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Risk Loving and Fat Tails in the Wealth Distribution

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ABSTRACT

We study the dynamic properties of the wealth distribution in an overlapping generation model with warm-glow bequests and heterogeneous attitudes towards risk. Some dynasties of agents are risk averters, and others are risk lovers. Agents can invest in two types of Lucas trees. The two types of trees are symmetric in the sense that one type has a high return in states where the other has a return of zero. This symmetry allows risk averters to perfectly ensure their future income and eliminates aggregate uncertainty in the model. Furthermore, risk lovers take extreme portfolio positions, which makes it easy for us to characterize the evolution of their wealth holdings over time. We show that the model has an equilibrium in which the aggregate wealth distribution converges to a unique invariant distribution. The invariant distribution of wealth of the risk lovers has fat tails for high bequest rates. The existence of fat tails is endogenously generated by the behavior of risk lovers rather than by the exogenous existence of fat tails in the endowments or in the returns of the assets. Therefore, the invariant distribution of wealth of risk averters does not have fat tails.

Keywords: Inequality; Overlapping generations; Invariant distribution; Fat tails; Risk loving.

JEL Codes: C62, D51, D53

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1. Introduction

We have written this paper to honor David K. Levine for his contributions to economics. Some of the other papers in this volume bear testimony to David's contributions in areas of economics where the coauthors of this paper have little experience. There are areas of overlap, however, of David's contributions with our own. Like David, Aloisio Araujo and Tim Kehoe have been active members of the Society for the Advancement of Economic Theory and have served as the Society's President. Like David, Aloisio and Tim have supervised graduate student research in economics over periods of many years. In fact, Tim was the co-supervisor of David's Ph.D. thesis at M.I.T. And the research project that has produced this paper started when Juan Pablo Gama was working on his Ph.D. at IMPA under Aloisio's supervision.

David's connection with this paper is even more direct, however. In papers written with Tim, David has used simple dynamic general equilibrium models as building blocks in constructing more general theories. In that sense, the simple models in Kehoe and Levine (2001, 2008) served as building blocks for the general theory of dynamic general equilibrium with constraints on the enforcement of debt contracts developed in Kehoe and Levine (1993), even though the order of publication of these papers does not indicate the direction of the dependence. In the same sense, the coauthors of this paper intend our simple model to be a building block in constructing a more general theory of dynamic general with agents with heterogeneous attitudes towards risk. Furthermore, the formal modeling of general equilibrium with time and uncertainty and even the notation used in this paper follow that in Kehoe and Levine (1993, 2001, 2008).

Different attitudes towards risk can generate an unequal allocation among agents both in the short run and in the long run. In the short run, different degrees of risk aversion generate different portfolio allocations causing unequal distribution of returns, as Buhlman (1980) has

shown. In the long run, these inequalities can generate unequal distributions of wealth and income. A number of researchers have studied the effects of entrepreneurs in economic models due to their tendency to take more risk than other type of agents, as Kihlstrom and Laffont (1979) have shown. In general, researchers have modeled entrepreneurs as risk-neutral or risk-averse agents. Nonetheless, other researchers have pointed out that some behavior of entrepreneurs is not consistent with risk aversion or risk neutrality. For more references see Shane, Locke and Collins (2003) and Araujo, Chateauneuf, Gama and Novinski (2018).

Araujo, Chateauneuf, Gama, and Novinski (2018) and Araujo, Gama, and Suarez (2022) have incorporated risk-loving agents into static general equilibrium models. In this paper, we analyze the properties of equilibria in dynamic general equilibrium models with both risk-averse and risk-loving agents. We analyze the existence of fat tails in the wealth distribution. We study an overlapping generation model with bequests as in Galor and Zeira (1993) and two types of Lucas trees. We show that in this sort of model there can be an invariant distribution of wealth distribution with fat tails due to the existence of risk-loving agents. To analyze the impact of different assets on the distribution of wealth, we study a simple class of models where our assumptions of symmetry on assets and probabilities of events rule out aggregate uncertainty. We start by considering two types of dynasties — risk-averse and risk-loving ones — that is, if one agent is risk averse, or risk lover, all her predecessors and successors have the same attitude towards risk. To ensure the existence of equilibrium in this economy, we assume that there is a continuum of each type of dynasties. Araujo, Chateauneuf, Gama, and Novinski (2018) and Araujo, Gama, and Suarez (2022) show that aggregate uncertainty conditions are necessary to ensure the existence of equilibrium with a finite number of agents. Additionally, due to the

behavior of risk lovers, wealth holdings tend to accumulate in the hands of a small proportion of these risk lovers, which generates fat tails distributions.

We start by briefly discussing the literature on wealth and income inequality. Piketty and Saez (2014) argue that the gap among the interest rate and the real GDP growth rate have a strong impact on inequality. Other authors as Lindert and Williamson (2016) study long term data to analyze the inequality in the U.S. economy since colonial times. Most of this research generate distributions of wealth and income similar to Pareto distributions. Moreover, Benhabib, Bisin, and Zhu (2011, 2015, 2016) find conditions to generate fat tails for transformation processes induced by investment risk. On the other hand, Beare and Toda (2017) show that tails of wealth distribution decay exponentially in a heterogeneous-agent dynamic general equilibrium model with idiosyncratic endowment risk. Additionally, Gouin-Bonenfant and Toda (2023) develop an analytical framework to analyze models with heterogeneous risk averse agents that endogenously generates fat tails wealth distributions. Our model has larger similarities with Benhabib, Bisin, and Zhu (2011, 2015, 2016), however, than with Beare and Toda (2017) or with Gouin-Bonenfant and Toda (2023) since the fat tails in our model are a consequence of the risk lovers' extreme specialization in their asset holdings and not a consequence of fat tails in the distribution of endowments.

In section 2, we set up the model. In subsection 2.1, we define the concept of equilibrium and characterize some basic properties of the equilibrium and the consumption plans of the agents. In section 3, we analyze the dynamic properties of the equilibrium. In subsection 3.1, we develop the analytical expressions for the invariant wealth distribution. In subsection 3.2, we present results on convergence to this distribution. In section 4, we generalize the model to allow dynasties to

change their attitude toward risk over time. Finally, in section 5, we provide some concluding remarks.

2. Model

We analyze an overlapping generation model with warm-glow bequests and uncertainty. Our model has two events, 1 and 2, that can occur in each period. As in Kehoe and Levine (1993), a state $s = (\eta_1, \eta_2, \dots, \eta_t)$ at date $t \geq 1$ is a history of events $\eta_k \in \{1, 2\}$ for $k \in \{1, 2, \dots, t\}$. Each event η_k is equally probable. In each state s there is only one consumption good. There are two types of agents with different attitudes towards risk, risk lovers and risk averters, each of which has measure 1. We denote risk lovers as $\{l_i\}_{i \in [0, 1]}$ and risk averse agents as $\{a_i\}_{i \in [0, 1]}$. There are two assets, two types of Lucas trees of measure 1, with real returns $(R_{1,1}, R_{1,2}) = (R, 0)$ for the asset 1 and $(R_{2,1}, R_{2,2}) = (0, R)$ for the asset 2 with $R > 0$ in each state with $t \geq 1$, where $R_{j,1}, R_{j,2}$ are the returns of asset j in event 1 and event 2, respectively. In $t = 1$, asset 1 has a return of $R_{1,1} = R$, and the asset 2 has no return, $R_{2,1} = 0$. In other words, we choose to start with $\eta_1 = 1$. In $t = 1$, the old generation, whom we refer to as generation 0 because we imagine them as having been born before the model starts, has initial amount of the assets: $(\theta_{1,0}^{a_i}, \theta_{2,0}^{a_i})$ for the risk averters and $(\theta_{1,0}^{l_i}, \theta_{2,0}^{l_i})$ for the risk lovers where $\theta_{j,0}^{a_i}$ is the asset holdings of tree j of risk averse agent a_i with $i \in [0, 1]$, and $\theta_{j,0}^{l_i}$ is the asset holdings of tree j of risk lover agent l_i with $i \in [0, 1]$. The initial asset holdings satisfy

$$\int_0^1 \theta_{j,0}^{a_i} di + \int_0^1 \theta_{j,0}^{l_i} di = 1 \text{ for } j = 1, 2.$$

A consumer i from the old generation uses the Lucas trees and their returns to purchase consumption, $c_{(1)}^i$, and to leave a bequest to his successor, $b_{(1)}^i$. In particular, risk-averse consumer $a_i \in [0,1]$ solves the problem

$$\begin{aligned} \max \quad & U_0^{a_i}(c, b) = ((1 - \delta) \log c + \delta \log b) \\ \text{s. t.} \quad & c + b \leq \sum_{j=1,2} (R_{j,(1)} + q_{j,(1)}) \theta_{j,0}^{a_i}, \\ & 0 \leq c, b. \end{aligned}$$

where $\delta \in (0,1)$ is the bequest rate and $q_{j,s}$ is the price of the asset j in state s . Bequests are transfers of wealth between the old generation to the new one. Note that the price of the consumption good is equal to 1 in all state. Then, the initial wealth of a risk averse agent a_i from the old generation is $w_0^{a_i} = \sum_{j=1,2} (R_{j,(1)} + q_{j,(1)}) \theta_{j,0}^{a_i}$. Similarly, the risk-loving consumer $l_i \in [0,1]$ solves the problem

$$\begin{aligned} \max \quad & U_0^{l_i}(c, b) = (c^{1-\delta} b^\delta)^2 \\ \text{s. t.} \quad & c + b \leq \sum_{j=1,2} (R_{j,(1)} + q_{j,(1)}) \theta_{j,0}^{l_i}, \\ & 0 \leq c, b. \end{aligned}$$

In the first period of his life, an agent receives a bequest $b_s^i \geq 0$ from his predecessor and an endowment $\omega > 0$, and the agent decides on purchases of assets that pay off in states $(s, 1)$ and $(s, 2)$. In second period of his life, the agent decides on consumption and the bequest that he leaves to his successor based on the returns of the Lucas trees $R_{j,\eta}$ for $j, \eta = 1,2$. A risk averter, $a_i \in [0,1]$, in state s at date $t \geq 1$ solves the problem

$$\begin{aligned} \max \quad & U^{a_i}(c, b) = \frac{1}{2} ((1 - \delta) \log c_1 + \delta \log b_1) + \frac{1}{2} ((1 - \delta) \log c_2 + \delta \log b_2) \\ & q_{1,s} \theta_1 + q_{2,s} \theta_2 \leq \omega + b_s^{a_i}, \\ \text{s. t.} \quad & c_\eta + b_\eta \leq \sum_{j=1,2} (R_{j,\eta} + q_{j,(s,\eta)}) \theta_j \quad \text{for } \eta = 1,2, \\ & 0 \leq c_1, c_2, b_1, b_2, \end{aligned}$$

where $q_{j,s}$ is the price of the asset j in state s .

Notice that bequests are in terms of the consumption good, and the price of the consumption good is normalized to 1 in each state. The solution to the problem is a consumption plan, a bequest plan, and a Lucas tree portfolio $(c_s^{a_i}, b_s^{a_i}, \theta_s^{a_i})$ that maximizes the utility of the agent a_i subject to his budget constraints in state s , state $(s, 1)$, and state $(s, 2)$.

A risk lover, $l_i \in [0,1]$, in state s at date $t \geq 1$ solves the problem

$$\begin{aligned}
 \max \quad & U^{l_i}(c, b) = \frac{1}{2}(c_1^{1-\delta} b_1^\delta)^2 + \frac{1}{2}(c_2^{1-\delta} b_2^\delta)^2 \\
 & q_{1,s}\theta_1 + q_{2,s}\theta_2 \leq \omega + b_s^{l_i}, \\
 \text{s. t.} \quad & c_\eta + b_\eta \leq \sum_{j=1,2} (R_{j,\eta} + q_{j,(s,\eta)})\theta_j \quad \text{for } \eta = 1,2, \\
 & 0 \leq c_1, c_2, b_1, b_2.
 \end{aligned}$$

Notice that the risk averters allocate the same proportions, $1 - \delta$ and δ , of wealth to consumption and bequests in each event. Risk lovers, however, have extreme behavior: They choose to allocate their consumption and bequests as much as possible to one event in their second period of life depending on the prices of the two assets available in the economy. If the prices are such that it is cheaper to invest in event 1 instead of event 2, all risk lovers specialize in event 1. If the prices of the two assets are the same, however, a positive measure of the agents can specialize in each of the events. Notice that, in the model, both types of the agents allocate the same proportions, $1 - \delta$ and δ , of their wealth to consumption and bequests.

2.1. Equilibrium and equilibrium allocation

We now define an equilibrium for the model:

Definition 1. A sequence $(q_s, (c_s^{a_i}, b_s^{a_i}, \theta_s^{a_i})_i, (c_s^{l_i}, b_s^{l_i}, \theta_s^{l_i})_i)_s$ is an equilibrium for the economy

if, for each state s ,

$$\begin{aligned} \int_i \theta_{j,s}^{a_i} di + \int_i \theta_{j,s}^{l_i} di &= 1 && \text{for } j = 1, 2, \\ \int_i c_s^{a_i} di + \int_i c_s^{l_i} di &= R + 2\omega \end{aligned}$$

where $(c_s^{a_i}, b_s^{a_i}, \theta_s^{a_i})$ is the optimal solution for the risk averse a_i , and $(c_s^{l_i}, b_s^{l_i}, \theta_s^{l_i})$ is the optimal solution for the risk lover l_i .

Lemma 1. In any equilibrium, the prices of two trees need to be identical at every state s :

$$q_{j,s} = \frac{2\omega + \delta R}{2(1 - \delta)}, \quad j = 1, 2.$$

This result, whose proof can be found in the Appendix, allows to use the agents' contingent-claims market problem to calculate the equilibrium allocations. Based on the state contingent market price vectors, the Lucas tree asset prices satisfy

$$q_{1,s} = p_{(s,1)}(R + q_{1,(s,1)}) + p_{(s,2)}(q_{1,(s,2)}) \quad (2.1)$$

and

$$q_{2,s} = p_{(s,1)}(q_{2,(s,1)}) + p_{(s,2)}(R + q_{2,(s,2)}). \quad (2.2)$$

Then, we can transform the budget constraints in state s , state $(s, 1)$, and state $(s, 2)$ into a single budget constraint in state s using the contingent-claims market price vector $(p_{(s,1)}, p_{(s,2)})$:

$$\begin{aligned} p_{(s,1)}(c_1 + b_1) + p_{(s,2)}(c_2 + b_2) &\leq \omega + b_s^i \\ c_1, c_2, b_1, b_2 &\geq 0, \end{aligned}$$

for a state of length $t \geq 1$.

Note that the Lucas trees are long-lived assets in positive net supply. Then, there is no asset pricing bubble for both assets in all state s , see Santos and Woodford (1997). Additionally, since we have not imposed any short sale constraints on purchases of the Lucas trees, we can do this conversion from the sequential markets budget constraints to contingent-claims market budget constraints as long as there is no state s in which the two trees have the same gross return in the next period, whether they bear fruit or not. Then, markets must be complete, that is, the rank of the matrix

$$V(s) = \begin{pmatrix} R + q_{1,(s,1)} & q_{2,(s,1)} \\ q_{1,(s,2)} & R + q_{2,(s,2)} \end{pmatrix}$$

has full rank at prices q for every state s , see Hernandez and Santos (1996).

Then, an optimal allocation of a risk averter satisfies,

$$\frac{\delta}{(1-\delta)} c_{(s,1)}^{a_i} = b_{(s,1)}^{a_i} \quad \text{and} \quad \frac{\delta}{(1-\delta)} c_{(s,2)}^{a_i} = b_{(s,2)}^{a_i} \quad (2.3)$$

for every state s . Therefore,

$$c_{(s,k)}^{a_i} = \frac{(1-\delta)}{2p_{(s,k)}} (\omega + b_s^{a_i}) \quad \text{for } k = 1, 2. \quad (2.4)$$

In contrast, each risk lover specializes as much as possible in one event, but he also distributes his consumption and bequests as in Equation (2.3) since both agents distribute their wealth between consumption and bequests in the same proportion. Therefore, if $p_{(s,1)} < p_{(s,2)}$,

$$c_{(s,1)}^{l_i} = \frac{(1-\delta)}{p_{(s,1)}} (\omega + b_s^{l_i}) \quad \text{and} \quad c_{(s,2)}^{l_i} = 0, \quad (2.5)$$

and analogously to the case in which $p_{(s,1)} > p_{(s,2)}$, for any state s . In these cases, all risk lovers invest in the same state generating problems with the symmetry with the aggregate endowments.

Lemma 2. There is no equilibrium in which the Arrow-Debreu state contingent prices satisfy $p_{(s,1)} < p_{(s,2)}$ or $p_{(s,1)} > p_{(s,2)}$ for any state s . Consequently, there is a unique sequence of Arrow-Debreu state contingent prices, and are given by

$$p_{(s,j)} = \frac{2\omega + \delta R}{2(2\omega + R)} \quad \forall s, j.$$

Then, due to the symmetry mentioned above, there is only a symmetric equilibrium in which $p_{(s,1)} = p_{(s,2)}$ and that half of the aggregate wealth of each type of type of agent is held in each type of asset. The risk averters individually choose to divide their asset holdings evenly between the two types of Lucas trees. The risk lovers individually choose to hold only one type of asset, but their aggregate asset holdings are split evenly between the two types of trees.

The optimality conditions of risk averters and risk lovers imply that aggregate consumption and aggregate bequests are both constant fractions of aggregate wealth. Since in every state half of aggregate wealth is transferred into each of the two future states, in equilibrium there is no aggregate uncertainty and aggregate bequests are constant across the states.

We can now characterize the equilibrium allocation in the sequential markets version of the model. Lemma 1 says that, for $j = 1, 2$, and all s

$$q_{j,s} = \frac{2\omega + \delta R}{2(1 - \delta)} .$$

Then, using the optimality conditions, the relationship between the Arrow-Debreu state contingent constraint, and the Lucas tree asset prices, the portfolio allocation of a risk averter a_i in the first period of his life is

$$\theta_{1,s}^{a_i} = \theta_{2,s}^{a_i} = \frac{(1 - \delta)(\omega + b_s^{a_i})}{2\omega + \delta R}.$$

In contrast, due to the symmetry of the returns of the Lucas trees and the asset prices, any risk lover l_i is indifferent between investing all the wealth in event 1 or in event 2. Then, the portfolio allocation in the first period of his life is either

$$\theta_{1,s}^{l_i} = \frac{(R + q_{1,s})(2(1 - \delta))(\omega + b_s^{l_i})}{R(2\omega + \delta R)}, \theta_{2,s}^{l_i} = -\frac{2q_{2,s}(1 - \delta)(\omega + b_s^{l_i})}{R(2\omega + \delta R)}$$

if agent l_i decides to invest all in event 1 (in this case, $c_{(s,2)}^{l_i} = b_{(s,2)}^{l_i} = 0$), or

$$\theta_{1,s}^{l_i} = -\frac{2q_{1,s}(1 - \delta)(\omega + b_s^{l_i})}{R(2\omega + \delta R)}, \theta_{2,s}^{l_i} = \frac{(R + q_{2,s})(2(1 - \delta))(\omega + b_s^{l_i})}{R(2\omega + \delta R)}$$

if agent l_i decides to invest all in event 2 (in this case, $c_{(s,1)}^{l_i} = b_{(s,1)}^{l_i} = 0$) in the second period of his life.

3. Analysis of equilibria and the invariant distribution of wealth

Before we analyze the invariant distribution of wealth, let us point out a particularity of the model: Although the equilibrium prices are unique, the equilibrium allocation is not unique. At each state, there is a continuum of distributions in which the risk lovers could specialize if $p_{(s,1)} = p_{(s,2)}$. We have argued that in equilibrium half of the aggregate wealth of risk lovers must have been invested in each type of tree, but this does not enable us to identify which risk lovers hold tree 1 and which hold tree 2.

Lemma 3. There is a continuum of equilibria for the economy.

3.1. Invariant distribution with constant distribution of endowment among time

For risk averters, the wealth distribution is given by $w_1^{a_i} = (q_{1,(1)} + R)\theta_{1,0}^{a_i} + q_{2,(1)}\theta_{2,0}^{a_i}$ in $t = 1$.

And therefore, $w_{(1,1)}^{a_i} = w_{(1,2)}^{a_i} = \omega + \delta w_1^{a_i} / (2p_{(1,1)})$ where

$$p_{(1,1)} = p_{(1,2)} = p_{(s,1)} = p_{(s,2)} = \pi = \frac{2\omega + \delta R}{2(2\omega + R)},$$

and $w_{(s,1)}^{a_i} = w_{(s,2)}^{a_i} = \omega + \delta w_s^{a_i} / (2p_{(s,1)})$ for all state s of length $t \geq 1$. Then,

$$w_{(s,1)}^{a_i} = w_{(s,2)}^{a_i} = \sum_{k=0}^t \left(\frac{\delta}{2\pi}\right)^k \omega + \left(\frac{\delta}{2\pi}\right)^t w_0^{a_i}$$

if

$$q_{1,1} = q_{1,2} = \frac{q_{1,1} + q_{1,2}}{2} = \frac{2\omega + \delta R}{2(1 - \delta)},$$

which implies that $w_{(s,1)}^{a_i} = w_{(s,2)}^{a_i} = w_{(s',1)}^{a_i} = w_{(s',2)}^{a_i}$ for all states s of length $t \geq 1$ and

$$\lim_{t \rightarrow \infty} w_{(s,1)}^{a_i} = \frac{2\pi}{(2\pi - \delta)} \omega = \frac{2\omega + \delta R}{2\omega(1 - \delta)} \omega = \frac{\omega + R/2}{(1 - \delta)},$$

the wealth distribution converges to a constant value that only depends on ω .

For risk lovers, we focus our attention on equilibria with a symmetry on how they specialize.

Assumption RL: If the risk lovers are indifferent between investing in either event, we assume that half of the agents specialize in event 1 and half in event 2. Moreover, we assume that half of the agents with the same level of wealth specialize in event 1 and half in event 2.

RL is an assumption that ensures that risk lovers invest symmetrically between the two states. To ensure that this assumption holds, we could assume that, when risk lovers are indifferent between

two asset holdings, they randomly choose one with probability 1/2. The advantage of studying equilibrium allocations that satisfy assumption RL is that we can characterize a unique invariant distribution of wealth holdings.

We can now characterize the wealth distribution for risk lovers:

- For $t = 1$, we have that $w_0^{l_i} = (q_{1,1} + R)\theta_{1,0}^{l_i} + q_{2,1}\theta_{2,0}^{l_i}$.
- For $t = 2$, we have that $w_{(1,1)}^{l_i} = \omega$ with measure 1/2 of the risk lovers and $w_{(1,1)}^{l_i} = \omega + \delta w_0^{l_i}/\pi$ with measure 1/2.
- For $t = 3$, we have, for a state s , $w_s^{l_i}$ has the following distribution:
 - ω with measure 1/2,
 - $\omega + \delta\omega/\pi$ with measure 1/4,
 - $\omega + \delta(\omega + \delta/\pi w_0^{l_i})/\pi$ with measure 1/4.
- For $t = 4$, we have, for a state s , $w_s^{l_i}$ has the following distribution:
 - ω with measure 1/2,
 - $\omega + \delta\omega/\pi$ with measure 1/4,
 - $\omega + \delta(\omega + \delta/\pi\omega)/\pi$ with measure 1/8,
 - $\omega + \delta\left(\omega + \delta/\pi(\omega + \delta/\pi w_0^{l_i})\right)/\pi$ with measure 1/8,
- Therefore, recursively, for any state s with length $t \geq 1$, $w_s^{l_i}$ has the distribution:
 - $\sum_{k=0}^m (\delta/\pi)^k \omega$ with measure $1/2^{m+1}$ for $m = 0, \dots, t-2$,
 - $\sum_{k=0}^{t-2} (\delta/\pi)^k \omega + (\delta/\pi)^{t-1} w_0^{l_i}$ with measure $1/2^{t-1}$.

Notice that the initial distribution of wealth $w_0^{l_i}$ disappears over time because all dynasties of risk lovers eventually have bad luck with probability that approaches 1. Consequently, the wealth distribution for risk lovers tends to

$$(w_\infty^{l_i})_i: \sum_{k=0}^n (\delta/\pi)^k \omega \text{ with measure (or proportion) } 1/2^{n+1} \text{ for } n \in \mathbb{N}.$$

Notice that RL implies the convergence to the invariant distribution for any initial wealth distribution and the uniqueness of the equilibrium if the initial wealth distribution of risk lovers is discrete. Also notice that although RL does not imply uniqueness of the equilibrium, it does ensure that all the possible wealth distributions in equilibrium converge to the unique invariant distribution w_∞ .

Proposition 4. Under RL, there is a unique invariant distribution for the agents $(w_\infty^{l_i})_i$ and any initial portfolio distribution converges to an invariant distribution, $(w_\infty^{l_i})_i$, in distribution.

Intuitively, in any given state, half of the risk lovers have no return from the Lucas trees, implying that their wealth from that state onwards does not depend on their initial wealth, $w_0^{l_i}$, in any way. Additionally, due to RL, with probability 1, all risk lovers fall to the bottom of the wealth distribution at least once in the long run. Consequently, the proportion the risk lovers that do not follow the distribution $(w_\infty^{l_i})_i$ at date t is bounded by $1/2^t$.

The dynamics of wealth distribution of the risk lovers have a positive reflecting barrier, see Benhabib, Bisin, and Zhu (2011, 2015, 2016), and, with probability 1 the wealth a poor risk lover can reach in finite time any threshold of the wealth process generated by risk lovers that were at the bottom of the distribution at least once. Therefore, due to all these properties, for any given

any initial wealth distribution, the wealth distribution converges to an invariant distribution that is also unique.

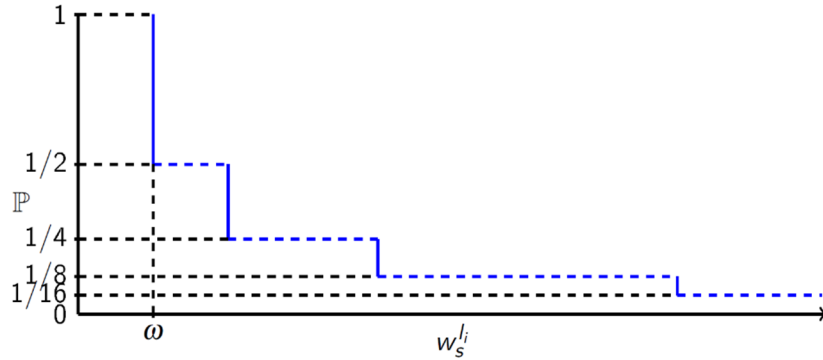


Figure 1. Invariant wealth distribution for risk lovers with high bequest rate ($\delta \geq 2\omega/(4\omega + R)$).

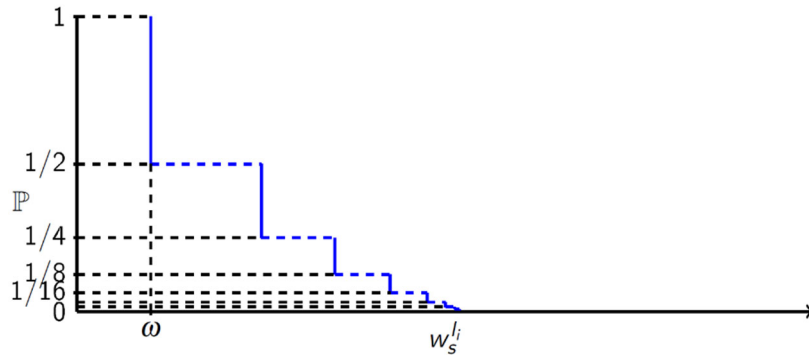


Figure 2. Invariant wealth distribution for risk lovers with low bequest rate ($\delta < 2\omega/(4\omega + R)$).

As we can see in Figure 1 and 2, the shape of the invariant distribution depends on the level of bequest of the agents. For high bequest rates, $\delta \geq \pi$, that is, $\delta \geq 2\omega/(4\omega + R)$, gains given by the extreme behavior of risk lovers make that the agents whose predecessors were lucky to have invested in the events that in fact occurred, then they become extremely wealthy over time. For low bequest rates, however, $\delta < 2\omega/(4\omega + R)$, the gains from having lucky ancestors have little impact on their successors' wealth. Therefore, when agents invest correctly, their successors'

wealth increase their wealth by amount that converge to zero in the long run. For an agent whose dynasty always bets correctly, his wealth converges to

$$\sum_{k=0}^{\infty} \left(\frac{\delta}{\pi}\right)^k \omega = \frac{\pi}{(\pi - \delta)} \omega = \frac{(2\omega + \delta R)}{((1 - 2\delta)2\omega - \delta R)} \omega.$$

Consequently, a low level of bequests implies bounded levels of wealth even for risk lovers whose predecessors were always lucky.

Comparing the two types of agents, we can observe that bequests affect the risk lovers more than they do the risk averters. The wealth of risk averse agents increases when the bequest rate increases. Their distribution is still constant, however, and their proportion of the aggregate wealth is also constant. On the other hand, wealth inequality among risk lovers increases as the rate of bequests increases. The poorest risk lovers always have the same level of wealth. Furthermore, the wealth of the other risk lovers is strictly higher when the bequest rate increases, and the increase in wealth of the risk lovers increases with the level of wealth. Since the aggregate consumption and wealth proportion of risk lovers are maintained with all levels of bequest rates, a low bequest rate implies a higher share of wealth in the hands of the poorest risk lovers and a lower share of wealth in the hands of the richest risk lovers (see Figure 2).

Notice that the invariant distribution of wealth depends on the return of the Lucas trees, the endowment of the agents, and the bequest rate of the agents. Larger bequest rate implies larger inequality levels. Moreover, if the bequest rate is lower than $2\omega/(4\omega + R)$, inequality is so low that the invariant distribution of wealth is bounded by $(2\omega + \delta R)\omega/((1 - 2\delta)2\omega - \delta R)$. Above that threshold, however, the invariant distribution of wealth is unbounded. One of the properties that can be analyzed in this case is the existence of fat tails in the invariant distribution of wealth. The following definition of a fat tail depends on whether the tail of the distribution has an

exponential decay. Usually, authors differentiate the tail of the distribution based on the speed of the decay, see Bryson (1974).

Definition 2. Given a random variable W , the distribution F_W has exponential fat tails if there exists $\alpha \geq 0$ such that for every w ,

$$\liminf_{w \rightarrow \infty} \frac{\log(1 - F_W(w))}{\log w} = -\alpha.$$

The following proposition ensures that for high bequest rates, the invariant distribution of wealth of the risk lovers has fat tails.

Proposition 5. If $\delta \geq 2\omega/(4\omega + R)$, the invariant distribution of the risk lovers, (w_∞^i) , has exponential fat tails with $\alpha = \log_2 2\beta$ with

$$\beta = \frac{\delta(2\omega + R)}{(2\omega + R) - (1 - \delta)R}$$

Moreover,

$$P[i: w_\infty^i > w] = \frac{1}{2^{n+2}} \propto \frac{1}{2^{\log_2 \beta w}}$$

where $w \in [\sum_{k=0}^n (\delta/\pi)^k \omega, \sum_{k=0}^{n+1} (\delta/\pi)^k \omega]$ for all $n \geq 1$.

The proof of Proposition 5 is in the Appendix.

Note that if $\delta \geq 2\omega/(4\omega + R)$, the constant $\beta \geq 1$. Moreover, β increases with the bequest rate. That is, fat tails become fatter as the bequest rate increases.

Several researchers have shown conditions for the existence of fat tails in the wealth distributions based on properties of the wealth transformation process (see Benhabib, Bisin, and Zhu, 2016). Other authors showed that tails of wealth distribution decay exponentially in a

heterogeneous-agent dynamic general equilibrium model with idiosyncratic endowment risk (see Beare and Toda, 2017). In our case, the existence of fat tails is due to the existence of risk loving agents who specialize in such a way that generate a large concentration of wealth in the long run.

4. Switching the type of agents: An example

Let us consider $\omega = 1$, $\delta = 0.5$, and $R = 1$. We analyze the case in which a proportion $p \in (0,1)$ of the successor of agents switch types: risk lovers to risk averse, and vice versa. For the case where types do not change from one generation to the next one, the agents give a proportion δ of their wealth as before. A risk lover agent who has a successor that is a risk averse decides to leave to his descendant the average bequest that risk-averse agents receive from their predecessors. A risk averse agent who has a successor that is a risk lover leaves no wealth to his descendant. Additionally, we assume that only successors of risk lovers who has been successful can become risk averse.

4.1. Case without switching

In the case without switching, that is, $p = 0$, the wealth distribution for risk lovers is

$$(w_{\infty}^{l_i})_i: \sum_{k=0}^n \left(\frac{\delta}{\pi}\right)^k \omega = \sum_{k=0}^n \left(\frac{6}{5}\right)^k \text{ with measure (or proportion) } 1/2^{n+1} \text{ for } n \in \mathbb{N},$$

the wealth distribution for the risk averters is

$$(w_{\infty}^{a_i})_i: 2\pi/(2\pi - 1/2) = 5/2 \text{ with measure (or proportion) } 1.$$

4.2. Case with switching

Let us now analyze the example in which a proportion of $p = 1/10$ of agents switches from entrepreneurs to risk averse, and vice versa. A successor of a risk-averse investor has a probability $1/10$ of becoming a risk lover, and the bequests received are equal to zero. A successor of a risk lover that received the high return at least twice in a row in the periods immediately before has a probability of $4/7$ of becoming a risk-averse investor. The bequests received by the new risk averse agent is equal to the average bequests of the risk-averse investors.

Let us denote $(\widehat{w}_\infty^i)_i$ as the invariant distribution of wealth. Note that the invariant distribution for the risk lovers, $(\widehat{w}_\infty^{li})_i$, has a similar form to Figure 1 and 2. However, the proportions of the investors are different.

We define y^a, y^l as the proportion of risk averters and risk lovers in the invariant distribution, respectively. If the proportion of risk lovers at the bottom $n + 1$ level of wealth is given by $y^{l^{n+1}}$, then the proportions $(y^{ln})_n$ satisfies that

- $y^{l^1} = (1/10)y^a + (1/2)(y^{l^1} + y^{l^2} + (3/7)(2 - y^a - y^{l^1} - y^{l^2}))$.
- $y^{l^n} = (1/2)y^{l^{n-1}}$ for $n = 2, 3$,
- $y^{l^n} = 1/2 (3/7)y^{l^{n-1}}$ for $n \geq 4$.

For the proportion of risk averters, we have that

- $y^a = (9/10)y^a + (4/7)(2 - y^a - y^{l^1} - y^{l^2})$.

Since $(y^{ln})_n$ and y^a are the proportion of the agents in the invariant distribution, we have that

$$y^l = y^a = 1.$$

If we define $\hat{\delta}$ as the aggregate bequest rate of the economy, we have that $\hat{\delta} \approx 0.449$ and the contingent-claims market prices are equal to

$$\pi_{\hat{\delta}} = \frac{2\omega + \hat{\delta}R}{2(2\omega + R)} = \frac{2 + \hat{\delta}}{6} \approx 0.408.$$

Then, the invariant distribution for the risk lovers, $(\hat{w}_{\infty}^{l_i})$, is given by:

- 1 with measure (or proportion) $y^{l^1} = 0.55$,
- $1 + (1/(2\pi_{\hat{\delta}})) \approx 2.224$ with measure (or proportion) $y^{l^2} = 0.275$,
- $1 + (1/(2\pi_{\hat{\delta}})) + (1/(2\pi_{\hat{\delta}}))^2 \approx 3.725$ with measure (or proportion) $y^{l^3} = 0.1375$,
- $\sum_{k=0}^{n-1} (1/(2\pi_{\hat{\delta}}))^k$ with measure (or proportion)

$$y^{l^n} = \frac{11}{80} \left(\frac{3}{14} \right)^{n-3},$$

with $n \geq 4$.

The invariant distribution for the risk averters is

$$(\hat{w}_{\infty}^{a_i})_i: \frac{2\pi_{\hat{\delta}}}{(2\pi_{\hat{\delta}} - 1/2)} \approx 2.58 \text{ for the risk averters with a proportion of } y^a = 1.$$

Note that all risk lovers that have a positive probability of becoming risk averse have wealth levels higher than the wealth of the risk averters. Then, the bequest rate of these risk lovers is lower than 0.5. For the other risk lovers, their bequest rate is equal to 0.5. For the risk averters, 9/10 of them have a bequest rate equal to 0.5, and the rest 1/10 have a bequest rate equal to zero. Then, the bequest rate of all agents is at most equal to 0.5, which explains that the aggregate bequest rate $\hat{\delta}$ is lower than 0.5. Therefore, contingent-claims market prices in this example, $\pi_{\hat{\delta}} \approx 0.41$, is a slightly lower than $\pi = 5/12 = 0.41\bar{6}$ in the case without switching.

Since the aggregate bequest rate of the economy is lower than 0.5, the aggregate wealth of the economy is also lower. However, the aggregate wealth of the risk averter is larger in this case

(the aggregate wealth of the risk averters without switching is 2.5). Additionally, all but the lowest risk-lovers' wealth levels are higher than the levels observed in the no switching case. These phenomena are a consequence of a higher proportion of risk lovers at the bottom of the distribution compared to a lower proportion of risk lovers at the top.

Finally, the invariant distribution of wealth has exponential fat tails with α given by

$$\alpha = \frac{\log_2\left(\frac{3}{2+\delta}\right)}{\log_2\left(\frac{14}{3}\right)} \approx 0.132$$

and in the case without switching, we have that $\alpha = \log_2(6/5) \approx 0.263$. Therefore, the invariant distribution with switching types has thinner tails than the case without switching. From this example we conclude that the slightly larger levels of wealth of the rich risk lovers in the case with switching do not compensate the smaller proportion of risk lovers at those levels of wealth.

5. Concluding remarks

We developed an overlapping generation model with risk-averse and risk-loving dynasties in an economy without aggregate uncertainty and no production. Due to specialization of the risk lovers, a large proportion of them will be at the bottom of the wealth distribution, while other risk lovers will be all the wealthiest agents of the economy. On the other hand, risk averse agent will be concentrated on the average wealth of the economy.

In any given state, a large proportion of the risk lovers have no return from the Lucas trees, implying that their wealth from that state onwards does not depend on their initial wealth in any way. Additionally, the proportion the risk lovers that have no return in each state converges to 0.5 since, with probability one, all risk lovers will be at the bottom of the wealth distribution.

Moreover, the dynamics of wealth distribution of the risk lovers have a positive reflecting barrier which is the initial endowment received in each period by the new generation. With probability 1, the wealth of a very poor risk lover can reach in finite time any threshold of the wealth process generated by risk lovers that were at the bottom of the distribution at least once. Therefore, due to all these properties, for any given any initial wealth distribution, the wealth distribution converges to an invariant distribution in distribution to an invariant distribution which is also unique.

The invariant distribution of wealth depends strongly on the return of the Lucas trees, the endowment of the agents, and the bequest rate of the agents. Larger bequest rate implies larger inequality levels. Moreover, if the bequest rate is lower than some threshold, inequality is so low that the invariant distribution of wealth is bounded. Above that threshold, however, the invariant distribution of wealth is a Pareto distribution.

When investors have a positive probability of switching their attitudes towards risk, the wealth distribution converges to an invariant distribution with a Pareto distribution for the risk averse investors as long as it has a Pareto distribution for the risk lovers.

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Appendix A. Proofs

Proof of Lemma 1. It is a direct consequence of Proposition 7 in Appendix B.

Proof of Lemma 2. Let us assume that there is an equilibrium with complete markets $(q_s, (c_s^{a_i}, b_s^{a_i}, \theta_s^{a_i})_i, (c_s^{l_i}, b_s^{l_i}, \theta_s^{l_i})_i)_s$ such that the Arrow-Debreu prices satisfy that $p_{(s,1)} < p_{(s,2)}$ for some state s . Due to the optimality conditions of risk averse investors and the absence of arbitrage opportunities, we have that the Arrow-Debreu prices and the equilibrium allocation satisfy Equation 2.1, 2.2, 2.3, and 2.4 for all state. Then $c_{(s,1)}^{a_i} > c_{(s,2)}^{a_i}$ for all risk averse a_i . Since $p_{(s,1)} < p_{(s,2)}$, risk lovers investors invest all their wealth in state $(s, 1)$ and zero in state $(s, 2)$ (see Equation 2.6 and 2.7, which implies that $c_{(s,1)}^{l_i} > c_{(s,2)}^{l_i} = 0$). Therefore, the aggregate consumption in state $(s, 1)$ is strictly larger than in state $(s, 2)$, which contradicts the aggregate endowment in each state is given $2\omega + R$.

Now, let us prove that

$$p_s = \frac{2\omega + \delta R}{2(2\omega + R)}, q_{k,s} = \frac{2\omega + \delta R}{2(1 - \delta)} \quad \forall s, k$$

are the state-contingent prices and asset prices at equilibrium, respectively. First note that

$$\begin{aligned} p_{(s,1)}q_{(s,1),1} + p_{(s,2)}(q_{(s,2),1} + R) &= \frac{2\omega + \delta R}{2(2\omega + R)} \left(\frac{2\omega + \delta R}{2(1 - \delta)} \right) + \frac{2\omega + \delta R}{2(2\omega + R)} \left(\frac{2\omega + \delta R}{2(1 - \delta)} + R \right) = \\ &= \frac{2\omega + \delta R}{2(2\omega + R)} \left(\frac{2\omega + \delta R}{(1 - \delta)} + R \right) = \frac{2\omega + \delta R}{2(2\omega + R)} \left(\frac{2\omega + R}{1 - \delta} \right) = \frac{2\omega + \delta R}{2(1 - \delta)} = q_{s,1}, \end{aligned}$$

and

$$\begin{aligned} p_{(s,1)}(q_{(s,1),2} + R) + p_{(s,2)}q_{(s,2),2} &= \frac{2\omega + \delta R}{2(2\omega + R)} \left(\frac{2\omega + \delta R}{2(1 - \delta)} + R \right) + \frac{2\omega + \delta R}{2(2\omega + R)} \left(\frac{2\omega + \delta R}{2(1 - \delta)} \right) = \\ &= \frac{2\omega + \delta R}{2(2\omega + R)} \left(\frac{2\omega + \delta R}{(1 - \delta)} + R \right) = \frac{2\omega + \delta R}{2(2\omega + R)} \left(\frac{2\omega + R}{1 - \delta} \right) = \frac{2\omega + \delta R}{2(1 - \delta)} = q_{s,2}. \end{aligned}$$

To see that $(p_s)_s$ are the state contingent market equilibrium, we must prove that the allocation described above is optimal for every agent and that it implements an equilibrium.

$$\begin{aligned}
q_{1,s}\theta_{1,s}^{l_i} + q_{2,s}\theta_{2,s}^{l_i} &= \frac{2\omega + \delta R}{2(1-\delta)} \left(\frac{2R + 2\omega - \delta R}{R(2\omega + \delta R)} (\omega + b_s^{l_i}) - \frac{(\omega + b_s^{l_i})}{R} \right) \\
&= \frac{2\omega + \delta R}{2(1-\delta)} (\omega + b^{l_i}) \left(\frac{2R + 2\omega - \delta R - (2\omega + \delta R)}{R(2\omega + \delta R)} \right) \\
&= \frac{2\omega + \delta R}{2(1-\delta)} (\omega + b^{l_i}) \left(\frac{2(1-\delta)R}{R(2\omega + \delta R)} \right) = \omega + b^{l_i},
\end{aligned}$$

and in state $(s, 1)$ we have that

$$(q_{1,(s,1)} + R)\theta_{1,s}^{l_i} + q_{2,(s,1)}\theta_{2,s}^{l_i} = (\omega + b^{l_i}) \left(1 + \frac{2R + 2\omega - \delta R}{(2\omega + \delta R)} \right)$$

and in state $(s, 2)$

$$q_{1,(s,2)}\theta_{1,s}^{l_i} + (q_{2,(s,2)} + R)\theta_{2,s}^{l_i} = \omega + b^{l_i} + R \left(-\frac{(\omega + b_s^{l_i})}{R} \right) = 0$$

or

$$(q_{1,(s,1)} + R)\theta_{1,s}^{l_i} + q_{2,(s,1)}\theta_{2,s}^{l_i} = 0$$

and in state $(s, 2)$

$$q_{1,(s,2)}\theta_{1,s}^{l_i} + (q_{2,(s,2)} + R)\theta_{2,s}^{l_i} = (\omega + b^{l_i}) \left(1 + \frac{2R + 2\omega - \delta R}{(2\omega + \delta R)} \right).$$

Since, for the risk lovers, we know that the optimal solutions are

$$\begin{aligned}
b_{(s,k)}^{l_i} &= c_{(s,k)}^{l_i} \frac{\delta}{(1-\delta)} = \frac{\delta}{p_{(s,k)}} (\omega + b_s^{l_i}) = \frac{\delta}{\pi} (\omega + b_s^{l_i}) \text{ or} \\
b_{(s,k)}^{l_i} &= c_{(s,k)}^{l_i} \frac{\delta}{(1-\delta)} = 0.
\end{aligned}$$

For risk averters, using the FOC we have that the optimal solution satisfies that

$$b_{(s,k)}^{a_i} = c_{(s,k)}^{a_i} \frac{\delta}{(1-\delta)} = \frac{\delta}{2p_{(s,k)}} (\omega + b_s^{a_i}) = \frac{\delta}{2\pi} (\omega + b_s^{a_i}) \quad \forall k = 1, 2.$$

Then, if risk lovers specialize symmetrically between the states, we have that

$$\int_i c_s^i di = 2\omega + R,$$

$$\begin{aligned}
\int_i b_s^i di &= \frac{\delta}{(1-\delta)} (2\omega + R), \\
\int_0^1 \theta_{j,s}^{a_i} di + \int_0^1 \theta_{j,s}^{l_i} di &= \int_0^1 \frac{(1-\delta)(\omega + b_s^{a_i})}{2\omega + \delta R} di + \int_0^1 \left(\frac{1-\delta}{(2\omega + \delta R)} \right) (\omega + b_s^{l_i}) di \\
&= \frac{(1-\delta)}{2\omega + \delta R} \left(2\omega + \int_0^1 b_s^{a_i} di + \int_0^1 b_s^{l_i} di \right) = \frac{(1-\delta)}{2\omega + \delta R} \left(2\omega + \frac{\delta}{(1-\delta)} (2\omega + R) \right) \\
&= \frac{(1-\delta)}{2\omega + \delta R} \left(\frac{2\omega(1-\delta) + \delta(2\omega + R)}{(1-\delta)} \right) = 1 \quad \forall j = 1,2,
\end{aligned}$$

for all state s , which concludes the proof.

Proof of Lemma 3. If there is an equilibrium with complete markets such that $p_{(s,1)} = p_{(s,2)}$ for some state s , all risk lovers are indifferent between investing in either of the Lucas trees. Note also that there is at least a continuum of family of sets $\{A_1, A_2\} \in \mathcal{P}([0,1])$ such that $\lambda(A_1) = \lambda(A_2) = 1/2$ and that $\int_{A_1} w^{l_i} di = \int_{A_2} w^{l_i} di$ where λ is the Lebesgue measure which concludes the proof.

Proof of Proposition 4. Note that any invariant distribution of wealth of the economy must satisfy that the poorest risk lovers are half of the risk lovers with ω as wealth. Using the same recursive process used to determine the invariant distribution, we have that any invariant distribution for the risk lovers coincides with the one mentioned in Subsection 3.1.

Note that for risk averse agents, any invariant distribution is constant since all of them have riskless returns and constant endowments. Additionally, since all agents have the same endowment $\omega > 0$, we have that $\bar{w}_\infty^l = \bar{w}_\infty^a$, which concludes the proof.

Proof of Proposition 5. Note that the invariant wealth distribution of the risk lovers $(w_\infty^{l_i})_{i \in [0,1]}$ is given by

$$\sum_{k=0}^n \left(\frac{\delta}{\pi}\right)^k \omega = \sum_{k=0}^n \left(\frac{2\delta(2\omega + R)}{(2\omega + R) - (1 - \delta)R}\right)^k \omega$$

with measure (or proportion) $1/2^{n+1}$ (for $n \in \mathbb{N}$). Given

$$w \in \left[\sum_{k=0}^n \left(\frac{2\delta(2\omega + R)}{(2\omega + R) - (1 - \delta)R}\right)^k \omega, \sum_{k=0}^{n+1} \left(\frac{2\delta(2\omega + R)}{(2\omega + R) - (1 - \delta)R}\right)^k \omega \right],$$

we have that $\lambda[i: w_\infty^{l_i} > w] = 1/2^{n+2}$ where $\lambda[\cdot]$ is the Lebesgue measure in $[0,1]$. Let us define

β as

$$\beta = \frac{\delta(2\omega + R)}{(2\omega + R) - (1 - \delta)R}$$

Since, $\delta \geq 2\omega/(4\omega + R)$, $\beta \in [1/2, 1)$. Let us suppose first that $\delta > 2\omega/(4\omega + R)$. Then, $\beta >$

0.5, and we have that

$$\begin{aligned} \sum_{k=0}^n (2\beta)^k \omega &\leq w \leq \sum_{k=0}^{n+1} (2\beta)^k \omega, \\ \frac{(2\beta)^{n+1} - 1}{2\beta - 1} \omega &\leq w \leq \frac{(2\beta)^{n+2} - 1}{2\beta - 1} \omega, \\ \frac{(2\beta)^{n+1} - 1}{2\beta - 1} &\leq \frac{w}{\omega} \leq \frac{(2\beta)^{n+2} - 1}{2\beta - 1}, \\ (2\beta)^n &\leq \frac{1 + (2\beta - 1)\frac{w}{\omega}}{2\beta} \leq (2\beta)^{n+1}, \\ n &\leq \log_{2\beta} \left(\frac{1 + (2\beta - 1)\frac{w}{\omega}}{2\beta} \right) \leq n + 1 \end{aligned}$$

Then,

$$\frac{1}{2^{\log_{2\beta}\left(\frac{1+(2\beta-1)\frac{w}{\omega}}{2\beta}\right)}} \geq \lambda[i: w_\infty^{l_i} > w] = \frac{1}{2^{n+1}} \geq \frac{1}{2^{\log_{2\beta}\left(\frac{1+(2\beta-1)\frac{w}{\omega}}{2\beta}\right)+1}}.$$

Therefore,

$$\frac{\log \lambda[i: w_\infty^{l_i} > w]}{\log(w)} \propto \frac{-\log_{2\beta}\left(\frac{w}{\omega}\right)}{\log_2 w} \propto -\log_2(2\beta),$$

which concludes the proof for $\beta > 0.5$. For $\beta = 0.5$, we have that $(n+1) \leq \frac{w}{\omega} \leq (n+2)$, then

$$\frac{2}{2^{w/\omega}} \geq \lambda[i: w_\infty^{l_i} > w] = \frac{1}{2^{n+1}} \geq \frac{1}{2^{w/\omega}},$$

which concludes the proof.

Appendix B. Incomplete market case

Let us analyze the case in which the financial market is incomplete in a state s in date $t \geq 1$, that is, the prices of the assets $q_{1,s}, q_{2,s}, q_{1,(s,1)}, q_{2,(s,1)}, q_{1,(s,2)}, q_{2,(s,2)} > 0$ are such that the vectors returns $(q_{1,(s,1)} + R, q_{1,(s,2)})$ and $(q_{2,(s,1)}, q_{2,(s,2)} + R)$ are co-linear. Note that due to absence of arbitrage in the economy we have that there is a $(\mu_s)_s$ a state price sequence such that

$$q_{1,s} = \mu_{(s,1)}(q_{1,(s,1)} + R) + \mu_{(s,2)}q_{1,(s,2)},$$

$$q_{2,s} = \mu_{(s,2)}(q_{2,(s,2)} + R) + \mu_{(s,1)}q_{2,(s,1)}.$$

Note that the optimal consumption and bequests of a risk averse agent a_i are given by

$$c_{(s,1)}^{a_i} = \frac{(1-\delta)(q_{1,(s,1)} + R)}{2q_{1,s}}(\omega + b_s^{a_i}), c_{(s,2)}^{a_i} = \frac{(1-\delta)q_{1,(s,2)}}{2q_{1,s}}(\omega + b_s^{a_i}),$$

$$b_{(s,1)}^{a_i} = \frac{\delta(q_{1,(s,1)} + R)}{2q_{1,s}}(\omega + b_s^{a_i}), \text{ and } b_{(s,2)}^{a_i} = \frac{\delta q_{1,(s,2)}}{2q_{1,s}}(\omega + b_s^{a_i}),$$

the optimal consumption and bequests of a risk lover l_i are given by

$$c_{(s,1)}^{li} = \frac{(1-\delta)(q_{1,(s,1)} + R)}{2q_{1,s}}(\omega + b_s^{li}) \text{ and } c_{(s,2)}^{li} = \frac{(1-\delta)q_{1,(s,2)}}{2q_{1,s}}(\omega + b_s^{li}),$$

$$b_{(s,1)}^{li} = \frac{\delta(q_{1,(s,1)} + R)}{2q_{1,s}}(\omega + b_s^{li}), \text{ and } b_{(s,2)}^{li} = \frac{\delta q_{1,(s,2)}}{2q_{1,s}}(\omega + b_s^{li}).$$

Since there is no aggregate uncertainty, in equilibrium we have that

$$\int_i c_{(s,1)}^i di = \frac{(1-\delta)(q_{1,(s,1)} + R)}{2q_{1,s}} \left(\int_i (\omega + b_s^i) di \right) = 2\omega + R,$$

$$\int_i c_{(s,2)}^i di = \frac{(1-\delta)q_{1,(s,2)}}{2q_{1,s}} \left(\int_i (\omega + b_s^i) di \right) = 2\omega + R.$$

Then,

$$\frac{q_{1,(s,1)} + R}{q_{1,s}} = \frac{q_{1,(s,2)}}{q_{1,s}} = \frac{q_{2,(s,1)}}{q_{2,s}} = \frac{q_{2,(s,2)} + R}{q_{2,s}} = \frac{2\omega + R}{2\omega + \delta R},$$

and

$$\mu_{(s,1)} = \mu_{(s,2)} = \mu = \frac{2\omega + \delta R}{2(2\omega + R)}.$$

If the markets are incomplete in date $t = 0$ we have that the aggregate consumption of the old generation satisfies in equilibrium that

$$\int_i c_{(0)}^i di = (1-\delta) \left((q_{1,(0)} + R) \left(\int_i \theta_{1,-1}^i di \right) + q_{2,(0)} \left(\int_i \theta_{2,-1}^i di \right) \right) = 2\omega + R,$$

which implies that

$$q_{1,(0)} + q_{2,(0)} = \frac{2\omega + \delta R}{1-\delta},$$

and $\mu_{(0,1)}$ and $\mu_{(0,2)}$ are given by

$$\mu_{(0,1)} = \mu_{(0,2)} = \frac{(1-\delta)\bar{q}}{2(2\omega + R)}.$$

In this case, the asset prices in date 0 and date 1 satisfies that

$$\frac{(q_{1,(0,1)} + R)}{q_{1,(0)}} = \frac{q_{1,(0,2)}}{q_{1,(0)}} = \frac{q_{2,(0,1)}}{q_{2,0}} = \frac{q_{2,(0,2)} + R}{q_{2,0}} = \frac{2\omega + R}{2\omega + \delta R}$$

Now we can show that incomplete markets cannot occur in equilibrium.

Proposition 6. There is no equilibrium with incomplete markets. Moreover, in any equilibrium the asset prices are given by

$$q_{j,s} = \frac{2\omega + \delta R}{2(1 - \delta)} \quad \forall j = 1,2, \text{ and } s.$$

Proof. Let us suppose that there is an equilibrium with incomplete markets in state s with date $t \geq 0$. Without loss of generality, we assume that

$$q_{1,(0)} \leq \frac{2\omega + \delta R}{2(1 - \delta)}.$$

We will show by induction in t that the price of asset 1 is negative for some state s' successor of s . If the market is incomplete in $t = 0$, we have that

$$q_{1,(0,1)} = \left(\frac{2\omega + R}{2\omega + \delta R} \right) q_{1,(0)} - R \leq \frac{2\omega - R + 2\delta R}{2(1 - \delta)} = \frac{2\omega + \delta R}{2(1 - \delta)} - \frac{R}{2} < \frac{2\omega + \delta R}{2(1 - \delta)}.$$

Since $q_{1,s} = \mu(q_{1,(s,1)} + R + q_{1,(s,2)})$ for all state s , consider the minimum between $q_{1,(s,1)}$ and $q_{1,(s,2)}$, then

$$\min\{q_{1,(s,1)}, q_{1,(s,2)}\} \leq \left(\frac{(2\omega + R)}{(2\omega + \delta R)} \right) q_{1,s} - R/2.$$

Let us consider $\{s^n\}_n$ the sequence of states where s^{n+1} is the immediate successor of s^n , $q_{1,s^{n+1}}$ is the minimum price of asset 1 between $q_{1,(s^n,1)}$ and $q_{1,(s^n,2)}$, and $s^1 = (0)$. Since

$$q_{1,(0,1)} \leq \frac{(2\omega + \delta R)}{(2(1 - \delta))} - R/2,$$

for n large enough we have that

$$q_{1,(s^n,1)} \leq \frac{2\omega + \delta R}{2(1-\delta)} - \frac{R}{2} \left(\frac{2\omega + R}{2\omega + \delta R} \right)^n < 0$$

which is a contradiction.

Let us assume that in state $s = (0, s_1, \dots, s_{t=1}, s_t)$ with date $t \geq 1$, Let us consider $\{s^n\}_n$ the sequence of states where s^{n+1} is the immediate successor of s^n , $q_{1,s^{n+1}}$ is the minimum price of asset 1 between $q_{1,(s^n,1)}$ and $q_{1,(s^n,2)}$, and $s^1 = s$. Since

$$\begin{aligned} q_{1,(s,1)} &\leq \left(\frac{2\omega + R}{2\omega + \delta R} \right) q_{1,s} - R \leq \left(\frac{2\omega + R}{2\omega + \delta R} \right) q_{1,(0,s_1,\dots,s_{t=1})} - R \leq \dots \leq \left(\frac{2\omega + R}{2\omega + \delta R} \right) q_{1,(0)} - R \\ &\leq \frac{2\omega - R + 2\delta R}{2(1-\delta)} = \frac{2\omega + \delta R}{2(1-\delta)} - \frac{R}{2} < \frac{2\omega + \delta R}{2(1-\delta)}, \end{aligned}$$

for n large enough we have that

$$q_{1,(s^n,1)} \leq \frac{2\omega + \delta R}{2(1-\delta)} - \frac{R}{2} \left(\frac{2\omega + R}{2\omega + \delta R} \right)^n < 0$$

which is a contradiction.

To conclude the proof, note that if

$$q_{j,(0)} < \frac{(2\omega + \delta R)}{(2(1-\delta))}$$

for some asset j , we have that there is a state s with date t large enough such that $q_{j,s} < 0$ since the first part of the proof also holds. Therefore, the only possible asset price is

$$q_{j,s} = \frac{2\omega + \delta R}{2(1-\delta)} \quad \forall j = 1,2, \text{ and } s.$$

Note that in this case, markets are complete, and we have that the portfolio allocation in a state s with date $t \geq 0$ of a risk averter a_i is given by

$$\theta_{1,s}^{a_i} = \theta_{2,s}^{a_i} = \frac{(1-\delta)(\omega + b_s^{a_i})}{2\omega + \delta R},$$

and the portfolio allocation of a risk lover l_i is given by

$$\theta_{1,s}^{l_i} = -\frac{(\omega + b_s^{l_i})}{R}, \theta_{2,s}^{l_i} = \frac{(2\omega + R(2 - \delta))(\omega + b_s^{l_i})}{2R(1 - \delta)}, \text{ or}$$

$$\theta_{1,s}^{l_i} = \frac{(2\omega + R(2 - \delta))(\omega + b_s^{l_i})}{2R(1 - \delta)}, \theta_{2,s}^{a_i} = -\frac{(\omega + b_s^{l_i})}{R}, \text{ and}$$

the portfolio allocation in $t = 0$ of a risk averter a_i is given by

$$\theta_{1,(0)}^{a_i} = \theta_{2,(0)}^{a_i} = \frac{(\theta_{1,0}^{a_i} + \theta_{2,0}^{a_i})}{2},$$

and the portfolio allocation of a risk lover l_i is given by

$$\theta_{1,s}^{l_i} = -\frac{(2\omega + \delta R)(\theta_{1,0}^{l_i} + \theta_{2,0}^{l_i})}{2R(1 - \delta)}, \theta_{2,s}^{l_i} = \frac{(2\omega + R(2 - \delta))(2\omega + \delta R)(\theta_{1,0}^{l_i} + \theta_{2,0}^{l_i})}{R}, \text{ or}$$

$$\theta_{1,s}^{l_i} = \frac{(2\omega + R(2 - \delta))(2\omega + \delta R)(\theta_{1,0}^{l_i} + \theta_{2,0}^{l_i})}{R}, \theta_{2,s}^{a_i} = -\frac{(2\omega + \delta R)(\theta_{1,0}^{l_i} + \theta_{2,0}^{l_i})}{2R(1 - \delta)},$$

which concludes the proof.

Appendix C. Switching type of agents

Let us suppose that $p \in [0, 1/7)$. We define $p_1 = 4p/(1 - 3p) \in [0, 1)$. Let us analyze how the proportion of risk lovers and risk averter changes. To do so, we define y_t^a, y_t^l as the mass of the risk averters and the risk lovers at date t , respectively, then $y_t^a = y_t^l = 1$. Note that for the mass of the risk lovers at the bottom of the distribution in date t , $y_t^{l^0}$, then the mass of risk lovers at the bottom of the distribution in date $t + 1$ satisfies that

$$y_{t+1}^{l^0} = py_t^{l^0} + \frac{1}{2}(y_t^{l^0} + y_t^{l^1} + (1 - p_1)(2 - y_t^a - y_t^{l^0} + y_t^{l^1})).$$

where consider $y_t^{l^k}$ as the mass of risk lovers that are at the bottom $k + 1$ level of wealth at date t . Then, the mass of the second poorest group of risk lovers we have that

$$y_{t+1}^{l^1} = \frac{1}{2}y_t^{l^0}, \quad y_{t+1}^{l^2} = \frac{1}{2}y_t^{l^1},$$

and for the other risk lovers we have that

$$y_{t+1}^{l^{k+1}} = \frac{y_t^{l^k}(1 - p_1)}{2},$$

For $k \leq t$. For the mass of risk averters we have that

$$y_{t+1}^a = (1-p)y_t^a + p_1(2 - y_t^a - y_t^{l^0} - y_t^{l^1}).$$

when $t \rightarrow \infty$, we have that

$$y_\infty^{l^0} = \frac{1+p}{2}, y_\infty^{l^1} = \frac{1+p}{4}, \text{ and } y_\infty^a = 1.$$

Then,

$$y_\infty^{l^{n+2}} = \frac{(1+p)}{4}(1-p_1)^n = \frac{(1+p)}{4} \left(\frac{1-7p}{2(1-3p)} \right)^n$$

for $t \geq 0$.

Finally, we can construct recursively the invariant distribution as in Proposition 1. In this case, the wealth of each group of risk lovers and the wealth of the risk averters in the invariant distribution follows the argument of Proposition 4. The reason for this is that any risk averter that the predecessor is a risk lover receives the average bequest of the risk averters. However, the aggregate bequest rate is different in this case. Let us consider $\pi_{\hat{\delta}}$ as

$$\pi_{\hat{\delta}} = \left(\frac{2\omega + R - (1-\hat{\delta})R}{2(2\omega + R)} \right)$$

with $\hat{\delta}$ as the aggregate bequest rate of the economy. Note that $\pi_{\hat{\delta}}$ is the analogous of π of the model without switching. Then

$$\frac{1-\hat{\delta}}{\pi_{\hat{\delta}}} \left(\int_i w_\infty^{a_i} di + \int_i w_\infty^{l_i} di \right) = \frac{1-\hat{\delta}}{\pi_{\hat{\delta}}} \left(\int_i (b_\infty^{a_i} + \omega) di + \int_i (b_\infty^{l_i} + \omega) di \right) = 2\omega + R,$$

that is, $\hat{\delta}$ is the solution of

$$\begin{aligned} \frac{1-\hat{\delta}}{\pi_{\hat{\delta}}} \left(\left(\frac{\pi_{\hat{\delta}}}{2\pi_{\hat{\delta}} - \delta} \omega \right) + \sum_{n=0}^{\infty} \frac{(1+p)}{4} \left(\frac{1-7p}{2(1-3p)} \right)^n \left(\sum_{k=0}^{n+2} \left(\frac{\delta}{\pi_{\hat{\delta}}} \right)^k \omega \right) + \left(\frac{1+p}{2} \right) \omega \right. \\ \left. + \left(\frac{1+p}{4} \right) \left(\omega + \left(\frac{\delta}{\pi_{\hat{\delta}}} \right) \omega \right) \right) = 2\omega + R. \end{aligned} \quad (\text{C.1})$$

Note that since for all $d \in [0,1]$ we have that

$$\left(1 + \frac{2\delta(2\omega + R)}{2\omega + dR} + \left(\frac{2\delta(2\omega + R)}{2\omega + dR} \right)^2 - \frac{2\omega + dR}{2(1-\delta)\omega + (d-\delta)R} \right) > 0,$$

In particular

$$\omega + \frac{2\delta(2\omega + R)}{2\omega + \hat{\delta}R} \omega + \left(\frac{2\delta(2\omega + R)}{2\omega + \hat{\delta}R} \right)^2 \omega > \frac{2\omega + dR}{2(1 - \delta)\omega + (\hat{\delta} - \delta)R} \omega,$$

which implies that the wealth of the risk lovers who receives the highest return twice in a row have a larger wealth than the risk averters. Therefore, $\hat{\delta} \leq \delta$.

In the numerical example, we have that $p = 0.1$, we estimate the proportion of agents in each wealth level using the formulas to obtain y_{∞}^{ln} , $\hat{\delta}$ using Equation C.1. The level of each group are given by

$$w_{\infty}^{ai} = \frac{\pi_{\hat{\delta}}}{2\pi_{\hat{\delta}} - \delta} \omega,$$

$$(w_{\infty}^{li})_i : \sum_{k=0}^n \left(\frac{\delta}{\pi_{\hat{\delta}}} \right)^k \omega \text{ with measure (or proportion) } y_{\infty}^{ln} \text{ for } n = 0, 1, \dots$$