Beyond Quasilinearity: Exploring Nonlinear Scoring Rules in Procurement Auctions*

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Abstract

This study examines procurement auctions in which bidders submit price and quality, and they are evaluated using the price-per-quality-ratio scoring rule. We formulate a model of scoring auctions in which bidders' cost is determined by unidimensional type and unidimensional quality. We characterize the equilibrium behavior for the first-score and second-score auctions. In contrast to the well-known quasilinear scoring rules in which price and quality are additively separable and score is linear in price, the equivalence theorem does not hold for the auction formats. We show that the second-score auction yields a lower (better) expected score than the first-score auction. Under certain conditions, the expected quality and price are higher in the first-score auction than in the second-score auction. We also argue how these results can be extended to other non-quasilinear scoring rules.

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1 Introduction

Due to huge public debts, the pressure to seek *value for money* in procurement has surged in many countries. While low-price auctions have been commonly used as a competitive, transparent, and accountable allocation mechanism, more procurement buyers have introduced mechanisms in which the buyers assess not only prices but also non-monetary attributes, such as delivery time, of proposals in the entire procurement cycle. The *scoring auction* is one of the prevailing mechanisms that aim at price competition and value for money at the same time.

In a scoring auction, each bidder submits a multidimensional bid that consists of price and non-monetary attributes (henceforth, *quality*). A pre-announced scoring rule assigns a score to the multidimensional bid to rank the bidders in the auction. The seminal paper by Che (1993) shows that the scoring auction under a symmetric independent private value setting can be reduced to a canonical model of auctions with unidimensional bid if the scoring rule is quasilinear (QL), i.e., score is linear in price and additively separable from quality.[1] Hence, the well-known revenue equivalence theorem applies to scoring auctions with QL scoring rules, and equivalence results with respect to price and quality hold between several auction formats. Since then, scoring auctions have attracted a growing number of theoretical and empirical studies.

In real-world procurement auctions, however, a much wider variety of scoring rules than QL scoring rules are adopted. A typical example is the "price-per-quality-ratio" (PQR) scoring rule, in which a score is given by the price bid divided by the quality bid. Many state departments of transportation (DOTs) in the United States,

[1] We use the abbreviation *QL* only for *quasilinear scoring rules.* We do not abbreviate the term and denote *quasilinear* for the quasilinearity of the payoff function.

including those in Alaska, Florida, Michigan, North Carolina, and South Dakota, have adopted the "adjusted bid," which is equivalent to PQR scoring rule. The Department of Health and Aging in Australia also employs a PQR awarding rule for contracts that need to achieve better returns on public investment (The Department of Health and Ageing, Australia, 2011). In addition, most public procurement contracts in Japan are allocated to the bidder with the highest price-per-quality bid ratio. Despite the frequent use of non-QL or PQR scoring rules in reality, little is known about the properties of such scoring auctions.[2]

To fill the gap, we examine scoring auctions with the PQR scoring rule. The cost function considered here consists of a unidimensional private signal and a unidimensional quality level. We focus on the bidding behavior in the following two scoring auction formats: the first-score (FS) and the second-score (SS) auctions. In both auctions, the winner is the lowest-score bidder.[3] In the FS auction, the winner delivers the quality at the price specified in its bid. In the SS auction, the winner is free to choose a price–quality pair such that the score given by the price–quality pair meets the minimal rival score.

Our main finding is to characterize the equilibrium of the PQR scoring auctions and to evaluate the performance of the FS and SS auctions with respect to equilibrium score, quality, and price. Similar to Che (1993), the scoring auctions are analyzed by transforming the multidimensional-bid auctions into a unidimensional score-bid auction game. Given a scoring rule, bidders choose their profit-maximizing contract (a price–quality pair) for each feasible score. Bidders' profit is reduced to an indirect payoff function in score, and bidders play an auction game with respect to score-bid. Under a QL scoring rule, Che (1993) shows that a scoring auction is

[2]While bid ranking is preserved in any monotonic transformation of the scoring rule, the transformation does not generally convert a non-QL scoring rule into a QL rule. If, for instance, we take a logarithm of the price-per-quality-ratio scoring rule, score is not linear in price anymore; a necessary condition for quasilinearity is thus violated.

[3]In the PQR scoring rule, a score is determined by the price-per-quality, thus that the lower score is more preferable for the buyer.

reduced to a score-bid auction in which bidders have a quasilinear indirect payoff function in score. Hence, the well-understood results of auction theory apply to the QL scoring auctions, such as the revenue equivalence theorem between FS and SS auctions. Under the PQR scoring rule, in contrast, the bidders' indirect payoff function is not quasilinear. We show that a symmetric Bayesian Nash equilibrium exists in the FS and SS auctions for a broad class of cost functions. The bidder of the lowest (efficient) type wins in equilibrium of both auctions.

We show that under the PQR scoring rule and certain conditions, the FS auction yields a higher (worse) expected score, a higher expected quality, and a higher expected price than the SS auction. Under the PQR scoring rule, bidders choose a higher quality as the score increases, and their winning profit is reduced to a convex function in score. Bidders are "risk-loving" in score and take risk on the larger winning profit in the FS auction. Hence, the FS auction yields a higher expected score than the SS auction in equilibrium, which is analogous to auctions with non-quasilinear payoffs such as Maskin and Riley (1984). Because the FS auction yields a higher score and the chosen quality is increasing in score, the FS auction is likely to provide a higher quality. Similarly, the expected price is likely higher in the FS auction than in the SS auction. However, the expected price ranking is more ambiguous than the expected quality ranking. Under certain specifications of cost function, the FS auction provides a higher quality than the SS auction, whereas the expected price is equivalent. The higher expected score in the FS auction does not necessarily mean the FS auction is worse for the buyer than the SS auction, unless PQR is the buyer's true objective function. In addition, we show that scoring auctions with the PQR scoring rule achieve a higher quality than price-only auctions at the expense of higher price, which is similar to the QL scoring rules.

We also discuss how the equilibrium properties under the PQR scoring rule can be generalized to other non-QL scoring rules. Given a scoring rule, the expected score in the FS auction is higher (lower) than the SS auction if the bidders' indirect payoff function is convex (concave) in score. We examine key properties on scoring rules for convex/concave indirect payoffs in score. Although the quality and price rankings are ambiguous under general scoring rules, we argue that the property of the optimal quality function is crucial for our results on the PQR scoring rule. We show that the equivalence theorem with respect to the expected score, quality, and price is a unique feature of the QL scoring rule.

Related Literature This paper belongs to the theoretical and empirical literature on scoring auctions. The theoretical study of scoring auctions is pioneered by Che (1993), which focuses on scoring auctions in which price enters linearly into the scoring rule. In Che (1993)'s approach, the model of scoring auctions in which bidders post price and quality reduces to the model of auctions in which bidders submit only scores as if it were a price-only auction. Che (1993) is extended to the cases of correlated signals (Branco, 1997), of multidimensional signals (Asker and Cantillon, 2008), and of multidimensional quality (Nishimura, 2015). Furthermore, Asker and Cantillon (2008), Awaya, Fujiwara and Szabo (2022), and Sano (2023) compare the performance between scoring auctions with the QL scoring rule and alternative mechanisms. Different from these previous studies, we concentrate our analysis on scoring auctions with the PQR scoring rule and compare the performance of the FS and SS auctions.

In contrast to the studies on scoring auctions with the QL scoring rule, there are a few papers that study scoring auctions in which price does not enter linearly into the scoring rule. Wang and Liu (2014) consider a scoring auction with a non-QL scoring rule where price and quality are additively separable. They characterize the equilibrium of the FS auction and examine the relation between the number of bidders and equilibrium price and quality. Dastidar (2014) analyzes the scoring auction with a general monotone scoring rule. He finds that the equilibrium bidding function of the FS auction is explicitly obtained if the cost function of each bidder is additively separable in quality and her private information. Different from these studies, we characterize the equilibrium of scoring auctions with the PQR scoring rule and identify the conditions FS auctions yield higher quality level but higher price than SS auctions.

Hanazono, Hirose, Nakabayashi and Tsuruoka (2020) study the models of scoring auctions with general scoring rules where bidders' private information about their cost is multidimensional. However, Hanazono, Hirose, Nakabayashi and Tsuruoka (2020) emphasize that with the exception of the QL scoring rule, it is difficult to analyze the properties of a monotone equilibrium. Thus, by focusing on the model where each bidder has a unidimensional signal, this paper examines the properties of the monotone equilibrium.

Empirical research on the scoring auction is growing in the literature as well (e.g., Lewis and Bajari, 2011; Koning and van de Meerendonk, 2014; Iimi, 2016; Takahashi, 2018; Huang, 2019; Krasnokutskaya, Song and Tang, 2020; Ryan, 2020; Kong, Perrigne and Vuong, 2022 and Allen, Clark, Hickman and Richert, 2023). In addition, building on the literature on scoring auctions, Bajari, Houghton and Tadelis (2014) and Bolotnyy and Vasserman (2023) develop structural auction models in which firms post unit price bids for each of items required to complete a construction project. Among these studies, Takahashi (2018) is the closest to our paper, who examines scoring auctions with the PQR scoring rule to quantify the impacts of uncertainty from reviewers' quality evaluation. Unlike these previous studies, this paper compares performance between the PQR scoring auctions and price-only auctions, and hence, the theoretical results presented here have policy implications and empirically testable predictions.

The remainder of this paper is organized as follows. Section 2 describes the model of scoring auctions. We show that scoring auctions are transformed into a unidimensional score-bid auction game. In Section 3, we focus on the PQR scoring rule and analyze symmetric equilibria in FS and SS auctions. We compare the expected winning score, quality, and price between the two auction formats. Also, we compare the performance of the PQR scoring auction with price-only auctions. Section 4 analyzes general scoring rules. We characterize the expected score rankings for FS and SS auctions. The final section is the conclusion.

2 Model

Consider that a procurement buyer auctions off a procurement contract to $n \geq 2$ risk-neutral bidders. All bidders are ex ante symmetric. Bidder i's private type is denoted by θ_i and is independently and identically drawn from a cumulative distribution F over $\Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$. The cumulative distribution F has a continuous density $f(\theta) > 0$ for every $\theta \in \Theta$. Let $q \in \mathbb{R}_+$ be non-monetary attribute (quality). Each bidder's production cost is given by $C(q, \theta_i)$. We impose several assumptions on the cost function. The basic assumptions are listed as follows.

- The cost function C is thrice differentiable and strictly increasing in both q and θ; C_q, $C_\theta > 0$.
- The cost function C is strictly convex in quality; $C_{qq} > 0$.
- There exists a sufficiently large number $B > 0$, and for all θ, $C_q(q, \theta) \geq B$ for some $q > \underline{q}$.
- The cost function C has non-decreasing differences; $C_{q\theta} \geq 0$.

Note that $C_\theta = \partial C / \partial \theta$, and that the other subscripts are defined in the same manner. The production cost is increasing in quality and type, so that a bidder of a lower type is more efficient. The third assumption guarantees that that the optimal quality exists under scoring auctions we consider. The last assumption means that a bidder of a lower type has a smaller marginal cost.[4]

When bidder i wins the auction and signs a contract with a price p and a quality q, their payoff is given by

$$p - C(q, \theta_i).$$

We suppose that every losing bidder's payoff is zero.

[4]In fact, the assumption $C_{q\theta} \geq 0$ is not necessary for most of the result. In particular, we can verify that the symmetric Bayesian Nash equilibrium exists for the FS auction under $C_{q\theta} \leq 0$.

In a scoring auction, each bidder submits a proposal (p, q), where $p \leq \bar{p}$ is a price bid and $q \geq \underline{q}$ is a quality bid. A reserve price and a minimum quality requirement are denoted by $\bar{p} > 0$ and $\underline{q} > 0$, respectively. Each proposal is evaluated by a pre-announced scoring rule $S : [0, \bar{p}] \times [\underline{q}, \infty) \to \mathbb{R}$. The scoring rule maps a multidimensional bid into a unidimensional score $s = S(p, q)$, and the lowest-score bidder wins. We assume that the scoring rule is sufficiently smooth and satisfies $S_p > 0$ and $S_q < 0$.

We study the first-score (FS) and the second-score (SS) auctions. In either auction, each bidder submits (p, q), and the bidder with the lowest score wins. In the FS auction, the winner's proposal is finalized as a contract. In the SS auction, the winner is required to match the highest rejected (the second lowest) score. To meet the score, the winner is free to choose any quality-price pair. Thus, the finalized contract generally differs from the winning bid (p, q) and the lowest losing bid in the SS auction.

Our model allows general scoring rules, but in Section 3 we will focus on the *price-per-quality ratio* (PQR) scoring rule:

$$S(p, q) = \frac{p}{q}, \tag{1}$$

with $p \leq \bar{p}$ and $q \geq \underline{q}$.[5] Score ranking is preserved in any monotone transformation of the scoring rule. Hence, most properties of scoring auctions, especially the equilibrium price and quality, do not change in monotone transformation of the scoring rule, except for the expected score ranking. The expected score ranking is preserved in affine transformation. For example, the quality-per-price-ratio rule $S = -\frac{q}{p}$ is a monotone (but not affine) transformation of the PQR scoring rule. Thus, the properties of the equilibrium price and quality presented in the next section will be invariant, but the expected score ranking will be different.

Remark 1 We do not formulate the buyer's preferences. One interpretation of PQR

[5] The quality q here is measured in terms of "quality score." One might consider a scoring rule $S(p, q) = p/V(q)$, where V is an increasing function. This is equivalent to the case in which a quality is defined as $\tilde{q} = V(q)$.

and other non-QL scoring rules is that they are the buyer's (true) objective function. However, as Che (1993) shows, the buyer may be better off using a scoring rule that differs from the true objective function. We aim to characterize and compare equilibrium scores, quality, and price under PQR and other non-QL rules without specifying the buyer's preferences.

2.1 Score-bid Auctions

Equilibrium of scoring auctions is derived in a similar manner to Che (1993). Given an arbitrary score s, every bidder will choose the optimal contract (p, q) that induces score s. The auction with multidimensional bid is reduced to a unidimensional auction in terms of score bid.

Suppose that the winner of type θ is required to enforce a contract that fulfills score s. The winner determines a contract (p, q) that solves

$$\begin{aligned} &\max_{(p,q)} p - C(q, \theta) \\ &\text{s.t. } S(p, q) = s, \\ &\quad p \leq \bar{p},\ q \geq \underline{q}. \end{aligned} \tag{2}$$

Throughout the analysis, we assume that the reserve price and the minimum quality are not binding at (2). By substituting the score constraint into the objective function, the payoff maximization is written as

$$\max_{q \geq \underline{q}} P(s, q) - C(q, \theta), \tag{3}$$

where P is the inverse function of S with respect to p. When the objective function in (3) is strictly concave in q, the maximization problem has a unique solution. The optimal quality is denoted by

$$q^*(s, \theta) \in \arg\max_{q} P(s, q) - C(q, \theta). \tag{4}$$

The indirect payoff function is denoted by

$$u(s, \theta) \equiv P\left(s, q^*(s, \theta)\right) - C\left(q^*(s, \theta), \theta\right). \tag{5}$$

Note that by $S_p > 0$, we have $P_s > 0$. By the envelope theorem, the indirect payoff u is strictly increasing in s and strictly decreasing in θ. Equilibrium of scoring auctions are derived by solving standard auctions in terms of score bid s, where each bidder has the winning profit $u(s, \theta)$.

Let $z(\theta)$ be the unique solution of

$$u(z(\theta), \theta) = 0.$$

That is, z is the score bid such that the winner's indirect payoff is zero. It is the maximum willingness to accept of a bidder of type θ in the auction. We call $z(\theta)$ *the break-even score for type* θ. Because u is increasing in s and decreasing in θ, z is increasing in θ.

Lemma 1 *Suppose that $P(s, q) - C(q, \theta)$ is strictly concave in q. Then, $z(\theta)$ is well-defined and strictly increassing in θ.*

Proof See Appendix.

2.2 QL Scoring Rule

In the seminal paper, Che (1993) studies the *quasilinear (QL) scoring rules* $S(p, q) = p - q$. Under the QL scoring rule, the optimal quality is given by the profit maximization problem

$$\max_q s + q - C(q, \theta).$$

When the optimal quality is determined by the first-order condition

$$1 - C_q(q^*, \theta) = 0,$$

q^* depends only on θ and is independent of s. The indirect payoff is reduced to quasilinear in score,

$$u(s, \theta) = s - k(\theta),$$

where

$$k(\theta) = -\max_q q - C(q, \theta).$$

Thus, the QL scoring auction is reduced to score-bid auctions with quasilinear payoffs. Hence, the revenue equivalence theorem applies. The expected score between FS and SS auctions are equivalent in equilibrium. Because the bidder of the lowest type wins in both auctions, the exercised quality is *ex post* equivalent between the two formats. Because of the score equivalence and $p = s + q$, the equivalence holds for the expected price too.

2.3 PQR Scoring Rule

Consider the PQR scoring rule $S(p, q) = p/q$. The inverse function of S with respect to p is given by $P(s, q) = sq$. The optimal quality is derived by the profit maximization problem

$$\max_{q \geq \underline{q}} sq - C(q, \theta). \tag{6}$$

It is clear that the objective function is strictly concave in q. We assume that the optimal quality q^* always lies in the interior. This is satisfied if the optimal quality at $(z(\theta), \theta)$ is not binding.

Assumption 1 In the PQR scoring rule, the optimal quality satisfies

$$q^* \left(z(\theta), \theta \right) > \underline{q}$$

for all θ.

When the optimal quality q^* lies in the interior, it is determined by the first-order condition

$$s - C_q(q^*, \theta) = 0. \tag{7}$$

By the implicit function theorem, we have

$$q^*_s = \frac{1}{C_{qq}} > 0,$$
$$q^*_\theta = -\frac{C_{q\theta}}{C_{qq}} \leq 0.$$

The optimal quality is increasing in score s and non-increasing in type θ. Because bidders have no incentive to submit a score lower than $z(\theta)$, Assumption 1 is sufficient for q^* to satisfy the first-order condition (7) in equilibrium. Assumption 1 is likely to hold if the fixed cost that is not sunk exists. Note that the indirect payoff u is increasing in s and that q^* is increasing in s (if it is determined by (7)). With the presence of a fixed cost, a higher score and quality is necessary to earn a non-negative profit for bidders.

Lemma 2 *Under the PQR scoring rule and Assumption 1, the indirect payoff function u is strictly convex in s.*

Proof By the envelope theorem, we have

$$u_s(s,\theta) = q^*(s,\theta) > 0, \tag{8}$$

$$u_{ss}(s,\theta) = q_s^*(s,\theta) > 0. \tag{9}$$

□

The score-bid auction game can be interpreted as a supplier's competition for a cost reimbursement contract. Under the PQR scoring function (contract), the reimbursement is made for each unit of quality provision at the price equal to the exercised score, $s = p/q$, and the bidder who offers the lowest quality price wins. Suppliers, who submit a quality price, are price makers under the PQR scoring rule.

The bidder's optimal quality choice $q^*(s,\theta)$ is given by $C_q(q,\theta) = s$. That is, each supplier chooses the optimal quality q^* such that the marginal cost equals the quality price. These facts give a view on why the optimal quality, q^*, depends on the score bid under the PQR scoring function. The dependence of the optimal quality choice results in the profit upon winning (indirect utility) rising faster than linearly in quality price; (i) the inframarginal profit increases linearly due to the increase in quality price, and (ii) there is an extramarginal profit due to the increase in optimal quality.

The PQR scoring rule is distinct from the QL scoring rule in two respects. First, the optimal quality under the PQR rule depends not only on the bidder's type but also on the required score s. Second, the indirect payoff function is not quasilinear; the revenue equivalence theorem is not applied to the PQR rule.

3 Equilibrium Analysis of the PQR Scoring Auctions

In this section, We focus on the PQR scoring rule and analyze the equilibrium properties of FS and SS auctions.

3.1 Equilibrium

We first characterize the equilibrium of the SS and FS auctions. We show that both auctions select the bidder of the lowest type as the winner.

In the SS auction, it is a weakly dominant strategy to bid $z(\theta)$ as in the standard second-price auction. The following proposition is shown in a standard manner and similar to Maskin and Riley (1984), Saitoh and Serizawa (2008), and Sakai (2008); thus, the proof is omitted.

Proposition 1 *In the SS auction, it is a weakly dominant strategy for each bidder to submit $s^{SS}(\theta) = z(\theta)$.*

As for the FS auction, Maskin and Riley (1984) and Athey (2001) show that the FS auction has a symmetric, monotone Bayesian Nash equilibrium if the payoff function u is log-supermodular:

$$\frac{\partial^2 \log u(s,\theta)}{\partial s \partial \theta} > 0. \tag{10}$$

To meet the log-supermodularity condition, we additionally impose technical conditions below.

Assumption 2 At least one of the following conditions holds.

1. C_θ / C_q is non-increasing in q, or

2. the cost function satisfies

$$1 + q\left(\frac{C_{qqq}}{C_{qq}} - \frac{C_{qq\theta}}{C_{q\theta}}\right) > 0. \tag{11}$$

A wide range of cost functions satisfy either of the above. The first case means $C_q C_{q\theta} - C_\theta C_{qq} \leq 0$. Roughly speaking, this condition is met when the marginal cost is more sensitive to the change in quality than to the change in type; i.e., C_{qq} is large and $C_{q\theta}$ is small. A special case is $C_{q\theta} = 0$; bidders' marginal cost is independent of θ. That is, bidders' variable costs for quality are identical but the fixed costs are heterogeneous.[6] The second case is likely satisfied when the cost function is polynomial in q and type θ does not depend on the coefficient of the maximum degree of q. For example, the condition is met if the cost function is quadratic and $C(q, \theta) = q^2 + \theta q + \kappa(\theta)$ with $\kappa(\theta) > 0$.

Under the log-supermodularity condition, the equilibrium bidding function is characterized by the first-order condition. Let $G(\theta) = 1 - (1 - F(\theta))^{n-1}$ be the distribution of the lowest order statistic of $n-1$ independent draws from F. In addition, let $g = G'$ be its density.

Proposition 2 *If Assumptions 1 and 2 hold, there exists a symmetric Bayesian Nash equilibrium in the FS auction. Equilibrium score-bidding function s^{FS} is characterized by*

$$(s^{FS})'(\theta) = \frac{u\left(s^{FS}(\theta), \theta\right)}{u_s\left(s^{FS}(\theta), \theta\right)} \cdot \frac{g(\theta)}{1 - G(\theta)} \tag{12}$$

with $s^{FS}(\bar{\theta}) = z(\bar{\theta})$.

Proof See Appendix.

To see how each supplier decides its score bid in FS auction, let us introduce the *pseudotype function* $k(\cdot, \theta)$ as

$$k(s, \theta) \equiv \frac{C(q^*(s, \theta), \theta)}{q^*(s, \theta)}. \tag{13}$$

That is, k is the supplier's average cost (AC) under the PQR scoring rule. By linearizing the indirect payoff in s at (s, θ), we have

$$\begin{aligned} u^l(\tilde{s}; s, \theta) &\equiv u_s(s, \theta)(\tilde{s} - s) + u(s, \theta) \\ &= u_s(s, \theta)\left(\tilde{s} - s + \frac{sq^*(s, \theta) - C(q^*(s, \theta), \theta)}{q^*(s, \theta)}\right) \\ &= u_s(s, \theta)(\tilde{s} - k(s, \theta)) \end{aligned}$$

[6]Dastidar (2014) focuses on this type of cost functions and examines the equilibrium of non-QL scoring auctions.

by $u_s(s,\theta) = q^*(s,\theta)$. This implies that $u^l(k(s,\theta); s,\theta) = 0$. Thus, $k(s,\theta)$ is a break-even score of the bidder that has the linearized utility at s. If $u(s,\theta)$ is already linear in s, $k(s,\theta)$ is independent of s and indeed coincides with the original break-even score, $z(\theta)$. This special case arises for the QL scoring rule and is extensively discussed in Che (1993) and Asker and Cantillon (2008), where $k(s,\theta) = z(\theta)$ is called the *productive potential* or the *pseudotype*, respectively.

Using $k(s,\theta)$, we can also characterize the equilibrium bidding behavior in the FS auction. Suppose hypothetically that a bidder with type θ had the linearized induced utility at $s = s^{FS}(\theta)$ and anticipated that all the other bidders would follow $s^{FS}(\cdot)$. Notice that the associated first-order condition is the same as with the original induced utility. Hence, each bidder's behavior can be alternatively described as if this bidder had the linearized indirect utility. Analogously to the equilibrium bid characterization in the first-price auction with risk-neutral bidders, we can express equilibrium bid in the FS auction by the order statistic of $k(\cdot)$.

Corollary 1 *In the FS auction, the equilibrium score bid is the conditional expectation of the second-lowest bidder's pseudotype function $k(s^{FS}(\cdot),\cdot)$:*

$$s^{FS}(\theta) = \int_{\theta}^{\bar{\theta}} \frac{(n-1)f(\tau)\,[1-F(\tau)]^{n-2}}{[1-F(\theta)]^{n-1}} k(s^{FS}(\tau),\tau)d\tau. \tag{14}$$

Proof See Appendix.

The supplier with the lowest k wins in equilibrium. This implies that a supplier wins if it has the lowest AC. In the SS competition, the amount of cost reimbursement is independent of its own offer. Hence, it is dominant for suppliers to bid their minimum AC (the break-even score).

In the FS competition, the amount of cost reimbursement is specified by its own offer. Hence, it is not a best response for each supplier to bid its minimum AC. The quality price is equal to the winning supplier's bid, thus that the quality level behind the bid is greater than the efficient scale (the one that minimizes AC). It follows that the lowest quality price is the expectation of the lowest rival's AC that is strictly greater than the rival's minimum AC. Hence, it is natural to see that the equivalence in the expected winning score fails between the FS and SS auctions under the PQR scoring rule.

3.2 Comparisons between FS and SS Auctions

We now compare the equilibrium performance between FS and SS auctions. In contrast to the QL scoring rule, the equivalence theorem between the two formats does not hold. We provide ranking results with respect to the expected score, expected quality, and the expected price.

3.2.1 Score Ranking

If PQR is the buyer's true objective function, they prefer an auction format that yields a lower (expected) score. The expected score ranking between the FS and SS auctions depends on the curvature of the bidder's indirect payoff u. Maskin and Riley (1984) show that if bidders' payoff function u is *concave* in payment, the expected revenue in the first-price auction is higher than that in the second-price auction. By Lemma 2, bidder's indirect payoff is *convex* in the PQR scoring auction. Hence, we have a similar but reverse expected score ranking, which is shown in an analogous manner to Maskin and Riley (1984). The following proposition implies that the buyer prefers the SS than the FS auction.

Proposition 3 *Suppose that Assumptions 1 and 2 hold. The expected score of the FS auction is higher than that of the SS auction. Moreover, for every winner's type θ, we have*

$$s^{FS}(\theta) \geq E[s^{SS}(\tilde{\theta}) \mid \tilde{\theta} > \theta], \tag{15}$$

where $\tilde{\theta}$ is the lowest order statistic of $n-1$ independent draws from F.

Proof This is shown in a parallel manner to Theorem 4 of Maskin and Riley (1984). Although Maskin and Riley (1984) consider concave payoff function, it is not necessary to assume the concavity for the existence of the symmetric equilibrium. □

Convex payoff function is similar to risk-loving bidders. When bidders are risk-loving, they take more risk on the winning: they want to increase the winning profit even if they lose more often. Hence, bidders submit a higher score (that gives a larger profit) than the case in which they are risk neutral. A similar ranking result for convex payoff is provided by Board (2007), which examines auctions of a risky asset with limited liability.

3.2.2 Quality Ranking

Because the optimal quality depends on score s and the score equivalence does not hold for the PQR scoring rule, the equilibrium quality also differs between the two auction formats. Note that the optimal quality function q^* is increasing in score. Because the FS auction yields a higher expected score than the SS auction, the FS auction is likely to provide a higher quality than the SS auction.

The expected quality is ranked under additional conditions. Note that in the FS auction, the winner's quality is deterministic at the bidding stage because the winner's quality bid is enforced in case of winning. In contrast, in the SS auction, the winner's quality is stochastic because the optimal quality depends on the second lowest score, which is uncertain. To have the expected quality ranking, we need a condition on the curvature of the optimal quality function q^*. The following theorem states that the FS auction provides a higher expected quality than the SS auction when the optimal quality q^* is weakly concave in score.

Theorem 1 *Consider the PQR scoring rule. Suppose that Assumptions 1 and 2 hold and that $C_{qqq} \geq 0$. Then, the expected quality in the FS auction is higher than that in the SS auction.*

Proof See Appendix.

The condition $C_{qqq} \geq 0$ means that the marginal cost is convex, and it implies that the optimal quality function q^* is weakly concave in s. The optimal quality is determined by (7); the unit price per quality equals the marginal cost. When marginal cost is convex, it rapidly increases as q increases. Hence, the optimal quality does not increase so much by increasing score = the unit price per quality, and it is weakly concave.

3.2.3 Price Ranking

The equilibrium price ranking is more ambiguous than the quality ranking. Under the PQR scoring rule, the equilibrium price is given by

$$\pi(s, \theta) \equiv sq^*(s, \theta).$$

Analogous to the quality ranking, we have an expected price ranking if the optimal price π is weakly concave in score. However, π is more likely to be convex because q^* is increasing in s. Thus, the concavity of π is more stringent than the concavity of q^*.

We provide two cases in which the expected prices between the FS and SS auctions are ranked. The first case is when the optimal price is weakly concave in score.

Theorem 2 *Consider the PQR scoring rule. Suppose that Assumptions 1 and 2 hold and*

$$C_q C_{qqq} \geq 2(C_{qq})^2. \tag{16}$$

Then, the expected price in the FS auction is higher than that in the SS auction.

Proof See Appendix.

The price function π is weakly concave under (16). An example of such a cost function is

$$C(q, \theta) = \log \frac{a}{a - \theta - q},$$

where $a > 0$ is constant. This cost function satisfies all the basic assumptions and Assumption 2. The optimal quality and price are given by

$$q^*(s, \theta) = a - \theta - \frac{1}{s}$$

and

$$\pi(s, \theta) = (a - \theta)s - 1,$$

respectively.

The second case of the price ranking is when the bidders' type represents their fixed costs, $C_{q\theta} = 0$. In this case, the optimal quality q^* is independent of type; $q^*(s, \theta) = q^*(s)$. Hence, the optimal price is also independent of type and $\pi(s) = sq^*(s)$. Because the quality function q^* is increasing in s, the optimal price $\pi(s) = sq^*(s)$ is increasing. Thus, the score has a one-to-one correspondence to the optimal price, so that we transform the indirect payoff function $u(s, \theta)$ in terms of s into one in terms of price p; $\hat{u}(p, \theta) \equiv u(\pi^{-1}(p), \theta)$. The payoff function $\hat{u}(p, \theta)$ is the winner's payoff when they sign a contract under which they optimally choose the price as p. The bidder of the lowest score-bid makes the lowest price-bid. Thus, the score-bid auction is transformed into a unidimensional price-bid auction. The equilibrium price is ranked when bidders' payoff is concave (or convex) for the associated price-bid auction.

Theorem 3 *Consider the PQR scoring rule. Suppose that Assumption 1 holds and $C_{q\theta} = 0$. The expected price in the FS auction is at least as high as that in the SS auction if qC_{qq}/C_q is nondecreasing in q, or equivalently,*

$$C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 \geq 0 \tag{17}$$

holds for all $q \geq \underline{q}$. The expected price in the SS auction is at least as high as that in the FS auction if qC_{qq}/C_q is nonincreasing in q, or equivalently,

$$C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 \leq 0 \tag{18}$$

holds for all $q \geq \underline{q}$.

Proof See Appendix.

Given $C_{q\theta} = 0$, the condition (17) is weaker than the concavity of π, (16). Indeed, when (16) holds, we have

$$\begin{aligned} C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 &= C_q C_{qq} + q\left(C_q C_{qqq} - 2(C_{qq})^2 + (C_{qq})^2\right) \\ &\geq (C_q + qC_{qq})C_{qq} \\ &> 0. \end{aligned}$$

Note that (17) for the price ranking is relatively stronger than that for the quality ranking. This reflects the fact that the price function π is more likely to be convex compared to the quality function. Although FS auction yields a higher expected score than SS auction, the convex price function could result in a higher expected price for SS auction than FS auction. Thus, while the expected quality is higher for the FS auction than the SS auction, the expected price may be equivalent between FS and SS auctions or even lower for FS auction.

To see this, consider a specific cost function $C(q, \theta) = q^a + bq + \theta$ with $a \geq 2$ and $b \in \mathbb{R}$.[7] Because $C_{qqq} \geq 0$, the expected quality is higher in the FS auction than in the SS auction. Also, we have

$$C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 = a(a-1)^2 bq^{a-2}.$$

Hence, the expected price is higher in the FS auction than in the SS auction if $b > 0$. Conversely, the expected price is lower in the FS auction if $b < 0$. For a particular case, suppose $b = 0$. Then, the optimal quality and price are given by

$$q^*(s) = a^{-\frac{1}{a-1}} s^{\frac{1}{a-1}}$$

and

$$\pi(s) = sq^*(s) = a^{-\frac{1}{a-1}} s^{\frac{a}{a-1}},$$

respectively. The indirect payoff function is

$$u(s, \theta) = s^{\frac{a}{a-1}} a^{-\frac{1}{a-1}} \left(1 - a^{-1}\right) - \theta.$$

[7] We focus on the region where the cost is increasing in q when $b < 0$.

This payoff function can be transformed into

$$\hat{u}(p,\theta) = \frac{a-1}{a}\left(p - \frac{a\theta}{a-1}\right),$$

where $p = \pi(s)$. That is, the score-bid auction is transformed into a price-bid auction with a quasilinear payoff function and a pseudotype $a\theta/(a-1)$. Thus, we can apply the revenue equivalence theorem, and the equilibrium price is equivalent between FS and SS auctions.

Corollary 2 *Consider the PQR scoring rule, and suppose that Assumption 1 and* $C_{q\theta} = 0$ *hold. If* $C_{qqq} \geq 0$ *and* qC_{qq}/C_q *is nonincreasing in* $q \geq \underline{q}$*, then the expected quality is higher in the FS auction than SS auction, and the expected price in the FS auction is at most as high as in the SS auction. Thus, the FS auction achieves a higher expected quality with a weakly lower expected price.*

Corollary 2 is somewhat surprising because the expected price per quality is lower in SS auction. Although the expected price per quality is higher, FS auction can lead to a higher expected quality and lower expected price than SS auction. If the buyer's true objective is to achieve a higher expected quality at a lower expense rather than to minimize PQR, FS auction is better than FS auction.

3.3 Comparison with Price-only Auctions

In addition to comparisons between FS and SS auctions, it is also important to compare scoring auctions to price-only auctions. With respect to empirical analysis, using the data of highway construction procurement in California, Lewis and Bajari (2011) shows that scoring auctions with the QL scoring rule result in shorter delivery time but higher winning prices relative to price-only auctions. For theoretical analysis, Asker and Cantillon (2008) show that scoring auctions with the QL scoring rule strictly dominate price-only auctions with minimum quality levels in terms of the expected buyer surplus. Awaya, Fujiwara and Szabo (2022) also show that SS auctions with the QL scoring rule achieve higher expected quality levels in comparison with price-only auctions when the buyer specify the quality requirement.

Here, we compare performance between scoring auctions with the PQR scoring rule and price-only auctions with minimum quality levels. The following proposition states that scoring auctions with the PQR scoring rule lead to higher winning prices but higher quality levels.

Proposition 4

i. Under Assumption 1, SS auctions with the PQR scoring rule lead to higher winning prices and quality levels than price-only auctions with minimum quality levels.

ii. Under Assumptions 1, 2, and the conditions presented in Theorems 2 or 3, FS auctions with the PQR scoring rule result in higher winning prices and quality levels than price-only auctions with minimum quality levels.

Proof See Appendix.

Thus, at the choice of auction formats, a buyer faces a trade-off between price and quality. This theoretical result has practical implications because not only price-only auctions but also scoring auctions with the PQR scoring rule are used worldwide.

4 General Scoring Rules

In this section, we consider general scoring rules and examine how the properties of the PQR scoring rule can be generalized to other non-QL scoring rules. The scoring rule S is increasing in p and decreasing in q. The inverse function in terms of p is denoted by $P(s, q)$, which is the price function given a required score s and quality q. It holds that $P_s > 0$ and $P_q > 0$. Bidders' indirect payoff function is given by

$$u(s, \theta) \equiv \max_{q \geq \underline{q}} P(s, q) - C(q, \theta). \tag{19}$$

We assume that the payoff function $P - C$ is strictly concave in q and that the payoff maximization problem always has a (unique) interior solution. That is, the optimal quality q^* is determined by the first order condition

$$P_q(s, q^*) - C_q(q^*, \theta) = 0. \tag{20}$$

We further assume that the indirect payoff function u is well behaved and satisfies the log-supermodularity condition $\partial^2 \log u / \partial s \partial \theta > 0$. The equilibrium of the SS and FS auctions is characterized in the same manner with the PQR scoring rule.

Proposition 5 *Suppose that $P(s, q) - C(q, \theta)$ is strictly concave in q and that the optimal quality q^* is determined by the first order condition (20). In the SS auction, it is a weakly dominant strategy for each bidder to submit $s^{SS}(\theta) = z(\theta)$. In the FS auction, the symmetric equilibrium score-bidding function s^{FS} is characterized by (12) with $s^{FS}(\bar{\theta}) = z(\bar{\theta})$ if u is log-supermodular.*

Proof The proof is the same with those of Propositions 1 and 2. See Appendix B for sufficient conditions for the log-supermodularity of u. □

4.1 Expected Score

The same argument as Section 3.2.1 can be directly applied to general scoring rules. Namely, the expected score is higher (lower) in the FS than in the SS auction if $u(s, \theta)$ is convex (concave) in s.

Proposition 6 *Suppose that the FS auction has a symmetric Bayesian Nash equilibrium. Then, the expected score in the FS auction is weakly higher (lower) than that in the SS auction if $u(s, \theta)$ is convex (concave) in s for all θ.*

Proof The proof is the same with Proposition 3. Given that the existence of symmetric equilibrium, the expected score ranking is shown in the same manner to Theorem 4 of Maskin and Riley (1984). ■

Note that bidders with a convex (concave) utility function place score bids less (more) aggressively in the FS auction than in the SS auction. This is analogous to the comparison of the bidding behaviors between first- and second-price auctions with non-risk-neutral bidders.

Two factors affect the curvature of the bidder's indirect payoff function: (i) the direct effect and (ii) the indirect effect of the scoring function (as well as the cost function for factor (ii)). More specifically, we have

$$u_{ss}(s,\theta) = P_{ss}(s, q^*(s,\theta)) + P_{sq}(s, q^*(s,\theta))q_s^*(s,\theta).$$

Factors (i) and (ii) are associated with the first and the second terms on the right-hand side of the equation, respectively. The direct effect captures the change in the marginal payments with respect to s given q, while the indirect effect captures the change in the marginal payments with respect to s through the change in q.

Regarding the direct effect, the curvature of the scoring function directly affects that of the bidder's induced utility function. Note that $P_{ss}(s,q) = -S_{pp}(P(s,q),q)/S_p(P(s,q),q)^3$. Therefore, as the scoring function is more concave (convex) in p ceteris paribus, $u(s,\theta)$ becomes more (less) convex in s. Note that the direct effect is independent of the property of the cost function.

On the other hand, the indirect effect, $P_{sq}(s, q^*(s,\theta))q_s^*(s,\theta)$, is always nonnegative. Indeed, by the first order condition (20) for the optimal quality and the implicit function theorem, we have

$$q_s^* = -\frac{P_{sq}}{P_{qq} - C_{qq}}. \tag{21}$$

Hence, $P_{sq}q_s^*$ is always nonnegative when the strict concavity of the payoff function in q; $P_{qq} - C_{qq} < 0$. Intuitively, with a scoring function in which the associated P_s falls (rises) as q rises, the bidder will optimally choose a smaller (larger) q as s becomes larger. Moreover, as the indirect effect increases ceteris paribus, $u(s,\theta)$ becomes more convex in s. Given the fact that the indirect effect is always nonnegative, $u(s,\theta)$ is convex if $S_{pp} \le 0$.

4.2 Expected Quality and Price

An interesting feature in the PQR scoring rule is that the optimal quality q^* is increasing in score s. This suggests that under the PQR scoring auction, the lower-type bidders compete on price at the expense of quality. Note that the lower-type bidder submits the lower score bid in equilibrium. Thus, bidders propose a lower quality with a much lower price as they gets more efficient. This property may not be desirable for the procurer unless the scoring function represents their true preferences over price-quality choice.

Note that the sign of q_s^* and P_{sq} coincide by (21). Also, we have

$$P_{sq} = -\frac{S_{pp}P_q + S_{pq}}{(S_p)^2} = \frac{S_{pp}S_q - S_pS_{pq}}{(S_p)^3}. \tag{22}$$

The sign of S_{pq} is crucial for the slope of the optimal quality in s. In particular, if the scoring rule is linear in p (i.e., $S_{pp} = 0$), then the sign of q_s^* is determined by $-S_{pq}$. In the PQR scoring rule, $S_{pq} < 0$ and the optimal quality is increasing in score s.[8]

A scoring rule with $S_{pq} < 0$ means that the marginal score with respect to price, S_p, increases as the quality decreases. That is, when quality is relatively low, a lower price can lower the score more significantly. In other words, the lower the quality, the more price competition will be encouraged. Although bidders of lower type choose higher quality, scoring rules such as the PQR rule are prone to price competition at the expense of quality.

The quality ranking between FS and SS auctions depends on the curvature of the quality function q^* too. When the indirect payoff u is convex, the FS auction yields a higher expected score than the SS auction. Similar to the previous section, the expected quality is higher in the FS than in the SS auction if q^* is increasing and weakly concave in s. Also, the SS auction provides a higher expected quality than

[8] As Che (1993) argues, the optimal quality is determined by the iso-bid curve and the seller's cost function. The optimal quality is not affected by any monotone transformation of scoring rule S. Hence, we can focus on scoring rules with $S_{pp} = 0$ because every reasonable scoring rule can be transformed into such one.

the FS auction if q^* is decreasing and weakly convex in s. However, the condition for the concavity or convexity of q^* is complicated, and it is difficult to have a clear result on the expected quality ranking between FS and SS auctions.

The price ranking between FS and SS auctions is more ambiguous than the quality. Let $\pi(s,\theta) \equiv P(s, q^*(s,\theta))$ be the price associated with score s and the optimal quality q^*. Then,

$$\pi_s = P_s + q_s^* P_q$$

and

$$\pi_{ss} = P_{ss} + q_{ss}^* P_q - \frac{(P_{sq})^2}{P_{qq} - C_{qq}} \left(2 - \frac{P_{qq}}{P_{qq} - C_{qq}} \right)$$

by (21). The last term of π_{ss} is positive if $P_{qq} \geq 0$. Hence, the optimal price π is likely to be convex. Then, the expected price ranking becomes ambiguous when u is convex and q^* is increasing in s. This is analogous to Theorem 2 for the PQR scoring rule.

For a scoring rule in which the associated optimal quality q^* is decreasing in s, the price function π may no longer be monotone. Thus, the price ranking is further ambiguous.

4.3 Characterization of the QL Scoring Rules

As Che (1993) shows, QL scoring rules are reduced to score-bid auction with quasi-linear indirect payoff. Thus, the equilibrium properties for QL scoring auctions are well understood, and we have the equivalence theorem with respect to price and quality. We also have a converse result: if a scoring rule induces the equivalence theorem with respect to price and quality, it is a QL scoring auction.

Note that if we consider the ex post buyer's preferences over (p, q), it does not lose the generality to suppose $S_{pp} = 0$. This is because the buyer's preferences over price-quality choice is characterized by the iso-score (indifference) curve. For a reasonable class of scoring rules S, we would be able to construct another scoring rule $\hat{S}$ which is affine in p, $\hat{S}_{pp} = 0$, by taking an appropriate monotone transformation on S.[9]

Suppose $S_{pp} = 0$. Then, a scoring rule is QL if and only if $S_{pq} = 0$. Because the slope of the optimal quality q^* is determined by (21) and (22), q^* is independent of s if and only if the scoring rule is QL. Also, the twice derivative of u in s is given by $u_{ss} = P_{sq} q^*_s$, thus that the indirect payoff u is quasilinear only for QL scoring rules. Thus, we have confirmed that the QL scoring rule is a unique rule that induces a quasilinear indirect payoff. Also, the QL scoring rule is a unique rule under which the optimal quality is independent of s.

Proposition 7 *Suppose $S_{pp} = 0$. Then, the following statements are equivalent.*

1. *The scoring function is QL; $S_{pq} = 0$.*
2. *The optimal quality q^* is independent of score s.*
3. *The indirect payoff u is quasilinear in s.*

5 Concluding Remarks

We have examined scoring auctions with the PQR and non-QL scoring rules. For the PQR scoring rule, we have characterized the equilibrium bidding strategies in FS and SS auctions. We have found that equivalence fails between FS and SS auctions, and showed that the expected score is lower in SS auction than in FS auction. In addition, under certain conditions, the expected quality and price are higher in the FS auction than in SS auction. Interestingly, we have provided an example in which the expected quality in FS auction is higher than in SS auction, whereas the expected price is equivalent. We have also showed that PQR scoring auction provides a higher quality at a higher cost than price-only auctions. These results suggest that if PQR is the procurement buyer's true preference, the SS auction is better for the buyer than the FS auction. Nevertheless, FS auction may perform better than SS auction with respect to the expected quality and price.

[9]However, it does lose the generality when we consider the expected score.

We have also examined other non-QL scoring rule. The expected score ranking is characterized by the curvature of the indirect payoff function. The payoff function is reduced to convex, and SS auction yields a lower expected score if the scoring function is weakly concave in price. The expected quality and price rankings are ambiguous because the curvature of the optimal quality function is complicated.

There are various potential extensions for further research. In this study, we have restricted our attention to scoring rules in which each bidder's score depends only on its price and quality. In practice, the buyer sometimes uses relative or *interdependent* scoring rules in which the score depends not only on the bidder's own price and quality bid but also on some or all competitors' price and quality bids. An important extension would make a theoretical consideration of the scoring auction with such an interdependent scoring rule. Another extension would be to incorporate the uncertainty of buyer's quality bid evaluation. While Che (1993) and our model assume certainty in how a bidder's quality proposal is evaluated by the buyer, in reality quality is evaluated by reviewers and includes noise (Takahashi (2018)). Theoretical analysis of auctions with such uncertainty in quality evaluation is also left to future research.

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A Proofs

A.1 Proof of Lemma 1

Consider the following minimization problem

$$\min_{q \geq \underline{q}} S(C(q,\theta), q)$$

Given an arbitrary q_0, set $s_0 = S(C(q_0,\theta), q_0)$. We can restrict the constraint set to $\left\{q \geq \underline{q} | S(C(q,\theta), q) \leq s_0\right\}$ without affecting the solution. We show that the restricted set is compact. Suppose not. Since the set is closed, it must be unbounded. Then we can take an arbitrarily large q_1 such that $S(C(q_1,\theta), q_1) \leq s_0$, which implies that $P(s_0, q_1) \geq C(q_1, \theta)$. Thus

$$\int_{q_0}^{q_1} \{P_q(s_0,\xi) - C_q(\xi,\theta)\} d\xi = P(s_0,q_1) - C(q_1,\theta) - \{\overbrace{P(s_0,q_0) - C(q_0,\theta)}^{0}\} \geq 0.$$

Because $P(s,q) - C(q,\theta)$ is strictly concave in q, $P_q(s_0,q) < C_q(q,\theta)$ for all $q > q_2$, where $P_q(s_0,q_2) = C_q(q_2,\theta)$. Therefore

$$\int_{q_0}^{q_2} \{P_q(s_0,\xi) - C_q(\xi,\theta)\} d\xi + \int_{q_2}^{q_1} \{P_q(s_0,\xi) - C_q(\xi,\theta)\} d\xi \geq 0. \tag{23}$$

The second term of the left-hand side is negative and has a sufficiently large absolute value as $q_1 \to \infty$, which is a contradiction to inequality (23). By Weierstrass Theorem, the solution to the score minimization exists, and the value is the break-even score.

To show that $z(\cdot)$ is strictly increasing, let $q^z(\theta)$ denote a solution to the above score-minimization problem. Then $z(\theta) = S(C(q^z(\theta),\theta), q^z(\theta))$. Note that $P(z(\theta), q) \leq C(q,\theta)$ for all q (with equality at $q = q^z(\theta)$). Consider $\tilde{\theta} > \theta$. Since $C(q,\theta) < C(q,\tilde{\theta})$, we must have $P(z(\theta), q) < C(q,\tilde{\theta})$ for all q, implying that there is no intersection between $P(z(\theta),\cdot)$ and $C(\cdot,\tilde{\theta})$. Since $P_s(s,q) > 0$ and $P(z(\tilde{\theta}), q^z(\tilde{\theta})) = C(q^z(\tilde{\theta}),\tilde{\theta})$, $z(\tilde{\theta}) > z(\theta)$. □

A.2 Proof of Proposition 2

Note that $q_s^* = 1/C_{qq}$ and $q_\theta^* = -C_{q\theta}/C_{qq}$. By differentiation, we have

$$\frac{\partial \log u(s,\theta)}{\partial s} = \frac{q^*(s,\theta)}{u(s,\theta)}$$

and

$$\begin{aligned}\frac{\partial^2 \log u(s,\theta)}{\partial s \partial \theta} &= \frac{1}{u(s,\theta)^2}\left(q_\theta^*(s,\theta)u(s,\theta) + q^*(s,\theta)C_\theta(q^*(s,\theta),\theta)\right) \\ &= \frac{1}{u(s,\theta)^2}\left(-q_s^*(s,\theta)C_{q\theta}(q^*,\theta)u(s,\theta) + q^*(s,\theta)C_\theta(q^*,\theta)\right).\end{aligned}$$

It is immediate that the log-supermodularity holds if $C_{q\theta} \leq 0$. In what follows, we assume $C_{q\theta} > 0$ and provide two sufficient conditions under which the log-supermodularity condition holds.

Condition 1. Suppose that C_θ/C_q is non-increasing in q. That is, we have

$$C_{q\theta}C_q - C_\theta C_{qq} \leq 0 \Leftrightarrow -\frac{C_{q\theta}}{C_{qq}} \geq -\frac{C_\theta}{C_q}$$

for all q and θ. By evaluating this at $q = q^*(s,\theta)$, we have

$$q_\theta^*(s,\theta) > -\frac{C_\theta(q^*,\theta)}{C_q(q^*,\theta)}. \tag{24}$$

Because $u(s,\theta) \geq 0$ for $s \geq z(\theta)$, we have

$$\begin{aligned}
\frac{\partial^2 \log u(s,\theta)}{\partial s \partial \theta} &= \frac{1}{u(s)^2}\left(q^*_\theta(s)u(s) + q^*(s)C_\theta(q^*(s))\right) \\
&\geq \frac{1}{u(s)^2}\left(-\frac{C_\theta(q^*(s))}{C_q(q^*(s))}u(s) + q^*(s)C_\theta(q^*(s))\right) \\
&= \frac{C_\theta(q^*(s))}{C_q(q^*(s))u(s)^2}\left(q^*(s)C_q(q^*(s)) - (sq^*(s) - C(q^*(s)))\right) \\
&> \frac{q^*(s)C_\theta(q^*(s))}{C_q(q^*(s))u(s)^2}(C_q(q^*(s)) - s) \\
&= 0.
\end{aligned} \tag{25}$$

Note that we omit the parameter θ from the exhibition. The second line is derived from (24). The third line comes from the definition of the indirect payoff $u(s,\theta)$. The strict inequality is due to $C(q^*,\theta) > 0$ under Assumption 1. Finally, the last line comes from the first order condition for the optimal quality $s - C_q(q^*,\theta) = 0$. Thus, the log-supermodularity condition holds.

Condition 2. Fix an arbitrary θ and define a function V of score s by[10]

$$V(s) \equiv -q^*_s(s)C_{q\theta}\left(q^*(s)\right)u(s) + q^*(s)C_\theta\left(q^*(s)\right).$$

What we want to show is $V(s) > 0$ for all $s \geq z(\theta)$. Note that $V(z(\theta)) = q^*C_\theta > 0$ by $u(z(\theta)) = 0$. Hence, it suffices to show that $V(s) = 0 \Rightarrow V'(s) > 0$ for every $s > z(\theta)$.

By differentiation, we have

$$\begin{aligned}
V'(s) &= -q^*_{ss}C_{q\theta}u - (q^*_s)^2C_{qq\theta}u - q^*_sC_{q\theta}q^* + q^*_sC_\theta + q^*q^*_sC_{q\theta} \\
&= q^*_sC_\theta - q^*_{ss}C_{q\theta}u - (q^*_s)^2C_{qq\theta}u.
\end{aligned} \tag{26}$$

Suppose $V(s) = 0 \Leftrightarrow u = q^*C_\theta/q^*_sC_{q\theta}$. By substituting it into (26), we have

$$V'(s)\big|_{V(s)=0} = q^*_sC_\theta - \frac{q^*_{ss}q^*C_\theta}{q^*_s} - \frac{q^*_sC_{qq\theta}q^*C_\theta}{C_{q\theta}}. \tag{27}$$

Note that $q^*_s = 1/C_{qq}$ and $q^*_{ss} = -C_{qqq}/(C_{qq})^3$. By substituting them into (27), we

[10] We omit the fixed parameter θ from the exhibition.

have

$$
\begin{aligned}
V'(s)\big|_{V(s)=0} &= \frac{C_\theta}{C_{qq}} + \frac{C_{qqq} q^* C_\theta}{(C_{qq})^2} - \frac{q^* C_{qq\theta} C_\theta}{C_{qq} C_{q\theta}} \\
&= \frac{C_\theta}{C_{qq}} \left(1 + q^* \left(\frac{C_{qqq}}{C_{qq}} - \frac{C_{qq\theta}}{C_{q\theta}} \right) \right).
\end{aligned} \tag{28}
$$

Because C_θ, $C_{qq} > 0$, we conclude that the log-supermodularity holds if

$$
1 + q \left(\frac{C_{qqq}}{C_{qq}} - \frac{C_{qq\theta}}{C_{q\theta}} \right) > 0
$$

for all q.

If the log-supermodularity condition (10) holds, there exists a monotone pure-strategy Bayesian Nash equilibrium in FPA (Athey, 2001). The equilibrium strategy is symmetric and characterized by the first-order condition as Maskin and Riley (1984, Theorem 2).[11] Suppose that the equilibrium is symmetric and let s^{FS} be the symmetric equilibrium strategy. Suppose that every bidder other than i follows s^{FS}. The interim expected payoff when bidder i makes an equilibrium bid of type τ is

$$
(1 - G(\tau))\, u\left(s^{FS}(\tau), \theta\right).
$$

The first-order condition for the payoff maximization is

$$
-g(\tau) u\left(s^{FS}(\tau), \theta\right) + (s^{FS})'(\tau)\,(1 - G(\tau))\, u_s\left(s^{FS}(\tau), \theta\right) = 0.
$$

Because the first-order condition should hold with $\tau = \theta$, we have

$$
-g(\theta) u\left(s^{FS}(\theta), \theta\right) + (s^{FS})'(\theta)((1 - G(\theta))\, u_s\left(s^{FS}(\theta), \theta\right) = 0,
$$

which is (12). The terminal condition for the differential equation is $u(s^{FS}(\bar{\theta}), \bar{\theta}) = 0$. Thus, $s^{FS}(\bar{\theta}) = z(\bar{\theta})$. Under the log-supermodularity condition (10), the monotonicity of a strategy and the first-order condition are sufficient for the best response. Hence, the strategy s^{FS} characterized by (12) is the symmetric equilibrium. □

[11] Although Maskin and Riley (1984) assume that U is concave, it is not used or necessary to have the equilibrium of FSA. Board (2007, Lemma 3) is an example of a convex payoff function.

A.3 Proof of Corollary 1

The first-order condition of the bidder's problem to

$$\max_{\tilde{\theta}} u(s^{FS}(\tilde{\theta}), \theta)[1 - F(\tilde{\theta})]^{n-1}$$

is given by

$$u_s(s^{FS}(\theta), \theta)[1 - F(\theta)]^{n-1} - u(s^{FS}(\theta), \theta)(n-1)f(\theta)[1 - F(\theta)]^{n-2}\frac{1}{\dot{s}^{FS}(\theta)} = 0,$$

where $\dot{s}^{FS}$ is the derivative of $s^{FS}(\theta)$.[12] Since $u_s > 0$, dividing both sides by u_s yields

$$[1 - F(\theta)]^{n-1} - (s^{FS}(\theta) - k(s^{FS}(\theta), \theta))(n-1)f(\theta)[1 - F(\theta)]^{n-2}\frac{1}{\dot{s}^{FS}(\theta)} = 0.$$

Solving the differential equation gives (14). □

A.4 Proof of Theorem 1

By the first-order condition for the optimal quality $s = C_q(q^*, \theta)$, we have

$$q^*_{ss} = -\frac{C_{qqq}}{(C_{qq})^3}.$$

Hence, the optimal quality function q^* is weakly concave if $C_{qqq} \geq 0$. Let $\theta_{(1)}$ and $\theta_{(2)}$ be the lowest and second lowest order statistics of bidder types. When q^* is weakly concave in s, we have

$$\begin{aligned} E\left[q^*\left(s^{SS}(\theta_{(2)}), \theta_{(1)}\right)\right] &= E_{\theta_{(1)}}\left[E_{\theta_{(2)}}\left[q^*\left(s^{SS}(\theta_{(2)}), \theta_{(1)}\right) \mid \theta_{(2)} > \theta_{(1)}\right]\right] \\ &\leq E\left[q^*\left(E\left[s^{SS}(\theta_{(2)}) \mid \theta_{(2)} > \theta_{(1)}\right], \theta_{(1)}\right)\right] \\ &\leq E\left[q^*\left(s^{FS}(\theta_{(1)}), \theta_{(1)}\right)\right]. \end{aligned}$$

Note that E_X means that we take expectation regarding X. The first inequality is Jensen's inequality. The second inequality comes from Proposition 3. □

[12] The first-order condition is the same as when the bidder's utility function is substituted with the linearized induced utility at $s = s^{FS}(\theta)$:

$$\max_{\tilde{\theta}} u^l(s^{FS}(\tilde{\theta}), \theta)[1 - F(\tilde{\theta})]^{n-1} \equiv \max_{\tilde{\theta}}[u_s(s^{FS}(\theta), \theta)(s^{FS}(\tilde{\theta}) - s^{FS}(\theta)) + u(s^{FS}(\theta), \theta)][1 - F(\tilde{\theta})]^{n-1}.$$

A.5 Proof of Theorem 2

Let $\pi(s, \theta) = sq^*(s, \theta)$ be the optimal price given score s and type θ. Then, by differentiation, we have

$$\pi_{ss}(s, \theta) = sq^*_{ss} + 2q^*_s.$$

Bu substituting $q^*_s = 1/C_{qq}$, $q^*_{ss} = -C_{qqq}/(C_{qq})^3$, and the first order condition $s = C_q$, we have

$$\pi_{ss} = \frac{2(C_{qq})^2 - C_q C_{qqq}}{(C_{qq})^3}.$$

Thus, the optimal price is weakly concave if (16) holds. When π is weakly concave in s, we have the expected price ranking in the same manner with the quality ranking Theorem 1. □

A.6 Proof of Theorem 3

Suppose that $C_{q\theta} = 0$. Then, it is clear that the optimal quality q^* is independent of θ and is denoted by $q^*(s)$. Let π be the optimal price function $\pi(s) = sq^*(s)$. Because q^* is increasing in s, π is also increasing in s. Thus, each price bid correspond to a score bid in the one-to-one sense. That is, for every score s, we have the unique associated price $p = \pi(s)$. We define a payoff function in terms of the price bid $\hat{u}$ as

$$\hat{u}(p, \theta) \equiv u(\pi^{-1}(p), \theta).$$

Abusing notations, the cost function is denoted by $C = C(q) + \theta$, where $C(q)$ is variable cost and θ is the fixed cost.[13] Then, we have

$$\hat{u}(p, \theta) = p - C\left(\frac{p}{\pi^{-1}(p)}\right) - \theta. \tag{29}$$

By differentiation, we have

$$\hat{u}_p = 1 - \left(\frac{p}{\pi^{-1}(p)}\right)' C_q \left(\frac{p}{\pi^{-1}(p)}\right) \tag{30}$$

[13]Because the cost function is increasing in θ (by assumption), it is without loss of generality to define θ by fixed cost.

and

$$\hat{u}_{pp} = -\left(\frac{p}{\pi^{-1}(p)}\right)'' C_q\left(\frac{p}{\pi^{-1}(p)}\right) - \left\{\left(\frac{p}{\pi^{-1}(p)}\right)'\right\}^2 C_{qq}\left(\frac{p}{\pi^{-1}(p)}\right). \tag{31}$$

By differentiation, we have

$$\left(\frac{p}{\pi^{-1}(p)}\right)' = \frac{\pi^{-1}(p) - p(\pi^{-1})'(p)}{(\pi^{-1}(p))^2} = \frac{\pi'(\pi^{-1}(p))\pi^{-1}(p) - p}{\pi'(\pi^{-1}(p))(\pi^{-1}(p))^2} \tag{32}$$

and

$$\begin{aligned}\left(\frac{p}{\pi^{-1}(p)}\right)'' &= \frac{1}{(\pi^{-1})^3}\left[-p(\pi^{-1})''\pi^{-1} - 2(\pi^{-1})'\pi^{-1} + 2p((\pi^{-1})')^2\right] \\ &= \frac{1}{(\pi^{-1})^3}\left[\frac{p\pi^{-1}\pi''}{(\pi')^3} - \frac{2\pi^{-1}}{\pi'} + \frac{2p}{(\pi')^2}\right] \\ &= \frac{1}{(\pi')^3(\pi^{-1})^3}\left[p\pi^{-1}\pi'' + 2p\pi' - 2\pi^{-1}(\pi')^2\right].\end{aligned} \tag{33}$$

Note that by definition, we have $p = \pi(s) = sq^*(s)$, $\pi^{-1}(p) = s$, $\pi'(s) = q^* + sq^*_s$, and $\pi''(s) = sq^*_{ss} + 2q^*_s$. By substituting them into the above, we have

$$\left(\frac{p}{\pi^{-1}(p)}\right)' = \frac{(q^* + sq^*_s)s - sq^*}{(q^* + sq^*_s)s^2} = \frac{q^*_s}{q^* + sq^*_s} \tag{34}$$

and

$$\begin{aligned}\left(\frac{p}{\pi^{-1}(p)}\right)'' &= \frac{1}{(q^* + sq^*_s)^3 s^3}\left[sq^* \cdot s(sq^*_{ss} + 2q^*_s) + 2sq^*(q^* + sq^*_s) - 2s(q^* + sq^*_s)^2\right] \\ &= \frac{q^* q^*_{ss} - 2(q^*_s)^2}{(q^* + sq^*_s)^3}.\end{aligned} \tag{35}$$

By the first-order condition for the optimal quality $s = C_q$, we have

$$\hat{u}_p = 1 - \frac{q^*_s C_q}{q^* + sq^*_s} = \frac{q^* C_{qq}}{q^* C_{qq} + C_q} > 0.$$

Also, we have

$$\begin{aligned}\hat{u}_{pp} &= \frac{2(q^*_s)^2 - q^* q^*_{ss}}{(q^* + sq^*_s)^3} C_q(q^*) - \frac{(q^*_s)^2}{(q^* + sq^*_s)^2} C_{qq}(q^*) \\ &= \frac{1}{(q^* + sq^*_s)^3}\left[2(q^*_s)^2 C_q - q^* q^*_{ss} C_q - (q^*_s)^2 C_q q(q^* + sq^*_s)\right] \\ &= \frac{1}{(q^* + sq^*_s)^3}\left[\frac{2C_q}{(C_{qq})^2} + \frac{qC_q C_{qqq}}{(C_{qq})^3} - \frac{qC_{qq} + C_q}{(C_{qq})^2}\right] \\ &= \frac{C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2}{(q^* + sq^*_s)^3 (C_{qq})^3}.\end{aligned} \tag{36}$$

The third line comes from $q^*_s = 1/C_{qq}$ and $q^*_{ss} = -C_{qqq}/(C_{qq})^3$. Hence, $\hat{u}$ is convex in p if $C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 \geq 0$. Then, the expected price in the FS auction is higher than in the SS auction, which is analogous to Proposition 3 and Maskin and Riley (1984). $\square$

A.7 Proof of Proposition 4

In the price-only auctions, it is clear that bidders choose the minimum quality $\underline{q}$ in equilibrium. Thus, it is immediate by Assumption 1 that the equilibrium quality in FS and SS auctions $q^*(s, \theta) > \underline{q}$, where $s \geq z(\theta)$ is the winning score. The price associated with the break-even score $z(\theta)$ is $\pi(z(\theta), \theta) = C(q^*(z(\theta), \theta), \theta)$. Note that the equilibrium bid in the second-price price-only auction is given by $p^{PO}(\theta) \equiv C(\underline{q}, \theta)$. Let bidders i and j be the lowest and the second lowest bidders, respectively. By Proposition 1, the winner i's price in the SS auction is

$$\begin{aligned} \pi(z(\theta_j), \theta_i) = z(\theta_j) q^*(z(\theta_j), \theta_i) &\geq z(\theta_j) q^*(z(\theta_j), \theta_j) \\ &= \pi(z(\theta_j), \theta_j) \\ &> p^{PO}(\theta_j). \end{aligned}$$

The first inequality comes from $q^*_\theta \leq 0$ and $\theta_j \geq \theta_i$. The last line comes from Assumption 1: $q^*(z(\theta), \theta) > \underline{q}$ and $C(q^*(z(\theta), \theta), \theta) > C(\underline{q}, \theta)$. Thus, the equilibrium price in the SS auction is higher than that in the second-price price-only auction for every type profile. Because the revenue equivalence theorem holds for price-only auctions, the expected price in the FS auction is higher than that in the first-price price-only auction if the conditions presented in Theorems 2 or 3 hold. $\square$

B Sufficient Conditions for the Equilibrium Existence of the FS Auction

In this Appendix, we explore a set of conditions on primitives that guarantees the log-supermodularity condition for general scoring rules. We can restrict the domain to $\{(s,\theta)|u(s,\theta)>0\}$, since otherwise, the score bid is clearly suboptimal for a type θ bidder. Suppose that $u_{s\theta}$ exists. Then the sorting condition holds if and only if

$$\frac{u_{s\theta}(s,\theta)}{u_s(s,\theta)} - \frac{u_\theta(s,\theta)}{u(s,\theta)} > 0. \tag{37}$$

The associated conditions in primitives differ depending on whether the second-step maximization has an interior or a corner solution.

First, we consider the case in which the solution to the second-step maximization is interior, i.e., $q^*(s,\theta) > \underline{q}$ and $P(s,q^*(s,\theta)) < \bar{p}$. Note that, by the envelope theorem, $u_s = P_s$, $u_\theta = -C_\theta$, and $u_{s\theta} = P_{sq}q^*_\theta$. Then, inequality (37) holds if

$$\frac{P_{sq}(s,q)}{P_s(s,q)}\frac{C_{q\theta}(q,\theta)}{P_{qq}(s,q)-C_{qq}(q,\theta)} > \frac{-C_\theta(q,\theta)}{P(s,q)-C(q,\theta)} \tag{38}$$

for all s, q, and θ.

Proposition 8 *The log-supermodularity condition holds if the optimal quality (and price) are not binding for all (s,θ) and*

1. $P_{sq} = 0$,
2. $P_{sq}C_{q\theta} \leq 0$,
3. $P_{sq} > 0$, $C_{q\theta} \geq 0$, P/P_s *weakly increasing in* q*,and* $C_{q\theta}/(C_{qq}-P_{qq}) < C_\theta/C_q$, *or*

Conditions 1 and 2 are immediately given from inequality (38).

In what follows, we provide the proof for condition 3. Generally, the sorting condition is

$$\frac{u(s,\theta)}{u_s(s,\theta)}P_{sq}(s,q^*(s,\theta))q^*_\theta(s,\theta) + C_\theta(q^*(s,\theta),\theta) > 0, \tag{39}$$

where $q^*_\theta(s,\theta) = -C_{q\theta}/[C_{qq}(q^*(s,\theta),\theta) - P_{qq}(s,q^*(s,\theta))]$. We assume that $C_{q\theta}(q,\theta) \geq 0$ and that $P(s,q)/P_s(s,q)$ is weakly increasing in q. Given that $C_{q\theta}(\cdot) \geq 0$, $q_\theta(\cdot) \leq 0$. Let us further assume that

$$C_q(\cdot)\left[-\frac{C_{q\theta}(\cdot)}{C_{qq}(\cdot)-P_{qq}(\cdot)}\right] + C_\theta(\cdot) > 0$$

holds for all q and θ. Then we evaluate this inequality at $q = q^*(s, \theta)$. Recall that the square-bracket term equals $q^*_\theta(s, \theta)$ if $q = q^*(s, \theta)$. Hence, we obtain

$$C_q(q^*(s, \theta), \theta) q^*_\theta(\cdot) + C_\theta(q^*(s, \theta), \theta) > 0. \tag{40}$$

Next, we show that if $P(s, q)/P_s(s, q)$ is weakly increasing in q for all s and q and $P_{sq}(\cdot) \geq 0$, then $[u(s, \theta)/u_s(s, \theta)]P_{sq}(\cdot) < C_q$. First, the condition that P/P_s is weakly increasing in q implies that

$$\frac{d}{dq}\frac{P(s, q)}{P_s(s, q)} = \frac{1}{(P_s(s, q))^2}\left[P_q(s, q)P_s(s, q) - P(s, q)P_{sq}(s, q)\right] \geq 0$$

for all s and q. Given the fact that $P_s > 0$, this inequality is equivalent to

$$\frac{P(s, q)}{P_s(s, q)} P_{sq}(s, q) \leq P_q(s, q) \text{ for all } s \text{ and } q.$$

Then we consider this (weak) inequality, replacing $P(s, q)$ with $P(s, q) - C(q, \theta)$ on the left-hand side. Given that $P_{sq} \geq 0$ and that $C(q, \theta)$ is nonnegative, the inequality implies that

$$\frac{[P(s, q) - C(q, \theta)]P_{sq}(s, q)}{P_s(s, q)} \leq P_q(s, q), \tag{41}$$

for all s and q.

Using expression (41), we now show that $(u/u_s)P_{sq}$ is bounded above by C_q. By definition, we have the following identity:

$$\frac{u(s, \theta)}{u_s(s, \theta)} \equiv \frac{P(s, q^*(s, \theta)) - C(q^*(s, \theta), \theta)}{P_s(s, q^*(s, \theta))}$$

for all s and θ. Multiplying by $P_{sq}(\cdot)$ on both sides gives

$$\frac{u(s, \theta)}{u_s(s, \theta)} P_{sq}(s, q^*(s, \theta)) \equiv \frac{P(s, q^*(s, \theta)) - C(q^*(s, \theta), \theta)}{P_s(s, q^*(s, \theta))} P_{sq}(s, q^*(s, \theta)).$$

Expression (41) suggests that the right-hand side of this identity is less than or equal to $P_q(s, q^*(s, \theta))$. Moreover, if $q = q^*(s, \theta)$, then $P_q(s, q) = C_q(q, \theta)$. Hence, we have

$$\frac{u(s, \theta)}{u_s(s, \theta)} P_{sq}(s, q^*(s, \theta)) \leq C_q(q^*(s, \theta), \theta). \tag{42}$$

Finally, we show that expressions (40) and (42) imply the sorting condition if $P_{sq}(s,q) > 0$. Multiplying by $q^*_\theta(s,\theta) < 0$ on both sides in expression (42) gives

$$\frac{u(s,\theta)}{u_s(s,\theta)} P_{sq}(s, q^*(s,\theta)) q^*_\theta(s,\theta) \geq C_q(q^*(s,\theta),\theta) q^*_\theta(s,\theta).$$

Thus, we bound the left-hand side of expression (39) from below as

$$\begin{aligned}\frac{u(s,\theta)}{u_s(s,\theta)} P_{sq}(s, q^*(s,\theta)) q^*_\theta(s, q^*(s,\theta)) + C_\theta(q^*(s,\theta),\theta)& \\ \geq C_q(q^*(s,\theta),\theta) q^*_\theta(s, q^*(s,\theta)) + C_\theta(q^*(s,\theta),\theta)&.\end{aligned}$$

Based on expression (40), the right-hand side of this inequality is strictly positive. Hence, the sorting condition holds.