



Redistribution Through Market Segmentation*

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Abstract

Consumer data can be used to sort consumers into different market segments, allowing a monopolist to charge different prices at each segment. We study consumer-optimal segmentations with redistributive concerns, i.e., that prioritize poorer consumers. Such segmentations are efficient but may grant additional profits to the monopolist, compared to consumer-optimal segmentations with no redistributive concerns. We characterize the markets for which this is the case and provide a procedure for constructing optimal segmentations given a strong redistributive motive. For the remaining markets, we show that the optimal segmentation is surprisingly simple: it generates one segment with a discount price and one segment with the same price that would be charged if there were no segmentation. We also show that a regulator willing to implement the redistributive-optimal segmentation does not need to observe precisely the composition and the frequency of each market segment, the aggregate distribution over prices suffices.

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1 Introduction

Consumers are continuously leaving traces of their identities on the internet, be it through social media activity, search-engine utilization, online-purchasing, etc. The vast amount of consumer data that is generated and collected has acquired the status of a highly-valued good, as it allows firms to tailor recommendations, advertisements, and prices to different consumers. In practice, the availability of consumer data enables platforms or firms to segment consumers by differentiating them based on their observed characteristics. Adjusting how coarse-grained is the information available about consumers has an impact on how they will be segmented, which leaves room for regulatory oversight. This may be of particular concern when consumer data are used to charge personalized prices.

As shown by [Bergemann, Brooks and Morris \(2015\)](#), consumer segmentation and price discrimination can induce a wide range of welfare outcomes. It can not only be used to increase social surplus—by creating segments with prices that allow more consumers to buy—, but can also be performed in a way that ensures that all created surplus accrues to consumers — that is, that maximizes consumer surplus. This is done by creating segments that pool together consumers with high and low willingness to pay, thus allowing higher willingness to pay consumers to benefit from lower prices. However, an important aspect of price discrimination that remains overlooked by the literature is its distributive effect: since different consumers pay different prices, this practice defines how surplus is distributed across consumers, raising questions about how it can benefit poorer consumers relative to richer ones. Indeed, if willingness to pay and wealth are positively related, segmentations that maximize total consumer surplus tend to benefit high willingness to pay consumers, that is, the richer ones.

In this paper we provide a normative analysis of the distributive impacts of market segmentation. Our aim is to study how this practice impacts different consumers and how it should be performed under the objective of increasing consumer welfare while prioritizing poorer consumers. Our results draw qualitative characteristics of segmentations that achieve this goal, which can be used to inform future regulation. Importantly, our analysis also shows that the prioritization of poorer consumers can be inconsistent with the maximization of total consumer surplus: raising the surplus of poorer consumers may only be possible while granting additional profits to the producer, at the expense of richer consumers.

We consider a setting in which a monopolist sells a good on a market composed of heterogeneous consumers, each of whom can consume at most one unit and is characterized by their willingness to pay for the good. A social planner can provide information about consumers' willingness to pay to the monopolist. The information provision strategy effectively divides the aggregate pool of consumers into different segments, each of which can be priced differently by the monopolist. The social planner's objective is to maximize a weighted sum of consumers'

surplus. As in [Dworczak](#) [Kominers](#) [Akbarpour](#) (2021), we consider weights that are decreasing on the consumers' willingness to pay, capturing the notion of a redistributive motive under the assumption that consumers with higher willingness to pay are on average richer than those with lower willingness to pay.

We first establish that optimal segmentations are Pareto efficient, such that satisfying a redistributive objective does not come at the expense of social surplus. [Bergemann et al. \(2015\)](#) show that, in the absence of redistributive concerns, consumer-optimal segmentations do not strictly benefit the monopolist: all of the surplus created by the segmentation accrues to consumers. In contrast, we show that once redistributive preferences are considered, consumer-optimal segmentations may imply additional profits to the monopolist. This happens because increasing the surplus of poor consumers is done by pooling them with even poorer consumers, such that they can benefit from lower prices. In doing so, richer consumers become more representative in other segments, which might increase the price they pay and result in a greater profit for the seller. We characterize the set of markets for which this is the case and denote them as rent markets. For no-rent markets, on the contrary, any redistributive objective can be met while still maximizing total consumer surplus. In this case, our analysis selects one among the many consumer-optimal segmentations established by [Bergemann et al. \(2015\)](#).

Our analysis also provides insights on how to construct optimal segmentations. We show that consumer-optimal segmentations under sufficiently strong redistributive preferences divide consumers into contiguous segments based on their willingness to pay, having consumers with the same willingness to pay belong to at most two different segments. This allows us first to construct a procedure that generates consumer-optimal segmentations under strong redistributive preferences, which is discussed in [Section 3.2](#). This also has an implication for the optimal segmentations in no-rent markets: they exhibit a stunningly simple form, simply dividing consumers into two segments: one where the price is the same that would be charged under no segmentation and one with a discount price.

Finally, we prove in [Section 4](#) that the optimal policy does not require the social planner to directly control or observe the market segmentation that is implemented on the market. Indeed, simply knowing the aggregate composition of the market, and being able to observe the aggregate distribution of prices induced by a segmentation is enough for the social planner to incentivize a third party (e.g. the seller herself or a data-gathering intermediary) to implement the optimal segmentation with redistributive concerns.

Related Literature. Third-degree price discrimination and its welfare effects are the subject of an extensive literature. Early analysis ([Pigou, 1920](#); [Robinson, 1933](#)) and subsequent development ([Schmalensee, 1981](#); [Varian, 1985](#)) considered exogenously fixed market segmentations and studied conditions under which such segmentations would increase or decrease total

surplus.

More recently, a literature incorporating an information design approach has revisited the question of welfare impacts of third-degree price discrimination by analyzing all feasible segmentations of a market. [Bergemann et al. \(2015\)](#) analyze a setting with a monopolist selling a single good and characterize attainable pairs of consumer and producer surplus, showing that any distribution of total surplus over consumers and producer that guarantee at least the uniform-price profit for the producer is attainable. In particular, they show that there are typically many consumer-optimal segmentations of a given market. Their analysis has been extended to multi-product settings by [Haghpanah and Siegel \(2022, 2023\)](#) and to imperfect competition settings by [Elliott, Galeotti, Koh and Li \(2021\)](#) and [Ali, Lewis and Vasserman \(2022\)](#). [Hidir and Vellodi \(2020\)](#) study market segmentation in a setting where the monopolist can offer one from a continuum of goods to each consumer, such that consumers, upon disclosing their information, face a trade-off between being offered their best option and having to pay a fine-tuned price. Finally, [Roesler and Szentes \(2017\)](#) and [Ravid, Roesler and Szentes \(2022\)](#) study the inverse problem of information design to a buyer who is uncertain about the value of a good. Our paper differs from these by focusing on how surplus is distributed *across* consumers, and by studying consumer-optimal segmentations when different consumers are assigned different welfare weights. We show that, once distributional preferences are taken into account, optimal segmentations might not coincide with consumer-optimal segmentations under uniform welfare weights. When they do, our analysis selects one among the many direct consumer-optimal segmentations established in [Bergemann et al. \(2015\)](#).

Our paper also dialogues with a recent literature on mechanism design and redistribution ([Condorelli, 2013, 2012](#)), most notably with [Dworczak \(R\) al. \(2021\)](#) and [Akbarpour \(R\) Dworczak \(R\) Kominers \(2023\)](#), who study the design of allocation mechanisms under redistributive concerns; and [Pai and Strack \(2022\)](#), who study the optimal taxation of a good with a negative externality when agents differ on their utility for the good, their disutility for the externality, and their marginal value for money. A key difference in the results obtained in these papers and ours is that, in their settings, redistributive mechanisms are not pareto-efficient: redistribution implies some loss in social surplus. This is not the case in our paper, where optimal redistributive segmentations always maximize total surplus.

Finally, our paper dialogues with [Dube and Misra \(2022\)](#), who study experimentally the welfare implications of personalized pricing implemented through machine learning. The authors find a negative impact of personalized pricing on total consumer surplus, but note that a majority of consumers benefit from price reductions under personalization, pointing that under some inequality-averse weighted welfare functions, data-enabled price personalization might increase welfare. Their paper shows experimentally how the implementation of market segmentations aimed at maximizing profits might generate, as a by-product, the redistribution

of surplus among consumers. Our paper, on the other hand, shows theoretically how consumer-optimal redistributive segmentations might grant additional profits for the firm.

2 Model

A monopolist (he) sells a good to a continuum of mass one of buyers, each of whom can consume at most one unit. We normalize the marginal cost of production of the good to zero. The consumers privately observe their type v , which represents their willingness to pay for the good, and which can take K possible values $\{v_1, \dots, v_K\} \equiv V$, where:

$$0 < v_1 < \dots < v_K.$$

A *market* μ is a distribution over the valuations and we denote the set of all possible markets:

$$M \equiv \Delta(V) = \left\{ \mu \in \mathbb{R}^K \mid \sum_{k=1}^K \mu_k = 1 \text{ and } \mu_k \geq 0 \text{ for all } k \in \{1, \dots, K\} \right\}.$$

Price v_k is **optimal for market** $\mu \in M$ if it maximizes the expected revenue of the monopolist when facing market μ^1 , that is:

$$v_k \sum_{i=k}^K \mu_i \geq v_j \sum_{i=j}^K \mu_i, \quad \forall j \in \{1, \dots, K\}.$$

Let M_k denote the set of markets where price v_k is optimal:

$$M_k = \left\{ \mu \in M \mid v_k \in \arg \max_{v_i \in V} v_i \sum_{j=i}^K \mu_j \right\}.$$

In the remaining of the paper we will hold an aggregate market fixed and denote it by $\mu^* \in M$.

Segmentation.. The consumers' types are perfectly observed by a social planner (she) who can **segment** consumers, that is, sort consumers into different (sub-)markets. The set of possible segmentations of a given aggregate market μ^* is:²

$$\Sigma(\mu^*) \equiv \left\{ \sigma \in \Delta(M) \mid \sum_{\mu \in \text{supp}(\sigma)} \mu \sigma(\mu) = \mu^*, |\text{supp}(\sigma)| < \infty \right\}.$$

¹Note that we can restrict the action set of the monopolist to be equal to V , since any price $p \notin V$ is dominated by some $v \in V$.

² $\text{supp}(x)$ denotes the support of a distribution x .

Formally, a segmentation is a probability distribution on M which averages to the aggregate market μ^* . The requirement that the different segments generated by a segmentation average to the aggregate market is a consequence of the Bayes plausibility condition (c.f. [Kamenica and Gentzkow, 2011](#)).

Given a segmentation σ , the monopolist can price differently at each segment μ in the support of σ . As will become clear in [Section 3](#), segments with more than one optimal price play a key role in our results, such that we focus on the following pricing rule $p : M \rightarrow V$ applied by the monopolist:

$$p(\mu) = \min \left\{ \arg \max_{v_k \in V} v_k \sum_{i=k}^K \mu_i \right\}.$$

That is, the monopolist charges at each segment the lowest among the optimal prices for that segment. This pricing rule makes the objective of the social-planner (which is stated below) upper semicontinuous and ensures the existence of an optimal segmentation.³

We can therefore write the **utility** of a consumer of type v_k in market μ as:

$$U_k(\mu) \equiv \max \{0, v_k - p(\mu)\}.$$

Social objective. The social planner's objective is to maximize a weighted sum of consumers' surplus, with positive weights $\lambda \in \mathbb{R}_+^K$, where each dimension λ_k of vector λ represents the welfare weight attributed to consumers of type v_k . For a given market μ , the weighted consumer surplus of such market is given by:

$$W(\mu) \equiv \sum_{k=1}^K \lambda_k \mu_k U_k(\mu).$$

Hence, for any aggregate market μ^* , the social planner's objective is given by the following well-defined maximization program, whose value is denoted $V(\mu^*)$:

$$\max_{\sigma \in \Sigma(\mu^*)} \sum_{\mu \in \text{supp}(\sigma)} \sigma(\mu) W(\mu). \quad (\text{S})$$

Given an aggregate market μ^* , a segmentation $\sigma \in \Sigma(\mu^*)$ is **optimal** if

$$\sum_{\mu \in \text{supp}(\sigma)} \sigma(\mu) W(\mu) = V(\mu^*).$$

³Although technically important, this pricing rule does not impact our results qualitatively. Indeed, any joint distribution of consumers and prices that can be induced by the social planner under this pricing rule could be approximated arbitrarily well by a social planner facing a monopolist who selects among optimal prices in some other way.

We focus on welfare weights that are decreasing on the consumer's willingness to pay, such that $\lambda_k \geq \lambda_{k'}$ for any $k < k' \leq K - 1$, and say that the social planner has **redistributive preferences** if the inequality holds strictly for some $k, k' \in \{1, \dots, K\}$. Under the assumption that consumers with lower willingness to pay are on average poorer than consumers with higher willingness to pay, this amounts to attributing a greater weight to surplus accruing to poorer consumers.

Efficiency. Every consumer has a value for the good that is strictly greater than the marginal cost of production. Hence, social surplus is maximized when every consumer buys the good. Hence we say that a market μ is **efficient** if every consumer can buy the good, that is if the lowest optimal price for the seller at that market allows everyone to consume: $p(\mu) = \min \text{supp}(\mu)$. A segmentation σ is **efficient** if it is only supported on efficient markets.

Informational Rents. We denote the profit of the monopolist at a given market μ as:

$$\pi(\mu) = p(\mu) \sum_{k : v_k \geq p(\mu)} \mu_k.$$

Similarly, the profit of the monopolist under a given segmentation σ is denoted as:

$$\Pi(\sigma) = \sum_{\mu \in \text{supp}(\sigma)} \sigma(\mu) \pi(\mu).$$

We know that a segmentation σ can only weakly increase the profit of a monopolist, such that, $\Pi(\sigma) \geq \pi(\mu^*)$, $\forall \sigma \in \Sigma(\mu^*)$. We say that a segmentation σ grants a **rent** to the monopolist whenever this inequality holds strictly.

Uniformly weighted consumer-optimal segmentations. If $\lambda_k = \lambda_{k'} > 0$ for all $k, k' \in \{1, \dots, K\}$, program (S) maximizes total (or average) consumer surplus over all possible segmentations. A segmentation that solves this optimization problem is named **uniformly weighted consumer-optimal**. As shown in [Bergemann et al. \(2015\)](#), uniformly weighted consumer-optimal segmentations are (i) efficient—and hence achieve the maximum feasible social surplus⁴—, and (ii) do not grant the monopolist any rent.

Typically, for an interior aggregate market μ^* , there exist infinitely many uniformly weighted consumer-optimal segmentations. In [Section 3.3](#), we characterize the set of aggregate markets for which consumer-optimal segmentations *with redistributive preferences* are

⁴For a given market μ , the maximum feasible social surplus is given by

$$s(\mu) = \sum_k \mu_k v_k.$$

Note that a segmentation of μ achieves $s(\mu)$ if and only if it is efficient.

also uniformly weighted consumer-optimal, thus providing a natural way to select among these segmentations for such markets.

3 Optimal Segmentations

We now turn to the analysis of the optimal segmentation. In [Section 3.1](#) we derive general properties of optimal segmentations — that is, characteristics that are present in optimal segmentations given any decreasing welfare weights λ . [Section 3.2](#) then constructs optimal segmentations under strongly redistributive preferences: when the weight assigned to lower types is sufficiently larger than the weight assigned to higher types. Finally, [Section 3.3](#) characterizes the set of aggregate markets for which satisfying a redistributive objective might require granting additional profits to the monopolist.

3.1 General Properties

Efficient segmentations. Our first result establishes that i) we can always restrict ourselves to efficient segmentations—as long as the weights are non-negative; ii) if the weights are all strictly positive (i.e. if $\lambda_K > 0$ under our assumption of decreasing weights), only efficient segmentations can be optimal.

Proposition 1. *For any aggregate market μ^* and any weights $\lambda \in \mathbb{R}_+^K$ (not necessarily decreasing), there exists an efficient optimal segmentation of μ^* . Furthermore, if every weight is strictly positive ($\lambda \in \mathbb{R}_{++}^K$), any optimal segmentation is efficient.*

This result is a direct consequence of Proposition 1 in [Haghpanah and Siegel \(2023\)](#)—which itself follows from the proof of Theorem 1 in [Bergemann et al. \(2015\)](#). Indeed, their result states that any inefficient market can be segmented in a Pareto-improving manner, that is, in a way that weakly increases the surplus of all consumers and strictly increases it for some consumers. This implies that, as long as the social planner does not assign a negative weight to any consumer, there must be an efficient optimal segmentation. As a consequence, the social planner’s redistributive objective never comes at the expense of efficiency.

Direct segmentations. A segmentation σ is **direct** if all segments in σ have different prices, that is, if for any $\mu, \mu' \in \text{supp}(\sigma)$, $\mu \neq \mu' \implies p(\mu) \neq p(\mu')$. Our next lemma shows that it is without loss of generality to focus on direct segmentations.

Lemma 1. *For any aggregate market μ^* and any segmentation $\sigma \in \Sigma(\mu^*)$, there exists a direct segmentation $\sigma' \in \Sigma(\mu^*)$ such that,*

$$\sum_{\mu \in \text{supp}(\sigma)} \sigma(\mu) W(\mu) = \sum_{\mu \in \text{supp}(\sigma')} \sigma'(\mu) W(\mu).$$

We further show that there always exists an optimal and direct segmentation that is only supported on the boundaries of price regions $\{M_k\}_k$. For this, denote for any aggregate market μ^* , $I(\mu^*) \equiv \{k \mid v_k \in \text{supp}(\mu^*)\}$, the set of indices k such that v_k is in the support of μ^* .

Lemma 2. *For any aggregate market μ^* that is not efficient, there exists an optimal direct segmentation supported on boundaries of sets $\{M_k\}_{k \in I(\mu^*)}$.*

3.2 Strongly Redistributive Social Preferences

In this section, we derive some characteristics of the optimal segmentation when the social planner's preferences are *strongly redistributive*, that is, when the weights λ are strongly decreasing with the types v_k .

Definition 1. *The weights λ are κ -strongly redistributive if, for any $k < k' \leq K - 1$, $\frac{\lambda_k}{\lambda_{k'}} \geq \kappa$.*

That is, a social planner exhibits κ -strongly redistributive preferences (κ -SRP) if the weight she assigns to a consumer of type v_k is at least κ times larger than the weight she assigns to any consumer of type greater than v_k .

Before stating the main result of this section, let us formally define the *dominance* ordering between any two sets.

Definition 2. *Let $X, Y \subset \mathbb{R}$, X **dominates** Y , denoted $X \succcurlyeq_D Y$, if for any $x \in X$ and any⁵ $y \in Y$, $x \geq y$.*

Proposition 2. *For any aggregate market μ^* in the interior of M , there exists $\underline{\kappa}$ such that if λ 's are $\underline{\kappa}$ -strongly redistributive, then for any optimal direct segmentation $\sigma \in \Sigma(\mu^*)$ and any markets $\mu, \mu' \in \text{supp}(\sigma)$, $\mu \neq \mu'$: either $\text{supp}(\mu) \succcurlyeq_D \text{supp}(\mu')$ or $\text{supp}(\mu') \succcurlyeq_D \text{supp}(\mu)$.*

The result stated above establishes that, when the social planner's preferences exhibit a sufficiently strong taste for redistribution, optimal segmentations divide the type space V into contiguous overlapping intervals, with the overlap between any two segments being composed of at most one type. The following corollary is a direct consequence of [Proposition 2](#):

Corollary 1. *For any aggregate market μ^* in the interior of M , there exists $\underline{\kappa}$ such that if λ 's are $\underline{\kappa}$ -strongly redistributive, then for any optimal direct segmentation $\sigma \in \Sigma(\mu^*)$, any market $\mu \in \text{supp}(\sigma)$, and any k such that $\min\{\text{supp}(\mu)\} < v_k < \max\{\text{supp}(\mu)\}$: $\sigma(\mu)\mu_k = \mu_k^*$.*

The above result states that any segment μ belonging to a segmentation that is optimal under strong redistributive preferences contains *all* of the consumers with types strictly in-between $\min\{\text{supp}(\mu)\}$ and $\max\{\text{supp}(\mu)\}$. Together with [Proposition 2](#), it implies that, under

⁵Note that this definition of dominance is stronger than the notion of dominance in the strong set order ([Topkis, 1998](#)).

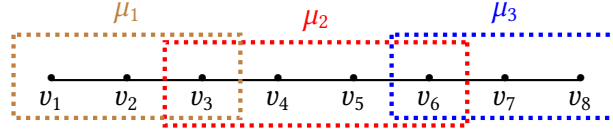


Figure 1: Structure of optimal segmentations under strong redistributive preferences.

κ -SRP optimal segmentations, every consumer type v will belong to *at most* two segments: either it will belong to the interior of the support of a segment μ , such that all consumers of this type have surplus $v - \min(\text{supp}(\mu))$, or it will be the boundary type between two segments μ and μ' , such that a fraction of these consumers (those belonging to segment μ) gets surplus $v - \min(\text{supp}(\mu))$ and the rest gets no surplus. The structure of optimal segmentations under strong redistributive preferences is illustrated in [Figure 1](#).

These results, along with [Proposition 1](#), completely pin down the κ -SRP optimal direct segmentation. One can construct it by employing the following procedure, presented as follows through steps:

- **Step i)** Start by creating a segment – call it μ_a – with all consumers of type v_1 .
- **Step ii)** Proceed to including in μ_a , successively, all consumers of type v_2 , then all of the types v_3 , and so on. From [Proposition 1](#) we know that μ_a must be efficient, meaning that we must have $p(\mu_a) = v_1$. As such, the process of inclusion of types higher than v_1 must be halted at the point in which adding a new consumer in μ_a would result in v_1 no longer being an optimal price in this segment. We denote as $v_{(a|b)}$ the type that was being included when the process was halted.
- **Step iii)** Create a new segment – call it μ_b – with all of the remaining types $v_{(a|b)}$.
- **Step iv)** Proceed to including in μ_b , successively, all of consumers of type $v_{(a|b)+1}$, then all of the types $v_{(a|b)+2}$, and so on. Halt this process at the point in which adding a new consumer in μ_b would result in $v_{(a|b)}$ no longer being an optimal price in this segment. We denote as $v_{(b|c)}$ the type that was being included when the process was halted.
- **Step v)** Create a new segment with all of the remaining types $v_{(b|c)}$. Repeat the process described in the last steps until every consumer has been allocated to a segment.

3.3 Optimal Segmentations and Informational Rents

This section explores the question of when an optimal segmentation maximizes total consumer surplus or, conversely, when it grants a rent for the monopolist.

Say that an aggregate market μ^* belongs to the **rent region** if there exists some $\underline{\kappa}$ such that if the social planner has $\underline{\kappa}$ -strongly redistributive preferences, the optimal segmentation

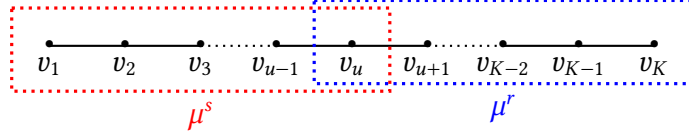


Figure 2: Segmentation σ^{NR} .

grants a rent to the monopolist. Conversely, define the **no-rent region** as the set of aggregate markets for which any optimal segmentation with redistributive preferences also maximizes total consumer surplus.

Before we characterize the rent and no-rent regions, we define a particular segmentation, which we will call σ^{NR} :

Definition 3. Let μ^* be an aggregate market with uniform price v_u . Call σ^{NR} the segmentation that splits μ^* into two segments μ^s and μ^r , with:

$$\mu^s = \left(\frac{\mu_1^*}{\sigma}, \frac{\mu_2^*}{\sigma}, \dots, \mu_u^s, 0, \dots, 0 \right),$$

$$\mu^r = \left(0, 0, \dots, \mu_u^r, \frac{\mu_{u+1}^*}{1-\sigma}, \dots, \frac{\mu_K^*}{1-\sigma} \right),$$

where $\mu_u^s = \frac{v_1}{v_u}$, $\mu_u^r = \frac{\mu_u^* - \sigma \mu_u^s}{1-\sigma}$ and $\sigma = \frac{v_u \sum_{i=1}^{u-1} \mu_i^*}{v_u - v_1}$.

Segmentation σ^{NR} is very simple and generates only two segments: one pooling all the consumers who would not buy the good on the unsegmented market (those with type lower than v_u) and another one pooling all the consumers who would buy the good on the unsegmented market (those with type higher than v_u). Under segmentation σ^{NR} , the only consumer type that gets assigned to two different segments is v_u .

Proposition 3. An aggregate market μ^* belongs to the no-rent region if and only if σ^{NR} is an efficient segmentation of μ^* .

Proposition 3 establishes a simple criterion that defines whether an aggregate market belongs to the no-rent region: it suffices to check if, under σ^{NR} , $p(\mu^s) = v_1$ and $p(\mu^r) = v_u$. Whenever this is not true, the aggregate market belongs to the rent region.

Corollary 2. Consider an aggregate market μ^* . If σ^{NR} is not an efficient segmentation of μ^* , then there exists $\underline{\kappa}$ such that, if welfare weights λ are $\underline{\kappa}$ -strongly redistributive, any optimal segmentation grants a rent to the monopolist.

The intuition for the results above is as follows. A market belongs to the no-rent region if, given any redistributive preferences, its optimal segmentation maximizes total consumer surplus. On one hand, we know from **Proposition 2** that, under strong redistributive preferences,

optimal segmentations divide the type space into overlapping intervals, with the overlap between two segments being comprised of at most one type. On the other hand, we have as a necessary and sufficient condition for total consumer surplus to be maximized that the segmentation is i) efficient and ii) the uniform price v_u is an optimal price at *every* segment generated by this segmentation. Condition i) ensures that total surplus is maximized, while condition ii) ensures that producer surplus is kept at its uniform price level, meaning that all of the surplus created by the segmentation goes to consumers. Since condition ii) can only be satisfied if type v_u belongs in the support of all segments, we get that the conditions for optimality under strong redistributive preferences and for total consumer surplus to be maximized can only be simultaneously met by a segmentation that only generates two segments, with v_u being the only type that is in the support of both segments.

Such a segmentation indeed maximizes total consumer surplus if it is efficient and if v_u is an optimal price in both segments. This is the case if v_1 and v_u are both optimal prices on the lower segment, and if v_u is an optimal price in the upper segment. Segmentation σ^{NR} is the *only* segmentation that can potentially satisfy all of these conditions at once, as it includes in the lower segment the exact proportion of types v_u that would make the monopolist indifferent between charging a price of v_1 or v_u . As such, segmentation σ^{NR} maximizes total consumer surplus if and only if it is efficient.

Corollary 3. *If an aggregate market μ^* belongs to the no-rent region, then σ^{NR} is its only direct optimal segmentation under any redistributive preferences.*

This result establishes that, for markets in the no-rent region, optimal segmentations have an extremely simple structure: they only generate a discount segment with price v_1 , pooling all the types who would not consume under the uniform price and some of the types v_u , and a residual segment with price v_u , containing all of the remaining consumers. Furthermore, this segmentation must be optimal under *any* decreasing welfare weights λ . As such, this result selects for the markets belonging to the no-rent region one among the many uniformly weighted consumer-optimal segmentations that were outlined in [Bergemann et al. \(2015\)](#).

Due to the structure of segmentation σ^{NR} , all of the surplus that is generated by the segmentation is given to consumers with types below or equal to v_u , all of which get the maximum surplus they could potentially get. Since it is impossible to raise the surplus of any type below v_u , and impossible to raise the surplus of types above v_u without redistributing from lower to higher types, this segmentation must be optimal whenever the weights assigned to different consumers are (weakly) decreasing on the type.

The results in this section establish that there are essentially two types of markets: those for which redistribution can be done only within consumers, while keeping total consumer surplus maximal, and those for which increasing the surplus of lower types past a certain

point necessarily decreases the total pie of surplus accruing to consumers and grants additional profits to the monopolist.

4 Implementation through the Distribution of Prices

Until now, the situation considered has been as if the social planner could directly decide which segmentation is implemented; or equivalently, this is akin to a situation where the regulator could contract with a data intermediary directly on the segmentations. We now consider a more realistic setting where the social planner does not directly observe the composition of each market segment, but only the distribution (i.e. the frequency) of the prices on the market. This information seems more plausibly verifiable by the regulator, for instance through surveys with the consumers or the analysis of selling data on internet platforms.

Formally, the social planner knows the aggregate market μ^* and proposes a segmentation $\sigma \in \Sigma(\mu^*)$ and a pricing rule $q : \text{supp}(\sigma) \rightarrow V$ to the seller. Together, such a pair (σ, q) induces a distribution over prices: that is, the aggregate proportion of price $v_k \in V$ on the market is equal to $\sum_{\mu \in \text{supp}(\sigma)} \sigma(\mu) \mathbb{1}_{q(\mu)=v_k}$.

Definition 4. A pair (σ, q) is **implementable** if the seller has no strict incentive to choose another pair of segmentation and pricing rule (σ', q') that induces the same distribution over prices.

Our final result simply states that the optimal segmentation of the social planner as defined by (S), together with the pricing rule p is implementable.

Proposition 4. For any optimal, direct and efficient segmentation σ that solves (S), the pair (σ, p) is implementable.

This result directly follows from two observations:

1. Given a distribution over prices $\gamma \in \Delta(V)$, the profit of the firm can be written as:

$$\sum_{v \in V} \gamma(v) \times v \times \text{mass of consumers consuming at market segment with price } v \varepsilon.$$

Therefore, the difference in profit for the firm between two pairs (σ, q) and (σ', q') that induces the same distribution over prices only depends on the mass of consumers that buy the good at each segment that is on the supports of σ and σ' .

2. From Proposition 1, we can restrict ourselves to efficient segmentation, which means that at each market, the mass of consumers that buy the good is equal to 1. Hence, the seller cannot improve his profit with another pair (σ, q) that induces the same price distribution.

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A Proof of section 3.1

Proof of lemma 1. Let $\sigma \in \Sigma$ and suppose that there exist $\mu, \mu' \in \text{supp}(\sigma)$ with $p(\mu) = p(\mu')$. Consider the following market:

$$\tilde{\mu} = \frac{\sigma(\mu)}{\sigma(\mu) + \sigma(\mu')}x + \frac{\sigma(\mu')}{\sigma(\mu) + \sigma(\mu')}x'.$$

By the convexity of $X_{p(\mu)}$, $p(\tilde{\mu}) = p(\mu)$. Define σ' in the following way: $\sigma'(\tilde{\mu}) = \sigma(\mu) + \sigma(\mu')$, $\sigma'(\mu) = \sigma'(\mu') = 0$ and $\sigma' = \sigma$ otherwise. Is it easy to check that $\sum_{\mu \in \text{supp}(\sigma)} \sigma(\mu)W(\mu) = \sum_{\mu \in \text{supp}(\sigma')} \sigma'(\mu)W(\mu)$. We can iterate this operation as many times as the number of pairs $v, v' \in \text{supp}(\sigma')$ such that $p(v) = p(v')$ to finally obtain the desired conclusion. \square

Proof of lemma 2. Let μ^* be an inefficient aggregate market, hence for any optimal segmentation $\sigma \in \Sigma(\mu^*)$, $|\text{supp}(\sigma)| \geq 2$. Let σ be a direct and optimal segmentation of μ^* and $\mu \in \text{supp}(\sigma)$ such that μ is in the interior of $X_{p(\mu)}$. Let v be any other market in the support of σ . Consider the market:

$$\xi = \frac{\sigma(\mu)}{\sigma(\mu) + \sigma(v)}\mu + \frac{\sigma(v)}{\sigma(\mu) + \sigma(v)}v.$$

Because μ^* is inefficient, it is without loss of generality to assume that ξ is also inefficient.

Denote $\bar{\mu}$ (resp. \bar{v}) the projection of ξ on the boundary of the simplex M in direction of μ (resp. v). For σ to be optimal, the segmentation of ξ between μ with probability $\frac{\sigma(\mu)}{\sigma(\mu)+\sigma(v)}$ and v with probability $\frac{\sigma(v)}{\sigma(\mu)+\sigma(v)}$ must be optimal. In particular, it must be optimal among any segmentation on $[\bar{\mu}, \bar{v}]$.

There exists a one-to-one mapping $f : [\bar{\mu}, \bar{v}] \rightarrow [0, 1]$ such that for any $\gamma \in [\bar{\mu}, \bar{v}]$, $\gamma = f(\gamma)\bar{\mu} + (1 - f(\gamma))\bar{v}$. Thus, the set $[\bar{\mu}, \bar{v}]$ can be seen as all the distributions on a binary set of states of the world $\{\bar{\mu}, \bar{v}\}$, where for any $\gamma \in [\bar{\mu}, \bar{v}]$, $f(\gamma)$ is the probability of $\bar{\mu}$.

Therefore, the maximization program,

$$\begin{aligned} & \max_{\sigma} \sum_{\gamma \in \text{supp}(\sigma)} \sigma(\gamma)W(\gamma) & (\bar{S}) \\ \text{s.t. } & \sigma \in \Sigma^{[\bar{\mu}, \bar{v}]}(\xi) \equiv \left\{ \sigma \in \Delta([\bar{\mu}, \bar{v}]) \mid \sum_{\gamma \in \text{supp}(\sigma)} \sigma(\gamma)\gamma = \xi, \text{supp}(\sigma) < \infty \right\}, \end{aligned}$$

is a bayesian persuasion problem (Kamenica and Gentzkow, 2011), with a binary state of the world and a finite number of actions. Hence, applying theorem 1 in Lipnowski and Mathevet (2017), there exists an optimal segmentation only supported on extreme points of sets $M \in \mathcal{M}^{[\bar{\mu}, \bar{v}]} \equiv \{M_k \cap [\bar{\mu}, \bar{v}] \mid k \in \{1, \dots, K\} \text{ and } M_k \cap [\bar{\mu}, \bar{v}] \neq \emptyset\}$. It happens that for any $M \in \mathcal{M}^{[\bar{\mu}, \bar{v}]}$, so that $M = M_k \cap [\bar{\mu}, \bar{v}]$ for some k , if γ is an extreme point of M , then it is on the boundary of (M_k) .

Let (μ', v') with respective probabilities $(\alpha, 1 - \alpha)$ be a solution to (\bar{S}) where μ' and v' are extreme points of some $M \in \mathcal{M}^{[\bar{\mu}, \bar{v}]}$. We now consider the segmentation $\bar{\sigma}$ such that $\bar{\sigma}(\gamma) = \sigma(\gamma)$ for all $\gamma \in \text{supp}(\sigma) \setminus \{\mu, v\}$, $\bar{\sigma}(\mu') = (\sigma(\mu) + \sigma(v))\alpha$, $\bar{\sigma}(v') = (\sigma(\mu) + \sigma(v))(1 - \alpha)$, and $\bar{\sigma} = 0$ otherwise. One can easily check that $\bar{\sigma} \in \Sigma(\mu^*)$. If *sig $\bar{\sigma}$* is not direct, that is, there exists $\gamma \in \text{supp}(\bar{\sigma})$ such that (w.l.o.g.) $p(\gamma) = p(\mu')$, then construct a direct segmentation $\bar{\bar{\sigma}}$ following the same process as in the proof of lemma 1. Then, if $\bar{\bar{\sigma}}$ is not only supported on boundaries of sets $\{M_k\}_{k \in I(\mu^*)}$, reiterate the same process as above, until you reach the desired conclusion. \square

B Proofs of Section 3.2.

Proof of proposition 2. Fix an aggregate market μ^* and let $\sigma \in \Sigma(\mu^*)$ be optimal and direct. Suppose by contradiction that there exist $\mu, \mu' \in \text{supp}(\sigma)$ such that $v_a := \min\{\text{supp}(\mu)\} < \max\{\text{supp}(\mu')\} =: v_d$ and $v_b := \min\{\text{supp}(\mu')\} < \max\{\text{supp}(\mu)\} =: v_c$. Assume further, without loss of generality, that $\min\{\text{supp}(\mu)\} < \min\{\text{supp}(\mu')\}$.

Define $\bar{\mu} := \frac{\sigma(\mu)}{\sigma(\mu)+\sigma(\mu')}\mu + \frac{\sigma(\mu')}{\sigma(\mu)+\sigma(\mu')}\mu'$. A consequence of σ being optimal is that $V(\bar{\mu}) = \frac{\sigma(\mu)}{\sigma(\mu)+\sigma(\mu')}W(\mu) + \frac{\sigma(\mu')}{\sigma(\mu)+\sigma(\mu')}W(\mu')$. The proof consists in showing that we can improve on this splitting of $\bar{\mu}$ and thus obtains a contradiction.

Define, for small $\epsilon > 0$, $\check{\mu}, \check{\mu}'$ as follows:

$$\check{\mu}_k = \begin{cases} \mu_k + \epsilon & \text{if } k = b \\ \mu_k - \epsilon & \text{if } k = c \\ \mu_k & \text{otherwise.} \end{cases}$$

$$\check{\mu}'_k = \begin{cases} \mu'_k - \frac{\sigma(\mu)}{\sigma(\mu)+\sigma(\mu')}\epsilon & \text{if } k = b \\ \mu'_k + \frac{\sigma(\mu)}{\sigma(\mu)+\sigma(\mu')}\epsilon & \text{if } k = c \\ \mu'_k & \text{otherwise.} \end{cases}$$

By construction, $\bar{\mu} = \frac{\sigma(\mu)}{\sigma(\mu)+\sigma(\mu')}\check{\mu} + \frac{\sigma(\mu')}{\sigma(\mu)+\sigma(\mu')}\check{\mu}'$. Note that v_a is still an optimal price for $\check{\mu}$. Indeed, for any $v_a \leq v_k \leq v_b$, the profit made by fixing price v_k is equal in markets μ and $\check{\mu}$ and for any $v_b < v_k \leq v_c$ the profit made by fixing price v_k is strictly lower in $\check{\mu}$ than in μ . On the contrary, $\phi(\check{\mu}') \geq \phi(\mu')$ and it is possible that the inequality holds strictly. In any case, it must be that $\phi(\check{\mu}') = v_e$ for $b \leq e \leq d$. Denote $\alpha := \frac{\sigma(\mu)}{\sigma(\mu)+\sigma(\mu')}$, hence $\frac{\sigma(\mu)}{\sigma(\mu')} = \frac{\alpha}{1-\alpha}$.

$$\alpha W(\check{\mu}) + (1-\alpha)W(\check{\mu}') - (\alpha W(\mu) + (1-\alpha)W(\mu')) \quad (1)$$

$$= \alpha(W(\check{\mu}) - W(\mu)) + (1-\alpha)(W(\check{\mu}') - W(\mu')) \quad (2)$$

$$= \alpha\epsilon(\lambda_b(v_b - v_a) - \lambda_c(v_c - v_a)) \quad (3)$$

$$+ (1-\alpha)\left(-\sum_{k>e} \lambda_k \mu'_k(v_e - v_b) - \sum_{b<k\leq e} \lambda_k \mu'_k(v_k - v_b) + \lambda_c \frac{\alpha}{1-\alpha} \epsilon(v_c - v_e)\right) \quad (4)$$

$$= \alpha\epsilon\lambda_b(v_b - v_a) - \alpha\epsilon\lambda_c(v_e - v_a) - (1-\alpha)\left(\sum_{k>e} \lambda_k \mu'_k(v_e - v_b) + \sum_{b<k\leq e} \lambda_k \mu'_k(v_k - v_b)\right) \quad (5)$$

$$> \alpha\epsilon\lambda_b(v_b - v_a) - \alpha\epsilon\lambda_{b+1}(v_e - v_a) - (1-\alpha)\left(\sum_{k>e} \lambda_{b+1} \mu'_k(v_e - v_b) + \sum_{b<k\leq e} \lambda_{b+1} \mu'_k(v_k - v_b)\right) \quad (6)$$

$$= \alpha\epsilon\lambda_b(v_b - v_a) - \lambda_{b+1}\left[\alpha\epsilon(v_e - v_a) - (1-\alpha)\left(\sum_{k>e} \mu'_k(v_e - v_b) + \sum_{b<k\leq e} \mu'_k(v_k - v_b)\right)\right] \quad (7)$$

Finally,

$$(7) \geq 0 \iff \frac{\lambda_b}{\lambda_{b+1}} \geq \kappa$$

where

$$\kappa = \frac{\alpha\epsilon(v_e - v_a) - (1-\alpha)\left(\sum_{k>e} \mu'_k(v_e - v_b) + \sum_{b<k\leq e} \mu'_k(v_k - v_b)\right)}{\alpha\epsilon(v_b - v_a)}$$

which ends the proof. \square

C Proofs of Section 3.3.

Proof of Proposition 3. As argued in the core of the text, all markets with uniform price v_u belonging to no-rent region must be optimally segmented by splitting μ^* between $\mu^s = (\frac{\mu_1^*}{\sigma}, \frac{\mu_2^*}{\sigma}, \dots, \mu_u^s, 0, \dots, 0)$ and $\mu^r = (0, 0, \dots, \mu_u^r, \frac{\mu_{u+1}^*}{1-\sigma}, \dots, \frac{\mu_K^*}{1-\sigma})$. Such a segmentation indeed gives no rents to the monopolist if v_u is an optimal price in both μ^s and μ^r . That is, if:

$$v_1 = v_u \mu_u^s \geq v_j \left(\sum_{i=j}^{u-1} \frac{\mu_i^*}{\sigma} + \mu_u^s \right) \quad \forall 2 \leq j \leq u-1 \quad (\text{NR-s})$$

$$v_u \geq v_j \left(\sum_{i=j}^K \frac{\mu_i^*}{1-\sigma} \right) \quad \forall u+1 \leq j \leq K \quad (\text{NR-r})$$

As such, any optimal segmentation under strong redistributive preferences that maximizes consumer surplus must have $\mu_u^s = \frac{v_1}{v_u}$, $\sigma = \frac{v_u}{v_u - v_1} \sum_{i=1}^{u-1} \mu_i^*$ and $\mu_u^r = \frac{\mu_u^* v_u - \sum_{i=1}^u \mu_i^* v_1}{\sum_{i=u}^K \mu_i^* v_u - v_1}$, which pins down segmentation σ^{NR} . Conditions (NR-s) and (NR-r) are satisfied whenever σ^{NR} is efficient, which concludes the proof.

It is also interesting to note that conditions (NR-s) and (NR-r) define the no-rent region inside M_u as a convex polytope. Indeed, we can rearrange both conditions and get:

$$0 \geq -\alpha(j) \sum_{i=1}^{j-1} \mu_i^* + (1 - \alpha(j)) \sum_{i=j}^{u-1} \mu_i^* \quad \forall 2 \leq j \leq u-1 \quad (\text{NR-s})$$

$$-\frac{v_1}{v_j(v_u - v_1)} \geq -\beta(j) \sum_{i=u}^{j-1} \mu_i^* + (1 - \beta(j)) \sum_{i=j}^K \mu_i^* \quad \forall u+1 \leq j \leq K \quad (\text{NR-r})$$

for $\alpha(j) = \frac{v_1(v_u - v_j)}{v_j(v_u - v_1)}$ and $\beta(j) = \frac{v_u^2}{v_j(v_u - v_1)}$.

The conditions expressed above define $K - 2$ half-spaces in \mathbb{R}^K . The no-rent region in M_u is thus given by the closed polytope defined by the intersection of such half-spaces. We can represent such polytope as follows:

$$NRR_u = \{\mu \in M_u : A\mu \leq z\},$$

with

$$A = \begin{bmatrix} S & O_S \\ O_R & R \end{bmatrix} \in \mathbb{R}^{K-2 \times K} \quad \text{and} \quad z = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{v_1}{v_{u+1}(v_u - v_1)} \\ \vdots \\ -\frac{v_1}{v_K(v_u - v_1)} \end{bmatrix} \in \mathbb{R}^{K-2}$$

where O_S and O_R are null matrices with, respectively, dimensions $(u-2) \times (u-1)$ and $(K-u) \times (K+1-u)$, and

$$S = \begin{bmatrix} -\alpha(2) & 1-\alpha(2) & \cdots & 1-\alpha(2) & 1-\alpha(2) \\ -\alpha(3) & -\alpha(3) & \cdots & 1-\alpha(3) & 1-\alpha(3) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha(u-2) & -\alpha(u-2) & \cdots & 1-\alpha(u-2) & 1-\alpha(u-2) \\ -\alpha(u-1) & -\alpha(u-1) & \cdots & -\alpha(u-1) & 1-\alpha(u-1) \end{bmatrix} \in \mathbb{R}^{(u-2) \times (u-1)},$$

$$R = \begin{bmatrix} -\beta(u+1) & 1-\beta(u+1) & \cdots & 1-\beta(u+1) & 1-\beta(u+1) \\ -\beta(u+2) & -\beta(u+2) & \cdots & 1-\beta(u+2) & 1-\beta(u+2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\beta(K-1) & -\beta(K-1) & \cdots & 1-\beta(K-1) & 1-\beta(K-1) \\ -\beta(K) & -\beta(K) & \cdots & -\beta(K) & 1-\beta(K) \end{bmatrix} \in \mathbb{R}^{(K-u) \times (K+1-u)}$$

for $\alpha(j) = \frac{v_1(v_u - v_j)}{v_j(v_u - v_1)}$ and $\beta(j) = \frac{v_u^2}{v_j(v_u - v_1)}$.

□