n_i^B \to 0$ or $n_i^A \to 1$. Suppose that there exists a platform with $n_i^B \to 0$. This implies that $\widehat{CS}^B(CS^A) \to \infty$, because $g_i^B \to \infty$ otherwise. However, then we must have $\widetilde{n}_j^B \to 0$ for all $j$, which contradicts the condition $\sum_{i=1}^M \widetilde{n}_j^B(CS^A, \widehat{CS}^B(CS^A)) = 1$. Thus, there can be no $i$ such that $n_i^B \to 0$ as $CS^A \to -\infty$. Therefore,

$$\lim_{CS^A \to -\infty} \widetilde{n}_i^A(CS^A, \widehat{CS}^B(CS^A)) = 1$$

for all $i \in \{1, \dots, M\}$. Hence, we have

$$\lim_{CS^A \to \infty} \sum_{i=1}^M \widetilde{n}_i^A(CS^A, \widehat{CS}^B(CS^A)) = 0 < 1,$$
$$\lim_{CS^A \to -\infty} \sum_{i=1}^M \widetilde{n}_i^A(CS^A, \widehat{CS}^B(CS^A)) = M > 1.$$

As the last step to establish equilibrium existence, the intermediate value theorem implies that there exists a solution to the equilibrium condition

$$\sum_{i=1}^M \widetilde{n}_i^A(CS^A, \widehat{CS}^B(CS^A)) = 1. \tag{21}$$

To establish that equilibria are ordered in terms of user surplus, we note that

$$\frac{\partial \widehat{CS}^B(CS^A)}{\partial CS^A} = -\frac{\sum_{i=1}^M \frac{\partial \widetilde{n}_i^B(CS^A, CS^B)}{\partial CS^A}}{\sum_{i=1}^M \frac{\partial \widetilde{n}_i^B(CS^A, CS^B)}{\partial CS^B}} \leq 0.$$

For any equilibrium values of $CS^A$, $CS_1^{A*}$ and $CS_2^{A*}$ such that $CS_1^{A*} > CS_2^{A*}$, we have

$$CS_1^{B*} = \widehat{CS}^B(CS_1^{A*}) < \widehat{CS}^B(CS_2^{A*}) = CS_2^{B*}.$$

Therefore, equilibria are ranked in terms of group-$A$ or group-$B$ user surplus.

There exists $CS^A$-maximal and $CS^A$-minimal equilibria, the former of which minimizes $CS^B$, and the latter maximizes it. These extremal equilibria are stable, as

$$\left(\sum_{i=1}^{M} \frac{\partial \widetilde{n}_i^A}{\partial CS^A}\right)\left(\sum_{i=1}^{M} \frac{\partial \widetilde{n}_i^B}{\partial CS^B}\right) > \left(\sum_{i=1}^{M} \frac{\partial \widetilde{n}_i^A}{\partial CS^B}\right)\left(\sum_{i=1}^{M} \frac{\partial \widetilde{n}_i^B}{\partial CS^A}\right)$$

holds at extremal equilibria. □

*Proof of Proposition 5.* Consider the type $(\tau_i^A, \tau_i^B)$ of a platform that is consistent with the network sizes $(n_i^A, n_i^B)$ and aggregates $(H^A, H^B)$. Solving explicitly for $(\tau_i^A, \tau_i^B)$ that is consistent with $(n_i^A, n_i^B)$ and $(H^A, H^B)$, we obtain the solution

$$\tau_i^k = \frac{\left(n_i^k H^k\right)^{1-\alpha^k}}{\left(n_i^l H^l\right)^{\beta^k}} \exp\left[\frac{1}{1-n_i^k}\left(1-\alpha^k-\beta^l \frac{n_i^l}{n_i^k}\right)\right].$$

This completes the first part of the proof.

Letting $n = (n_i^A, n_i^B)_{i\in\{1,\ldots,M\}}$, we can write

$$\sum_{j=1}^{M} \tau_j^k = \mathcal{T}^k(n)\left(H^k\right)^{1-\alpha^k}\left(H^l\right)^{-\beta^k},$$

where

$$\mathcal{T}^k(n) = \sum_{j=1}^{M} \left(n_j^k\right)^{1-\alpha^k}\left(n_j^l\right)^{-\beta^k} \exp\left[\frac{1}{1-n_j^k}\left(1-\alpha^k-\beta^l \frac{n_j^l}{n_j^k}\right)\right]$$

For the chosen $(\bar{\tau}^A, \bar{\tau}^B)$, set

$$H^k = \left(\frac{\bar{\tau}^k}{\mathcal{T}_k(n)}\right)^{\Gamma^{kk}}\left(\frac{\bar{\mathcal{T}}^l}{\mathcal{T}^l(n)}\right)^{\Gamma^{kl}},$$

we obtain the type profiles $(\tau_i^A, \tau_i^B)_{i\in\{1,\ldots,M\}}$ such that $\sum_{j=1}^{M} \tau_j^k = \bar{\tau}^k$, and the network sizes are consistent with the aggregates $(H^A, H^B)$, which completes the second part of the proof. □

*Proof of Proposition 6.* Using the definitions of $\widetilde{f}_i^k$ from the proof of Lemma 3, we have

$$\frac{\partial \widetilde{f}_i^A}{\partial v_i^A} = 1 - n_i^A,$$
$$\frac{\partial \widetilde{f}_i^B}{\partial v_i^A} = 0$$

By applying the implicit function theorem, it can easily be shown that, as an equilibrium property, $\partial h_i^A/\partial v_i^A > 0$ and $\partial h_i^B/\partial v_i^A \geq 0$, as shown in the proof of Lemma 2. We note that $\partial h_i^B/\partial v_i^A > 0$ if and only if $\beta^A$ or $\beta^B$ is strictly positive. Since $n_i^k = h_i^k/H^k$, the result follows. $\square$

*Proof of Proposition 7.* We consider $v_i^A > v_j^A$, $v_i^B = v_j^B$. Denote price-cost margin for group $k$ as $\mu_i^k = p_i^k - c_i^k$. Differences across platform depend only on cost-adjusted quality offered to group $A$, $v_i^A = a_i^A - c_i^A$. We can write
$\mu^k(v_i^A) = \mu^k(v_j^A) + \int_{v_j^A}^{v_i^A} \frac{d\mu^k(v^A)}{dv^A} dv^A$.

Hence, $\mu^k(v_i^A) > \mu^k(v_j^A)$ is implied by $\frac{d\mu^k(v^A)}{dv^A} > 0$. We express price-cost margins using the formula from Lemma 1.

$$p_i^B(h_i^A, h_i^B) - c_i^B = v_i^B - (1-\alpha^B)\log h_i^B + \beta^B \log h_i^A.$$

As in Proposition 6, we use the definitions of $\widetilde{f}_i^k$ from the proof of Lemma 3,Sign $\left(\frac{d\left[p_i^B(h_i^A,h_i^{B+}) - c_i^B\right]}{dv_i^A}\right) =$ Sign $\left(\frac{\partial h_i^A}{\partial v_i^A}\frac{\beta^B}{h_i^A} - \frac{\partial h_i^B}{\partial v_i^A}\frac{1-\alpha^B}{h_i^B}\right)$ which is positive if and only if

$$\beta^B \left\{ \frac{(1-n_i^B)^2 + n_i^B}{1-n_i^B}(1-\alpha^B) - \beta^A \frac{n_i^A}{1-n_i^B} + \beta^B \frac{n_i^A}{n_i^B} \right\} - (1-\alpha^B)\left[\beta^B(1-n_i^B) + \beta^A \frac{n_i^A}{n_i^B}\right]$$

is positive. This simplifies to

$$\frac{n_i^B}{1-n_i^B}\beta^B(1-\alpha^B) - \beta^A\beta^B \frac{n_i^A}{1-n_i^B} + (\beta^B)^2 \frac{n_i^A}{n_i^B} - (1-\alpha^B)\beta^A \frac{n_i^A}{n_i^B} > 0.$$

We now prove the statement of the proposition. (i) If $\beta^A > 0$ and $\beta^B = 0$, the above expression is negative, so $p_i^B - c_i^B$ is decreasing in $v_i^A$, implying that $p_i^B - c_i^B < p_j^B - c_j^B$ when $v_i^A > v_j^A$ and $v_i^B = v_j^B$. (ii) If $\beta^B > 0$ and $\beta^A = 0$, the above expression is positive, so $p_i^B - c_i^B$ is increasing in $v_i^A$, implying that $p_i^B - c_i^B > p_j^B - c_j^B$ when $v_i^A > v_j^A$ and $v_i^B = v_j^B$. $\square$

*Proof of Proposition 8.* Similar to the proof of Proposition 7, this proof relies on small variations of the cost-adjusted platform quality, to establish how the price-cost margin given in

Lemma 1, $p_i^A(h_i^A, h_i^B) - c_i^A = v_i^A - (1-\alpha^A)\log h_i^A + \beta^A \log h_i^B$, reacts. We have

$$
\begin{aligned}
& \text{Sign}\left(\frac{d[p_i^A(h_i^{A+}, p_i^{B+}) - c_i^A]}{dv_i^A}\right) \\
= \quad & \text{Sign}\left(1 - (1-n_i^A)\frac{-(1-\alpha^A)\frac{1}{h_i^A}\frac{\partial \tilde{f}_i^B}{\partial h_i^B} - \beta^A \frac{1}{h_i^B}\frac{\partial \tilde{f}_i^B}{\partial h_i^A}}{\frac{\partial \tilde{f}_i^A}{\partial h_i^A}\frac{\partial \tilde{f}_i^B}{\partial h_i^B} - \frac{\partial \tilde{f}_i^A}{\partial h_i^B}\frac{\partial \tilde{f}_i^B}{\partial h_i^A}}\right)
\end{aligned}
$$

After some calculations, it turns out that the above expression has the same sign as

$$
\begin{aligned}
& \left[n_i^A(p_i^A - c_i^A) + \beta^B \frac{n_i^B}{n_i^A}\right]\left[n_i^B(p_i^B - c_i^B) + \beta^A \frac{n_i^A}{n_i^B} + (1-\alpha^B)(1-n_i^B)\right] \\
& - \beta^B\left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \\
= & \left[\frac{n_i^A}{1-n_i^A}(1-\alpha^A) + \beta^B \frac{n_i^B}{n_i^A}\left(1 - \frac{n_i^A}{1-n_i^A}\right)\right]\left[\left(\frac{n_i^B}{1-n_i^B} + 1 - n_i^B\right)(1-\alpha^B) + \beta^A \frac{n_i^A}{n_i^B}\left(1 - \frac{n_i^B}{1-n_i^B}\right)\right] \\
& - \beta^B\left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \\
\geq & \left[\beta^B \frac{(1-n_i^B)n_i^B + (n_i^A - n_i^B)^2}{(1-n_i^A)n_i^A}\right]\left[\left(\frac{n_i^B}{1-n_i^B} + 1 - n_i^B\right)(1-\alpha^B) + \beta^A \frac{n_i^A}{n_i^B}\left(1 - \frac{n_i^B}{1-n_i^B}\right)\right] \\
& - \beta^B\left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right]
\end{aligned}
$$

where we obtain the last expression from the following calculation:

$$
\begin{aligned}
& \frac{n_i^A}{1-n_i^A}(1-\alpha^A) + \beta^B \frac{n_i^B}{n_i^A}\left(1 - \frac{n_i^A}{1-n_i^A}\right) \\
= \quad & \frac{(n_i^A)^2(1-\alpha^A) + \beta^B n_i^B(1-2n_i^A)}{(1-n_i^A)n_i^A} \\
\geq \quad & \beta^B \frac{(n_i^A)^2 + n_i^B - 2n_i^A n_i^B}{(1-n_i^A)n_i^A} \\
= \quad & \beta^B \frac{(1-n_i^B)n_i^B + (n_i^A - n_i^B)^2}{(1-n_i^A)n_i^A}.
\end{aligned}
$$

In the following cases, $d(p_i^A - c_i^A)/dv_i^A$ positive:

1. When $\beta^B = 0$: this is straightforward.

2. When $\beta^A = 0$: to see this, for $\beta^A = 0$, the expression under consideration is

$$\begin{aligned}
& \left[\frac{n_i^A}{1-n_i^A}(1-\alpha^A) + \beta^B \frac{n_i^B}{n_i^A}\left(1 - \frac{n_i^A}{1-n_i^A}\right)\right]\left(\frac{n_i^B}{1-n_i^B} + 1 - n_i^B\right)(1-\alpha^B) \\
& -(\beta^B)^2 \frac{n_i^B}{n_i^A}(1-n_i^B) \\
> \; & (\beta^B)^2 \frac{n_i^B}{n_i^A} \frac{n_i^B + (1-n_i^B)^2 - (1-n_i^B)}{(1-n_i^A)n_i^A} \\
> \; & 0.
\end{aligned}$$

3. When $n_i^A \geq n_i^B$: to see this, the expression under consideration is greater than

$$\begin{aligned}
& \beta^B \frac{(1-n_i^B)n_i^B}{(1-n_i^A)n_i^A}\left[\beta^A \frac{(1-n_i^A)n_i^A}{(1-n_i^B)n_i^B} + \beta^B(1-n_i^B)\right] \\
= \; & \beta^B \left[\beta^A + \frac{(1-n_i^B)}{(1-n_i^A)} \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \\
\geq \; & \beta^B \left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right]
\end{aligned}$$

because $(1-n_i^B)/(1-n_i^A) \geq 1$ when $n_i^A \geq n_i^B$.

Therefore, if corresponding values of $(n^A, n^B)$ satisfy $n^A \geq n_i^B$ for all $v^A \in [v_j^A, v_i^A]$, we have the desired result. To establish this, it suffices to examine that $d(n^A/n^B)/dv^A > 0$ at $v^A$ such that $n^A = n^B$. To see this, note that

$$\begin{aligned}
& \text{Sign}\left[\frac{d}{dv_i^A}\left(\frac{n_i^A}{n_i^B}\right)\right] \\
= \; & \text{Sign}\left[\frac{1}{h_i^A}\frac{\partial h_i^A}{\partial v_i^A} - \frac{1}{h_i^B}\frac{\partial h_i^B}{\partial v_i^A}\right] \\
= \; & \text{Sign}\left(-\frac{1}{h_i^A}\frac{\partial \widetilde{f}^B}{\partial h_i^B} - \frac{1}{h_i^B}\frac{\partial \widetilde{f}_i^B}{\partial h_i^A}\right) \\
= \; & \text{Sign}\left[n_i^A(p_i^A - c_i^A) + (1-n_i^B)(1-\alpha^B-\beta^B)\right],
\end{aligned}$$

which is positive as long as $p_i^A - c_i^A \geq 0$, which always holds when $n_i^A \geq n_i^B$. Therefore, along the path from $v_j^A$ to $v_i^A$, if $n_j^A \geq n_j^B$, $n^A \geq n^B$ always holds for the intermediate values of $v^A$, holding other parameters fixed. Therefore, $p^A - c^A$ also monotonically increases on this path, establishing that $p_i^A - c_i^A > p_j^A - c_j^A$.

4. When $p_i^A - c_i^A \geq 0$ and $p_i^B - c_i^B \geq 0$: we consider the case where $n_i^B > n_i^A$ because we have established the desired result in the case where $n_i^A \geq n_i^B$. First, it is straightforward that a small change in $v_i^A$ increases $p_i^A - c_i^A$ in this case. What needs to be shown is that $p_i^A - c_i^A$

continuously increases as $v_i^A$ increases more. To see this, note that a small increase in $v_i^A$ increases $n_i^A/n_i^B$ if $p_i^A - c_i^A \geq 0$. Therefore, the sign of $p_i^A - c_i^A$, which is determined by the sign of $1 - \alpha^A - \beta^B n_i^B/n_i^A$, is positive for all $v \in [\underline{v}_i^A, \overline{v}_i^A]$ as long as $p_i^A - c_i^A \geq 0$ at $\underline{v}_i^A$. Furthermore, because $p_i^B - c_i^B \geq 0$ whenever $n_i^B > n_i^A$. Put together, at any point on the path $[\underline{v}_i^A, \overline{v}_i^A]$, the signs of $p_i^A - c_i^A$ and $p_i^B - c_i^B$ are positive, implying that the local increase in $p_i^A - c_i^A$ continues until the end. This establishes that $p_i^A - c_i^A > p_j^A - c_j^A$.

The remaining case is $\beta^A > 0$, $\beta^B > 0$, and $n_i^B > n_i^A$. It turns out that $d(p_i^A - c_i^A)/dv_i^A < 0$ may hold in this case. We show this by example. Suppose that $\beta^A = \beta^B = 0.995$, $\alpha^A = \alpha^B = 0$, $n_i^A = 0.2$, and $n_i^B = 0.25$. By Proposition 5, we can obtain these participation levels with appropriate choices of the primitives of the model. Then,

$$
\begin{aligned}
&\left[\frac{n_i^A}{1-n_i^A}(1-\alpha^A) + \beta^B \frac{n_i^B}{n_i^A}\left(1 - \frac{n_i^A}{1-n_i^A}\right)\right]\left[\left(\frac{n_i^B}{1-n_i^B} + 1 - n_i^B\right)(1-\alpha^B) + \beta^A \frac{n_i^A}{n_i^B}\left(1 - \frac{n_i^B}{1-n_i^B}\right)\right] \\
&- \beta^B \left[\beta^A + \beta^B \frac{n_i^B}{n_i^A}(1-n_i^B)\right] \\
&= -0.000911406.
\end{aligned}
$$

□

*Proof of Proposition 9.* We show each of the statements of the proposition.

1. Fix the characteristics of an entrant $(a_E^A, c_E^A, a_E^B, c_E^B)$. Then, we show that if $\max\{\beta^A, \beta^B\}$ is sufficiently small, the user surpluses for both user groups increase with entry. To see this, consider the limit case of zero cross-group network effects (i.e. $\beta^A = \beta^B = 0$). When $\beta^A = \beta^B = 0$, $\widetilde{n}_i^k(CS^A, CS^B)$ depends only on $CS^k$. We have $\widetilde{n}_i^A(CS^A, CS^B) = \widetilde{n}_i^A(CS^A, 0)$ and $\widetilde{n}_i^B(0, CS^B)$. Given the pre-entry equilibrium user surplus $(CS^{A*}, CS^{B*})$ and

$$
\begin{aligned}
\sum_{i=1}^M \widetilde{n}_i^A(CS^{A*}, 0) + \widetilde{n}_E^A(CS^{A*}, 0) > 1, \\
\sum_{i=1}^M \widetilde{n}_i^B(0, CS^{B*}) + \widetilde{n}_E^B(0, CS^{B*}) > 1,
\end{aligned}
$$

   the post-entry user surplus $(CS^{A**}, CS^{B**})$ must satisfy $CS^{A**} > CS^{A*}$ and $CS^{B**} > CS^{B*}$. Because of the continuity of the model in parameters, we obtain the statement.

2. Take group-A optimal equilibrium $(CS^{A*}, CA^{B*})$, which is also the equilibrium that minimizes the group-B user surplus. Let $CS^{A**} > CS^{A*}$ and $CS^{B**} = \widehat{CS}^B(CS^{A**}) + \epsilon$,

where $\epsilon > 0$ is a sufficiently small positive number such that $\widehat{CS}^B(CS^{A**}) + \epsilon < CS^{B*}$. Then, at $(CS^{A**}, CS^{B**})$,

$$\sum_i \widetilde{n}_i^A(CS^{A**}, CS^{B**}) < 1,$$
$$\sum_i \widetilde{n}_i^B(CS^{A**}, CS^{B**}) < 1.$$

This pair of $(CS^{A**}, CS^{B**})$ is consistent with entry of platform $E$ with post-entry market shares

$$n_E^A = 1 - \sum_i \widetilde{n}_i^A(CS^{A**}, CS^{B**}) > 0,$$
$$n_E^B = 1 - \sum_i \widetilde{n}_i^B(CS^{A**}, CS^{B**}) > 0.$$

Then, the proof of Proposition 5 implies that there exists a type of platform that is consistent with $(n_E^A, n_E^B)$ and $(CS^{A**}, CS^{B**})$. Hence, there exists platform entry that induces $(CS^{A**}, CS^{B**})$ as an equilibrium outcome, and the lowest equilibrium group-B user surplus is lower after the entry than the pre-entry level.

3. Consider the entry of new platform $E$ with characteristics $(a_E^A, c_E^A, a_E^B, c_E^B)$. At any pre-entry equilibrium user surplus $(CS^A, CS^B)$, we have

$$\sum_{i=1}^{M} \widetilde{n}_i^k(CS^A, CS^B) + \widetilde{n}_E^k(CS^A, CS^B) = 1 + \widetilde{n}_E^k(CS^A, CS^B) > 1$$

for $k = A, B$. Since $\widetilde{n}_j^k(CS^A, CS^B)$ is decreasing in $(CS^A, CS^B)$, $CS^A$ or $CS^B$ must be greater than the pre-entry level. Now, take the group-$A$ optimal equilibrium user surplus $(CS^{A*}, CS^{B*})$. Then, there must be a post-entry equilibrium user surplus $(CS^{A**}, CS^{B**})$ that satisfies $CS^{A**} > CS^{A*}$ or $CS^{B**} > CS^{B*}$, implying that maximal group-$A$ user surplus or minimal group-$B$ user surplus increases with entry.

$\square$

*Proof of Proposition 10.* Define $\Pi_E(n_E^A, n_E^B)$ by

$$\begin{aligned}\Pi_E(n_E^A, n_E^B) &= \frac{n_E^A}{1-n_E^A}\left(1 - \alpha^A - \beta^B \frac{n_E^B}{n_E^A}\right) + \frac{n_E^B}{1-n_E^B}\left(1 - \alpha^B - \beta^A \frac{n_E^A}{n_E^B}\right) \\ &= \frac{n_E^A(1-\alpha_E^A) - n_E^B\beta^B}{1-n_E^A} + \frac{n_E^B(1-\alpha^B) - n_E^A\beta^A}{1-n_E^B}\end{aligned}$$

The free entry condition is $\Pi_E(n_E^A, n_E^B) - K = 0$.

The equilibrium surplus of group-$k$ users can be written as the function of the entrant's characteristics $v_E^k = a_E^k - c_E^k$ and their equilibrium network sizes $(n_E^A, n_E^B)$:

$$CS^k = v_E^k + \beta^k \log n_E^l - (1-\alpha^k)\log n_E^k - \frac{1-\alpha^k - \beta^l \frac{n_E^l}{n_E^k}}{1-n_E^k}.$$

Suppose that $n_E^A \geq n_E^B$. We write $n_E^A$ and $n_E^B$ as $n_E^B = n$ and $n_E^A = n\theta$ noting that $\theta \geq 1$ and $n \leq 1/\theta$. We define

$$\overline{\Pi}_E(\theta, n) := \Pi_E(\theta n, n).$$

Surplus of the two user groups is

$$\begin{aligned}
CS^A &= v_E^A - (1-\alpha^A-\beta^A)\log n - (1-\alpha^A)\log\theta - \frac{1-\alpha^A - \frac{\beta^B}{\theta}}{1-\theta n}, \\
CS^B &= v_E^B - (1-\alpha^B-\beta^B)\log n + \beta^B\log\theta - \frac{1-\alpha^B-\beta^A\theta}{1-n}.
\end{aligned}$$

The free entry condition $\overline{\Pi}(\theta, n) - K = 0$ has a unique solution in $n(\theta) \in (0, 1/\theta)$ such that $n'(\theta) \in (-n/\theta, 0)$ given $\theta \geq 1$, as shown next.

Noting that
$$\frac{\partial \Pi_E}{\partial n_E^k}(n_E^A, n_E^B) = \frac{1-\alpha^k - n_E^l\beta^l}{(1-n_E^k)^2} - \frac{\beta^k}{1-n_E^l}.$$
and using the implicit function theorem, we have

$$n'(\theta) = -\frac{W(\theta)}{V(\theta)},$$

where

$$\begin{aligned}
W(\theta) = \frac{\partial \overline{\Pi}_E}{\partial \theta}(\theta, n) &= n\frac{\partial \Pi_E}{\partial n_E^A}(\theta n, n) \\
&= n\left[\frac{1}{(1-\theta n)^2}(1-\alpha^A - n\beta^B) - \frac{\beta^A}{1-n}\right] \\
&\geq \frac{n}{(1-\theta n)^2}\left[1-\alpha^A - n\beta^B - (1-\theta n)\beta^A\right] \\
&\geq \frac{n}{(1-\theta n)^2}\left[n(1-\alpha^A-\beta^B) + (1-\theta n)(1-\alpha^A-\beta^A)\right] \\
&> 0
\end{aligned}$$

for all $\theta \in [1, 1/n)$, and

$$\begin{aligned} V(\theta) &= \frac{\partial \overline{\Pi}_E}{\partial n}(\theta, n) \\ &= \theta \frac{\partial \Pi_E}{\partial n_E^A}(\theta n, n) + \frac{\partial \Pi_E}{\partial n_E^B}(\theta n, n) \\ &= \frac{\theta}{n} W(\theta) + X, \end{aligned}$$

where

$$X := \frac{\partial \Pi_E}{\partial n_E^B}(\theta n, n) = \frac{1 - \alpha^B - n\theta\beta^A}{(1-n)^2} - \frac{\beta^B}{1-\theta n}.$$

Hence, if $X > 0$, we have

$$n'(\theta) = -\frac{W(\theta)}{\frac{\theta}{n} W(\theta) + X} \in \left(-\frac{n}{\theta}, 0\right).$$

We now show that under the condition stated in Proposition 10, $X > 0$ holds. First, note that by Proposition 6, we have $n_i^k \geq n_E^k$ for all $i = 1, \dots, M_I$ because we assume $v_i^k \geq v_E^k$ for all $i = 1, \dots, M_I$, implying that $\theta n \leq 1/(M_I + M_E)$ . Next, we have

$$\begin{aligned} X &= \frac{1 - \alpha^B - n\theta\beta^A}{(1-n)^2} - \frac{\beta^B}{1-\theta n} \\ &\geq \frac{1}{(1-n)^2}\left[1 - \bar{\alpha} - \bar{\beta}\left(\theta n + \frac{(1-n)^2}{1-\theta n}\right)\right] \\ &\geq \frac{1}{(1-n)^2}\left[1 - \bar{\alpha} - \bar{\beta}\left(\frac{1}{M_I + M_E} + \frac{(1-n)^2}{1 - \frac{1}{M_I + M_E}}\right)\right] \\ &\geq \frac{1}{(1-n)^2}\left[1 - \bar{\alpha} - \bar{\beta}\frac{(M_I + M_E)^2 + M_I + M_E - 1}{(M_I + M_E)(M_I + M_E - 1)}\right], \end{aligned}$$

where $\bar{\alpha} := \max\{\alpha^A, \alpha^B\}$ and $\bar{\beta} := \max\{\beta^A, \beta^B\}$, and we used $\theta n \leq 1/(M_I + M_E)$ to show the second inequality and $(1-n)^2 \leq 1$ to show the last inequality. Defining

$$\zeta(M) := \frac{M^2 + M - 1}{M(M-1)}, \tag{22}$$

we have that if $1 - \bar{\alpha} \geq \bar{\beta}\zeta(M_I + M_E)$, it must hold that $X > 0$ for all relevant values of $(\theta, n)$.

Finally, we show that $\zeta(M)$ is decreasing in $M$ and $\zeta(M) \in (1, 5/2]$. At $M = 2$, we have

$$\zeta(2) = \frac{2^2 + 1}{2} = \frac{5}{2}.$$

The derivative of $\zeta$ is

$$\zeta'(M) = \frac{(2M+1)M(M-1) - (2M-1)(M^2+M-1)}{[M(M-1)]^2} = \frac{-2M^2+2M-1}{M^2(M-1)^2} < 0.$$

Finally, we have

$$\lim_{M\to\infty} \zeta(M) = \lim_{M\to\infty} \left( \frac{1+\frac{1}{M}-\frac{1}{M^2}}{1-\frac{1}{M}} \right) = 1.$$

This establishes that $\zeta(M)$ is decreasing and takes values $\zeta(M) \in (1, 5/2]$.

Put together, under the condition that $1-\bar{\alpha} \geq \bar{\beta}\zeta(M_I + M_E)$, we have $n'(\theta) \in (-n/\theta, 0)$.

Using this result, we show that $dCS^A/d\theta < 0$ and $dCS^B/d\theta > 0$. To see that $dCS^A/d\theta < 0$, note that

$$\frac{dCS^A}{d\theta} = \frac{\partial CS^A}{\partial \theta} + n'(\theta)\frac{\partial CS^A}{\partial n}.$$

We have

$$\frac{\partial CS^A}{\partial \theta} = -\frac{1-\alpha^A}{\theta} - \frac{\beta^B}{1-\theta n}\frac{1}{\theta^2} - \frac{n}{(1-\theta n)^2}\left(1-\alpha^A - \frac{\beta^B}{\theta}\right),$$

and

$$\frac{\partial CS^A}{\partial n} = -\frac{1-\alpha^A-\beta^A}{n} - \frac{\theta}{(1-\theta n)^2}\left(1-\alpha^A-\frac{\beta^B}{\theta}\right) < 0.$$

Hence, we have

$$\begin{aligned}
\frac{dCS^A}{d\theta} &= -\frac{1-\alpha^A}{\theta} - \frac{\beta^B}{1-\theta n}\frac{1}{\theta^2} - \left(1+\frac{\theta}{n}n'(\theta)\right)\frac{n}{(1-\theta n)^2}\left(1-\alpha^A-\frac{\beta^B}{\theta}\right) - n'(\theta)\frac{1-\alpha^A-\beta^A}{n} \\
&\leq -\frac{1-\alpha^A}{\theta} - \frac{\beta^B}{1-\theta n}\frac{1}{\theta^2} - \left(1+\frac{\theta}{n}n'(\theta)\right)\frac{n}{(1-\theta n)^2}\left(1-\alpha^A-\frac{\beta^B}{\theta}\right) + \frac{1-\alpha^A-\beta^A}{\theta} \\
&\leq -\frac{\beta^A}{\theta} - \frac{\beta^B}{(1-\theta n)\theta^2} < 0,
\end{aligned}$$

because $n'(\theta) \in (-n/\theta, 0)$. Next, we show that $dCS^B/d\theta > 0$. To see this, note that

$$\frac{dCS^B}{d\theta} = \frac{\partial CS^B}{\partial \theta} + n'(\theta)\frac{\partial CS^B}{\partial n},$$

$$\frac{\partial CS^B}{\partial \theta} = \frac{\beta^B}{\theta} + \frac{\beta^A}{1-n} > 0,$$

and

$$\frac{\partial CS^B}{\partial n} = -\frac{1-\alpha^B-\beta^B}{n} - \frac{1-\alpha^B-\beta^A\theta}{(1-n)^2}.$$

If $\partial CS^B/\partial n < 0$, $dCS^B/d\theta > 0$. If $\partial CS^B/\partial n > 0$, we have that

$$\begin{aligned}
\frac{dCS^B}{d\theta} &= \frac{\beta^B}{\theta} + \frac{\beta^A}{1-n} + n'(\theta)\left[-\frac{1-\alpha^B-\beta^B}{n} - \frac{1-\alpha^B-\beta^A\theta}{(1-n)^2}\right] \\
&\geq \frac{\beta^B}{\theta} + \frac{\beta^A}{1-n} + \frac{n}{\theta}\left[\frac{1-\alpha^B-\beta^B}{n} + \frac{1-\alpha^B-\beta^A\theta}{(1-n)^2}\right] \\
&= \frac{\beta^A}{1-n} + \frac{n}{\theta}\left[\frac{1-\alpha^B}{n} + \frac{1-\alpha^B-\beta^A\theta}{(1-n)^2}\right] \\
&\geq \frac{\beta^A}{\theta}\left[1 + \frac{\theta}{1-n} - \frac{n(\theta-1)}{(1-n)^2}\right] \\
&= \frac{\beta^A}{\theta(1-n)^2}\left[(1-n)^2 + \theta - 2\theta n + n\right] \\
&= \frac{\beta^A}{\theta(1-n)^2}\underbrace{\left(1-n+n^2+\theta-2\theta n\right)}_{Z_{CS}(\theta,n)},
\end{aligned}$$

where $Z_{CS}$ is decreasing in $n \in [0, 1/\theta]$ and, thus, minimized at $n = 1/\theta$ in the range $[0, 1/\theta]$ for any given $\theta$. At the minimum it takes value

$$\begin{aligned}
Z_{CS}(\theta, 1/\theta) &= -1 - \frac{1}{\theta} + \frac{1}{\theta^2} + \theta \\
&= \frac{\theta^3 - \theta^2 - \theta + 1}{\theta^2} \\
&= \frac{(\theta-1)(\theta^2-1)}{\theta^2} \geq 0
\end{aligned}$$

for all $\theta \geq 1$, implying that $dCS^B/d\theta > 0$. Therefore, along the $\Pi_E = K$ curve with $n_E^A/n_E^B \geq 1$, $CS^B$ is increasing in $(n_E^A/n_E^B)$ and $CS^A$ is decreasing in $(n_E^A/n_E^B)$.

Next, consider the case that $n_E^B > n_E^A$. Then, we can write $(n_E^A, n_E^B)$ as $(n_E^A, n_E^B) = (n, \theta' n)$, where $\theta' > 1$. Using the same logic, we can show that $CS^A$ increases with $\theta'$ and $CS^B$ decreases with $\theta'$. Therefore, along the $\Pi_E = K$ curve with $n_E^A/n_E^B \leq 1$, again $CS^B$ is increasing in $(n_E^A/n_E^B)$ and $CS^A$ is decreasing in $(n_E^A/n_E^B)$.

Finally, compare the long-run equilibria with the same parameters $(\alpha^A, \alpha^B, \beta^A, \beta^B, a_E^A, a_E^B, c_E^A, c_E^B, K)$ but different parameters other than that. Then, the two equilibria are characterized by the network sizes of entrants $(n_E^{A*}, n_E^{B*})$ and $(n_E^{A*}, n_E^{B*})$ along the $\Pi_E = K$ curve. Let $(CS^{A*}, CS^{B*})$ and $(CS^{A**}, CS^{B**})$ be the surplus of the two user groups in the respective equilibria. Suppose without loss of generality that $n_E^{A*}/n_E^{B*} \geq n_E^{A**}/n_E^{B**}$. Then, we have $CS^{A*} \leq CS^{A**}$ and $CS^{B*} \geq CS^{B**}$ with equality if and only if $(n_0^A, n_0^B) = (n_1^A, n_1^B)$. Therefore, we have

$$(CS^{A*} - CS^{A**})(CS^{B*} - CS^{B**}) < 0,$$

which completes the proof. $\square$

*Proof of Proposition 11.* The market share of platform $i$ changes with $\tilde{\alpha}^A$

$$\begin{aligned}\frac{d\widetilde{n}_i^A}{d\tilde{\alpha}^A} &= \frac{v_i^A}{(1-\tilde{\alpha}^A)^2} \frac{1-n_i^A}{\frac{1}{1-n_i^A} + \frac{1-n_i^A}{n_i^A}} \\ &= -\frac{H^A v_i^A}{(1-\tilde{\alpha}^A)^2} \frac{\partial \widetilde{n}_i^A}{\partial H^A}.\end{aligned}$$

The aggregate $H^{A*}$ changes with $\tilde{\alpha}^A$ as follows:

$$\begin{aligned}\frac{dH^{A*}}{d\tilde{\alpha}^A} &= \frac{-\sum_{i=1}^M \frac{\partial \widetilde{n}_i^A}{\partial \tilde{\alpha}^A}}{\sum_{i=1}^M \frac{\partial \widetilde{n}_i^A}{\partial H^A}} \\ &= \frac{\sum_{i=1}^M \frac{H^A v_i^A}{(1-\tilde{\alpha}^A)^2} \frac{\partial \widetilde{n}_i^A}{\partial H^A}}{\sum_{i=1}^M \frac{\partial \widetilde{n}_i^A}{\partial H^A}}.\end{aligned}$$

We have

$$\begin{aligned}\frac{dn_i^{A*}}{d\tilde{\alpha}^A} &= \frac{\partial \widetilde{n}_i^A}{\partial \tilde{\alpha}^A} + \frac{dH^{A*}}{d\tilde{\alpha}^A} \frac{\partial \widetilde{n}_i^A}{\partial H^A} \\ &= \left[-\frac{\partial \widetilde{n}_i^A}{\partial H^A}\right] \left(\frac{H^A v_i^A}{(1-\tilde{\alpha}^A)^2} - \frac{dH^{A*}}{d\tilde{\alpha}^A}\right) \\ &= \left[-\frac{\partial \widetilde{n}_i^A}{\partial H^A}\right] \frac{H^A}{(1-\tilde{\alpha}^A)^2} \left(v_i^A - \frac{\sum_{j=1}^M v_j^A \left[-\frac{\partial \widetilde{n}_j^A}{\partial H^A}\right]}{\sum_{j=1}^M \left[-\frac{\partial \widetilde{n}_j^A}{\partial H^A}\right]}\right) \\ &= \underbrace{\left[-\frac{\partial \widetilde{n}_i^A}{\partial H^A}\right]}_{>0} \frac{H^A}{(1-\tilde{\alpha}^A)^2} \left(\frac{\sum_{j=1}^M (v_i^A - v_j^A) \left[-\frac{\partial \widetilde{n}_j^A}{\partial H^A}\right]}{\sum_{j=1}^M \left[-\frac{\partial \widetilde{n}_j^A}{\partial H^A}\right]}\right).\end{aligned}$$

Therefore, there exists a critical platform $\hat{j} \leq M-1$ such that for all $j > \hat{j}$, $dn_j^{A*}/d\tilde{\alpha}^A > 0$, and for all $j \leq \hat{j}$ with $dn_j^{A*}/d\tilde{\alpha}^A \leq 0$. This last inequality must be strict for $j = 1$. It must also be strict for all $j < \hat{j}$ in platform oligopoly provided that $v_j^A < v_{\hat{j}}^A$. Since an increase in the degree of compatibility implies a decrease in $\tilde{\alpha}^A$, the result follows. $\square$

*Proof of Proposition 12.* We first show Proposition 12-2 and then show Proposition 12-1. Here we make use of a derivation in the proof of Proposition 13 below, which does not rely on any of the results obtained in the current proof.

Supposing that $\beta^A = \beta^B = 0$, we can simplify $\Omega^A$, which appears in equation (23) in the proof of Proposition 13, to

$$\begin{aligned}\tilde{\Omega}^A(n_1^A, \Delta v^A) &= \frac{\Delta v^A}{1-\tilde{\alpha}^A} - [\log n_1^A - \log(1-n_1^A)] - \left(\frac{1}{1-n_1^A} - \frac{1}{n_1^A}\right)\\ &= \Delta\tilde{v}^A - [\log n_1^A - \log(1-n_1^A)] - \left(\frac{1}{1-n_1^A} - \frac{1}{n_1^A}\right),\end{aligned}$$

where $\Delta v^A := v_1^A - v_2^A$ and $\Delta\tilde{v}^A := \Delta v^A/(1-\tilde{\alpha}^A)$. The proof of Proposition 13 shows that the equilibrium market share of platform 1 is given by $\Omega^A(n_1^A, \Delta v^A)$. Since $\frac{\partial\tilde{\Omega}^A}{\partial\Delta\tilde{v}^A} = 1$ and

$$\begin{aligned}\frac{\partial\tilde{\Omega}^A}{\partial n_1^A} &= -\frac{1}{n_1^A} - \frac{1}{1-n_1^A} - \frac{1}{(1-n_1^A)^2} - \frac{1}{(n_1^A)^2}\\ &= -\frac{1}{n_1^A(1-n_1^A)} - \frac{2(n_1^A)^2 - 2n_1^A + 1}{(n_1^A)^2(1-n_1^A)^2}\\ &= -\frac{(n_1^A)^2 - n_1^A + 1}{(n_1^A)^2(1-n_1^A)^2},\end{aligned}$$

we can write

$$\frac{dn_1^{A*}}{d\Delta\tilde{v}^A} = \frac{(n_1^A)^2(1-n_1^A)^2}{(n_1^A)^2 - n_1^A + 1}.$$

Next, noting that

$$\frac{\partial\Delta\tilde{v}^A}{\partial\tilde{\alpha}^A} = \frac{\Delta v^A}{(1-\tilde{\alpha}^A)^2} = \frac{1}{1-\tilde{\alpha}^A}\Delta\tilde{v}^A$$

and

$$CS^A = v_1^A - (1-\tilde{\alpha}^A)\log n_1^A - \frac{1-\tilde{\alpha}^A}{1-n_1^A},$$

we have

$$\begin{aligned}\frac{\partial CS^{A*}}{\partial\tilde{\alpha}^A} &= \log n_1^A + \frac{1}{1-n_1^A} - \frac{dn^{A*}}{d\Delta\tilde{v}^A}\Delta\tilde{v}^A\left[\frac{1}{n_1^A} + \frac{1}{(1-n_1^A)^2}\right]\\ &= \log n_1^A + \frac{1}{1-n_1^A} - \frac{n_1^A}{(n_1^A)^2 - n_1^A + 1}\left((1-n_1^A)^2 + n_1^A\right)\Delta\tilde{v}^A\\ &= \log n_1^A + \frac{1}{1-n_1^A} - n_1^A\Delta\tilde{v}^A.\end{aligned}$$

Since

$$\Delta\tilde{v}^A = \log n_1^A - \log(1-n_1^A) + \frac{1}{1-n_1^A} - \frac{1}{n_1^A},$$

we can write

$$\frac{\partial CS^{A*}}{\partial \tilde{\alpha}^A} = (1-n_1^A)\log n_1^A + 2 + n_1^A \log(1-n_1^A).$$

When $n_1^A = 1/2$, the inequality

$$\frac{\partial CS^{A*}}{\partial \tilde{\alpha}^A} = -\log 2 + 2 > 0$$

holds. As $n_1^A \to 1$, we have the limit result $\frac{\partial CS^{A*}}{\partial \tilde{\alpha}^A} \to -\infty$. Finally, since

$$\frac{\partial^2 CS^{A*}}{\partial \tilde{\alpha}^A \partial n_1^A} = -\left[\log n_1^A - \log(1-n_1^A)\right] + \frac{1-n_1^A}{n_1^A} - \frac{n_1^A}{1-n_1^A} < 0,$$

there exists a critical value $\hat{n}_{1,CS}^A \in (1/2, 1)$ of the market share of platform 1 such that user surplus is decreasing in $\tilde{\alpha}^A$ if and only if $n_1^{A*} > \hat{n}_{1,CS}^A$. Because $n_1^{A*}$ is increasing in $\Delta\tilde{v}^A$, which is increasing in $\Delta v^A$, there exists $\Delta_{CS} v^A > 0$ such that $dp_1^{A*}/d\tilde{\alpha}^A > 0$ if and only if $v_1^A - v_2^A > \Delta_{CS} v^A$. Because $\tilde{\alpha}^A$ is decreasing in $\lambda$, we obtain Proposition 12-2

Next, consider the impact of $\tilde{\alpha}^A$ on prices. Since $p_1^A = \frac{1-\tilde{\alpha}^A}{1-n_1^A}$, the effect of $\tilde{\alpha}^A$ on the equilibrium prices is given by

$$\begin{aligned}
\frac{dp_1^{A*}}{d\tilde{\alpha}^A} &= -\frac{1}{1-n_1^A} + \Delta\tilde{v}^A \frac{(n_1^A)^2}{(n_1^A)^2 - n_1^A + 1} \\
&= \frac{1}{(n_1^A)^2 - n_1^A + 1}\left[(n_1^A)^2\left(\log n_1^A - \log(1-n_1^A) + \frac{1}{1-n_1^A} - \frac{1}{n_1^A}\right) - \frac{(n_1^A)^2 - n_1^A + 1}{1-n_1^A}\right] \\
&= \frac{1}{(n_1^A)^2 - n_1^A + 1}\left[(n_1^A)^2\left(\log n_1^A - \log(1-n_1^A) - \frac{1}{n_1^A}\right) - 1\right]
\end{aligned}$$

The function

$$(n_1^A)^2\left(\log n_1^A - \log(1-n_1^A)\right) - n_1^A$$

has the derivative

$$\begin{aligned}
& 2n_1^A\left(\log n_1^A - \log(1-n_1^A)\right) + n_1^A + \frac{(n_1^A)^2}{1-n_1^A} - 1 \\
= \; & 2n_1^A\left(\log n_1^A - \log(1-n_1^A)\right) + \frac{(2n_1^A - 1)}{1-n_1^A} > 0
\end{aligned}$$

for all $n_1^A > 1/2$. Thus, there exists a critical value $\hat{n}_{1,p}^A \in (1/2, 1)$ of the market share of platform 1 such that $p_1^{A*}$ is increasing in $\tilde{\alpha}^A$ if and only if $n_1^{A*} > \hat{n}_{1,p}^A$. Therefore, there exists $\Delta_p v^A > 0$ such that $dp_1^{A*}/d\tilde{\alpha}^A > 0$ if and only if $v_1^A - v_2^A > \Delta_p v^A$. Because $\tilde{\alpha}^A$ is decreasing in $\lambda$, we obtain Proposition 12-1.

$\square$

*Proof of Proposition 13.* Suppose that $M = 2$, that is, platforms are duopolists. Also suppose that $v_1^k := a_1^k - c_1^k > a_2^k - c_2^k =: v_2^k$ for $k = A, B$; that is, platform 1 is more efficient than platform 2 is. In this setting, the equilibrium object can be summarized by $(n_1^A, n_1^B)$, $n_2^k = 1 - n_1^k$ for $k = A, B$.

Noting that, from the optimality conditions (18) and (19), we have

$$CS^A = v_2^A - (1-\tilde{\alpha}^A)\log n_2^A + \tilde{\beta}^A \log n_2^B - \frac{1}{1-n_2^A}\left(1-\tilde{\alpha}^A - \tilde{\beta}^B \frac{n_2^B}{n_2^A}\right),$$
$$CS^B = v_2^B - (1-\tilde{\alpha}^B)\log n_2^B + \tilde{\beta}^B \log n_2^A - \frac{1}{1-n_2^B}\left(1-\tilde{\alpha}^B - \tilde{\beta}^A \frac{n_2^A}{n_2^B}\right),$$

the equilibrium condition for the market share of platform 1, $(n_1^A, n_1^B)$, is given by the system of equations:

$$\Omega^A(n_1^A, n_1^B, \Delta v^A) = 0$$
$$\Omega^B(n_1^A, n_1^B, \Delta v^A) = 0,$$

where

$$\begin{aligned}
\Omega^A(n_1^A, n_1^B, \Delta v^A) &= v_1^A - (1-\tilde{\alpha}^A)\log n_1^A + \tilde{\beta}^A \log n_1^B - CS^A - \frac{1}{1-n_1^A}\left(1-\tilde{\alpha}^A - \tilde{\beta}^B \frac{n_1^B}{n_1^A}\right) \\
&= v_1^A - v_2^A - (1-\tilde{\alpha}^A)\log\frac{n_1^A}{n_2^A} + \tilde{\beta}^A \log\frac{n_1^B}{n_2^B} \\
&\quad - \frac{1}{1-n_1^A}\left(1-\tilde{\alpha}^A - \tilde{\beta}^B \frac{n_1^B}{n_1^A}\right) + \frac{1}{1-n_2^A}\left(1-\tilde{\alpha}^A - \tilde{\beta}^B \frac{n_2^B}{n_2^A}\right) \\
&= \Delta v^A - (1-\tilde{\alpha}^A)\left(\log n_1^A - \log(1-n_1^A)\right) + \tilde{\beta}^A\left(\log n_1^B - \log(1-n_1^B)\right) \\
&\quad - \frac{1}{1-n_1^A}\left(1-\tilde{\alpha}^A - \tilde{\beta}^B \frac{n_1^B}{n_1^A}\right) + \frac{1}{n_1^A}\left(1-\tilde{\alpha}^A - \tilde{\beta}^B \frac{1-n_1^B}{1-n_1^A}\right) \qquad (23)
\end{aligned}$$

and $\Omega^B(n_1^A, n_1^B, \Delta v^B)$ is analogously defined.

From Proposition 7, we know that any solution to this system of equations lies in $(1/2, 1)^2$. We can rewrite $\Omega^A$ as

$$\begin{aligned}
\Omega^A(n_1^A, n_1^B, \Delta v^A) =& \Delta v^A - (1-\tilde{\alpha}^A)[\log n_1^A - \log(1-n_1^A)] + \tilde{\beta}^A[\log n_1^B - \log(1-n_1^B)] \\
& - \frac{1}{n_1^A(1-n_1^A)}\left[(1-\tilde{\alpha}^A)(2n_1^A - 1) - \tilde{\beta}^B(2n_1^B - 1)\right],
\end{aligned}$$

Thus, we have the partial derivatives with respect to $n_1^A$ and $n_1^B$:

$$\begin{aligned}
\frac{\partial\Omega^A}{\partial n_1^A} &= -(1-\tilde{\alpha}^A)\frac{1}{n_1^A(1-n_1^A)} - \frac{(2n_1^A-1)}{[n_1^A(1-n_1^A)]^2}\left[(1-\tilde{\alpha}^A)(2n_1^A-1) - \tilde{\beta}^B(2n_1^B-1)\right] \\
&\quad -\frac{2(1-\tilde{\alpha}^A)}{n_1^A(1-n_1^A)} \\
&= -\frac{1}{(n_1^A)^2(1-n_1^A)^2}\left[(1-\tilde{\alpha}^A)\left[3n_1^A(1-n_1^A)+(2n_1^A-1)^2\right] + (2n^A-1)(2n_1^B-1)\tilde{\beta}^B\right] \\
&< -\frac{1}{(n_1^A)^2(1-n_1^A)}(1-\tilde{\alpha}^A)[2-n_1^A] < 0. \\
\frac{\partial\Omega^A}{\partial n_1^B} &= \tilde{\beta}^A\frac{1}{n_1^B(1-n_1^B)} + \tilde{\beta}^B\frac{2}{n_1^A(1-n_1^A)} > 0.
\end{aligned}$$

The partial derivative with respect to $\lambda$ is:

$$\frac{\partial\Omega^A}{\partial\lambda} = -\,\alpha^A\log\left(\frac{n_1^A}{1-n_1^A}\right) - \beta^A\log\left(\frac{n_1^B}{1-n_1^B}\right) - \frac{\alpha^A(2n_1^A-1)+\beta^B(2n_1^B-1)}{n_1^A(1-n_1^A)}$$

To conduct comparative statics with respect to $\lambda$, we can write

$$\frac{dn_1^A}{d\lambda} = \frac{-\overbrace{\frac{\partial\Omega^A}{\partial\lambda}}^{(-)}\overbrace{\frac{\partial\Omega^B}{\partial n_1^B}}^{(-)} + \overbrace{\frac{\partial\Omega^B}{\partial\lambda}}^{(-)}\overbrace{\frac{\partial\Omega^A}{\partial n_1^B}}^{(+)}}{\frac{\partial\Omega^A}{\partial n_1^A}\frac{\partial\Omega^B}{\partial n_1^B} - \frac{\partial\Omega^A}{\partial n_1^B}\frac{\partial\Omega^B}{\partial n_1^A}}$$

Therefore, once we establish that

$$\frac{\partial\Omega^A}{\partial n_1^A}\frac{\partial\Omega^B}{\partial n_1^B} - \frac{\partial\Omega^A}{\partial n_1^B}\frac{\partial\Omega^B}{\partial n_1^A} > 0, \tag{24}$$

we know that $dn_1^A/d\lambda < 0$ and analogously $dn_2^A/d\lambda < 0$.

To show this, we write the left-hand side of inequality (24) as

$$\frac{\partial\Omega^A}{\partial n_1^A}\frac{\partial\Omega^B}{\partial n_1^B} - \frac{\partial\Omega^A}{\partial n_1^B}\frac{\partial\Omega^B}{\partial n_1^A} = \frac{Z_1}{(n_1^A)^2(1-n_1^A)^2(n_1^B)^2(1-n_1^B)^2},$$

where

$$\begin{aligned}
Z_1 &= \Big[(1-\tilde{\alpha}^A)\left[3n_1^A(1-n_1^A)+(2n_1^A-1)^2\right]+(2n_1^A-1)(2n_1^B-1)\tilde{\beta}^B\Big] \\
&\quad \times \Big[(1-\tilde{\alpha}^B)\left[3n_1^B(1-n_1^B)+(2n_1^B-1)^2\right]+(2n_1^B-1)(2n_1^A-1)\tilde{\beta}^A\Big] \\
&\quad - \Big[\tilde{\beta}^A n_1^A(1-n_1^A)+2\tilde{\beta}^B n_1^B(1-n_1^B)\Big]\Big[\tilde{\beta}^B n_1^B(1-n_1^B)+2\tilde{\beta}^A n_1^A(1-n_1^A)\Big] \\
&> \Big[\max\{\tilde{\beta}^A,\tilde{\beta}^B\}\Big]^2 \times Z_2,
\end{aligned}$$

with

$$\begin{aligned}
Z_2 &= 3n_1^A(1-n_1^A)(2n_1^B-1)^2+3n_1^B(1-n_1^B)(2n_1^A-1)^2+(2n_1^A-1)^2(2n_1^B-1)^2 \\
&\quad -2\left[n_1^A(1-n_1^A)-n_1^B(1-n_1^B)\right]^2.
\end{aligned}$$

Inequality (24) is satisfied if and only if $Z_1>0$.

Without loss of generality, suppose that, on the larger platform, there are weakly more group-$A$ users than group-$B$ users, $n_1^A \geq n_1^B > 1/2$. Then, we have

$$\begin{aligned}
\frac{\partial Z_2}{\partial n_1^B} &= 4(2n_1^B-1)[n_1^B(1-n_1^B)-n_1^A(1-n_1^A)] \\
&\quad +12n_1^A(1-n_1^A)(2n_1^B-1)+(2n_1^B-1)(2n_1^A-1)^2>0.
\end{aligned}$$

Hence, if $Z_2 \geq 0$ at $n_1^B=1/2$, $Z_2>0$ for all $n_1^B \in (1/2, n_1^A]$. At $n_1^B=1/2$, we have

$$\begin{aligned}
Z_2\big|_{n_1^B=1/2} &= \frac{3}{4}(2n_1^A-1)^2-2\left[n_1^A(1-n_1^A)-\frac{1}{4}\right]^2 \\
&= \frac{5-16Z_3}{8},
\end{aligned}$$

where $Z_3$ is defined as

$$Z_3 = n_1^A[1-(2-n_1^A)(n_1^A)^2].$$

Function $Z_3$ has the first-order and second-order derivatives with respect to $n_1^A$:

$$\frac{\partial Z_3}{\partial n_1^A} = 1-(n_1^A)^2(6-4n_1^A),$$
$$\frac{\partial^2 Z_3}{\partial (n_1^A)^2} = -12n_1^A(1-n_1^A)<0.$$

Noting that $\partial Z_3/\partial n_1^A = 0$ at $n_1^A=1/2$, $Z_3$ is maximized at $n_1^A=1/2$ with maximum value

$Z_3|_{n_1^A=1/2} = 5/16$. Hence, $5 - 16Z_3$ is minimized at $n_1^A = 1/2$, with the minimum

$$(5 - 16Z_3)\big|_{n_1^A=1/2} = 0.$$

This establishes that $Z_1 > 0$ for all $n_1^A$ and $n_1^B \in (1/2, n_1^A]$ and, thus, inequality (24) is satisfied. $\square$

*Proof of Proposition 14.* Group-$k$ demand of each platform is implicitly defined by

$$n_i^k = \frac{h_i^k(p_i^k, p_i^l)}{H_{0i}^k(y_0^k, n_i^k, n_i^l, p_i^k, p_i^l) + H^k(\mathbf{p})}, \tag{25}$$

where

$$\begin{aligned} H_{0i}^k(y_0^k, n_i^k, n_i^l, p_i^k, p_i^l) &:= y_0^k \frac{\exp[(\Gamma_{kk} - 1)(a_i^k - p_i^k) + \Gamma_{kl}(a_i^l - p_i^l)]}{(n_i^k)^{\alpha_k}(n_i^l)^{\beta_k}} \\ &= y_0^k \frac{h_i^k}{\exp(a_i^k - p_i^k)(n_i^k)^{\alpha_k}(n_i^l)^{\beta_k}}. \end{aligned} \tag{26}$$

Equations (25) and (26), along with the fact that $n_i^k/n_j^k = h_i^k/h_j^k$ implies that $H_{0i}^k = H_{0j}^k$ for all $i, j = 1, \ldots, M$. This implies that there exist $\phi^k$, $k \in \{A, B\}$, such that

$$\frac{\exp[(\Gamma_{kk} - 1)(a_i^k - p_i^k) + \Gamma_{kl}(a_i^l - p_i^l)]}{(n_i^k)^{\alpha_k}(n_i^l)^{\beta_k}} = \phi^k.$$

This equation can be rewritten as

$$\begin{aligned} \alpha_A \log n_i^A + \beta_A \log n_i^B + \log \phi^A - (\Gamma_{AA} - 1)(a_i^A - p_i^A) - \Gamma_{AB}(a_i^B - p_i^B) = 0, \\ \alpha_B \log n_i^B + \beta_B \log n_i^A + \log \phi^B - (\Gamma_{BB} - 1)(a_i^B - p_i^B) - \Gamma_{BA}(a_i^A - p_i^A) = 0. \end{aligned}$$

Solving for this system of equations, we obtain

$$\begin{aligned} &\log n_i^A \\ =&\frac{\beta_A \log \phi^B - \alpha_B \log \phi^A + [\alpha_B(\Gamma_{AA} - 1) - \beta_A\Gamma_{BA}](a_i^A - p_i^A) + [\alpha_B\Gamma_{AB} - \beta_A(\Gamma_{BB} - 1)] \exp(a_i^B - p_i^B)}{\alpha_A\alpha_B - \beta_A\beta_B} \end{aligned}$$

Noting that

$$\begin{aligned} \alpha_B(\Gamma_{AA} - 1) - \beta_A\Gamma_{BA} &= \frac{\alpha_B[(1 - \alpha_B)\alpha_A + \beta_A\beta_B] - \beta_A\beta_B}{(1 - \alpha_A)(1 - \alpha_B) - \beta_A\beta_B} = (\alpha_A\alpha_B - \beta_A\beta_B)\Gamma_{AA}, \\ \alpha_B\Gamma_{AB} - (\beta_A\Gamma_{BB} - 1) &= \frac{\alpha_B\beta_A - \beta_A[(1 - \alpha_A)\alpha_B + \beta_A\beta_B]}{(1 - \alpha_A)(1 - \alpha_B) - \beta_A\beta_B} = (\alpha_A\alpha_B - \beta_A\beta_B)\Gamma_{AB}, \end{aligned}$$

we have

$$\log n_i^A = \log h_i^A + \frac{\beta_A \log \phi^B - \alpha_B \log \phi^A}{\alpha_A \alpha_B - \beta_A \beta_B},$$
$$\log n_i^B = \log h_i^B + \frac{\beta_B \log \phi^A - \alpha_A \log \phi^B}{\alpha_A \alpha_B - \beta_A \beta_B}.$$

Because equation (25) can also be written as

$$\log n_i^A = \log h_i^A - \log \left( y_0^A \phi^A + H^A \right),$$
$$\log n_i^B = \log h_i^B - \log \left( y_0^B \phi^B + H^B \right),$$

we can write the system of equations that determine the values of $(\phi^A, \phi^B)$ as a function of $(H^A, H^B)$:

$$F^A = \log \phi^A - \alpha_A \log(y_0^A \phi^A + H^A) - \beta_A \log(y_0^B \phi^B + H^B) = 0,$$
$$F^B = \log \phi^B - \alpha_B \log(y_0^B \phi^B + H^B) - \beta_B \log(y_0^A \phi^A + H^A) = 0.$$

Let $\phi^A(H^A, H^B)$, $\phi^B(H^B, H^A)$ be the solution to this system of equations.

Group-$k$ demand of platform $i$ is now written as

$$n_i^A(h_i^A, H^A, H^B) = \frac{h_i^A}{y_0^A \phi^A(H^A, H^B) + H^A}.$$

□

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