# Hotelling Revisited - The Price-then-Location Model

Markus Reisinger * Christian Seel † Frauke Stehr ‡

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**Abstract**

In several markets, such as the magazine or restaurant market, firms choose prices for a longer time horizon than product content, which can be varied more flexibly. In this paper, we analyze the pricing and content choices of competitive firms in such a setting. We consider a two-stage game in which two firms first choose prices and then locations on the Hotelling line, allowing for differences in firms' costs. We derive the complete solution for moderate differences in cost. At equilibrium, firms choose pure strategies at the price stage and mix in terms of location, with the more efficient firm locating closer to the middle. For sufficiently symmetric production costs, any subgame-perfect equilibrium involves mixing at both stages.

KEYWORDS: *Hotelling model, price competition, location choice, mixed-strategy equilibrium*

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*Frankfurt School of Economics, E-Mail: M.Reisinger@fs.de

†Maastricht University, E-Mail: c.seel@maastrichtuniversity.nl

‡Maastricht University, E-Mail: f.stehr@maastrichtuniversity.nl

# 1 Introduction

Product differentiation is one of the most important and long-standing concepts in Industrial Organization, dating back at least to Hotelling (1929) and Chamberlin (1933). Indeed, in almost all markets, firms have the choice to offer products that are different to their competitors'. A classic framework to study horizontal product differentiation is Hotelling's (1929) model, in which two firms compete for consumers distributed on a line and a consumer's location represents her preferences over the offered products.[1] Firms choose where to locate on the line, which is equivalent to the product variant the firm offers. This framework is the workhorse model to study location choices by firms.

Many influential papers (e.g., d'Aspremont et al., 1979; Mullainathan and Shleifer, 2005) consider a move order where firms first make their location choices and then compete in prices. This move order is often motivated by the reasoning that prices may be more flexible to adjust than product characteristics or content (e.g., Tirole, 1988), which implies that the decision which product variant to offer is a more long-term decision than the choice which price to set.[2]

However, there are markets that do not fit into this category. In these markets, product variants can be changed more flexibly compared to prices. This holds in all markets where prices are widely advertised and known to consumers well in advance of purchase, which makes it costly for firms to change them. Product characteristics, on the other hand, can only be found out by consumers shortly before the purchasing decision and vary over time, which implies that a firm's price level is a more long-term decision than the product variant.

Consider, for instance, the market for magazines. In this market, depending on publication frequency, publishers weekly or bi-weekly choose the content of an issue (e.g., cover page,

[1]The framework for modelling vertical differentiation—i.e., firms offer variants with different qualities and consumers have different preferences for quality—were developed later by e.g., Mussa and Rosen (1978), Gabszewicz and Thisse (1979), and Shaked and Sutton (1982).

[2]In these papers, under some conditions, firms choose maximal differentiation at equilibrium, that is, they position themselves at opposite ends of the Hotelling line, to dampen price competition.

headlines, stories, etc.). Prices are usually very stable over time, as documented by e.g., Willis (2006). A similar phenomenon can be observed for restaurants. Lunch menus often change every day or week, whereas prices in a dish category are unchanged for relatively long time periods. Also, in the market for simple art—e.g., posters or printed t-shirts—the poster or t-shirt variant on offer changes frequently, whereas prices are adjusted rarely. Interestingly, as mentioned by Anderson and de Palma (1992), also the often-cited example for Hotelling's (1929) model with two ice-cream vendors choosing their locations on a beach perhaps better fits to a situation of price-then-location choice as opposed to the reverse order of moves.

In this paper, we analyze the price-then-location game in Hotelling's (1929) classic framework. Although—as explained in the examples above—such a competitive situation is relevant in several markets, to the best of our knowledge, the game has not been solved so far. As shown in previous literature—e.g., Anderson and de Palma (1992) and Aragonès and Palfrey (2002)—the game is difficult to analyze as it does not have a pure-strategy equilibrium. The contribution of our paper is to, first, provide a full solution for the case in which firms are relatively asymmetric (i.e., the difference in their production costs is sufficiently large). Interestingly, we find that the resulting subgame-perfect equilibrium involves randomization only between location choices, but a pure-strategy equilibrium in the pricing stage. In addition, if firms are relatively symmetric, we are able to provide bounds on the firms' payoffs in the location stage, which allows us to show that mixing will then occur in both stages in equilibrium.

To be specific, we consider a price-then-location game on a Hotelling line: Two firms with potentially different production costs simultaneously choose their prices, and, after observing both prices, simultaneously choose their locations. Consumers are uniformly distributed on the line. After prices and locations are set, each consumer chooses her preferred firm, taking into account price differences and transport costs.

At the location stage, whenever firms set different prices, the equilibrium is in mixed-strategies (unless the price difference is extreme). If price differences between firms are moderate, we are able to characterize the equilibrium in closed form. In this location equilibrium, the firm setting the lower price randomizes its location close to the middle, while the other firm randomizes close to the boundaries of the line. The firm that enters the location stage with a competitive advantage thereby protects the center and obtains a large demand. The firm with a disadvantage can only secure some demand by choosing locations close to the boundaries.

For small price differences, we cannot characterize the equilibrium in closed form. However, we show equilibrium existence and we are able to provide bounds for equilibrium demand. In addition, we establish that the equilibrium demands are continuous. In particular, when approaching the case with equal prices, each firms obtains half of the demand.

Equipped with these results, we are then able to fully characterize the subgame-perfect equilibrium of the game for sufficiently asymmetric production costs between firms. We find that it features *pure strategies* at the pricing stage, with the more efficient firm choosing a lower price. The lower price provides the more efficient firm with a head start advantage for the location stage. It randomizes its location close to the middle, while the less efficient firm randomizes closer to both boundaries.

If firms are less asymmetric in terms of production costs, there is no equilibrium in pure strategies at the pricing stage. Intuitively, even if firms are fully symmetric, a symmetric pure-strategy equilibrium cannot occur. First, if prices were relatively high, a firm can profitably deviate by reducing its price, as the demand elasticity is very high for small price differences. Second, if prices were close to production cost, a firm can profitably deviate by setting a higher price and then benefit from the mixed-strategy equilibrium in the location stage, where also the disadvantaged firm obtains positive demand in expectation. We establish the existence of a mixed equilibrium in both stages in this case.

## 1.1 Related Literature

Anderson and de Palma (1992) introduce our setting and find that no subgame-perfect equilibrium exists if firms are symmetric in costs and restricted to pure strategies. In particular, in subgames after both firms choose moderately different prices, no pure-strategy equilibrium exists. Our results for symmetric costs extend their findings: even when mixing is allowed at the location stage, there is no equilibrium in pure strategies at the pricing stage. However, we show that if firms are sufficiently asymmetric, the subgame-perfect equilibrium exhibits pure strategies at the pricing stage.

In Aragonès and Xefteris (2012), two office-motivated politicians compete in an election by choosing a political location on the Hotelling line. One politician has a head start in terms of popularity. Their model resembles our second stage, with two main differences. First, they consider quadratic instead of linear transport costs. Second and more importantly, the main results are obtained for a sufficiently large number of voters—an assumption that helps to obtain a concave payoff function—whereas our setting would be equivalent to that of a single voter. Aragonès and Xefteris (2012) obtain equilibria in which the advantaged politician locates exactly in the middle, while the disadvantaged politician mixes between two points that are equidistant from the middle.

Aragonès and Palfrey (2002) also consider a one-stage model with location-based competition between two politicians where one of them has a head start – analogous to the second stage of our model. However, they focus on the case with a finite number of locations, and determine properties of the resulting mixed-strategy equilibrium. As an extension, they analyze continuous locations and show that only a mixed strategy equilibrium exists. For large enough head starts, we provide a characterization of this mixed strategy equilibrium. Additionally, we endogenize the head start by adding the pricing stage before the location stage.

The seminal paper by d'Aspremont et al. (1979) analyzes a symmetric location-then-price model under quadratic transport costs. This assumption leads to a pure-strategy equilibrium in both stages, in which firms choose maximal differentiation in location. Osborne and Pitchik (1987) consider linear transport costs in the location-then-price model and find that only a mixed-strategy equilibrium exists in the pricing stage. Using numerical techniques, they are able to determine that pure- and mixed strategy equilibria can exist in the first stage (location) and provide a characterization of them.[3] In the location-then-price model, the distinction between linear and quadratic (i.e., convex) transport costs is crucial for whether a pure- or a mixed-strategy equilibrium exists. As we will explain, in the price-then-location model, instead, a mixed-strategy equilibrium in the location stage emerges *regardless* of the assumption on the transport costs.[4]

In terms of idea, Ganuza and Hauk (2006) consider a similar setting where two competing firms first choose effort instead of price and then compete in location on the second stage. However, they limit the decision at each stage to two possible actions. Our techniques might prove useful in extending their setting and similar other settings to continuous action spaces.

# 2 Model

Consider a model with two firms $i = 1, 2$. In stage 1, each firm chooses a price $p_i \in \mathbb{R}_+$. In stage 2, after observing both prices, each firm chooses a location $l_i \in [0, 1]$ on the Hotelling line. A pure strategy of firm $i$ is thus given by $s_i = (p_i, l_i(p_1, p_2))$. A mixed strategy $\sigma_i(s_i)$ is a probability distribution over the pure strategies.

There is a mass one of consumers whose location is uniformly distributed on the Hotelling line. The utility of a consumer located at $l$ when buying from firm $i$ is $v - t|l - l_i| - p_i$, where $v$ is the gross utility from consuming the good and $t > 0$ is the consumer's transport cost.[5] We assume that $v$ is large enough so that the market is covered in equilibrium.[6]

[3]Xefteris (2013) also considers the case with linear transport costs but supposes, in contrast to Osborne and Pitchik (1987), that a firm's profit in case it is a monopolist goes to infinity. He shows that this feature gives rise to a pure-strategy equilibrium, in which minimal differentiation occurs.

[4]de Palma et al. (1985) analyze the case in which price and location is chosen at the same time. They find that if consumers perceive firms to be sufficiently heterogeneous, all firms choose to locate at the center, and prices are above marginal costs. Instead, if the heterogeneity between firms is only small, no pure-strategy equilibrium exists.

[5]The utility specification implies that transport costs enter linearly. We discuss the case of quadratic transport cost in Section 3.

[6]If $t = 0$, we are in the standard Bertrand model, and if $t \to \infty$, stage 2 converges to the Hotelling (1929) model, in which prices are absent. The assumption of full market coverage is standard in the related literature;

Therefore, consumer $l$ buys from Firm 1 if

$$p_1 + t|l - l_1| < p_2 + t|l - l_2|$$

and from Firm 2 if the above inequality is reversed. Unless otherwise stated, an indifferent consumer chooses each firm with equal probability.

Firms have constant marginal costs $c_i$, where $c_i \geq c_j \geq 0$. The payoff of firm $i$ is $\pi_i = (p_i - c_i)D_i(p_i, p_j, l_i, l_j)$, where $D_i(p_i, p_j, l_i, l_j)$ is the mass of consumers which buys from firm $i$ given price and location choices. Each firm maximizes its expected payoff. Our solution concept is subgame-perfect Nash equilibrium.

# 3 Equilibrium Analysis - Location Choices

In what follows, we denote the price difference weighted by the inverse of transport costs by $d = \frac{1}{t}(p_2 - p_1)$. This weighted price difference serves as a head start for the firm with the lower price, w.l.o.g. Firm 1. If the distance between locations $l_1$ and $l_2$ is smaller than $d$, Firm 1 obtains the entire demand. If the distance is larger than $d$, there is a cutoff between $l_1$ and $l_2$ and Firm 2 receives the demand on their side of the cutoff. Figure 1 illustrates two cases.

Formally, if there are no ties, for $d > 0$ the demand function of Firm 1 is given by:

$$D_1(l_1, l_2, d) = \begin{cases} 1 & \text{for } |l_1 - l_2| < d \\ \frac{l_1+d+l_2}{2} & \text{for } l_2 - l_1 > d \\ 1 - \frac{l_1-d+l_2}{2} & \text{for } l_1 - l_2 > d. \end{cases}$$

and the demand function of Firm 2 is given by $1 - D_1(l_1, l_2, d)$.

As the prices are given from stage 1, maximizing (expected) demand $D_i$ yields the same solution as maximizing (expected) profit $\pi_i = (p_i - c_i)D_i$ as long as $p_i > c_i$. Thus, to keep

see, e.g., d'Aspremont et al. (1979) for the location-then-price model and Aragonès and Xefteris (2012) in a political economy setting, where the assumption means that each voter casts a vote with probability 1.


**(a)** Distance between $l_1$ and $l_2$ is larger than $d$.

**(b)** Distance between $l_1$ and $l_2$ is smaller than $d$.

**Figure 1** Examples for demand faced by the two firms depending on the distance between their locations. The solid line shows which consumers are served by Firm 1; the dashed line shows Firm 2's demand. In the examples, the difference in prices (weighted by the inverse of the transport cost) is $d = 0.2$.

notation simple, we write the maximization problem in terms of demand $D_i$. Furthermore, $D_1 + D_2 = 1$, i.e., demand maximization is a constant-sum game and we thus obtain uniqueness of the derived equilibrium demands.

To proceed with our analysis, we successively analyze all cases in which $d \geq 0$. The analysis is exhaustive as analogous symmetric arguments with firm labels reversed apply for $d \leq 0$.

**Case 1:** $d = 0$

If $p_1 = p_2$, i.e., $d = 0$, we are back to the textbook model in which $l_1 = l_2 = 0.5$ is the unique equilibrium (this result relies on random tie-breaking). The demand of each firm is 0.5 and the profit is $\pi_i = \frac{p_i - c_i}{2}$.

**Case 2:** $d \geq 0.5$

Firm 1 can attract all consumers by locating at $l_1 = 0.5$. Thus, in any equilibrium $\pi_1 = p_1 - c_1$ and $\pi_2 = 0$.[7]

**Case 3:** $d \in [d^{crit}, 0.5)$

For $d$ below 0.5, Firm 1 can no longer guarantee to win by locating at $l_1 = 0.5$. However, if $d$ is sufficiently large (i.e., above a threshold value denoted by $d^{crit}$), we are still able to fully characterize the equilibrium. This equilibrium is in mixed strategies, which makes

[7]For $d = 0.5$, a tie occurs when $l_1 = 0.5$ and $l_2 = 0$ or $l_2 = 1$. Yet, only consumers at 0 or 1 affected by the tie. Thus, the mass of affected consumers is zero.

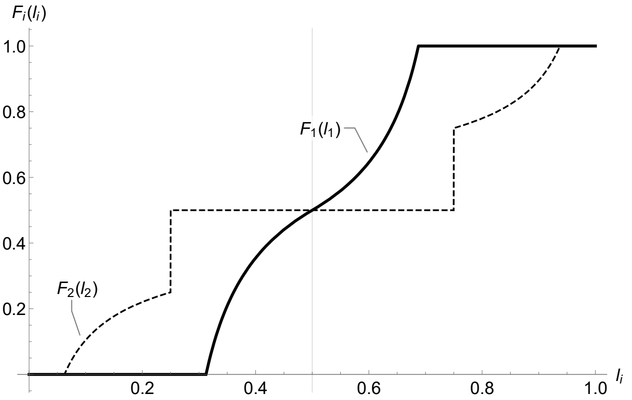


**Figure 2** Equilibrium CDFs $F_1(l_1)$ and $F_2(l_2)$ for $d = 0.25$.

the construction of the equilibrium substantially more complicated than in the previous cases. The following proposition states the expected equilibrium profits and formula for the threshold value $d^{crit}$.

**Proposition 1.** Suppose $d \in [d^{crit}, 0.5)$, where $d^{crit} \approx 0.1914$ is the solution to

$$\frac{1}{2} + (1-2d)(\frac{9}{8} - \frac{\sqrt{1-2d}}{\sqrt{1-4d}}) = \frac{5-10d}{16}.$$

The equilibrium demands are $D_1 = \frac{11+10d}{16}$ and $D_2 = \frac{5-10d}{16}$.

In the appendix, we derive the equilibrium distributions $F_1$ and $F_2$ in closed form. We provide a graphical illustration in Figure 2. To construct the distributions, we set up a differential equation for each firm that keeps the payoff change equal to zero inside the supports. We then verify that there are no profitable deviations outside the support, such as Firm 2 deviating to the middle. The difficult part is to guess the respective supports and find the appropriate mass points of Firm 2.

As the figure shows, Firm 1 randomizes continuously between locations that are symmetrically distributed around $\frac{1}{2}$. Firm 2 randomizes on a disconnected support which is also symmetric around $\frac{1}{2}$. In particular, denote the lower bound of Firm 1's distribution by $\underline{l}_1$ and its upper bound by $\bar{l}_1$. Then, Firm 2's lower bound is $\underline{l}_1 - d$ and its upper bound is $\bar{l}_1 + d$, but Firm 2 does not randomize on the interval $(\frac{1}{2} - d, \frac{1}{2} + d)$.

We are able to fully characterize the equilibrium for $d \geq d^{crit}$ due to the following reason: For these values of $d$, we can show that in equilibrium $\underline{l}_1 + d > \frac{1}{2} - d$. The inequality implies that for each location of Firm 1 in its mixing range, Firm 2 only gets demand on one side of the line. For instance, if Firm 2 locates to the left of the middle, then for any point in the equilibrium distribution of Firm 1, only consumers located between 0 and the marginal consumer buy from Firm 2, which implies that Firm 2's demand is on a connected interval.[8] Firm 2's demand for $l_2 \in (\underline{l}_2, \frac{1}{2} - d)$ is therefore

$$D_2(l_2) = \int_{l_2+d}^{1-d-l_2} \frac{l_1 - d + l_2}{2} dF_1(l_1).$$

In this case, there is a single demand segment, which allows us to explicitly solve the differential equation and determine the equilibrium distributions.

The intuition behind the equilibrium distributions is as follows: Firm 1 can ensure a demand of at least $\frac{1}{2} + d$ by locating in the middle. At equilibrium, Firm 1 extends its expected demand further by randomizing close to the middle. The randomization of Firm 1 is symmetric around $\frac{1}{2}$ to be unpredictable. In particular, Firm 1 deters Firm 2 from choosing a location close to the middle and makes Firm 2 indifferent between more distant locations. For these locations, Firm 2 faces a trade-off: locating closer to the middle increases the chance of getting no demand (when Firm 1's location is too close), but also increases the demand whenever Firm 1 locates further away. The randomization of Firm 1 is chosen such that the two effects offset each other.

The distribution of Firm 2 is also symmetric around $\frac{1}{2}$. Firm 2 chooses a two-fold strategy. On the one hand, placing mass close to the middle, i.e., at $\frac{1}{2} - d$ and $\frac{1}{2} + d$, makes it more attractive for Firm 1 to locate close to the middle. On the other hand, a continuous randomization away from $\frac{1}{2}$ makes it more attractive for Firm 1 to locate farther away from $\frac{1}{2}$. Again, the distribution of Firm 2 is such that the effects cancel each other out.

The equilibrium just described exists if Firm 1's advantage over Firm 2 is sufficiently large (i.e., $d$ is high enough). When $d$ becomes smaller (i.e., below $d^{crit}$), Firm 1 does not

[8] Similarly, if Firm 2 locates to the right of the middle, then for any point in the distribution of Firm 1, only consumers located to the right of the middle buy from Firm 2.

"protect" the middle sufficiently any longer with the distribution above, i.e., Firm 2 would want to deviate to $l_2 = 0.5$. This leads us to the next case.

**Case 4:** $d \in (0, d^{crit})$

Suppose that for small $d$ firms were to use similar distributions as in Case 3 with $\underline{l}_1 = \underline{l}_2 + d$. Then, in contrast to Case 3, for sufficiently small $d$, we would obtain $\underline{l}_1 + d < \frac{1}{2} - d$. As Firm 2 randomizes up to $\frac{1}{2} - d$, this would lead to a change in the demand function of Firm 2 for $l_2 \in (\underline{l}_1 + d, \frac{1}{2} - d)$, as Firm 2 would also get a positive demand when Firm 1 is below $l_2 - d$. More formally, the demand for $l_2 \in (\underline{l}_1 + d, \frac{1}{2} - d)$ would be

$$D_2(l_2) = \int_{l_2+d}^{1-d-l_2} \frac{l_1 - d + l_2}{2} dF_1(l_1) + \int_{\underline{l}_2+d}^{l_2-d} \left(1 - \frac{l_1 + l_2 + d}{2}\right) dF_1(l_1),$$

where the second term has not been present in Case 3.

To tackle this problem, we could use the above new demand function for the indifference condition on the given interval. If we take the same approach as before and set the derivative of the demand function equal to zero, the change in payoff depends on $F_1(l_2+d)$ and $f_1(l_2+d)$ as before. However, due to fact that there are two demand segments, it additionally depends on $F_1(l_2 - d)$ and $f_1(l_2 - d)$. Intuitively, this requires an increased density of Firm 1 close to the middle and thus a shortened interval on which Firm 1 randomizes.

Mathematically, this type of equation is called a linear neutral delay differential equation (linear NDDE) or differential-difference equation - a differential equation which depends both on past and present values of the function as well as the derivative; see, e.g., Myshkis (1951) and Bellman and Cooke (1963) for early seminal results. Instead of an initial (boundary) condition as in an ordinary differential equation (here $F_1(l_1)$), a history function describing the delay term needs to be specified (here $F_1(l_2 - d)$). In our case, a natural candidate for the history function is the distribution which forms an equilibrium for slightly larger $d$; i.e., the solution in Case 3 on the aforementioned interval $[\underline{l}_1, \underline{l}_1 + d]$, where no delay term occurs.

Unfortunately, closed-form solutions to NDDE's are difficult to obtain. A numerical approach led to an oscillating density (see Figure 3 for an illustration), which is typical of the solution to such equations; e.g., see Ladas and Ficas (1986).[9] The main problem in

[9]Baye et al. (1994) obtain a mildly related result with an oscillation in $f$ for the discrete all-pay auction.

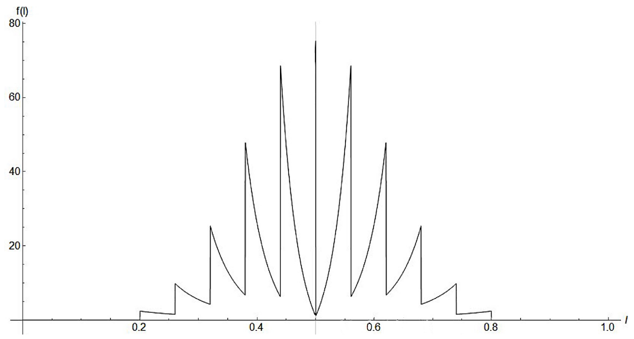


**Figure 3** Illustration of an oscillating density, based on numerical results for the linear NDDE.

showing that such a profile is indeed an equilibrium is that we cannot verify that the payoff of Firm 2 is maximized for any $l_2 \in [\underline{l}_1, \frac{1}{2} - d]$ in this case due to the lack of a closed-form solution.

In addition, a second problem arises when $d$ is below $d^{crit}$: Firm 1 needs more "protection" of the middle, i.e., Firm 1 might set a small mass point at $\frac{1}{2}$, which would further complicate the derivation of a closed-form solution for some $d$. We thus leave the derivation of a closed-form solution for future research.

We note that the problem which prevents us from obtaining an analytical solution—i.e., that the demand consists of two segments which gives rise to the NDDE—does not depend on the linearity of the transport cost but would also occur of transport costs were e.g. quadratic. For small values of $d$, the demand function of Firm 2 at a point $l_2$ still has two parts when considering all locations in the distribution function of Firm 1, which leads to a NDDE. Therefore, the reason why it is difficult to construct an equilibrium in the price-then-location game is *different* from that in a location-then-price game. In the latter case, as was shown by d'Aspremont et al. (1979) and Osborne and Pitchik (1987), the demand function is not concave with linear transport cost, but can be made concave with quadratic transport costs. Instead, in the price-then-location game, The source of the problem of obtaining a closed-form solution is different and occurs independently of the functional form of transport

Yet, the delay difference equation in their paper is due to the tie-breaking rule on a discrete strategy space. We are not aware of similar results for games on continuous strategy spaces.

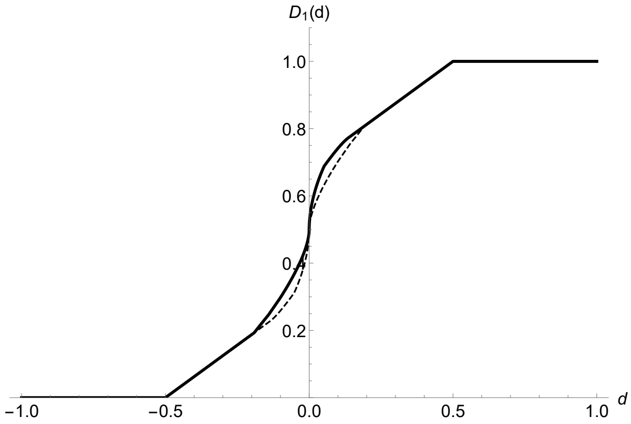


**Figure 4** Plot of the equilibrium demand of Firm 1 depending on d. The dashed line uses our lower bound for $d > 0$, while the solid line uses the upper bound.

cost.

We proceed by providing an equilibrium existence result from the literature. In a second step, we provide bounds on the equilibrium payoffs on the seconds stage, i.e., equilibrium demands. These bounds will be crucial for our partial characterization of an equilibrium on the pricing stage.

**Lemma 2** (Theorem 5 in Aragonès and Palfrey, 2002)**.** The location game admits a Nash equilibrium in mixed strategies.

Note that whenever an equilibrium exists, all other equilibria are payoff-equivalent by the constant-sum nature of the game. Moreover, the equilibrium payoffs satisfy $\Pi_1^*(d) = \Pi_2^*(-d)$.

Endowed with the existence and uniqueness of equilibrium demands, we derive bounds on these demands in Lemmas 7 and 8 of the appendix. To do so, we fix a (suboptimal) strategy of one firm and compute the supremum payoff which the other firm can obtain against that strategy. This provides an upper bound on the equilibrium demand of the rival as we analyze a constant sum game. Figure 4 illustrates the bounds for different $d$.

The corresponding construction in the appendix is tedious as it uses different bounds for different sub-regions of $[-d^{crit}, d^{crit}]$. Furthermore, note that the lower bound and the upper bound converge to $\frac{1}{2}$ as $d \to 0$ and to the equilibrium demands as $d \to d^{crit}$.

The continuity at $d = 0$ is not obvious ex-ante. To derive it, in Lemma 9 in the appendix, we construct a continuous distribution for the disadvantaged firm such that the support of the distribution converges to $\frac{1}{2}$ as $d \to 0$. At the same time, the derivative of each bound as $d \to 0^+$ tends to infinity, i.e., at $d = 0$, the demand is extremely sensitive to small changes in $d$ (and hence prices). As we shall see in the next section, this property upsets a potential pure-strategy equilibrium on the pricing stage when firms are symmetric.

# 4 Equilibrium Analysis - Pricing

Remember that $d = \frac{1}{t}(p_2 - p_1)$ captures difference in prices weighted by the inverse of the transport cost. From the second stage, we have demands for $|d| \geq d^{crit}$ and bounds for $|d| < d^{crit}$. We use these cases to derive some equilibria depending on the cost parameters $c_1, c_2$ and $t$.

Without loss of generality, let $c_2 - c_1 \geq 0$. Firms maximize profits in the first stage, given their production costs $c_i$, the consumer's transport cost $t$, and equilibrium location strategies in the second stage. Depending on these parameters, we can characterize different equilibria in the first stage.

**Case 1: Large differences in production cost**

For large enough cost differentials (or conversely, low enough transport costs), Firm 1 prices aggressively enough such that Firm 2 cannot attract any consumers when pricing at marginal cost. Firm 1 (the firm with lower production costs) attracts all consumers by locating in the middle at the location stage.

**Proposition 3.** Suppose $c_2 - c_1 \geq 2.1t$. Then there exists a subgame-perfect equilibrium where $p_1 = c_2 - \frac{t}{2}$ and $p_2 = c_2$. The equilibrium profits are $\pi_1 = p_1 - c_1$ and $\pi_2 = 0$.

**Case 2: Medium differences in production cost**

As cost differences become smaller, Firm 1 no longer finds it optimal to completely shut down the demand of Firm 2. Instead, Firm 1 allows some competition by raising prices, but still gets a larger share of the market.

**Proposition 4.** Suppose $c_2 - c_1 \in [x^{crit}, 2.1t)$, with $x^{crit} < 1.35t$. Then there exists a subgame-perfect equilibrium with pure strategies on the pricing stage. In equilibrium, we have $p_1 = \frac{1}{3}(2c_1 + c_2) + \frac{9}{10}t$, and $p_2 = \frac{1}{3}(c_1 + 2c_2) + \frac{7}{10}t$. Equilibrium payoffs are $\pi_1(p_1, p_2) = \frac{(27t+10(c_2-c_1))^2}{1440t}$ and $\pi_2 = \frac{(21t-10(c_2-c_1))^2}{1440t}$.

The proposition allows us to make a number of observations. First, from an economic point of view, having an equilibrium in pure strategies on stage 1 and in mixed strategies on stage 2 is appealing. Given the pricing decision of the rival, even if a firm could change short-term prices before choosing the location, it would not be willing to do so. For instance, competing restaurants keep their pricing strategies unchanged over a longer time horizon, but vary their lunch dishes (location) over time.

Second, both firms' prices are above marginal costs, allowing them to make positive profits in equilibrium. Thus, in contrast to Case 1 and the standard model of Bertrand competition, the less cost-efficient firm is now able to benefit from the differentiation in location. In particular, large differences in costs lead to relatively small differences in prices. For example, for $t = 1$, cost differences have to be between 1.35 and 2.1 to maintain an equilibrium with price differences between 0.19 and 0.5.

Third, both firms' profits are increasing in transport costs. Intuitively, higher transport costs decrease the competitive pressure on the other price component. At the same time, as transport costs increase, larger differences in production costs are necessary to maintain an equilibrium in this range.

Fourth, from a consumer's point of view, an increase of transport costs implies a double burden: first, the direct burden of the higher cost and second an additional price increase by both firms due to lower competitive pressure.

When the difference in production costs increases, price differences also increase. Thus, more consumers choose the more efficient firm in equilibrium. A first intuition might suggest that this increases the expected difference in location to the preferred firm. However, the following proposition shows that this is not necessarily the case:

**Proposition 5.** The expected location difference to the preferred firm is non-monotone in the production cost difference.

Intuitively, this is driven by the effect of different prices on behavior on the location stage. When $\frac{1}{t}(p_2 - p_1) = d \geq 0.5$, the advantaged Firm 1 locates at $l_1 = 0.5$ which minimizes location difference for cases with only one firm. For $d < 0.5$, the second firm obtains a share of the market as well. In this case, there are two counteracting effects. On the one hand, the second firm might generate a new option for the consumers, thereby (weakly) reducing the distance to their own location. On the other hand, the advantaged firm does not locate in the middle any more, as it tries to be unpredictable for the disadvantaged firm. With some probability, the advantaged firm remains a monopolist, but at a less attractive location.

**Case 3: Small or no differences in production cost**

When firms are (approximately) symmetric in terms of production cost, a pure strategy equilibrium on the pricing stage cannot be sustained any more.

**Proposition 6.** Let $c_1 = c_2 + \epsilon$. Then there exists an $\epsilon' > 0$ such that if $\epsilon \in [0, \epsilon')$, the game has no subgame-perfect equilibrium in which both firms play a pure strategy in stage 1.

The most intuitive candidate for an equilibrium in pure strategies on the pricing stage when $c_1 = c_2$ would be a symmetric pricing $p_1 = p_2$. Analyzing potential deviations from the symmetric pricing profile is analogous to the analysis of deviations in a Bertrand game with differentiated goods. Note that in our case the demand is not exogenously given, but taken from the equilibrium on the second stage. The literature on differentiated Bertrand games discusses whether symmetric prices do or do not form a Nash equilibrium and provides conditions on the demand elasticity; see e.g., Bester (1992). In particular, for symmetric prices to form a Nash equilibrium, the demand elasticity needs to be finite and sufficiently small as the price difference goes to 0. In our case, for both bounds on demand function, the demand elasticity tends to infinity as $d \to 0$. Thus, a small downward deviation in prices results in a large gain in demand, thereby upsetting a potential symmetric equilibrium. This is illustrated in Figure 5, which shows that the demand increase from a marginal reduction of $p_1$ at $p_1 = p_2$ is very steep.

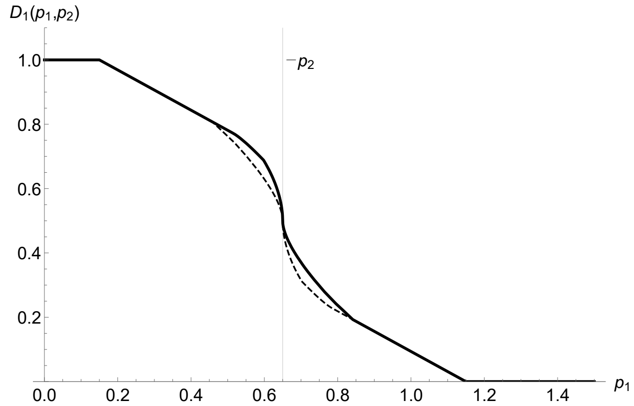


**Figure 5** Plot of Firm 1's equilibrium demand depending on $p_1$, for the given equilibrium play in stage 2 and $p_2 = 0.65$ and $t = 1$. The dashed graph uses the lower bound above 0.65, while the solid graph uses the upper bound.

Let us summarize the potential equilibria in the pricing stage from this section. When production cost differences are large, the more cost-efficient firm chooses a price that cuts the demand of the less efficient firm to zero. For moderate differences, the more efficient firm accommodates some demand of the less efficient firm. Both firms choose pure strategies on the pricing stage and mixed strategies in the resulting location game. When production cost differences are small, there is no longer a pure strategy equilibrium on the pricing stage, i.e., equilibria involve mixing on both stages.

# 5 Conclusion

In this paper, we analyze the price-then-location model in Hotelling's classical framework. We provide a closed-form characterization of the second stage when the price differences is sufficiently large, which—to the best of our knowledge—has not been solved so far. Our discussion identifies the technical problems in finding a closed-form solution when the price difference is small. These problems are different compared to the ones that arise in the location-then-price game and cannot be solved with changing assumptions on transport costs. We nevertheless derive payoff bounds for that case.

We also show that if firms are sufficiently heterogeneous, a full characterization of the subgame-perfect equilibrium is possible. In this equilibrium, firms play a pure strategy in the pricing stage but mix at the location stage. Instead, if firms are more symmetric, mixing occurs at both stages in equilibrium due to a high demand elasticity for approximately symmetric prices. A complete solution of this case would require a complete solution of the location stage. We leave this problem for future research.

From an economic point of view, the price-then-location model is a natural description in applications where price is a more long-term decision variable, whereas product variant (location) can be changed more often. Examples abound, including newspapers, lunch menus in restaurants, or the market for simple art. Arguably the timing is even more realistic in the famous toy example of ice-cream vendors on the Hotelling beach. In line with the long-term nature of the pricing decision, our pure-strategy solution on the pricing stage is particularly appealing: even if firms had the option of costlessly changing their price upon observing their rival's price prior to the location stage, they would not do so. This is in line with infrequent changes in prices, but frequent changes in "location" (i.e., characteristics) for t-shirts, magazines, or lunch menus.

Our results also provide a contribution to the political economy literature. Specifically, our stage 2 location game is identical to the continuous strategy case of Downsian voting with a favored candidate, first discussed in Aragonès and Palfrey (2002). They describe the solution to the continuous case as "a natural next step", which has not been taken. We have provided a closed-form characterization for the case when favoritism is sufficiently large.

In the political economy context, stage 1 could be seen as investments into building a reputation of competence on issues with broad consensus (valence issues) as discussed in Stokes (1963), where the politician who is considered more competent at the valence issues gets a head start in the campaign. In that interpretation, the "transport cost" on stage 2 represent the importance of the valence dimension to voters, relative to the distance of the politicians to the own position. To further pursue this approach, one might consider different cost and payoff functions for a politician.

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# 6 Appendix

*Proof of Proposition 1:* We construct equilibrium distribution functions with the following properties: Firm 1 randomizes continuously (with a density $f_1$) on an interval $[\underline{l}_1, 1-\underline{l}_1]$ and Firm 2 randomizes continuously on two intervals, $[\underline{l}_1 - d, \frac{1}{2} - d]$ and $[1-(\frac{1}{2}-d), 1-(\underline{l}_1 - d)]$, and places point mass at $\frac{1}{2} - d$ and $1-(\frac{1}{2}-d)$. For this construction to be an equilibrium, we need to show that any point in the randomization of one firm is a best response to the distribution chosen by the other firm.

Moreover, each of the constructed distributions is symmetric around $\frac{1}{2}$. Thus, it suffices to check payoffs for locations such that either $l_i \geq \frac{1}{2}$ or $l_i \leq \frac{1}{2}$, as the other half of the interval follows by symmetry.

Finally, recall that for $p_i > c_i$ maximizing expected demand is equivalent to maximizing profit in the second stage, and we therefore write the maximization problems in terms of demand.

**Distribution of Firm 1**

Let us write down the demand of Firm 2 from choosing a strategy $l_2 \in [\underline{l}_2, \frac{1}{2} - d]$. If $\underline{l}_1 \geq \frac{1}{2} - 2d$, Firm 2 only obtains a positive demand if $l_1 > l_2 + d$. In this case, we obtain

$$D_2(l_2) = \int_{l_2+d}^{1-d-\underline{l}_2} f_1(l_1)\frac{l_1 - d + l_2}{2} dl_1$$

We further know that the demand of Firm 2 has to be constant for every $l_2$ contained in its randomization. Thus:

$$D_2'(l_2) = \int_{l_2+d}^{1-d-\underline{l}_2} \frac{1}{2} f_1(l_1) dl_1 - l_2 f_1(l_2+d) = F_1(1-d-\underline{l}_2) - F_1(l_2+d) - 2l_2 f_1(l_2+d) = 0$$

Using $F_1(1-d-\underline{l}_2) = 1$ and rearranging, we obtain the following differential equation:

$$f_1(l_2+d) = \frac{1-F_1(l_2+d)}{2l_2}.$$

Let $l_1 = l_2 + d$. Solving the differential equation yields:

$$F_1(l_1) = 1 + \frac{c}{\sqrt{2d - 2l_1}}$$

for $l_1 \in [\underline{l}_1, \frac{1}{2}]$. Imposing the boundary condition $F_1(\frac{1}{2}) = \frac{1}{2}$, we find the constant $c$ and obtain

$$F_1(l_1) = 1 - \frac{\sqrt{\frac{1}{2} - d}}{2\sqrt{l_1 - d}}.$$

In the next step, we calculate $\underline{l}_1$ as the value of $l_1$ s.t. $F(l_1) = 0$. We obtain $\underline{l}_1 = \frac{6d+1}{8}$. This implies that the inequality $\underline{l}_1 \geq \frac{1}{2} - 2d$ is fulfilled for all $d \geq 3/22$. Since $3/22 < d^{crit}$, our starting assumption holds for all $d > d^{crit}$. Summing up the previous calculations and imposing symmetry, we obtain the distribution

$$F_1^*(l_1) \quad = \quad \begin{cases} 0 & \text{for } l_1 \leq \frac{6d+1}{8} \\ 1 - \frac{\sqrt{\frac{1}{2} - d}}{2\sqrt{l_1 - d}} & \text{for } l_1 \in (\frac{6d+1}{8}, \frac{1}{2}) \\ \frac{\sqrt{\frac{1}{2} - d}}{2\sqrt{1 - l_1 - d}} & \text{for } l_1 \in [\frac{1}{2}, 1 - \frac{6d+1}{8}) \\ 1 & \text{for } l_1 \geq 1 - \frac{6d+1}{8}. \end{cases}$$

Now consider Firm 2's demand given $F_1^*$—denoted by $D_2(l_2|F_1^*)$—for any $l_2 \in [0, \frac{1}{2}]$. We need to distinguish cases depending on the value of $d$. First suppose $\underline{l}_1 = \frac{6d+1}{8} \geq \frac{1}{2} - d$, or equivalently $d \geq \frac{3}{14}$. In this case, Firm 2 can only obtain a positive demand if $l_1 > l_2 + d$. We obtain

$$D_2(l_2|F_1^*) \quad = \quad \begin{cases} \int_{\frac{1+6d}{8}}^{\frac{1}{2}} \frac{\sqrt{\frac{1}{2} - d}}{4(l_1 - d)^{\frac{3}{2}}} \frac{l_1 + l_2 - d}{2} dl_1 + \int_{\frac{1}{2}}^{\frac{7-6d}{8}} \frac{\sqrt{\frac{1}{2} - d}}{4(1 - l_1 - d)^{\frac{3}{2}}} \frac{l_1 + l_2 - d}{2} dl_1 & \text{for } l_2 \in [0, \frac{1-2d}{8}] \\ \int_{l_2 + d}^{\frac{1}{2}} \frac{\sqrt{\frac{1}{2} - d}}{4(l_1 - d)^{\frac{3}{2}}} \frac{l_1 + l_2 - d}{2} dl_1 + \int_{\frac{1}{2}}^{\frac{7-6d}{8}} \frac{\sqrt{\frac{1}{2} - d}}{4(1 - l_1 - d)^{\frac{3}{2}}} \frac{l_1 + l_2 - d}{2} dl_1 & \text{for } l_2 \in (\frac{1-2d}{8}, \frac{1}{2} - d] \\ \max\{\int_{l_2 + d}^{\frac{7-6d}{8}} \frac{\sqrt{\frac{1}{2} - d}}{4(1 - l_1 - d)^{\frac{3}{2}}} \frac{l_1 + l_2 - d}{2} dl_1, 0\} & \text{for } l_2 \in (\frac{1}{2} - d, \frac{1}{2}]. \end{cases}$$

The first interval consists of locations such that Firm 2 wins with positive probability against any location in the randomization of Firm 1. The second interval with $l_2 \in [\frac{1-2d}{8}, \frac{1}{2} -$

$d]$ describes Firm 2's demand in the support of its equilibrium strategy. The third interval considers higher locations, where a location of Firm 2 close to $\frac{1}{2}$ leads to a payoff of zero.

For $d \in [d^{crit}, \frac{3}{14})$, a location of Firm 2 which is close to $\frac{1}{2}$ entails a positive probability to win if Firm 1 chooses a location close to $\underline{l}_1$, as $\underline{l}_1$ is now smaller than $\frac{1}{2} - d$. This leads to a fourth interval in the demand function:

$$D_2(l_2|F_1^*) \quad = \quad \begin{cases} \int_{\frac{1+6d}{8}}^{\frac{1}{2}} \frac{\sqrt{\frac{1}{2}-d}}{4(l_1-d)^{\frac{3}{2}}} \frac{l_1+l_2-d}{2} dl_1 + \int_{\frac{1}{2}}^{\frac{7-6d}{8}} \frac{\sqrt{\frac{1}{2}-d}}{4(1-l_1-d)^{\frac{3}{2}}} \frac{l_1+l_2-d}{2} dl_1 & \text{for } l_2 \in [0, \frac{1-2d}{8}] \\ \int_{l_2+d}^{\frac{1}{2}} \frac{\sqrt{\frac{1}{2}-d}}{4(l_1-d)^{\frac{3}{2}}} \frac{l_1+l_2-d}{2} dl_1 + \int_{\frac{1}{2}}^{\frac{7-6d}{8}} \frac{\sqrt{\frac{1}{2}-d}}{4(1-l_1-d)^{\frac{3}{2}}} \frac{l_1+l_2-d}{2} dl_1 & \text{for } l_2 \in (\frac{1-2d}{8}, \frac{1}{2} - d] \\ \int_{l_2+d}^{\frac{7-6d}{8}} \frac{\sqrt{\frac{1}{2}-d}}{4(1-l_1-d)^{\frac{3}{2}}} \frac{l_1+l_2-d}{2} dl_1 & \text{for } l_2 \in (\frac{1}{2} - d, \frac{14d+1}{8}] \\ \int_{\frac{1+6d}{8}}^{l_2-d} \frac{\sqrt{\frac{1}{2}-d}}{4(l_1-d)^{\frac{3}{2}}} (1 - \frac{l_1+l_2+d}{2}) dl_1 + \int_{l_2+d}^{\frac{7-6d}{8}} \frac{\sqrt{\frac{1}{2}-d}}{4(1-l_1-d)^{\frac{3}{2}}} \frac{l_1+l_2-d}{2} dl_1 & \text{for } l_2 \in (\frac{14d+1}{8}, \frac{1}{2}] \end{cases}$$

We first compute Firm 2's demand for $l_2 \in [\frac{1-2d}{8}, \frac{1}{2} - d]$ and then show that it is weakly higher than demand for all other $l_2 \leq \frac{1}{2}$. Solving the integrals and simplifying, we get

$$\begin{aligned} D_2(l_2|F_1^*) =& \frac{\sqrt{\frac{1}{2}-d}}{4\sqrt{\frac{1}{2}-d}} (\frac{1}{2} - d - l_2) - \frac{\sqrt{l_2}}{4\sqrt{l_2}} (l_2 + d - d - l_2) \\ &+ \frac{\sqrt{\frac{1}{2}-d}}{4\sqrt{1-d-\frac{7-6d}{8}}} (\frac{7-6d}{8} - l_2 + 3d - 2) - \frac{\sqrt{\frac{1}{2}-d}}{4\sqrt{1-d-\frac{1}{2}}} (\frac{1}{2} - l_2 + 3d - 2) \\ =& \frac{1}{8}(1 - 2d - 2l_2) + \frac{1}{16}(3 - 6d + 4l_2) \\ =& \frac{5}{16}(1 - 2d) \end{aligned}$$

which is constant in $l_2$.

We now move on to show that Firm 2's demand in the support of its strategy is higher than for all possible deviations. The demand is increasing on the first interval as the number of consumers served increases for any $l_1$ in the support of Firm 1. On that interval, the maximum $\frac{5}{16}(1 - 2d)$ is thus reached at $l_2 = \frac{1-2d}{8}$.

Demand on the third interval, for $l_2 > \frac{1}{2} - d$ can be rewritten to

$$D_2(l_2|F_1^*) = \frac{\sqrt{\frac{1}{2}-d}}{4\sqrt{1-d-\frac{7-6d}{8}}}\left(\frac{7-6d}{8} - l_2 + 3d - 2\right) - \frac{\sqrt{\frac{1}{2}-d}}{4\sqrt{1-2d-l_2}}(4d-2)$$
$$= \frac{\sqrt{\frac{1}{2}-d}}{8}\left(\frac{8d-4}{\sqrt{1-l_2-2d}} + \frac{9-18d+8l_2}{\sqrt{2-4d}}\right)$$

which is strictly decreasing in $l_2$ for $d \in [d^{crit}, \frac{3}{14})$.

Recall that the fourth interval exists only if $d < \frac{3}{14}$. To show that Firm 2 does not have a profitable deviation in this interval, we show that its demand in this interval is increasing in $l_2$ and that the demand at $l_2 = \frac{1}{2}$ is weakly below its equilibrium demand. The partial derivative of Firm 2's demand in the fourth interval is equal to

$$D_2'(l_2|F_1^*) = \frac{\sqrt{\frac{1}{2}-d}(1-2d)}{4(1-l_2-2d)^{\frac{3}{2}}(l_2-2d)^{\frac{3}{2}}}((1-l_2-2d)^{\frac{3}{2}} - (l_2-2d)^{\frac{3}{2}})$$

which is greater than 0 in the given interval. To see this, first note that the fraction is positive for any $l_2 > 2d$, which is true as $d < \frac{3}{14}$. The difference in the brackets is positive if $l_2 \in [2d, \frac{1}{2})$ and $d < \frac{1}{4}$, which finishes the argument. We now check that the demand from deviating to $l_2 = \frac{1}{2}$ is below $\frac{5}{16}(1-2d)$. The deviation demand can be written as

$$D_2(\frac{1}{2}|F_1^*) = \frac{1}{2} + (1-2d)(\frac{9}{8} - \frac{\sqrt{1-2d}}{\sqrt{1-4d}}) < \frac{5}{16}(1-2d)$$

This inequality holds for all $d > d^{crit} \approx 0.1914$.

**Distribution of Firm 2**

We start by writing down the demand of Firm 1, $D_1$, from choosing a strategy inside its support, i.e., $l_1 \in [\underline{l}_2 + d, \frac{1}{2}]$, assuming Firm 2 also places mass at $\frac{1}{2} - d$ and $\frac{1}{2} + d$ and that $\underline{l}_1 \geq \frac{1}{2} - 2d$:

$$D_1(l_1) = \int_{\underline{l}_2}^{l_1-d} f_2(l_2)(1 - \frac{l_1 + l_2 - d}{2})dl_2 + \int_{l_1-d}^{\frac{1}{2}-d} f_2(l_2)dl_2 + m_2(\frac{1}{2} - d) + m_2(\frac{1}{2} + d)\frac{l_1 + d + l_2}{2}$$
$$+ \int_{\frac{1}{2}+d}^{1-\underline{l}_2} f_2(l_2)\frac{l_1 + d + l_2}{2}dl_2$$

Using $m_2(\frac{1}{2} + d) = \frac{1}{2} - \int_{\underline{l}_2}^{\frac{1}{2}-d} f_2(l_2)dl_2$, we take the derivative and set it equal to zero, as Firm 1's demand needs to be constant on the interval on which it randomizes.

$$D_1'(l_1) = \int_{\frac{1}{2}+d}^{1-\underline{l}_2} \frac{1}{2}f_2(l_2)dl_2 - \int_{\underline{l}_2}^{l_1-d} \frac{1}{2}f_2(l_2)dl_2 - f_2(l_1-d)(l_1-d) + \frac{1}{2}\left(\frac{1}{2} - \int_{\underline{l}_2}^{\frac{1}{2}-d} f_2(l_2)dl_2\right) = 0$$

This can be simplified to

$$D_1'(l_1) = \frac{1}{4} - \int_{\underline{l}_2}^{l_1-d} \frac{1}{2}f_2(l_2)dl_2 - f_2(l_1 - d)(l_1 - d) = 0$$

Solving the integral, we get

$$D_1'(l_1) = \frac{1}{4} - \frac{1}{2}F_2(l_1 - d) + \frac{1}{2}F_2(\underline{l}_2) - f_2(l_1 - d)(l_1 - d) = 0$$

Using the boundary condition $F_2(\underline{l}_2) = 0$ gives us the following differential equation

$$f_2(l_1 - d) = \frac{\frac{1}{2} - F_2(l_1 - d)}{2(l_1 - d)}$$

Let $l_2 = l_1 - d$ to get

$$f_2(l_2) = \frac{\frac{1}{2} - F_2(l_2)}{2l_2}$$

Solving the differential equation gives us

$$F_2(l_2) = \frac{1}{2} + \frac{c}{\sqrt{l_2}}.$$

Using the boundary condition $F_2(\underline{l}_2) = F_2(\underline{l}_1 - d) = F_2(\frac{1-2d}{8}) = 0$, we can find $c = -\frac{\sqrt{1-2d}}{4\sqrt{2}}$. This gives us $F_2(l_2)$ on $[\frac{1-2d}{8}, \frac{1}{2} - d)$:

$$F_2(l_2) = \frac{1}{2} - \frac{\sqrt{2-4d}}{8\sqrt{l_2}}.$$

Then find $m_2(\frac{1}{2}-d) = \frac{1}{2} - \int_{\frac{1-2d}{8}}^{\frac{1}{2}-d} f_2(l_2)dl_2 = \frac{1}{4}$ or alternatively say $F_2(\frac{1}{2}-d) = 0.5$. Summing up the previous calculations and imposing symmetry, we obtain

$$F_2^*(l_2) \quad = \quad \begin{cases} 0 & \text{for } l_2 \leq \frac{1-2d}{8} \\ \frac{1}{2} - \frac{\sqrt{2-4d}}{8\sqrt{l_2}} & \text{for } l_2 \in (\frac{1-2d}{8}, \frac{1}{2}-d) \\ \frac{1}{2} & \text{for } l_2 \in [\frac{1}{2}-d, \frac{1}{2}+d) \\ \frac{1}{2} + \frac{\sqrt{2-4d}}{8\sqrt{1-l_2}} & \text{for } l_2 \in [\frac{1}{2}+d, 1-\frac{1-2d}{8}) \\ 1 & \text{for } l_2 \geq 1 - \frac{1-2d}{8}. \end{cases}$$

Using Firm 2's distribution, we can now calculate Firm 1's demand, $D_1(l_1|F_2^*)$, and show that it is constant on the support of its mixed strategy, $[\frac{6d+1}{8}, 1-\frac{6d+1}{8}]$. Firm 1's demand for $l_1$ in $(\frac{1+6d}{8}, 0.5]$ is:[10]

$$\begin{aligned} D_1(l_1|F_2^*) = & \int_{\frac{1-2d}{8}}^{l_1-d} \frac{\sqrt{2-4d}}{16l_2^{\frac{3}{2}}}(1 - \frac{l_1-d+l_2}{2})\, dl_2 + \int_{l_1-d}^{\frac{1}{2}-d} \frac{\sqrt{2-4d}}{16l_2^{\frac{3}{2}}}\, dl_2 + m_2(\frac{1}{2}-d) \\ & + m_2(\frac{1}{2}+d)\frac{l_1+d+\frac{1}{2}+d}{2} + \int_{\frac{1}{2}+d}^{\frac{7+2d}{8}} \frac{\sqrt{2-4d}}{16(1-l_2)^{\frac{3}{2}}}\frac{l_1+d+l_2}{2}\, dl_2. \end{aligned}$$

Solving the integrals and using $m_2(\frac{1}{2}+d) = \frac{1}{4}$, this can be simplified to

$$\begin{aligned} D_1(l_1|F_2^*) = & \frac{1}{32}\left(17 + 6d - 8l_1 - \frac{4\sqrt{2-4d}}{\sqrt{l_1-d}}\right) + \frac{1}{8}\frac{\sqrt{2-4d}}{\sqrt{l_1-d}} + \frac{2l_1+4d+1}{16} + \frac{3+6d+4l_1}{32} \\ = & \frac{11+10d}{16} \end{aligned}$$

[10] Here we use random $50-50$ tie-breaking for $l_1 = 0.5$.

Consider Firm 1's demand for $l_1 \in [0, \frac{1}{2}]$, given $F_2^*$. We need to distinguish two cases depending on the value of $d$. First, suppose $\frac{1}{2} - 2d < 0$, or equivalently, $d > \frac{1}{4}$. In this case, for any $l_1 \leq \underline{l}_1$, Firm 1 wins for sure against all $l_2 < \frac{1}{2}$. We obtain

$$D_1(l_1|F_2^*) = \begin{cases} \frac{1}{2} + \frac{1}{4}\frac{l_1+d+\frac{1}{2}+d}{2} + \int_{\frac{1}{2}+d}^{\frac{7+2d}{8}} \frac{\sqrt{2-4d}}{16(1-l_2)^{\frac{3}{2}}}\frac{l_1+l_2+d}{2}dl_2 & \text{for } l_1 \in [0, \frac{6d+1}{8}) \\ \frac{11+10d}{16} & \text{for } l_1 \in [\frac{6d+1}{8}, \frac{1}{2}]. \end{cases}$$

The demand is increasing on $[0, \frac{6d+1}{8}]$ (and continuous on the boundaries) as the second and third summand are strictly increasing in $l_1$ and first summand is constant.

For $d \in [d^{crit}, \frac{1}{4})$, there is a third interval: for small $l_1$, Firm 1 does no longer win with certainly anymore if $l_2 < \frac{1}{2}$. We obtain

$$D_1(l_1|F_2^*) = \begin{cases} \int_{\frac{1-2d}{8}}^{l_1+d} \frac{\sqrt{2-4d}}{16l_2^{\frac{3}{2}}}dl_2 + \int_{l_1+d}^{\frac{1}{2}-d} \frac{\sqrt{2-4d}}{16l_2^{\frac{3}{2}}}\frac{l_1+d+l_2}{2}dl_2 + \frac{1}{4}\frac{l_1+d+\frac{1}{2}-d}{2} + x & \text{for } l_1 \in [0, \frac{1}{2} - 2d) \\ \frac{1}{2} + x & \text{for } l_1 \in [\frac{1}{2} - 2d, \frac{6d+1}{8}) \\ \frac{11+10d}{16} & \text{for } l_1 \in [\frac{6d+1}{8}, \frac{1}{2}], \end{cases}$$

where $x = \frac{1}{4}\frac{l_1+d+\frac{1}{2}+d}{2} + \int_{\frac{1}{2}+d}^{\frac{7+2d}{8}} \frac{\sqrt{2-4d)}}{16(1-l_2)^{\frac{3}{2}}}\frac{l_1+l_2+d}{2}dl_2$ is the payoff Firm 1 gets for $l_2 > \frac{1}{2}$. Again, for $l_1 \in [0, \frac{6d+1}{8} - 2d)$, the demand is increasing as every summand is non-decreasing and some summands are increasing. The demand is continuous at $\frac{6d+1}{8}$, which completes the proof.

$\square$

**Demand Bounds for Small $d$**

We now derive the upper and lower bounds used in Figures 4 and 5, for small $d = \frac{p_2 - p_1}{t}$, i.e., for small differences in prices, or - equivalently - for large transport costs $t$.

**Lemma 7.** The equilibrium demand of the advantaged firm is bounded from below by

$$D_1^*(d) = D_2^*(-d) \quad \geq \quad \begin{cases} 1 & \text{for } d \geq 0.5 \\ \frac{11+10d}{16} & \text{for } d \in [d^{crit}, 0.5) \\ 1 - \frac{1}{16}(1-2d)(5 + m_1(2+m_1)) & \text{for } d \in [d', d^{crit}) \\ 1 - \frac{3-\sqrt{d(d^3-6d+6)}-2d(2+d)}{2(3+d)} & \text{for } d \in [0, d') \end{cases}$$

with $m_1 = 53(d^{crit})^3 - 53d^3$ and $d' \approx 0.136069$.

*Proof.* To derive a lower bound on the demand the advantaged firm (for the rest of the proof Firm 1 w.l.o.g.) can achieve, we derive the supremum demand Firm 2 can get, given some strategy of Firm 1. By the constant sum property of the game, this automatically gives us a lower bound on the equilibrium demand of Firm 1. We vary the distribution of Firm 1 depending on $d$.

We start with finding a payoff bound for $d$ slightly below $d^{crit}$. Remember, that at $d = d^{crit}$ deviating to $l_2 = \frac{1}{2}$ becomes optimal for Firm 2. Firm 1 can make it less attractive for Firm 2 to play $l_2 = \frac{1}{2}$ by placing mass at $\frac{1}{2}$ itself. Suppose Firm 1 puts mass $m_1 > 0$ on $l_1 = \frac{1}{2}$ and otherwise randomizes in a similar way as in the equilibrium for larger d, described in Proposition 1. Such a mass point $m_1$ needs to be sufficiently high for positive $d$ to make the deviation to $l_2 = \frac{1}{2}$ unattractive, and converge to zero for $d \to d^{crit}$. One point which fulfills this condition is $m_1 = 53(d^{crit})^3 - 53d^3 \approx 0.371 - 53d^3$. This yields the following distribution for Firm 1:

$$F_1(l_1) = \begin{cases} 0 & \text{for } l_1 \leq \underline{l}_1 \\ 1 - \frac{\sqrt{\frac{1}{2}-d}(1+m_1)}{2\sqrt{l_1-d}} & \text{for } l_1 \in (\underline{l}_1, \frac{1}{2}) \\ \frac{\sqrt{\frac{1}{2}-d}(1+m_1)}{2\sqrt{(1-l_1)-d}} & \text{for } l_1 \in [\frac{1}{2}, 1-\underline{l}_1) \\ 1 & \text{for } l_1 \geq 1-\underline{l}_1. \end{cases}$$

with $\underline{l}_1 = \frac{1}{2}((1-2d)(\frac{m_1}{2}+\frac{1}{2})^2 + 2d)$ and $m_1 = 53(d^{crit})^3 - 53d^3$. The corresponding demand faced by Firm 2 when playing a distribution between $\underline{l}_2$ and $\frac{1}{2} - d$ for $l_2 < \frac{1}{2}$ (and symmetrically for $l_2 > \frac{1}{2}$) is then given by

$$\begin{aligned} D_2(l_2|F_1(l_1)) = &\int_{l_2+d}^{\frac{1}{2}} -\frac{\sqrt{\frac{1}{2}-d}(1+m_1)}{4(l_1-d)^{3/2}} \frac{l_1-d+l_2}{2}\, dl_1 + \\ & m_1 \frac{\frac{1}{2}-d+l_2}{2} + \\ & \int_{\frac{1}{2}}^{1-\underline{l}_1} -\frac{\sqrt{\frac{1}{2}-d}(1+m_1)}{4(1-l_1-d)^{3/2}} \frac{l_1-d+l_2}{2}\, dl_1 \\ = & \frac{1}{16}(1-2d)(5+m_1(2+m_1)) \end{aligned}$$

We need to check for which $d$ this is better for Firm 2 than playing $l_2 = \frac{1}{2}$ for sure. The demand that can be obtained by this deviation is given by

$$\begin{aligned} D_2(l_2 = \frac{1}{2}|F_1(l_1)) = &2\int_{\underline{l}_1}^{\frac{1}{2}-d} -\frac{\sqrt{d-\frac{1}{2}}(m_1+1)}{4(d-l_1)^{3/2}} \left(1 - \frac{\left(l_1+d+\frac{1}{2}\right)}{2}\right) dl_1 \\ = &\frac{13+(1-2d)m_1(m_1+2)-18d}{8} - (1-2d)(m_1+1)\frac{\sqrt{1-2d}}{\sqrt{1-4d}} \end{aligned}$$

which is smaller than the demand Firm 2 can obtain by choosing any $l_2 \in (\underline{l}_2, \frac{1}{2}-d)$.

We now try improve the lower bound on the equilibrium demand of Firm 1 for very small $d$ to obtain the last part of our bound. We want to find a distribution such that Firm 2 is indifferent between playing $\underline{l}_2$ and $l_2 = \frac{1}{2}$. Suppose Firm 1 plays a uniform distribution around $\frac{1}{2}$ with continuous support on $(\frac{1}{2} - a(d), \frac{1}{2} + a(d))$, and mass $\frac{d}{2}$ on the edges of the support. Such mass placed at the ends of the distribution ensures that Firm 2 does not want to play a location $l_2 \in (\underline{l}_2, \frac{1}{2})$. This distribution is formally described by

$$F_1(l_1) = \begin{cases} 0 & \text{for } l_1 \in [0, \frac{1}{2} - a) \\ \frac{2a+d-1}{4a} + \frac{(1-d)}{2a} l_1 & \text{for } l_1 \in (\frac{1}{2} - a, \frac{1}{2} + a) \\ 1 & \text{for } l_1 \in (\frac{1}{2} + a, 1] \end{cases}$$

with $a = \frac{\sqrt{d(d^3-6d+6)}-d}{d+3}$.

We derive $a(d)$ such that Firm 2 is indifferent between playing $\underline{l}_2$ and $\frac{1}{2}$. In the calculation, we assume that ties are broken in favor of player 2 at $\underline{l}_2$ as we are interested in the supremum payoff player 2 can get.

In particular, Firm 2's demand when playing $l_2 = \frac{1}{2}$ is given by

$$\begin{aligned} D_2(l_2 = \frac{1}{2} | F_1(l_1)) =& 2\left(\int_{\frac{1}{2}+d}^{\frac{1}{2}+a} \frac{1-d}{2a} \frac{l_1 - d + \frac{1}{2}}{2} \, dl_1 + \frac{d}{2} \frac{\frac{1}{2} + a - d + \frac{1}{2}}{2}\right) \\ =& \frac{a^2(d+1) - 2a(d-1) - d\left(d^2 - 3d + 2\right)}{4a} \end{aligned}$$

At the same time, Firm 2's demand when playing $l_2 = \underline{l}_2 = \underline{l}_1 - d$ against the distribution above is given by

$$D_2(l_2 = \frac{1}{2} - a - d | F_1(l_1)) = \frac{\left(-a - d + \frac{1}{2}\right) + \left(\frac{1}{2} - d\right)}{2}$$

Setting these two demands equal and re-arranging yields $a = \frac{\sqrt{d(d^3-6d+6)}-d}{d+3}$, which yields the second part of the demand bound

$$D_2(l_2 = \frac{1}{2} | F_1(l_1)) = \frac{3 - \sqrt{d\left(d^3 - 6d + 6\right)} - 2d(2+d)}{2(3+d)}$$

To finish the proof, we need to verify that it is not better for Firm 2 to play some $l_2 \in (\underline{l}_2, \underline{l}_2 + 2d)$. Firm 2's demand for these $l_2$, given Firm 1's distribution is given by

$$\begin{aligned} D_2(l_2|F_1(l_1)) =& \int_{l_2+d}^{\frac{1}{2}+a} \frac{1-d}{2a} \frac{l_1 - d + l_2}{2} \, dl_1 + \frac{d}{2} \frac{\left(\frac{1}{2}+a\right) - d + l_2}{2} \\ =& \frac{(1-d)(2a-2d-2l_2+1)(2a-2d+6l_2+1)}{32a} + \frac{d\left(\frac{1}{2}+a-d+l_2\right)}{4} \end{aligned}$$

Solving the first order condition yields $l_2' = \frac{1}{6}\left(1 - \frac{2a}{d-1} - 2d\right)$ at which demand is maximized. We check for which $d$ a deviation to $l_2' > \underline{l}_2$ would be profitable. Note that as $D_2(l_2|F_1(l_1))$ jumps upwards at $l_2 = \underline{l}_2$, we need to compare Firm 2's demand given these two strategies and cannot simply find the $d$ for which $l_2' < \underline{l}_2$. Using $a$ as derived above, we derive Firm 2's demand of playing $l_2 = l_2'$, which yields

$$\begin{aligned} D_2(l_2'|F_1(l_1)) =& \frac{1}{24(d-1)(d+3)\left(d - \sqrt{d\left(d^3-6d+6\right)}\right)} \times \\ & \left(9 + d^6 + 12d^5 - 12d^4 - 66d^3 - 4\left(10\sqrt{d\left(d^3-6d+6\right)} + 9\right)d + \right. \\ & \left. 12\sqrt{d\left(d^3-6d+6\right)} + 94d^2 + 14\sqrt{d^7\left(d^3-6d+6\right)} + 12\sqrt{d^5\left(d^3-6d+6\right)}\right) \end{aligned}$$

Finally, we check for which $d$, Firm 2's demand at $l_2 = \underline{l}_2$ is higher than its demand from playing $l_2'$, which is true for $d < 0.145271$. For small $d$, Firm 1's payoff is therefore bounded from below by

$$D_1(d) \geq 1 - D_2(\underline{l}_2|F_1(l_1)) = 1 - \frac{3 - \sqrt{d\left(d^3-6d+6\right)} - 2d(2+d)}{2(3+d)}$$

It is straightforward to verify that given this strategy, we have $\lim_{d\to 0^+} D_1'(d) = \infty$. As a last step, we calculate that the two demand bounds intersect at $d \approx 0.136069$. □

**Lemma 8.** The equilibrium demand of the advantaged firm is bounded from above by

$$D_1^*(d) = D_2^*(-d) \leq \begin{cases} 1 & \text{for } d \geq 0.5 \\ \frac{11+10d}{16} & \text{for } d \in [\frac{3}{22}, 0.5] \\ \frac{3}{8} + \frac{9d}{4} + (\frac{1}{4} + \frac{d}{2})\sqrt{3 + \frac{2}{2d-1}} & \text{for } d \in [0.0522, \frac{3}{22}] \\ \frac{1}{2} - \frac{3}{4}d^{1.5} + \frac{d}{2} + \frac{3}{4}\sqrt{d} & \text{for } d \in [0, 0.0522] \end{cases}$$

*Proof.* We do the reverse to derive the upper demand bound. We calculate the supremum demand Firm 1 can get for a given strategy of Firm 2. As with the lower bound, what is a (reasonably) good strategy for Firm 2 depends on $d$.

The case with $d \geq d^{crit}$ is the equilibrium case described above. For $d < d^{crit}$, suppose Firm 2 uses a similar strategy as in Proposition 1, i.e., placing probability mass on $\frac{1}{2} - d$ (and symmetrically on $\frac{1}{2} + d$), with the same randomization below $\frac{1}{2} - d$. The corresponding distribution becomes

$$F_2(l_2) = \begin{cases} 0 & \text{for } l_2 \leq \frac{1-2d}{8} \\ \frac{1}{2} - \frac{\sqrt{2-4d}}{8\sqrt{l_2}} & \text{for } l_2 \in [\frac{1-2d}{8}, \frac{1}{2} - d) \\ \frac{1}{2} & \text{for } l_2 \in [\frac{1}{2} - d, \frac{1}{2} + d) \\ \frac{1}{2} + \frac{\sqrt{2-4d}}{8\sqrt{1-l_2}} & \text{for } l_2 \in [\frac{1}{2} + d, 1 - \frac{1-2d}{8}) \\ 1 & \text{for } l_2 \geq 1 - \frac{1-2d}{8}. \end{cases}$$

For $d \geq \frac{3}{22}$, we obtain $\frac{1-2d}{8} \geq \frac{1}{2} - 3d$, i.e., the payoff of Firm 1 is computed as in Proposition 1. When $d$ is smaller, $\underline{l}_2$ would be below $\frac{1}{2} - 3d$. This would change the payoff computation for Firm 1 as for $l_1$ below $\frac{1}{2} - 2d$, Firm 1 would not obtain a demand of 1 against $l_2 = \frac{1}{2} - d$. In this case, Firm 2 places mass on $\frac{1}{2} - 3d$ (and $\frac{1}{2} + 3d$) as well.[11] This leads to the distribution:

[11]Note that $\pi_1(l_1)$ jumps upward at $l_1 = \frac{1}{2} - 2d$ for all $d \in [0, \frac{3}{22}]$, i.e., Firm 1 has no profitable downward deviation.

$$F_2(l_2) = \begin{cases} 0 & \text{for } l_2 \leq \frac{1}{2} - 3d \\ \frac{1}{2} - \frac{\sqrt{2-4d}}{8\sqrt{l_2}} & \text{for } l_2 \in [\frac{1}{2} - 3d, \frac{1}{2} - d) \\ \frac{1}{2} & \text{for } l_2 \in [\frac{1}{2} - d, \frac{1}{2} + d) \\ \frac{1}{2} + \frac{\sqrt{2-4d}}{8\sqrt{1-l_2}} & \text{for } l_2 \in [\frac{1}{2} + d, 1 - \frac{1}{2} + 3d) \\ 1 & \text{for } l_2 \geq \frac{1}{2} + 3d. \end{cases}$$

The maximal demand that Firm 1 can get against this distribution is given in the third case of the demand bound.

Finally, we provide a better bound for very small $d$. Suppose Firm 2 randomizes uniformly on the interval $l_2 \in [\frac{1}{2} - \sqrt{d}, \frac{1}{2} + \sqrt{d}]$. The best Firm 1 can do is to put all mass on $l_1 = \frac{1}{2}$, which leads to the demand in the fourth case of the demand bound. For $d \in [0, 0.0522)$, straightforward computation shows that this is a better strategy for Firm 2 than the one described above. □

**Lemma 9.** The equilibrium payoff is continuous at $d = 0$.

*Proof.* At $d = 0$, the equilibrium payoff is equal to 0.5. Firm 1 can guarantee a payoff of $0.5 + d$ for any $d \geq 0$ by choosing $l_1 = 0.5$. As such, it suffices that as $d \to 0^+$, Firm 2 has a strategy that guarantees a limit payoff of 0.5. It is readily verified that the uniform distribution on $l_2 \in [\frac{1}{2} - \sqrt{d}, \frac{1}{2} + \sqrt{d}]$ which we used for the payoff bound guarantees a limit payoff of $\frac{1}{2}$ of Firm 2 as $d \to 0$. □

*Proof of Propositions 3 and 4:* Suppose a pricing equilibrium exists in which $d \in [d^{crit}, 0.5)$. In this case, payoffs are given by the following functions:

$$\pi_1(p_1, p_2) = (p_1 - c_1)\left(\frac{11}{16} + \frac{10}{16}\frac{p_2 - p_1}{t}\right)$$
$$\pi_2(p_1, p_2) = (p_2 - c_2)\left(\frac{5}{16} - \frac{10}{16}\frac{p_2 - p_1}{t}\right)$$

Note that each $\pi_i$ is concave in $p_i$. Maximizing the respective functions and solving for the Nash equilibrium yields prices $p_1^* = \frac{1}{3}(2c_1 + c_2) + \frac{9}{10}t$, and $p_2^* = \frac{1}{3}(c_1 + 2c_2) + \frac{7}{10}t$.

To check for which combinations of cost parameters $c_1, c_2$ and $t$ the difference in prices is indeed in the region specified above, we calculate $d^* = \frac{p_2^* - p_1^*}{t}$. The resulting $d^* = \frac{c_2 - c_1}{3t} - \frac{1}{5}$ is increasing in Firm 1's cost advantage $c_2 - c_1$ and decreasing in transport costs $t$. For $c_2 - c_1 \in [(\frac{3}{5} + 3d^{crit})t, \frac{21}{10}t)$, we get that $d^* \in [d^{crit}, 0.5)$.

If, instead, $c_2 - c_1 \geq \frac{21}{10}t$, $p_2^*$ would be below $c_2$, which would yield a negative payoff. Instead, any $p_2 \geq c_2$ leads to a payoff of 0 and, in particular, $p_2 = c_2$ is optimal for Firm 2. To determine the best response of Firm 1, we can use the upper bound on its payoff for small $d$ to obtain that the solution to its optimization problem is $p_1(p_2 = c_2) = c_2 - \frac{t}{2}$. For the (weakly lower) real payoff when $d$ is small, the same solution $p_1^*$ is optimal. Firm 2 then has no profitable deviation since any lower price would lead to a non-positive profit and any larger price would not yield a positive market share in stage 2. Therefore, the prices $p_1 = c_2 - \frac{t}{2}$ and $p_2 = c_2$ constitute an equilibrium for $c_2 - c_1 \geq \frac{21}{10}t$, which establishes Proposition 3.

We next turn to the range $c_2 - c_1 \in [(\frac{3}{5} + 3d^{crit})t, \frac{21}{10}t)$. For these cost parameters, the transport cost-weighted difference in equilibrium prices, $d^* = \frac{p_2^* - p_1^*}{t}$, is indeed in the range $d \in [d^{crit}, 0.5)$, that is, the range for which equilibrium prices maximize profits given the assumed equilibrium behavior in the location stage. To establish that the profit maximizing prices, $p_1^*$ and $p_2^*$, indeed form an equilibrium, we need to check that no firm has a profitable deviation in the pricing stage. We start by checking whether Firm 2 has a profitable deviation. Note that any potentially profitable deviation for Firm 2—the firm with the higher price—will be to a lower price. This might result in $d < d^{crit}$, a range for which we only have bounds on the demands. We therefore use an upper bound for Firm 2's demand for differences in prices below $d^{crit}$. The range we specify here gives thus a bound for the cost parameter combinations of $c_1, c_2$ and $t$ for which the prices specified in Proposition 4 indeed form an equilibrium.

We take simple new demand bound for Firm 2 in the form of the straight line connecting 0.5 and Firm 2's demand at $d = d^{crit}$, which can be described by $D_2(d) \leq 0.5 - 1.60467d$. Replacing $d = \frac{p_2 - p_1}{t}$, substituting Firm 1's equilibrium price and then maximizing this new profit bound yields a best response price $p_2^{dev} = \frac{1}{3}(c_1 + 2c_2) + 0.605795t$, which is about 0.1 below $p_2^*$. Firm 2's deviation profit when playing this price against Firm 1's equilibrium price is then given by

$$\pi_1(p_1^*, p_2^{dev}) \approx \frac{(t - 0.55(c_2 - c_1))^2}{1.698t},$$

which is smaller than the equilibrium payoff for $c_2 - c_1 > \Delta c^{crit} \approx 1.34819t$. This bounds the cost parameter combinations from below for which prices in Proposition 4 are indeed optimal. It can easily be verified that Firm 2 cannot gain by deviating to an even smaller $p_2 \leq p_1^*$, even when assuming that Firm 2 would gain full demand in this case. As a last step, it is easy to verify that Firm 1 has no profitable deviation using the upper demand bound described in Lemma 8, which establishes the equilibrium for $c_2 - c_1 \in [\Delta c^{crit}, 2.1t)$ and completes the proof of Proposition 4. □

*Proof of Proposition 5:* Denote the expected location difference to the preferred firm by $\mathbb{E}[\Delta(l)]$. We have to distinguish cases based on whether one firm caters to the entire market or whether the market is split between two firms (as indicated by the two indicator functions). W.l.o.g. we take $l_2 \leq \frac{1}{2}$ again (multiplying the respective probabilities by 2 due to symmetry). This yields:

$$\begin{aligned}
\mathbb{E}[\Delta(l)] =& \mathbb{1}_{\{l_1 - l_2 < d\}} \Bigg( \int_{\underline{l}_2}^{\frac{1}{2}-d} \int_{\underline{l}_1}^{\bar{l}_1} \left( \frac{l_1^2}{2} + \frac{(1-l_1)^2}{2} \right) 2 f_2(l_2) f_1(l_1) dl_1 dl_2 + \\
& \qquad 2m_2(\frac{1}{2} - d) \int_{\underline{l}_1}^{\bar{l}_1} \left( \frac{l_1^2}{2} + \frac{(1-l_1)^2}{2} \right) f_1(l_1) dl_1 \Bigg) + \\
& \mathbb{1}_{\{l_1 - l_2 > d\}} \Bigg( \int_{\underline{l}_2}^{\frac{1}{2}-d} \int_{\underline{l}_1}^{\bar{l}_1} \left( \frac{l_2^2}{2} + \frac{(l_2^* - l_2)^2}{2} + \frac{(l_1 - l_2^*)^2}{2} + \frac{(1-l_1)^2}{2} \right) 2 f_2(l_2) f_1(l_1) dl_1 dl_2 + \\
& \qquad 2m_2(\frac{1}{2} - d) \int_{\underline{l}_1}^{\bar{l}_1} \left( \frac{l_2^2}{2} + \frac{(l_2^* - l_2)^2}{2} + \frac{(l_1 - l_2^*)^2}{2} + \frac{(1-l_1)^2}{2} \right) f_1(l_1) dl_1 \Bigg) \\
=& \int_{\underline{l}_2}^{\frac{1}{2}-d} \int_{\underline{l}_1}^{l_2+d} \left( \frac{l_1^2}{2} + \frac{(1-l_1)^2}{2} \right) 2 f_2(l_2) f_1(l_1) dl_1 dl_2 +
\end{aligned}$$

$$2m_2(\frac{1}{2}-d)\ \int_{\underline{l}_1}^{\frac{1}{2}}\left(\frac{l_1^2}{2}+\frac{(1-l_1)^2}{2}\right)f_1(l_1)dl_1+$$
$$\int_{\underline{l}_2}^{\frac{1}{2}-d}\int_{l_2+d}^{\bar{l}_1}\left(\frac{l_2^2}{2}+\frac{(l_2^*-l_2)^2}{2}+\frac{(l_1-l_2^*)^2}{2}+\frac{(1-l_1)^2}{2}\right)2f_2(l_2)f_1(l_1)dl_1dl_2+$$
$$2m_2(\frac{1}{2}-d)\ \int_{\frac{1}{2}}^{\bar{l}_1}\left(\frac{l_2^2}{2}+\frac{(l_2^*-l_2)^2}{2}+\frac{(l_1-l_2^*)^2}{2}+\frac{(1-l_1)^2}{2}\right)f_1(l_1)dl_1$$
$$=\frac{37d^2}{64}-\frac{17d}{64}+\frac{61}{256}$$

where $l_2^*=\frac{l_1-d+l_2}{2}$ is the cutoff value beyond which the consumer will go to Firm 1. We update the integral bounds on the distribution of Firm 1 to resolve the indicator functions. The total expected location difference is a convex function of $d$, in the relevant range $d\in(d^{crit},\frac{1}{2})$. The minimum of the expected location difference is reached at $d_{min_{tc}}=\frac{17}{74}\approx 0.22973$. With $d\to\frac{1}{2}$ the expected location difference converges to 0.25. $\square$

*Proof of Proposition 6:* Suppose first $\epsilon=0$, i.e., $c_1=c_2=c$. There is no equilibrium with $c>\min\{p_1,p_2\}$ as this would lead to negative profits. We distinguish the following cases:

Case 1: $p_1=p_2$

(i) If $p_1=p_2=c$, both firms make 0 profits. Yet, Firm 1 could deviate and choose $p_1=c+0.25t$ resulting in $d=\frac{p_2-p_1}{t}=0.25$ with a positive demand and a positive profit for Firm 1.

(ii) Suppose $p_1=p_2>c$. Denote the lower bound of Firm 1's demand by $D_{1_{LB}}$. From the proof of Lemma 7, we know that $\lim_{d\to 0^+}D'_{1_{LB}}(d)=\infty\leq\lim_{d\to 0^+}D'_1(d)$. Thus, Firm 1 has a profitable deviation by slightly decreasing its price.

Case 2: $p_1\neq p_2$

Suppose there is an equilibrium with different prices, without loss of generality $p_2>p_1$. We write $D_1(p_1,p_2,t)=\frac{1}{2}+\Delta$. Then each firm prefers the own price over matching the price of the rival and getting half the market. Thus:

$$p_1(\frac{1}{2}+\Delta)\geq\frac{p_2}{2}\text{ and }\frac{p_1}{2}\leq p_2(\frac{1}{2}-\Delta),$$

where the first inequality is the incentive constraint of Firm 1 and the second inequality is the incentive constraint of Firm 2. Rewritten in terms of $\Delta$, we have

$$\Delta \geq \frac{p_2}{2p_1} - \frac{1}{2} \text{ and } \Delta \leq \frac{1}{2} - \frac{p_1}{2p_2}$$

Thus,

$$\frac{1}{2} - \frac{p_1}{2p_2} \geq \frac{p_2}{2p_1} - \frac{1}{2} \iff (p_1 - p_2)^2 \leq 0,$$

which leads us to $p_1 = p_2$, a contradiction.

The arguments extend in a straightforward way when $c_1 = c_2 + \epsilon$ for $\epsilon$ is sufficiently small: following the same steps, the difference in prices is of order $\epsilon$. This leads to a contradiction for small enough $\epsilon$ due to a large demand elasticity for approximately equal prices. $\square$