Polarization indices: Calibrating the sensitivity in the distance

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Abstract

The present document is a summary of the talk we propose for the *48th Simposio de la Asociación Española de Economía (SAEe).* There is a classic result due by Esteban and Ray characterizing the polarization indices when the population groups are described by quantitative variables (such as income). These, the Esteban-Ray indices, are used by most practitioners when they study phenomena such as conflict, diversity or war in economy and politics.

During our research on multidimensional polarization indices, we saw that the family of Esteban-Ray indices is much larger than the one described in its theorem. We have identified the reason behind this and provided a method to find new indices. This new family of indices are more versatile and appropriate for empirical studies since the weight of the distance in each index can be modulated.

Measuring the polarization in societies is a keystone problem in the study of many phenomena as war, diversity or innovation in societies. Roughly speaking, it measures how far the actual distribution of the population in predetermined groups is from a bimodal distribution (the most polarized one) [6]. When the population groups are described by a qualitative variable, for example, defined by an ethnic or a religious profile, then the indices are quite simple because polarization only depends on the number of individuals in each group [7].

However, in societies whose individuals are identified by a quantitative variable admitting a distance (such as the level of income), a new problem arises. Here the polarization depends on the size of each group as well as on the distance between them. As Wolfson pointed out, polarization is closely related to middle-class disappearance and he showed that inequality indices are insufficient to study such phenomena so, specific polarization measures are required [9]. There have been different proposals for quantitative polarization measures [2, 4, 8], but one of the most used indices are the Esteban-Ray family [1, 3]. Their model goes as follows:

Given a population of N individuals separated in n groups of size $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ with income levels $\boldsymbol{y} = (y_1, \ldots, y_n)$, Esteban and Ray proposed a polarization measure P satisfying Equation 1:

$$P(\boldsymbol{\pi}, \boldsymbol{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \pi_j \theta(\pi_i, |y_i - y_j|) \tag{1}$$

where θ is a real valued continuous function measuring the effective antagonism between groups. The quantity $\theta(\pi_i, |y_i - y_j|)$ is the antagonism felt by an individual in the group i towards an individual in j, whereas two individuals in the same group feel no antagonism. So, the polarization $P(\boldsymbol{\pi}, \boldsymbol{y})$ is the sum of individual antagonisms between every pair of individuals. The function θ captures the intra-group identification in the first component (the bigger the group one belongs, the bigger the antagonism one feels toward individuals in other groups). The second component represents alienation (the farther two individuals are, the bigger the antagonism is). So, θ must be increasing and $\theta(0, d) = \theta(\pi, 0) = 0$ for all number of individuals π and distance d.

Esteban and Ray proposed in their paper [3] four reasonable axioms for polarization measures. Moreover, they proved that assuming a polarization measure as in expression 1 and satisfying their four axioms, then it had to be as one of the indices described in Theorem 1:

Theorem 1 (Theorem 1 [3])**.** *A polarization measure of the family defined in 1 satisfies the four axioms if and only if it is of the form*

$$P(\boldsymbol{\pi}, \boldsymbol{y}) = K \sum_{i=1}^{n} \sum_{j=1}^{n} {\pi_i}^{\alpha+1} \pi_j |y_i - y_j|$$

for some constants $K > 0$ and $\alpha \in (0, \alpha^*]$ where $\alpha^* \approx 1.6$.

Notice these indices have a modulation on the weight of group size parameterized by α, but in all of them, the polarization depends linearly on the distance between groups without admitting any modulation.

We started our work in polarization looking for multidimensional polarization measures. However, we discover that there are more functions satisfying the Esteban-Ray model. For instance, antagonism functions as $\theta(\pi, d) = \pi d^2$ or $\theta(\pi, d) = \pi(e^d - 1)$ also satisfy the axioms. For that reason, revising their proof, we detected they implicitly use a domain extension on the population in the sense that the number of people in a group can be as small as one wants. Unfortunately, they assumed such extension without discussing it properly in their papers.
We have proved that, without this extension, the family of polarization indices derived from their axioms is much larger. Particularly, the indices allow a modulation in the weight of the distance between groups. So, as we have proved in the Theorem 2 below, the family of Esteban-Ray polarization measures is much larger and then their functional expressions are richer for empirical purposes.

Theorem 2. *Let P be a polarization measure of the family defined in (1). If $\theta(\pi, d) = \pi^\alpha f(d)$, being f convex and $\frac{f(2d)}{f(d)} > M_\alpha$[1] for every distance d, then P satisfies the four Esteban-Ray axioms.*

The theorem above shown that Esteban-Ray polarization measures are more versatile than they proved. They allow a modulation in the weight of the identification as well as in the alienation.

[1] M_α is the maximum of the function $g_\alpha(z) = \frac{2-(\alpha+1)z+2(\alpha+1)z^\alpha - z^{\alpha+1}}{\alpha+2}$ defined in $\mathbb{R}_{\geq 0}$.

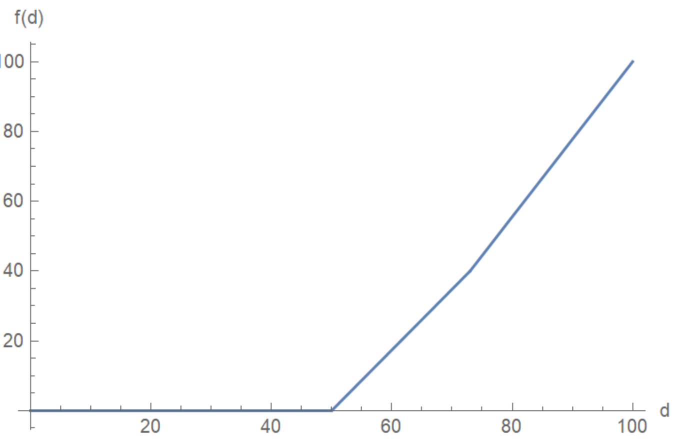

Figure 1: The alienation function $\theta(\pi, d) = \pi f(d)$ induces a polarization measure. When the distance between two groups is less than 50, their mutual antagonism is negligible.

There have been other revisions on Esteban-Ray indices but, as far as we know, no one of them have extended the family significantly. For instance, Kawada et al. detected a mistake in Esteban and Ray's proof [5]. However, they proposed a slight modification on the first axiom to obtain the same family, and they did not notice the implicit domain extension.

In the same line, Duclos et al. proposed an equivalent axiomatization for continuous population groups [1]. They also proved a theorem characterizing the family of polarization indices satisfying their axioms. However, their proof used the same domain extension as Esteban and Ray in [3]. So, we expect that the family satisfying the continuous model to be richer than the one described in [1, Theorem 1].

Moreover, we have revisited the literature and we have checked that in most of the studies in which polarization is measured, the domain extension does not play any role and is not needed. The family of indices derived by us, since it satisfies Estaban-Ray axioms and also allows a modulation in the distance, is more versatile for empirical purposes. Furthermore, if we modulate the distance's weight, we can better adjust the index to the society we are studying. For instance, we can capture in our index the idea that antagonism between close groups (inside a distance threshold) is negligible (see Figure 1).

In this talk, we will start exposing the Esteban-Ray model. After, we will discuss the domain extension in Esteban and Ray's proof. We will continue exposing the new Esteban-Ray indices we have obtained. We will end the talk by discussing the applicability of the new indices using some examples.

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