Setting Interim Deadlines to Persuade*

Maxim Senkov †   Yiman Sun ‡

September, 2023

Abstract

A principal funds a multistage project and retains the right to cut the funding if it stagnates at some point. An agent wants to convince the principal to fund the project as long as possible, and can design the flow of information about the progress of the project in order to persuade the principal. If the project is sufficiently promising ex ante, then the agent commits to providing only the good news that the project is accomplished. If the project is not promising enough ex ante, the agent persuades the principal to start the funding by committing to provide not only good news but also the bad news that a project milestone has not been reached by an interim deadline. We demonstrate that the outlined structure of optimal information disclosure holds irrespective of the agent's profit share, benefit from the flow of funding, and the common discount rate.

Keywords: dynamic information design, informational incentives, interim deadline, multistage project.

JEL Classification Numbers: D82, D83, G24, 031.

---

*Thanks to Pavel Kocourek, Jan Zápal, Inés Moreno de Barreda, Jeff Ely, Jean Tirole, Péter Esö, Margaret Meyer, Colin Stewart, Francesc Dilmé, Ansgar Walther, Ole Jann, Maxim Ivanov, Egor Starkov, Ludvig Sinander, Maxim Bakhtin, Rastislav Rehák, Vladimír Novák, Arseniy Samsonov, and the audiences at OLIGO 2022, SING17, CMiD2022, EEA-ESEM Congress 2022, and EWMES 2022 for helpful comments. Funding by the Czech Academy of Sciences through the Lumina Quaeruntur fellowship (LQ300852101) is gratefully acknowledged.

†Email: maxim.senkov@eruni.org. European Research University, U Haldy 200/18, 700 30 Ostrava, Czech Republic.

‡Email: yiman.sun@cerge-ei.cz. CERGE-EI, a joint workplace of Center for Economic Research and Graduate Education, Charles University and the Economics Institute of the Czech Academy of Sciences, Politických vězňů 7, P.O. Box 882, 111 21 Prague 1, Czech Republic

1 Introduction

The development of any innovation requires investment of both time and capital, while the outcome of this investment is inherently stochastic. Usually, the investor, being the principal, retains the option to stop funding the innovative project if at some point it proves unsuccessful. It is widely documented that the agent running the project tends to prefer the principal to postpone the stopping of the funding to enjoy either the extra funds or an additional chance to turn her research idea into a success story.[1] In such an agent-principal relationship, the agent's technological expertise and the quality of her innovative idea often allow her to manipulate the principal by designing how and when the outcomes of the research and development process are announced.

Recently, venture capital firms have started to pour billions into startups focused on the development of quantum computers, which are known for their technological complexity and difficulty of construction. The economic viability of quantum computing is questioned by a number of experts; however, the startups promise the investors a completed product in the foreseeable future.[2] For instance, a quantum startup PsiQuantum announced to potential investors that it hopes to develop a commercially-viable quantum computer within five years and managed to raise more than \$200 million in 2019.[3]

This paper studies the implications of the agent's control of information during the progress of a research and development project when the agent and the principal disagree about the timing of when to abandon the research idea. We ask: What is the degree of transparency to which an agent should commit before starting to work on an innovative project? In particular, which terms for self-reporting on the progress of the project should a startup propose while discussing the term sheet with a venture capitalist? As we show, depending on the properties of the project, the startup would strategically choose both the timing for the disclosure of updates on the progress of the project and the type of news it discloses - either good or bad.

We study the investor's dynamic information design problem. The startup controls the information on the progress of the project and has the power to propose the terms for self-reporting on it to the venture capitalist.[4] The startup has an intertemporal commitment power and commits to a dynamic information policy, which can be interpreted as designing the terms of the contract specifying how the information on the progress of the project is disclosed over time as the project unfolds. In return, the investor continuously provides funds and chooses when to stop funding the project.

The project has two stages and evolves stochastically over time toward completion,

[1]Agency conflict in which the agent prefers the principal to postpone abandoning the project that the agent is working on is studied in Admati and Pfleiderer (1994); Gompers (1995); Bergemann and Hege (1998, 2005); Cornelli and Yosha (2003).

[2]"The Quantum Computing Bubble." *Financial Times*, August 25, 2022.

[3]"Bristol Professor's Secretive Quantum Computing Start-Up Raises £179m." *The Telegraph*, November 16, 2019.

[4]We discuss the reasoning behind this assumption in Section 3.3.

conditional on continuous investment in it. The completion of each of the stages of project occurs according to a Poisson process. The completion of the first stage serves as a milestone, such as the development of a prototype, while completion of the second stage achieves the project. The investor gets a lump-sum project completion profit if and only if he stops investing after the project is completed and before an exogenous project completion deadline, and the startup prefers the principal to postpone stopping the funding.[5]

As the investor receives the reward only after a prolonged period of investment, he initially invests without being able to see if the investment is worthwhile. Hence, it is individually rational for the investor to start investing only if he is sufficiently optimistic regarding the future of the project. An important feature of the setting that we consider is that the *information is symmetric at the outset*: not only the investor, but also the startup is unable to find out if the project will bring profit to the investor or not - this can be inferred only as time goes on and some evidence is accumulated. The only tool that the startup has for persuading the investor to start investing is the promise of future reports on the progress of the project.

Clearly, the good news about the completion of the project is valuable to the investor as it helps him to stop investing in a timely manner. Further, as evidence regarding the project accumulates over time, failure to pass the milestone in a reasonable time makes the project unlikely to be accomplished in time - and the investor prefers to stop investing after observing such bad news. When designing the information policy, the startup chooses optimally between the provision of these two types of evidence in order to postpone the investor's stopping decision for as long as possible.

We show that the startup's choice of information policy depends on the ex ante attractiveness of the project for the investor. The attractiveness is captured by the *flow cost-benefit ratio of the project.* Thus, a project is relatively more attractive ex ante to the investor when its flow investment cost is lower, its project completion profit is higher, or the Poisson rate, at which completion of one stage of the project occurs, is higher.

When the project is sufficiently attractive ex ante to the investor, promises to provide information only on the completion of the project serve as a sufficiently strong incentive device to motivate the investor to start the funding at the outset. Further, the future news on the completion of the project does not harm the total expected surplus generated by the interaction of the startup and investor, while the future news on the project being poor decreases the surplus that the startup can potentially extract from the investor. Accordingly, the startup commits to providing only the good news, but not the bad news on the project in the future: it *discloses the completion of the project and postpones the disclosure* in order to ensure the extraction of as much surplus as possible from the investor. In the context of quantum computing, the startup optimally chooses and announces to the venture capitalist the date by which it plans to have a fully developed quantum computer. When the date comes, the startup reports completion if the quantum computer has been completed; if not, the startup reports the completion as soon as it occurs.

[5]We discuss the reasons for the presence of the project completion deadline in Section 3.1.

The situation changes when the project does not look promising to the investor ex ante. In that case, if the startup commits to disclosing only the completion of the project, the investor will not have the sufficient motivation to start investing in it. Thus, the startup extends the information policy to encompass not only the good news but also the bad. As in the case of the promising project, the startup discloses the project's completion and does so without any postponement, thereby fully exploiting its preferred incentive tool. In addition, the startup sets a date at which the bad news is released if the milestone of the project has not yet been reached - this date is *the interim reporting deadline*.

Setting the interim deadline, the startup chooses a deterministic date, which it optimally postpones. As the startup prefers the investor to postpone stopping the funding, it prefers the interim deadline to be at a later expected date. Further, the completion of the stages of project according to a Poisson process makes both the startup and the investor risk-averse with respect to the date of the interim deadline. Thus, the startup prefers to set the interim deadline at a deterministic date and to postpone it as late in time as possible in order to extract all the surplus from the investor. In the context of quantum computing, the startup optimally chooses and announces a fixed date by which it plans to have a prototype of the quantum computer. When the date comes, reporting having the prototype at hand convinces the investor to continue the funding, and reporting not having the prototype leads to termination of the project.

Finally, we demonstrate that the outlined structure of the optimal information disclosure holds for a broad class of preferences of the startup and the investor. We allow for profit-sharing between the startup and the investor, varying degrees of the startup's benefit from the flow of funding, and exponential discounting, and show that the startup prefers not to set any interim deadlines whenever the project is sufficiently promising to the investor. The future disclosure of the completion of the project promises investor profit in exchange for a prolonged investment, while the disclosure of the stagnation of the project at the interim deadline promises investor only saved costs, as further investment stops. Thus, when the project is attractive, the startup can make the funding and the beneficial experimentation relatively longer by setting no interim deadlines.

2 Related literature

Our paper is related to the *literature on dynamic information design*. The closest paper in this strand of literature is by Ely and Szydlowski (2020). Similarly to our paper, they study the optimal persuasion of a receiver facing a lump-sum payoff to incur costly effort for a longer time. In our model, as in theirs, the sender is concerned to satisfy the beginning-of-the-game individual rationality constraint of the receiver when choosing the information policy. Further, the "leading on" information policy in Ely and Szydlowski (2020) has a similar flavor to the "postponed disclosure of completion" information policy in our paper: promises of news on completion of the project serve as an incentive device sufficient to satisfy the receiver's individual rationality constraint.

However, there are several substantial differences between Ely and Szydlowski (2020) and our paper. While in their model the state of the world is static and drawn at $t = 0$, in our model it evolves endogenously over time, given the receiver's investment. As a result, the initial disclosure used in the "moving goalposts" policy in Ely and Szydlowski (2020) cannot provide additional incentives for the receiver in our model. The sender in our model uses another incentive device to incentivize the receiver to opt in at the initial period: she commits to an interim deadline at which she discloses that the first stage of the project is not completed.

Another closely related paper is by Orlov et al. (2020). The main similarity to our paper lies in the sender's incentive to postpone the receiver's irreversible stopping decision. The sender in their paper prefers to backload the information provision, which is in line with the properties of the optimal information policy in our paper. However, there are a number of substantial differences between our papers. In Orlov et al. (2020), the sender does not have the intertemporal commitment power; further, the receiver potentially obtains a non-negative payoff at each moment of time, and thus the sender does not need to persuade the receiver to opt in at $t = 0$.

Ely (2017); Renault et al. (2017); Ball (2019) also analyze dynamic information design models. However, their papers focus on persuading a receiver who chooses an action and receives a payoff at each moment of time, whereas in our paper the receiver takes an irreversible action and receives a lump-sum project completion payoff. Henry and Ottaviani (2019) consider a dynamic Bayesian persuasion model in which, similarly to our model, the receiver needs to take an irreversible decision. However, the incentives of the sender and receiver differ from our model: the receiver wants to match the static state of the world and the sender is concerned with both the receiver's action choice and with the timing of that choice. Basak and Zhou (2020) study dynamic information design in a regime change game. The optimal information policy in their model resembles the interim deadline policy in our model: at a fixed date, the principal sends the recommendation to attack if the regime is substantially weak by that time.

Our paper is also related to the *literature on the dynamic provision of incentives for experimentation* (Bergemann and Hege, 1998, 2005; Curello and Sinander, 2020; Madsen, 2022). The closest papers in this strand of literature are by Green and Taylor (2016) and Wolf (2017). Similarly to our model, both papers consider design of a contract regarding a two-stage project, in which the completion of stages arrives at a Poisson rate. In Green and Taylor (2016), there is no project completion deadline and the quality of the project is known to be good, while in Wolf (2017) the quality of the project is uncertain. In contrast to our paper, both papers focus on a canonical moral-hazard problem and give the power to design the terms of the contract to the investor (principal) rather than the startup (agent). In particular, the contract in Green and Taylor (2016) specifies the terms for the agent's reporting on the completion of the first stage of the project. Similarly to our model, the optimal reporting takes the form of a deterministic interim deadline: at a principal-chosen date, the agent truthfully reports if she has already completed the first stage, which determines the further funding of the project.[6]

3 The model

3.1 The setup

We consider an environment with an agent (she, sender) and a principal (he, receiver). Time is continuous and there is a publicly observable deadline T, $t \in [0, T]$.[7] For each t, *the principal* chooses sequentially to invest in the project ($a_t = 1$) or not ($a_t = 0$). The flow cost of the investment is constant and given by c. The choice of $a_t = 0$ at some t is irreversible and induces the end of the principal-agent relationship.[8]

The assumption that the project needs to be completed in finite time is natural in many economic settings. The main interpretation for T is an expected change in market conditions that renders the project unprofitable. In the context of a research and development project, T could stand for the date at which the competitor's innovative product is expected to enter the market, or the date at which the competitor is expected to get a patent on the competing innovation.

The *state of the world* at time t is captured by the number of stages of the project *completed* by t, x_t, and the project has two stages, $x_t \in \{0, 1, 2\}$. The state process begins at the state $x_0 = 0$ and, conditional on the continuation of the investment by the principal, it increases at a Poisson rate $\lambda > 0$. Information on the number of stages completed is controlled by *the agent*. Thus, when making investment decisions, the principal relies on the information provided by the agent.

The project brings the profit v if and only if the *second stage of the project* has been completed by the time of stopping, and a payoff of 0, otherwise. We assume that all of the profit goes to the principal. This assumption simplifies the exposition without affecting the main results of the paper. We relax this assumption and consider the profit-sharing between the agent and the principal in Section 6.

There is a *conflict of interest* between the agent and the principal as the agent benefits from using the funds provided by the principal for running the project, possibly diverting them for her benefit. Thus, the agent faces the flow payoff of c and wants the principal to postpone his irreversible decision to stop as long as possible.

We assume that the agent has intertemporal commitment power and study the agent's

[6] In a broad sense, our paper also relates to the small strand of theoretical literature on dynamic startup-investor and startup-worker relations under information asymmetry (Kaya, 2020; Ekmekci et al., 2020). However, while these papers focus on the signaling of the type of startup, we study the provision of information by the startup on the progress of the project.

[7] The results for the setting without a deadline are easily obtained by considering $T \to \infty$. They are presented in Appendix C.

[8] The absence of the principal's commitment to an investment policy and the irreversibility of the stopping decision capture the venture capitalist's option to abandon the project, e.g., in the case of its negative net present value.

dynamic information design problem. The agent chooses an information policy, which is a rule that specifies a probability distribution on the exogenously given and sufficiently rich set of messages M for each date and for each past history. We apply the revelation principle and without loss of generality restrict attention to information policies, which provide action recommendations to the principal at each date. Formally, $M = \{0,1\}$. Further, $\hat{a}_t \in \{0,1\}$ denotes the action recommended at date t. $\mathcal{H}^t$ denotes the set of histories up to date t with a typical element $h^t = \left\{\{x_s\}_{s=0}^t, \{\hat{a}_s\}_{s=0}^t, \{a_s\}_{s=0}^t\right\}$, i.e., history includes all realizations of the state process, all recommendations, and all of the principal's action choices up to date t. Given this, a *pure information policy* is given by $\sigma = (\sigma_t)_{t\in[0,T]}$, $\sigma_t : \mathcal{H}^{t-} \times \{0,1,2\} \to \{0,1\}, \forall t$, i.e., at each date t, σ_t maps from history up to, but not including, date t and date t draw of state process, x_t, to an action recommendation. Timing within some date t is such that first x_t is drawn, then $\hat{a}_t$ is determined according to σ, and, finally, a_t is chosen by the principal. A *mixed information policy* is a probability distribution over pure information policies σ. The mixed information policy induces stopping time τ, which is the first date at which $\hat{a}_t = 0$ is drawn according to the mixed information policy.

Given this formalism, it is straightforward to write out the long run payoffs of the agent and the principal. $\mathrm{P}(x_\tau = 2)$ captures the ex ante probability that two stages of the project will be completed by the first date at which the stopping recommendation is drawn, according to the mixed information policy. Further, $\mathrm{E}[\tau]$ captures the $t = 0$ perspective on the expectation of the first date at which stopping is recommended, according to the mixed information policy. The long-run payoff of the agent and the principal are given, respectively, by

$$W(\tau) := \mathrm{E}[\tau]\, c,$$
$$V(\tau) := \mathrm{P}(x_\tau = 2)\, v - \mathrm{E}[\tau]\, c.$$

Throughout the paper, we assume that whenever the principal is indifferent about investing or not, he chooses to invest. Finally, we use the following notational convention: for any two stopping times, S and τ,

$$S \wedge \tau := \min\{S, \tau\},$$
$$S \vee \tau := \max\{S, \tau\}.$$

3.2 Agent's problem

We start with Lemma 1 that presents the agent's problem of choosing the mixed information policy. This choice is formulated in terms of choosing the distribution of the stopping time τ induced by the mixed information policy. Without loss of generality, we formalize the choice of this distribution using the choice of conditional distributions of τ. $F_0(t)$ is the cdf of the stopping time $t \in [0,T]$ when $x_t = 0$, $F_1(t|t_1)$ is the cdf of the stopping time $t \in [t_1, T]$ when $x_s = 0$ for $s \in [0, t_1)$ and $x_s = 1$ for $s \in [t_1, t]$, $F_2(t|t_1, t_2)$ is the cdf of the stopping time $t \in [t_2, T]$ when $x_s = 0$ for $s \in [0, t_1)$, $x_s = 1$ for $s \in [t_1, t_2)$, and $x_s = 2$ for $s \in [t_2, T]$.

The agent's *mixed information policy* is given by the collection of conditional distributions

$$\sigma^{\mu} := \{F_0, F_1, F_2\}.$$

Lemma 1. *The agent's problem can be formulated as*

$$\begin{aligned} &\max_{F_0,F_1,F_2} \{c \cdot \mathrm{E}[\tau]\} \\ &s.t.\ [\mathrm{P}(x_\tau = 2|t < \tau)]\, v - \mathrm{E}[\tau - t|t < \tau]\, c \geq \mathrm{P}(x_t = 2|t < \tau)\, v, \forall t < \tau. \end{aligned} \tag{1}$$

To grasp the intuition behind Lemma (1), it is useful to note that, becasue the mixed information policy is a recommendation policy, the action recommendations generated by this policy have to be obedient to the principal. In other words, at each date and for each possible history the action recommendations drawn from the conditional distributions σ^{μ} have to be optimal for the principal. A useful object for characterizing if the policy σ^{μ} generates obedient action recommendations is given by the principal's continuation value at some interim date t given the mixed information policy. This continuation value depends on the beliefs of the principal.

The principal updates his belief given policy σ^{μ} and assesses the costs and benefits of either further following the recommendations drawn from the policy or deviating from them. The information disclosed by the agent up to date t serves as a source of inference for the principal. First, he forms a belief regarding the number of completed stages of the project by t, conditional on no stopping recommendation being drawn by t, $\mathrm{P}(x_t = n|t < \tau)$. Second, given the information available up to t, he forms a belief regarding the number of completed stages of the project at the random date τ when the stopping recommendation will be drawn in future, $\mathrm{P}(x_\tau = n|t < \tau)$.

The principal's *continuation value at t* given the mixed information policy σ^{μ} is the difference between the expected payoff promised by the policy and the expected payoff from stopping at t, we denote it by $V_t(\tau)$:

$$V_t(\tau) := [\mathrm{P}(x_\tau = 2|t < \tau) - \mathrm{P}(x_t = 2|t < \tau)]\, v - \mathrm{E}[\tau - t|t < \tau]\, c. \tag{2}$$

The system of constraints in the agent's problem (1) ensures that at each date before the stopping recommendation is drawn according to σ^{μ}, the principal's continuation value must be non-negative. As the principal's choice to postpone the stopping of funding is costly, it is natural to interpret the system of constraints in (1) as the *system of the principal's individual rationality constraints.*

3.3 Discussion of assumptions

The main interpretation of the considered dynamic information design problem is the contracting between the agent (startup) and the principal (investor) on the terms of reporting on the completion of stages of the project that are not publicly observed. The terms could take the form of a proposed formal reporting schedule or a schedule of meetings with the investor. Non-observability of the stage completions stems from the fact that, while the technology is being developed in the lab, the principal either does not have sufficient expertise to assess the progress or the full access to the lab.

We assume that the principal does not have the power to propose the terms for reporting to the agent and, e.g., make her fully disclose the progress achieved in the lab. The most natural interpretation of such an asymmetry in the bargaining power is the asymmetry in the market for private equity: there are sufficiently many investors willing to invest in a particular technology or sufficiently few startups working on the technology.[9] For instance, investors' interest in quantum computing has grown markedly in recent years, while there are reports of a shortage of human capital in this industry.[10][11] Another example is the communication software industry, which has recently experienced increased investment activity.[12]

As the agent enjoys the power of full control over the information on the progress of the project, she is completely free to offer what is disclosed and when. In particular, the contract between the agent and the principal can specify that the completion of the second stage of the project is disclosed with a delay rather than immediately. The agent who has an advantage in expertise over the principal can rationalize such a condition by saying that before the success is reported to the principal, it is worth re-checking the data, which takes time.

Even though the principal can not dictate to the agent which information and how she should disclose, the principal can potentially hire an external monitor who would visit the lab and prepare an additional report on the progress of the project. In that case, the contract signed between the agent and the principal will account for both free information that the agent promised to provide and additional costly information which the principal obtains with the help of a monitor. In the baseline version of the model, we assume that the principal can not use the help of a monitor. This can be rationalized by the shortage of experts in the field, which makes hiring a monitor prohibitively costly. Alternative interpretation is that the agent restricts the principal's access to additional information on the progress of the project by stating that a potential information leak would put the technology being developed at risk.[13]

The information policy relies upon the agent's commitment power, which holds not

[9]In the alternative interpretation of the model, contracting concerns internal corporate research and development and takes place between the leading researcher and the headquarters of a company. The leading researcher's bargaining power in proposing the terms for disclosure again stems from the market asymmetry: the specialists having the desired level of expertise might be in a short supply.

[10]"The Quantum Computing Bubble." *Financial Times*, August 25, 2022.

[11]"Quantum Computing Funding Remains Strong, but Talent Gap Raises Concern", a report by McKinsey Digital, https://www.mckinsey.com/business-functions/mckinsey-digital/our-insights/quantum-computing-funding-remains-strong-but-talent-gap-raises-concern/.

[12]"This Is Insanity: Start-Ups End Year in a Deal Frenzy." *Best Daily Times*, December 07, 2020.

[13]In particular, this rationale was used to restrict the investors' access to information on the progress of the project in the case of Theranos, see "What Red Flags? Elizabeth Holmes Trial Exposes Investors' Carelessness." *The New York Times*, November 04, 2021.

only within each date but also between the dates. The agent's commitment within each date follows from prohibitively high legal costs of cooking up the evidence. The agent's intertemporal commitment stems from the rigidity of terms and form of reporting fixed in the contract that the agent and the principal sign at $t = 0$.

4 No-information and full-information benchmarks

4.1 No-information benchmark

First, we consider the simple case when the information policy is given by σ^{NI}: the same message m is sent for all histories $h(t)$ and all dates t. Thus, the agent provides no information regarding the progress of the project. As we demonstrate, in this case the principal starts investing in the project if and only if it is sufficiently promising for the principal from the ex ante perspective and invests until a deterministic interior date.

As no news arrives, the principal bases his decision about when to stop investing on his unconditional belief regarding the completion of the second stage of the project. We denote the unconditional belief that n stages of the project were completed by t, by $p_n(t)$, i.e., $p_n(t) := \mathrm{P}(x_t = n)$. The state of the world is fully determined by $p(t)$ given by

$$\begin{aligned} p_0(t) &= e^{-\lambda t}, \\ p_1(t) &= \lambda t e^{-\lambda t}, \\ p_2(t) &= 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}. \end{aligned}$$

The principal's sequential choice of $a_t \in \{0, 1\}$ can be restated equivalently as the choice of deterministic stopping time $S^{NI} \in [0, T]$ chosen at $t = 0$.[14] Given the principal's continuous investment, the probability of completion of the second stage of the project, $p_2(t)$, increases monotonously over time, making obtaining the payoff v more likely. However, postponing the stopping is costly.

To decide on S^{NI}, the principal trades off the flow benefits and flow costs of postponing the stopping decision, while keeping the individual rationality constraint in mind. The flow cost of postponing the stopping for Δ_t is given by $c \cdot \Delta_t$ and the flow benefit is given by $v \cdot p_1(t) \lambda \Delta_t$.[15] Thus, the necessary condition for the principal's stopping at some interior moment of time $(0 < S < T)$ is given by

$$v \cdot p_1(S) \lambda = c. \tag{3}$$

Let

$$\kappa := \frac{c}{v\lambda},$$

[14]Note that the dynamic belief system that he faces is deterministic in a sense of being fully specified from $t = 0$ perspective.

[15]To observe this, note that the probability of the completing both the first and second stages within a very short time Δ_t is negligibly small; thus, during some Δ_t, the principal receives the project completion payoff v iff the first stage has already been completed.

the ratio of the flow cost of investment c to the gross project payoff v normalized using λ, the rate at which a project stage is completed in expectation. The intuitive interpretation of κ is the *flow cost-benefit ratio of the project.* κ is an inverse measure of how ex ante promising the project is for the principal. (3) is equivalently given by[16]

$$\underbrace{p_1(S)}_{\text{flow benefit of waiting}} = \underbrace{\kappa}_{\text{flow cost of waiting}} \tag{4}$$

and presented graphically in Figure 1. As the state process transitions monotonously from 0 to 1 and then to 2, $p_1(t)$ first increases and after some time starts to decrease. Thus, the principal has two candidate interior stopping times satisfying (4), $\bar{S}$ and $\bar{S}^{NI}$. The principal prefers to postpone stopping to $\bar{S}^{NI}$, as during $\left(\bar{S}, \bar{S}^{NI}\right)$ the flow benefits are higher than the flow costs.

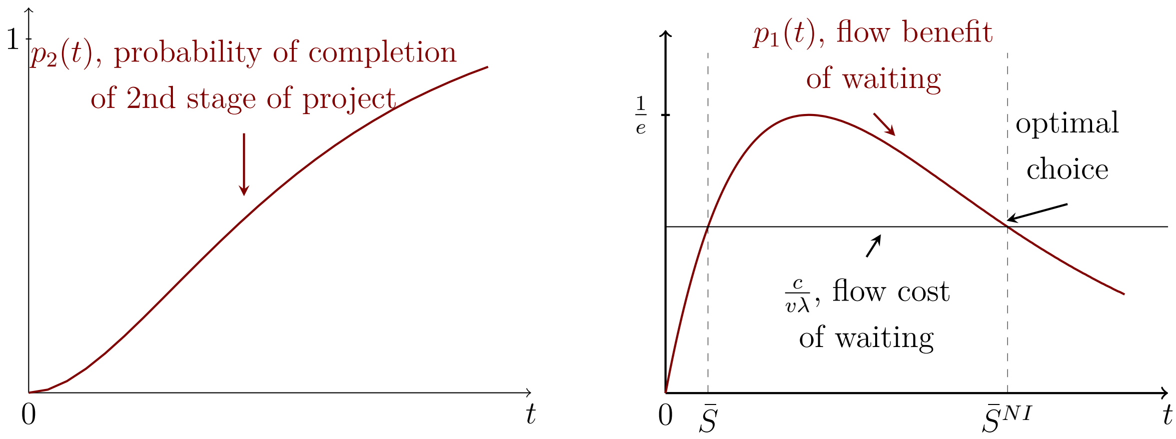

Figure 1: Principal's choice under no information:
left plot: postponing stopping increases the chance of getting a project payoff v;
right plot: principal trades off costs and benefits and optimally stops at $\bar{S}^{NI}$.

The forward-looking principal can guarantee himself a payoff of 0 if he does not start investing at $t = 0$. Thus, he will choose to start investing at $t = 0$ only if his flow gains accumulated up to $T \wedge \bar{S}^{NI}$ are larger than his flow losses, and his expected payoff is given by

$$V^{NI} := \max\left\{0, \int_0^{T\wedge\bar{S}^{NI}} \left(v \cdot p_1(s)\lambda - c\right) ds\right\}. \tag{5}$$

Geometrically, the integral in (5) represents the difference between the shaded areas in Figure 2 that correspond to the accumulated gains and losses. The principal starts investing at $t = 0$ if, given T and λ, the normalized cost-benefit ratio κ is low enough, so that the shaded area of the accumulated gains is at least as large as that of the accumulated losses. We denote such a cutoff value of κ by $\kappa^{NI}(T, \lambda)$ and summarize the principal's choice without information in Lemma 2.

[16]Here we WLOG express the flow benefits and flow costs of investing for the principal in different units of measurement.

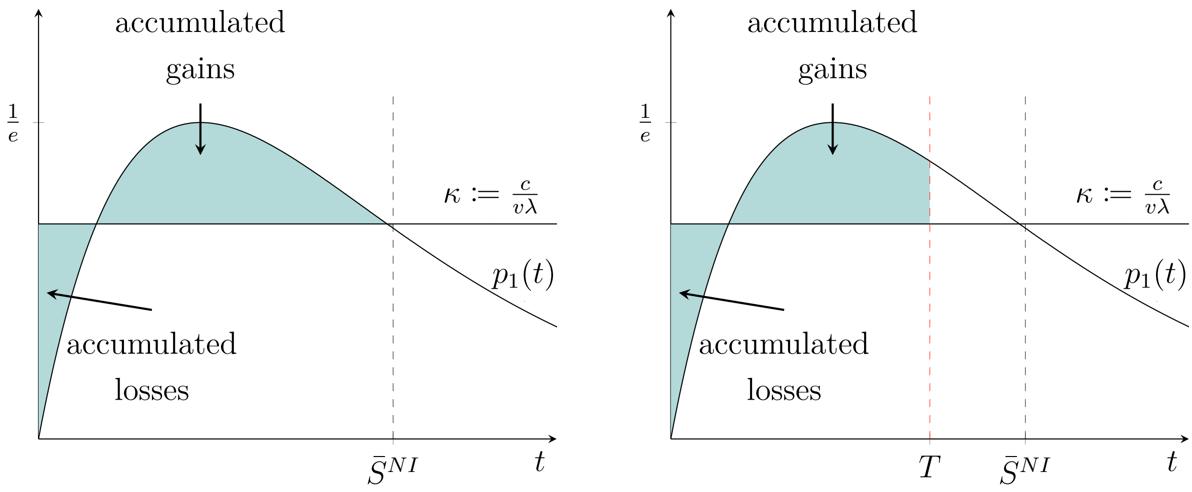

Figure 2: Principal's choice to start investing at $t = 0$ or not under no information:
left plot: $T > \bar{S}^{NI}$; the project deadline is distant and decision-irrelevant;
right plot: $T \leq \bar{S}^{NI}$; the project deadline is close, which leads to lower expected benefits of investing.
In both plots the expected accumulated gains are higher than the losses, so the principal starts to invest at $t = 0$.

Lemma 2. *Assume no information regarding the progress of the project arrives over time. Denote the time at which the principal stops investing by S^{NI}. If $\kappa > \kappa^{NI}(T, \lambda)$, then the principal does not start investing in the project, i.e., $S^{NI} = 0$. If $\kappa \leq \kappa^{NI}(T, \lambda)$, then the principal's choice of stopping time is given by*

$$S^{NI} = \begin{cases} \bar{S}^{NI}, & \text{if } \frac{1}{\lambda} \leq T \text{ and } \kappa \geq e^{-\lambda T}\lambda T \\ T, & \text{otherwise} \end{cases}, \tag{6}$$

the closed-form expressions for $\bar{S}^{NI}$ and $\kappa^{NI}(T, \lambda)$ are presented in the proof in Appendix B.

4.2 Full-information benchmark

Here, we consider the case in which the information policy is given by σ^{FI}: $M = \{m_0, m_1, m_2\}$ and the message m_n is sent for all t such that $x_t = n$, $n \in \{0, 1, 2\}$. Thus, the principal fully observes the progress of the project at each t. We characterize the cutoff level of the cost-benefit ratio below which the principal is willing to start investing. Further, we show that the principal chooses to stop when no stages of the project are completed and the project completion deadline T is sufficiently close.

At each t, the principal uses the information on the number of stages completed to decide either to stop investing or to postpone stopping. News of the completion of the second stage of the project causes the principal stop immediately, so that he immediately receives the project payoff v and stops incurring the costs of further investment. If only the first stage of the project is completed, the principal faces the following trade-off. The instantaneous probability that the second stage will be completed during Δ_t is given by $\lambda\Delta_t$. Thus, the expected benefit of postponing the stopping for Δ_t is given by

$$v \cdot \lambda\Delta_t + V^{FI}_{t+\Delta_t|1} \cdot (1 - \lambda\Delta_t)\,, \tag{7}$$

where $V^{FI}_{t|1}$ is the continuation value of the principal at time t under full information and conditional on $x_t = 1$. Meanwhile, the cost of postponing the stopping is given by $c \cdot \Delta_t$. If $\kappa < 1$, then $v \cdot \lambda\Delta_t > c \cdot \Delta_t$. Further, it can be shown that in this case $V^{FI}_{t|1} > 0, \forall t$.[17] Thus, the principal who knows that the first stage of the project has already been completed invests either until the second stage is complete or until the project deadline T is reached. If $\kappa > 1$, then the expected benefit (7) is smaller than the cost $c \cdot \Delta_t$ for all $t < T$, which implies that the principal chooses not to start investing at $t = 0$ under full information. To rule out this trivial case, we assume $\kappa \leq 1$.

Assumption 1. $\kappa \leq 1$.

We now consider the case in which the principal knows that the first stage has not yet been completed. The principal's trade-off with respect to the stopping decision is now more complex. Postponing the stopping for Δ_t leads to completion of the first stage of the project with instantaneous probability $\lambda\Delta_t$. Completion of the first stage of the project at some t implies that the principal receives $V^{FI}_{t|1}$ (rather than v). Thus, the expected benefit of postponing the stopping for Δ_t is now given by

$$V^{FI}_{t|1} \cdot \lambda\Delta_t + V^{FI}_{t+\Delta_t|0} \cdot (1 - \lambda\Delta_t)\,, \tag{8}$$

where $V^{FI}_{t|0}$ is continuation value of the principal at time t under full information and conditional on $x_t = 0$. The cost of postponing the stopping is, as before, given by $c \cdot \Delta_t$. In contrast to (7), where the principal obtains the completion payoff v, which is constant over time, now the principal obtains the continuation value, $V^{FI}_{t|1}$, which shrinks over time because there is less time left to complete the second stage before T. It turns out that there exists a date $t < T$ sufficiently close to the final date T such that the optimal policy prescribes the principal to stop at such date if the first stage of the project is still incomplete. We denote this date by S^P_0. The economic interpretation of S^P_0 is that it is *the interim deadline* that the principal sets for the project. Further, if the first stage is completed by S^P_0, then the optimal policy prescribes the principal to continue until either the second stage is completed or T is reached.

Finally, given the optimal policy, the principal chooses to opt out of investing at $t = 0$ (i.e., $S^P_0 = 0$) if the cost-benefit ratio of the project, κ, is sufficiently high. We denote the upper bound for κ such that the principal starts investing at $t = 0$ by $\kappa^{FI}(T, \lambda)$. Intuitively, $\kappa^{FI}(T, \lambda) > \kappa^{NI}(T, \lambda)$: whenever the principal is willing to start investing under no information, he is also willing to start under full information. We summarize the principal's choice under full information in Lemma 3.

[17] See the derivation in the proof of Lemma 3 in the Appendix.

Lemma 3. *Assume that the progress of the project is fully observable at each moment in time. If* $\kappa > \kappa^{FI}(T,\lambda)$*, where* $\kappa^{FI}(T,\lambda) > \kappa^{NI}(T,\lambda)$*, then the principal does not start investing in the project. If* $\kappa \leq \kappa^{FI}(T,\lambda)$*, the principal invests either until the random date at which the second stage of the project is completed,* $t = \tau_2$*, or until the interim deadline,* $t = S_0^P$*, at which he stops if the first stage has not yet been completed. Formally, the time at which the principal stops investing is a random variable* τ^{FI} *given by:*

$$\tau^{FI} = \begin{cases} \tau_2 \wedge T, & \text{if } x_{S_0^P} \neq 0 \\ S_0^P, & \text{otherwise} \end{cases},$$

where $S_0^P = T + \frac{1}{\lambda}\log\left(\frac{1-2\kappa}{1-\kappa}\right)$ *and the expression for* $\kappa^{FI}(T,\lambda)$ *is presented in the proof in Appendix B.*

Assume now that the agent chooses which information to provide to the principal. As for $\kappa > \kappa^{FI}(T,\lambda)$ the principal is not willing to start investing even under full information, there is no way in which the agent can strategically conceal the information to her benefit. In Section 5, We assume $\kappa \leq \kappa^{FI}(T,\lambda)$ and analyze how the agent can strategically provide information on the progress of the project and extract the principal's surplus.

5 Agent's choice of information policy

In this Section, we present how the agent's choice of information policy changes with the ex ante attractiveness of the project, which is captured by the cost-benefit ratio κ. In Section 5.1, we provide the big picture of the solution to the agent's problem. In Sections 5.2-5.3, we introduce the results formally and discuss the economic mechanisms that determine the outlined structure of the optimal information policy.

5.1 The structure of optimal information disclosure

The structure of optimal information disclosure is formally established in Propositions 1 and 2 (see Sections 5.2 and 5.3) as the solution to the agent's problem (1). In this Section, we put the results of these two Propositions together to present an overview of optimal information disclosure. It follows the simple and intuitive pattern. There exist cost-benefit ratio cutoffs $\kappa^{ND}(T,\lambda), \kappa^{ND}(T,\lambda) < \kappa^{NI}(T,\lambda)$, and $\tilde{\kappa}(T,\lambda), \kappa^{NI}(T,\lambda) < \tilde{\kappa}(T,\lambda) < \kappa^{FI}(T,\lambda)$. $\kappa^{ND}(T,\lambda)$ is defined as follows: for any $\kappa \leq \kappa^{ND}(T,\lambda)$, the principal invests until T in the no-information benchmark. $\tilde{\kappa}(T,\lambda)$ is defined in Lemma 4. Depending on the cost-benefit ratio of the project, the optimal information policy has the following form:

1. when $\kappa \leq \kappa^{ND}(T,\lambda)$, the agent provides no information and the principal invests until T;
2. when $\kappa^{ND}(T,\lambda) < \kappa \leq \tilde{\kappa}(T,\lambda)$, the agent discloses only the completion of the second stage of the project and does that with the postponement;

3. when $\tilde{\kappa}(T,\lambda) < \kappa < \kappa^{FI}(T,\lambda)$, the agent immediately discloses the completion of the second stage of the project whenever it occurs and specifies a deterministic interim deadline, at which it discloses if the first stage is already completed;
4. when $\kappa \geq \kappa^{FI}(T,\lambda)$, the agent provides no information as the principal's long-run payoff is non-positive even under full information.

Figure 3 illustrates the optimal structure of information disclosure and presents the partition of the cost-benefit ratio space based on the corresponding forms of the optimal information policy.

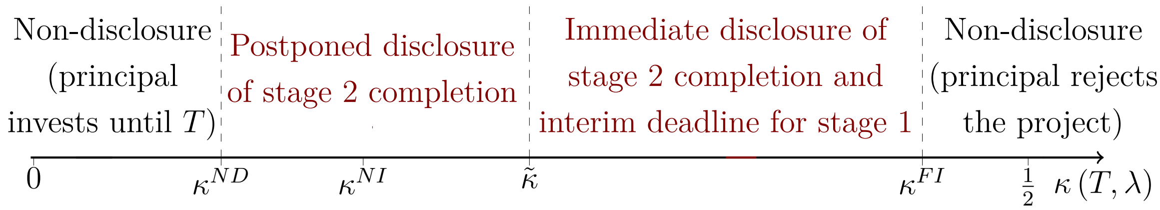

Figure 3: Comparative statics of the form of optimal information policy with respect to the cost-benefit ratio of the project, $\kappa(T,\lambda)$.

The lower is the value of cost-benefit ratio, the higher is ex ante attractiveness of the project to the principal. First, for $\kappa \leq \kappa^{ND}(T,\lambda)$, the project is so attractive that the principal is willing to keep investing until the project deadline T even in the no-information benchmark. Thus, there is no need to disclose any information. For the higher values of κ, there emerges a room for strategic disclosure, and the higher is the value of κ (i.e., the lower is the ex ante attractiveness of the project), the more information the agent has to disclose to incentivize the principal. For $\kappa \geq \kappa^{FI}(T,\lambda)$, the project gets so unattractive that the principal can not strictly benefit from investing even in the full-information benchmark. In this extreme case, the agent chooses not to disclose any information.

From Figure 3, one can see which additional pieces of information the agent chooses to disclose and when she chooses to discloses them as κ gets higher and higher. When $\kappa \in (\kappa^{ND}(T,\lambda), \tilde{\kappa}(T,\lambda)]$, the agent discloses only the completion of the second stage of the project and does not promise any information on the completion of the first stage of the project. Further, as κ increases from $\kappa^{ND}(T,\lambda)$ to $\tilde{\kappa}(T,\lambda)$, the agent adjusts the timing of the disclosure: she postpones the disclosure of the second stage completion less and less and discloses immediately for $\tilde{\kappa}(T,\lambda)$. For $\kappa \in (\tilde{\kappa}(T,\lambda), \kappa^{FI}(T,\lambda))$, the agent not only discloses the completion of the second stage of the project immediately, but also provides information on the completion of the first stage at the interim deadline that she optimally chooses.

Throughout the analysis, we maintain the following technical assumption:

Assumption 2. $e^{\lambda T} - \lambda T(\lambda T + 1) > 1$.

This assumption imposes a lower bound on T and rules out the case in which T is so low that whenever the principal is willing to start investing in the no-information benchmark, he invests until T. As a result, $\kappa^{ND} < \kappa^{NI}$, which allows for a richer comparative statics analysis in Section 5.2. We formally demonstrate the implications of this assumption for the optimal structure of information disclosure in Appendix B.

In Sections 5.2 and 5.3, we formally establish the comparative statics results presented in Figure 3. We start the discussion from the optimal information policy under $\kappa \in (\kappa^{ND}(T,\lambda), \tilde{\kappa}(T,\lambda)]$. The trivial case of non-disclosure under $\kappa \leq \kappa^{ND}(T,\lambda)$ is analysed in the discussion of Assumption 2 in Appendix B.

5.2 Postponed disclosure of project completion

In this Section, we restrict attention to $\kappa \in (\kappa^{ND}(T,\lambda), \tilde{\kappa}(T,\lambda)]$ and explain why the optimal information policy is such that the agent discloses only the completion of the project and does this with the postponement. The agent's problem is complex, and thus we solve it in steps. First, we characterize the information policy, which solves the *relaxed* version of (1) with the principal's individual rationality constraints only for some initial periods. Second, we demonstrate that there exists an information policy solving the *relaxed* agent's problem and satisfying the *full* system of the principal's individual rationality constraints in (1).

5.2.1 Solution to the agent's relaxed problem

In this Section, we consider the agent's relaxed problem and discuss its solution. This sheds light on the technical intuition behind the key properties of the optimal information policy. The agent's relaxed problem for the parametric case of $\kappa \in (\kappa^{ND}(T,\lambda), \kappa^{NI}(T,\lambda)]$ is given by (1) with the *principal's individual rationality constraint only for* $t \in [0, \bar{S}^{NI}]$. The agent's relaxed problem for the parametric case of $\kappa \in (\kappa^{NI}(T,\lambda), \tilde{\kappa}(T,\lambda)]$ is given by (1) with the *principal's individual rationality constraint only for* $t = 0$.

Consider $W(\tau)$, the agent's long-run payoff given some mixed information policy, represented by a stopping time τ. This can be restated equivalently as follows:

$$\begin{aligned} W(\tau) &= [W(\tau) + V(\tau)] - V(\tau) \\ &= \underbrace{\mathrm{P}(x_\tau = 2)\, v}_{\text{total surplus}} - \underbrace{[\mathrm{P}(x_\tau = 2)\, v - \mathrm{E}[\tau]\, c]}_{\text{principal's surplus}}. \end{aligned} \tag{9}$$

The solution to the agent's relaxed problem for both considered parametric cases follows a simple idea: the optimal information policy *ensures that the total surplus is maximal and that the principal's surplus is minimal.* Consider τ such that the stopping occurs after the completion of the second stage of the project, unless the project deadline T was hit, i.e., the policy satisfies the condition $\tau \geq \tau_2 \wedge T$. Such a policy leads to

$$\mathrm{P}(x_\tau = 2) = \mathrm{P}(x_T = 2). \tag{10}$$

Given a mixed information policy, represented by τ, satisfying (10), if τ is individually rational for the principal at date $t = 0$ then the total surplus generated achieves its upper bound and is given by $\mathrm{P}(x_T = 2)\, v$, which depends on the exogenously given project deadline T and the profit v. However, the stopping only after the second stage completion is not individually rational for the principal at $t = 0$ when the cost of funding is sufficiently high, the profit is sufficiently low, or the expected time until a project stage completion is sufficiently high.

Lemma 4 elaborates on the cost-benefit ratio cutoff value $\tilde{\kappa}(T, \lambda)$: it distinguishes the case in which stopping only after the second stage completion is individually rational at $t = 0$ from the case in which it is not. Based on this partition, when $\kappa \in (\kappa^{ND}, \tilde{\kappa}(T, \lambda)]$, we call the project *ex ante promising* for the principal.

Lemma 4. *For each* (T, λ) *there exists* $\tilde{\kappa}(T, \lambda)$, $\kappa^{NI}(T, \lambda) < \tilde{\kappa}(T, \lambda) < \kappa^{FI}(T, \lambda)$, *such that if* $\kappa \leq \tilde{\kappa}(T, \lambda)$ $(\kappa > \tilde{\kappa}(T, \lambda))$ *then an information policy, represented by* τ, *in which stopping after* $\tau_2 \wedge T$ *happens with probability one is individually rational at* $t = 0$ *(not individually rational at* $t = 0$*) for the principal.*

For $\kappa \in (\kappa^{ND}(T, \lambda), \tilde{\kappa}(T, \lambda)]$, the stopping time $\tau \geq \tau_2 \wedge T$ is individually rational for the principal at $t = 0$, and it maximizes the total surplus. In addition to choosing $\tau \geq \tau_2 \wedge T$, it is optimal for the agent to choose the stopping time with a higher expected date of stopping the funding to extract all the principal's surplus subject to his individual rationality constraints. For $\kappa \in (\kappa^{NI}(T, \lambda), \tilde{\kappa}(T, \lambda)]$, the agent chooses such τ that the principal's individual rationality constraint at $t = 0$ is binding. As a result, $V(\tau) = V^{NI}$, i.e., the principal gets his no-information benchmark payoff given by 0.

For $\kappa \in (\kappa^{ND}(T, \lambda), \kappa^{NI}(T, \lambda)]$, as in the no-information benchmark the principal invests until $\bar{S}^{NI}$ with certainty, the agent chooses the information policy as to postpone the start of information provision at least until $\bar{S}^{NI}$. Further, the agent chooses τ with a higher expected date of stopping so that the principal's individual rationality constraint at $t = \bar{S}^{NI}$ is binding. The absence of stopping until at least $\bar{S}^{NI}$ and the fact that individual rationality constraint binds at $t = \bar{S}^{NI}$ taken together imply that $V(\tau) = V^{NI}$, i.e., from $t = 0$ perspective, the principal gets her no-information benchmark payoff, which is non-negative and given by (5).

The next Lemma summarizes the *necessary* conditions for an information policy to solve the agent's relaxed problem when the project is promising. These conditions are shared both by the relaxed problem formulated for the case of $\kappa \in (\kappa^{ND}(T, \lambda), \kappa^{NI}(T, \lambda)]$ and the relaxed problem formulated for the case of $\kappa \in (\kappa^{NI}(T, \lambda), \tilde{\kappa}(T, \lambda)]$. The conditions that are *both necessary and sufficient* for an information policy to solve the agent's relaxed problem are presented in the Proof of Lemma 5.

Lemma 5. *Assume* $\kappa \in (\kappa^{ND}, \tilde{\kappa}(T, \lambda)]$. *If an information policy, represented by* τ, *solves agent's relaxed problem, then*

1. *with probability one, stopping occurs after* $\tau_2 \wedge T$;

2. $V(\tau) = V^{NI}$, *where* V^{NI} *is the principal's expected payoff in the no-information benchmark, given by (5).*

5.2.2 Optimal information policy

In this Section, we show that there exists an information policy that both solves the agent's relaxed problem and satisfies the full system of the individual rationality constraints. Given this, as Lemma 5 describes the solution to the relaxed problem, it also sheds light on the properties of the optimal information policy for the case of a promising project.

As the stopping time τ is induced by a direct recommendation policy σ^{μ}, it is clear from Lemma 5 that the *optimal information policy* has to satisfy the following conditions. *First*, whenever the agent recommends the principal to stop, the second stage of the project is already completed. *Second*, the recommendation to stop is postponed so that the principal's individual rationality constraint is binding, which manifests in $V(\tau) = V^{NI}$. The first condition presents the key feature of the optimal information policy for the case of promising project: the agent discloses the completion of the second stage of the project, but *stays silent regarding the completion of the first stage of the project*. The intuition behind the agent's choice is simple: a recommendation to stop when no stages of the project are completed and the project deadline T is close does indeed incentivize the principal; however, it also reduces the total surplus generated that can be extracted via the agent's control of information. Meanwhile, the recommendation to stop when the two stages of the project are completed incentivizes the principal without reducing the total surplus generated. When $\kappa \leq \tilde{\kappa}(T, \lambda)$, a partially informative policy that discloses only the completion of the second stage provides sufficient incentives to the principal, and thus the agent uses it.[18]

We proceed with obtaining an information policy that not only satisfies the conditions in Lemma 5 and solves the relaxed problem, but also satisfies the full system of the principal's individual rationality constraints in Lemma 1. Ensuring both is non-trivial. For instance, consider a policy solving the agent's relaxed problem and assume it recommends to continue for $t \in [0, S^*)$, then at S^* recommends stopping if the second stage is already completed, but recommends to continue at all the subsequent dates $t \in (S^*, T]$. A no stopping recommendation drawn at S^* reveals that the state is either 0 or 1. Clearly, after sufficient time passes after S^*, the principal would attach a high probability to the second stage already being completed and would potentially be tempted to deviate from the recommendation to continue.[19] However, the optimal policy satisfying the full system of constraints exists. We present it in Proposition 1.

Proposition 1. *Assume $\kappa \in (\kappa^{ND}(T, \lambda), \tilde{\kappa}(T, \lambda)]$. The optimal information policy is a direct recommendation mechanism that does not provide a recommendation to stop during $t \in [0, S^*)$. At $t = S^*$, if the second stage of the project is already completed, then the mechanism recommends the principal to stop. If the second stage of the project is not yet completed, then the mechanism recommends the principal to stop at the moment of its*

[18] The "leading on" information policy in Ely and Szydlowski (2020) is similar: the only information that the policy provides is that the task is already completed and, thus, it is time to stop investing.

[19] In other words, $V_t(\tau)$ drifts down over time and can get negative at some date.

completion $t = \tau_2$. Corresponding optimal information policy is

$$\tau = S^* \vee (\tau_2 \wedge T),$$

where S^* is a deterministic date chosen such that $V(\tau) = V^{NI}$, i.e., the respective constraint in the system of principal's individual rationality constraints is binding.

The recommendation mechanism starting from S^* generates recommendations to stop if the second stage is completed. As the recommendation to stop comes immediately at the completion of the second stage for all $t > S^*$, hearing no recommendation to stop reveals that the state is either 0 or 1. Further, as time goes on, the principal attaches a higher and higher probability to the state being 1, which ensures obedience to the recommendation to continue at each date. Further, the start of information provision S^* is sufficiently postponed to ensure that the principal's individual rationality constraint is binding either at $t = \bar{S}^{NI}$ or at $t = 0$.

The choice of S^* is driven by extraction of the principal's surplus and depends on κ in an intuitive way. First, consider the case $\kappa \in (\kappa^{ND}, \kappa^{NI}(T, \lambda)]$, the principal is willing to start investing and invests until $t = \bar{S}^{NI}$ in the no-information benchmark. The agent's optimal choice is to set $S^* > \bar{S}^{NI}$. Given such an information policy, the principal does not stop at $\bar{S}^{NI}$, the date of stopping in the no-information benchmark, and with probability one continues to invest during $t \in [\bar{S}^{NI}, S^*)$ even though the mechanism provides absolutely no information for all $t < S^*$. This is driven by the fact that the expected benefit from stopping at some future date $t \in [S^*, T]$ and obtaining the project payoff v with certainty compensates the flow losses of investing during $t \in [\bar{S}^{NI}, S^*)$.[20] Further, the agent sufficiently postpones S^* to ensure that she extracts the principal's surplus and the principal gets precisely $V^{NI} \geq 0$.

In the case $\kappa \in (\kappa^{NI}(T, \lambda), \tilde{\kappa}(T, \lambda)]$, the principal is not willing to start in the no-information benchmark as his expected payoff from investing is negative. Thus, the agent chooses S^* to guarantee that the principal gets his reservation value $V^{NI} = 0$ and is thus willing to start investing at $t = 0$. The value of S^* is relatively lower as compared to the previous case: as the project is less attractive, to provide the principal sufficient incentives, the agent needs to start the information provision regarding the completion of the project earlier.

Finally, there exist many information policies that both solve the agent's relaxed problem and satisfy the full system of constraints in (1). This constitutes an important advantage for the agent: she can choose an optimal policy that is easier to implement from the real-world perspective, depending on the particular environment. In the optimal mechanism from Proposition 1, the recommendation to stop at some date t depends only on the current state of the world x_t. In an alternative delayed disclosure mechanism, the recommendation

[20] Similarly to the "leading on" information policy in Ely and Szydlowski (2020), the promises of future disclosure of the completion of the project are used as a "carrot" to make the receiver continue investing beyond the point at which he stops in the no-information benchmark.

to stop arrives with a pre-specified delay after the second stage was completed. Thus, the recommendation depends only on the past history and not on the current state of the world. In an optimal delayed disclosure mechanism, the delay becomes smaller as more time passes.[21]

Recall that, as Lemma 5 suggests, the key idea of the optimal information policy is that the agent postpones the disclosure of the completion of the project to extract more surplus, which makes the principal's individual rationality constraint bind. The higher the cost-benefit ratio of the project κ becomes, the higher additional value the agent's information policy needs to provide to the principal to ensure that his active individual rationality constraint is satisfied. The implication of this for the optimal information policy is presented in Lemma 6.

Lemma 6. *Assume $\kappa \in (\kappa^{ND}(T,\lambda), \tilde{\kappa}(T,\lambda)]$. Given the direct recommendation mechanism inducing optimal τ, for a fixed Poisson rate λ, the expected length of investment $\mathrm{E}[\tau]$ decreases in the cost-benefit ratio κ.*

The intuition is that the higher the cost-benefit ratio of the project becomes, the sooner after the second stage of the project is completed the agent recommends the principal to stop. For the cost-benefit ratio as high as $\tilde{\kappa}(T,\lambda)$, the agent provides the recommendation to stop immediately at the date of completion of the second stage of the project. Further, for $\kappa > \tilde{\kappa}(T,\lambda)$, the optimal information policy satisfying the conditions in Lemma 5 ceases to be individually rational for the principal. As we show in the next Section, for $\kappa > \tilde{\kappa}(T,\lambda)$, in addition to immediate disclosure of the project completion, the agent provides the information regarding the completion of the first stage of the project.

5.3 Immediate disclosure of completion and an interim deadline

When $\kappa > \tilde{\kappa}(T,\lambda)$, the project is not promising for the principal and any information policy in which stopping occurs after $\tau_2 \wedge T$ with probability one violates the principal's individual rationality constraint. In other words, from the ex ante perspective the future reports disclosing only the completion of the project do not motivate the principal to start investing. Thus, an information policy that provides an individually rational expected payoff to the principal should assign a positive probability not only to stopping after the completion of the project, but also to stopping in either state 0, when no stages of the project are completed, or state 1, when only the first stage of the project is completed. We present the necessary conditions for an information policy to be optimal when the project is not promising in Lemma 7.

Lemma 7. *Assume $\kappa \in (\tilde{\kappa}(T,\lambda), \kappa^{FI}(T,\lambda))$. If a mixed information policy σ^{μ} solves agent's problem, then it satisfies the conditions*

[21] The rich set of optimal direct recommendation mechanisms in our model encompasses both mechanisms in which the information provision depends only on the current state, similarly to the optimal mechanism in Ely and Szydlowski (2020), and the mechanisms that use delay, similarly to the delayed beep from Ely (2017).

1. $F_0(t) > 0$ *for some* $t < T$*;*

2. $F_1(t|t_1) = 0$ *for all* $t \in [t_1, T)$*;*

3. $F_2(t|t_1, t_2) = 1$ *for all* $t \in [t_2, T]$.

The condition on $F_0(t)$ implies that conditional on no completed stages of the project, stopping of the funding happens with a positive probability before T. The condition on $F_1(t|t_1)$ implies that conditional on one completed stage of the project, stopping of the funding never occurs before T. Finally, the condition on $F_2(t|t_1, t_2)$ implies that conditional on two completed stages of the project, stopping of the funding happens immediately, i.e., at $t = \tau_2$. We proceed discussing the intuition behind these necessary conditions for optimality.

Stopping when only the first stage of the project is already completed is clearly inefficient. In state 1, the principal prefers to continue investing until the completion of the second stage and this principal's incentive to wait is aligned with the agent's incentive to postpone the stopping. Further, stopping in state 1 does not help overcome the problem of the violated individual rationality constraint under $\kappa > \tilde{\kappa}(T, \lambda)$. Meanwhile, assigning a positive probability to stopping when no stages are completed helps to overcome the problem of violated individual rationality constraint, as the principal benefits from stopping at some date t when the first stage of the project is not yet completed and the project deadline T is sufficiently close. Further, the agent chooses to induce stopping of funding after the completion of the second stage rather than in state 0 as the former does not harm the total surplus generated. Thus, a policy that is optimal assigns probability 1 to immediate stopping when the second stage is completed.

Lemma 7 implies that in an information policy, optimal for the agent, stopping after the completion of the second stage of the project happens immediately and stopping also happens given that no stages of the project are completed - i.e., at the *interim deadline chosen by the agent, which we denote by* S_0^A*, and which is distributed according to* F_0. Thus, Lemma 7 drastically simplifies the strategy space in the agent's design problem: it is only left to characterize the optimal distribution F_0. At $t = 0$, the agent publicly chooses a distribution F_0, then an interim deadline is drawn according to it and privately observed by the agent. Next, the information starts to flow. The action recommendation to stop the funding satisfies the following stopping time

$$\tau = \begin{cases} S_0^A, & \text{if } x_{S_0^A} = 0 \\ \tau_2 \wedge T, & \text{otherwise,} \end{cases} \tag{11}$$

where the principal knows only the distribution F_0, but not the draw of S_0^A.

Given that the completion of the second stage of the project is disclosed immediately, stopping at the interim deadline in state 0 leads to a loss of expected further investment flow for the agent, and a potential savings from abandoning a "stagnating" project for the principal. The agent's payoff can be without loss of generality restated as the expected loss in future investment due to stopping at the interim deadline S_0^A in state 0 (rather than at $\tau_2 \wedge T$). Given this, the agent's problem can be expressed as

$$\min_{F_0} \mathrm{E}_{F_0} \left[\underbrace{\mathrm{P}\left(x_{S_0^A} = 0\right) \mathrm{E}\left[\tau_2 \wedge T - S_0^A | x_{S_0^A} = 0\right]}_{\text{expected loss in future investment given } S_0^A} \right], \tag{12}$$

subject to the system of the principal's individual rationality constraints, which also have a natural interpretation as the expectation of principal's savings on the future investment, which discontinues at S_0^A in state 0, minus the loss in the project completion profit due to stopping the funding at S_0^A in state 0.[22]

Inspecting the agent's expected loss in future investment in (12) reveals that if the agent postpones the interim deadline S_0^A, then two effects arise. First, the probability that stopping at the interim deadline will happen decreases. Second, the expected loss in total surplus due to stopping at the interim deadline rather than at $\tau_2 \wedge T$ decreases. Thus, the agent's expected loss in future investment is decreasing in the date of interim deadline and the agent prefers an interim deadline with a later expected date.

Agent's choice of the interim deadline distribution F_0 is affected by the two factors. First, as the expected loss in future investment in (12) is decreasing and convex in the date of the interim deadline, and thus the agent is risk-averse with respect to random interim deadlines. Thus, given some random interim deadline, the agent *directly* benefits from inducing a mean-preserving contraction. Second, the agent benefits from inducing a mean-preserving contraction *indirectly*. Inspecting the principal's long-run payoff for some fixed S_0^A reveals that the principal is also risk-averse with respect to random interim deadlines. Thus, inducing a mean-preserving contraction makes the principal better-off and relaxes the individual rationality constraint at $t = 0$, hence, allowing the agent to postpone the expected interim deadline further. As a result the optimal for the agent interim deadline takes the form of a *deterministic date*. In other words, it is optimal for the agent to *publicly announce the interim deadline S_0^A at the outset*, so that the principal knows it.

The agent has an incentive to postpone the interim deadline and uses her control of the information environment to postpone the deadline as much as possible so that the principal's individual rationality constraint at $t = 0$ binds. Figure 4 demonstrates the principal's long-run payoff as a function of the interim deadline, which we denote by S_0. It is maximized at the principal-preferred interim deadline S_0^P, which was characterized in Lemma 3. The agent-preferred interim deadline S_0^A yields the principal's expected payoff of 0.

The next Proposition summarizes the optimal information policy, which can be without loss of generality implemented using a direct recommendation mechanism:

Proposition 2. *Assume $\kappa \in (\tilde{\kappa}(T, \lambda), \kappa^{FI}(T, \lambda))$. The optimal information policy is given by a direct recommendation mechanism that generates*

(a) the recommendation to stop at the moment of completion of the second stage of the project, $t = \tau_2$, and

[22] The principal's individual rationality constraint is presented in (64).

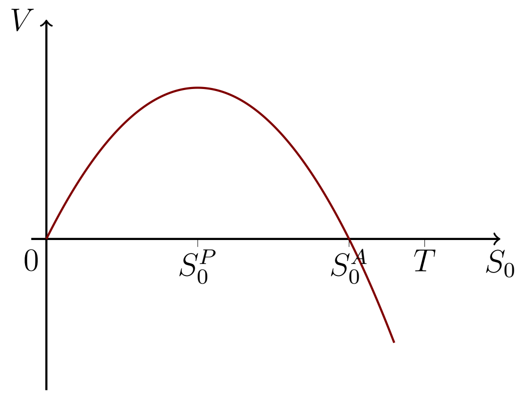

Figure 4: Principal's long-run payoff, V, as a function of an interim reporting deadline chosen by the agent, S_0.

(b) a conditional recommendation to stop at the publicly announced interim deadline $t = S_0^A$. At S_0^A, stopping is recommended with certainty if the first stage of the project has not yet been completed.

Formally,

$$\tau = \begin{cases} S_0^A, & \text{if } x_{S_0^A} = 0 \\ \tau_2 \wedge T, & \text{otherwise,} \end{cases}$$

where S_0^A is chosen so that the principal's individual rationality constraint at $t = 0$ is binding, i.e., $V(\tau) = 0$.

A stopping recommendation at any date other than the interim deadline $t = S_0^A$ fully reveals that project is accomplished. Further, observing a recommendation to stop at the interim deadline, the principal learns that the milestone of the project has not yet been reached and becomes sufficiently pessimistic that the project will be completed by T.

A notable feature of the optimal information policy when the project is ex ante unattractive is its uniqueness. The only optimal instrument through which the agent fine tunes the incentive provision to the principal is the choice of interim deadline, and there is a unique optimal way to set the deadline to make the principal's individual rationality constraint bind.

We proceed by considering the comparative statics of the interim deadline. Both the agent-preferred and the principal-preferred interim deadline, S_0^A and S_0^P, respectively, increase in the exogenous deadline T. This is because less time pressure relaxes the principal's individual rationality constraint and allows the agent to postpone the deadline further in order to extract the principal's surplus.

As the cost-benefit ratio increases up to κ^{FI}, the agent-preferred deadline converges to the principal-preferred deadline. An increase in the cost-benefit ratio of the project makes the principal's individual rationality constraint tighter.[23] As a result, for a higher κ, in the

[23] This is because the increase in κ makes the principal's instantaneous benefit from waiting decrease, and the normalized instantaneous cost of waiting becomes higher.

absence of completion of the first stage, the principal is willing to wait for a shorter time before stopping. Thus, both the interim deadline preferred by the principal S_0^P and the interim deadline chosen by the agent S_0^A are lower for a higher κ. Further, for a higher κ the agent has to choose an information policy relatively closer to the full-information benchmark to ensure that the individual rationality constraint at $t = 0$ is satisfied. Hence, the agent-chosen interim deadline S_0^A approaches S_0^P, the interim deadline preferred by the principal. The comparative statics of S_0^P and S_0^A with respect to the cost-benefit ratio of the project κ are presented in Figure 5.

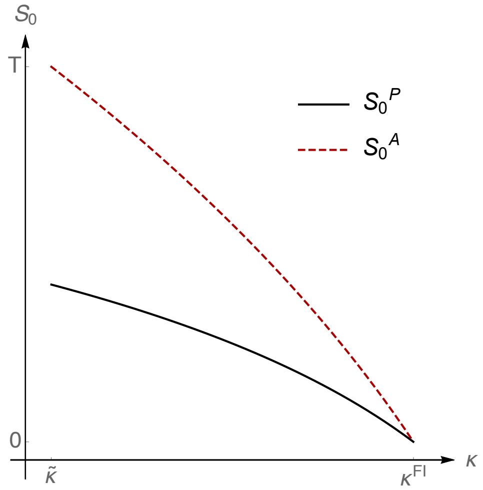

Figure 5: Interim deadline chosen by the agent S_0^A **(dashed)** and preferred by the principal S_0^P **(thick)**, as functions of the cost-benefit ratio of the project κ.

6 General preferences

In this Section, we allow for profit-sharing between the agent and the principal, varying degree of the agent's benefit from the flow of funds, and exponential discounting, and demonstrate that the optimal information policy still has the same properties as in the baseline model.

First, we assume that the agent and the principal share the project completion profit v: the principal gets $\alpha \cdot v$, while the agent gets $(1 - \alpha) \cdot v$, $\alpha \in (0, 1]$. Thus, now the agent benefits not only from the flow of funds provided by the principal for running the project but also from the share in the profit. The assumption that the agent gets a share in the project completion profit is natural in many situations. In particular, the research documents that the entrepreneurs in innovative startups are up to some extent driven by giving vent to their entrepreneurial mindset and bringing their innovative ideas to life (Gundolf et al., 2017). In such a context, a positive profit share of the agent captures that the agent is motivated by the success of the project.

Second, we assume that given a flow cost of c incurred by the principal, the agent obtains a flow benefit $\beta c, \beta \geq 0$. β can be interpreted as the agent's marginal benefit from using the funds provided by the principal for funding the project. Alternatively, for $\beta \in [0,1]$ the loss of $1-\beta$ of the amount of the transfer at each date can be interpreted as the transaction costs. Finally, setting $\beta = 0$ for some $\alpha < 1$ allows for abstracting from the agent's motives for diverting the funds and considering the agent motivated only by the success of the project.

Third, we allow for exponential discounting at a rate $r > 0$. Thus, the present value of a profit obtained at a date t is given by ve^{-rt} and the present value of a stream of funding up to date t is given by $\frac{1}{r}\left(1-e^{-rt}\right)c$. The following Proposition demonstrates that given the more general preference specification, the structure of the optimal disclosure, present in the baseline model, preserves.

Proposition 3.

(a) When the cost-benefit ratio of the project is low, $\kappa \leq \tilde{\kappa}(T, \lambda, r, \alpha)$, the optimal information policy, represented by the stopping time τ, satisfies $\tau \geq \tau_2 \wedge T$, i.e., the agent recommends the principal to stop only after the completion of the second stage of the project.

(b) When $\kappa > \tilde{\kappa}(T, \lambda, r, \alpha)$, the optimal τ assigns positive probability both to the stopping in state 2 and state 0, i.e., the agent not only discloses the completion of the second stage of the project, but also specifies an interim deadline for the completion of the first stage.

Similarly to the baseline model, allowing the principal to stop after the project completion brings profit to the principal and thus leads to a relatively higher total surplus, which the agent can extract. Meanwhile, allowing the principal to stop at the interim deadline does not increase total surplus and serves solely as an expected payoff transfer from the agent to the principal. To see that, note that stopping when the first stage of the project is still incomplete allows the principal to save on the further costs of funding the project when over time the project proves to be "unsuccessful". This can not be beneficial for the agent as she does not internalize the costs of running the project. Further, stopping at the interim deadline is strictly detrimental for the agent as she strictly prefers the principal to postpone the stopping of funding when no stages of the project are completed.[24]

When the project is sufficiently ex-ante attractive, the agent can motivate the principal to start funding the project without promising to stop the stagnant project at the interim deadline, and this is strictly beneficial for the agent. Thus, when the project is promising, the agent sets no interim deadlines, which in expectation gives her more funds and more experimentation for free.

[24]The probability of project success and stock of obtained funds are non-decreasing in the date of stopping irrespective of the number of the completed stages of the project.

7 Conclusion

A transparent flow of information is crucial for the successful management of any innovative project. However, the researcher, who controls the information on the progress of the project, often tends to have different motives than the investor. This leads to the question of how a researcher chooses the transparency of the flow of information about the progress of a project in order to manipulate the investor's funding decisions. We address this question in a dynamic information design model in which the agent commits to providing information to the principal with an incentive to postpone the principal's irreversible stopping of the funding.

We contribute to the dynamic information design literature by studying the problem of the dynamic provision of information regarding the progress of a multistage project, which evolves endogenously over time and needs to be completed before a deadline. We show that the agent's choice of which pieces of information to provide and when depends on the project being either ex ante attractive for the principal or not. In the case of a promising project, the agent provides only the good news that the project is completed and postpones the reports. In the case of an unattractive project, to motivate the principal to start funding the project the agent not only reports the completion of the project, but also helps the principal to find out when the project stagnates. To achieve this, the agent announces an interim deadline for the project – a certain date at which she recommends the principal to cut the funding of the project if the milestone of the project has not been reached.

References

Admati, A. R. and Pfleiderer, P. (1994). Robust financial contracting and the role of venture capitalists. *The Journal of Finance*, 49(2):371–402.

Ball, I. (2019). Dynamic information provision: Rewarding the past and guiding the future. *Available at SSRN 3103127.*

Basak, D. and Zhou, Z. (2020). Panics and early warnings. *PBCSF-NIFR Research Paper.*

Bergemann, D. and Hege, U. (1998). Venture capital financing, moral hazard, and learning. *Journal of Banking & Finance*, 22(6-8):703–735.

Bergemann, D. and Hege, U. (2005). The financing of innovation: Learning and stopping. *RAND Journal of Economics*, pages 719–752.

Cornelli, F. and Yosha, O. (2003). Stage financing and the role of convertible securities. *The Review of Economic Studies*, 70(1):1–32.

Curello, G. and Sinander, L. (2020). Screening for breakthroughs. *arXiv preprint arXiv:2011.10090.*

Ekmekci, M., Gorno, L., Maestri, L., Sun, J., and Wei, D. (2020). Learning from manipulable signals. *arXiv preprint arXiv:2007.08762*.

Ely, J. C. (2017). Beeps. *American Economic Review*, 107(1):31–53.

Ely, J. C. and Szydlowski, M. (2020). Moving the goalposts. *Journal of Political Economy*, 128(2):468–506.

Gompers, P. A. (1995). Optimal investment, monitoring, and the staging of venture capital. *The Journal of Finance*, 50(5):1461–1489.

Green, B. and Taylor, C. R. (2016). Breakthroughs, deadlines, and self-reported progress: Contracting for multistage projects. *American Economic Review*, 106(12):3660–99.

Gundolf, K., Gast, J., and Géraudel, M. (2017). Startups' innovation behaviour: An investigation into the role of entrepreneurial motivations. *International Journal of Innovation Management*, 21(07):1750054.

Henry, E. and Ottaviani, M. (2019). Research and the approval process: The organization of persuasion. *American Economic Review*, 109(3):911–55.

Kaya, A. (2020). Paying with information. *Available at SSRN 3661779*.

Madsen, E. (2022). Designing deadlines. *American Economic Review*, 112(3):963–97.

Orlov, D., Skrzypacz, A., and Zryumov, P. (2020). Persuading the principal to wait. *Journal of Political Economy*, 128(7):2542–2578.

Renault, J., Solan, E., and Vieille, N. (2017). Optimal dynamic information provision. *Games and Economic Behavior*, 104:329–349.

Wolf, C. (2017). Informative milestones in experimentation. *University of Mannheim, Working Paper*.

Appendix

A Notational conventions

Throughout Appendix B, the following notational conventions are used:

1. We denote the random time at which the nth stage of the project is completed by τ_n. Formally, $\tau_n \in \mathbb{R}_+$ is a continuously distributed random variable that represents the first hitting time of $x_t = n$.

2. The continuation values of the agent and principal at time t, respectively, given τ, and conditional on information disclosed up to t are given by

$$W_t(\tau) := \mathrm{E}[\tau - t | t < \tau] c,$$
$$V_t(\tau) := [\mathrm{P}(x_\tau = 2 | t < \tau) - \mathrm{P}(x_t = 2 | t < \tau)] v - \mathrm{E}[\tau - t | t < \tau] c.$$

3. Shorthand for posterior beliefs:

$$q_n(t) := \mathrm{P}(x_t = n | t < \tau),$$
$$r_n(t) := \mathrm{P}(x_\tau = n | t < \tau).$$

B Proofs

Proof of Lemma 2. The beliefs regarding the number of stages of the project completed by time t, x_t, evolve according to the Poisson process. The principal's unconditional beliefs are given by $p_0(0) = 1$ and for any t such that the stopping still has not occurred,

$$\begin{aligned} \dot{p}_0(t) &= -\lambda p_0(t), \\ \dot{p}_1(t) &= \lambda(p_0(t) - p_1(t)), \\ \dot{p}_2(t) &= \lambda p_1(t), \end{aligned} \tag{13}$$

where $p_0(t) = e^{-\lambda t}$ and $p_1(t) = \lambda t e^{-\lambda t}$, $p_2(t) = 1 - p_0(t) - p_1(t)$. The principal's problem is given by

$$\max_{S \in [0,T]} \{v \cdot p_2(S) - c \cdot S\}. \tag{14}$$

We start with analyzing the choice of S for the interior solution case, $S \in (0, T)$. F.O.C. for (14) is given by

$$v \cdot \dot{p}_2(S) = c, \tag{15}$$

or, equivalently, $p_1(S) = \kappa$. There are two values satisfying (15): $\bar{S}$ and $\bar{S}^{NI}$, $\bar{S} < \bar{S}^{NI}$. At each $t \in \left(\bar{S}, \bar{S}^{NI}\right)$ the principal receives a net positive payoff flow. Thus, stopping at $\bar{S}$ is not optimal and the only candidate for optimal stopping is $\bar{S}^{NI}$.[25] Further, one can obtain the closed form expression for the interior stopping time $\bar{S}^{NI}$ from (15):

$$\bar{S}^{NI} = -\frac{1}{\lambda}\mathcal{W}_{-1}(-\kappa), \tag{16}$$

where $\mathcal{W}_{-1}(x)$ denotes the negative branch of the Lambert W function. $\bar{S}^{NI}$ is well-defined for any $\kappa < e^{-1}$.

Thus, the solution to (14) could potentially be 0, $\bar{S}^{NI}$, or T. We proceed with a useful lemma.

Lemma 8. *The following is true regarding the principal's continuation value in the no-information benchmark, $\overline{V}_t^{NI}$: if $\overline{V}_t^{NI} \geq 0$, for some $t \in \left[0, \bar{S}^{NI} \wedge T\right]$, then $V^{NI}(s) \geq 0$, for all $s \in \left[t, \bar{S}^{NI} \wedge T\right]$.*

[25] $\bar{S}$ is a local minimum of the objective.

Proof. The principal's continuation value in the no-information benchmark is given by

$$\overline{V}_t^{NI} = \left[p_2\left(T \wedge \bar{S}^{NI}\right) - p_2(t)\right] v - \left(T \wedge \bar{S}^{NI} - t\right) c. \tag{17}$$

Further,

$$\dot{V}^{NI}(t) = v\lambda\left(\kappa - e^{-\lambda t}\lambda t\right) = v\lambda\left(\kappa - p_1(t)\right).$$

$p_1(t) \leq \kappa$ for all $t \in \left[0, \bar{S}\right]$ and $p_1(t) \geq \kappa$ for all $t \in \left[\bar{S}, \bar{S}^{NI} \wedge T\right]$. Thus, $\overline{V}_t^{NI}$ increases for $t \in \left[0, \bar{S}\right]$, decreases for $t \in \left[\bar{S}, T \wedge \bar{S}^{NI}\right]$, and $V^{NI}\left(T \wedge \bar{S}^{NI}\right) = 0$, which establishes the result. □

Lemma 8 implies that if $V^{NI}(0) \geq 0$ and the principal chooses to opt in at $t = 0$, then $\overline{V}_t^{NI} \geq 0$, $t \in \left[0, \bar{S}^{NI} \wedge T\right]$, i.e., he invests until $t = T \wedge \bar{S}^{NI}$. This implies that the solution to (14) is either $T \wedge \bar{S}^{NI}$ or 0.

Finally, at $t = 0$ the principal chooses to start investing or not. The condition for the principal to start investing at $t = 0$ is given by

$$V^{NI} \geq 0. \tag{18}$$

To specify the set of parameters for which (18) is satisfied, we obtain the cutoff value of κ given T and λ. Such a parameterization is intuitive: κ above the cutoff level corresponds to a project with sufficiently high normalized cost-benefit ratio and implies that the principal does not opt in. We denote this cutoff by $\kappa^{NI}(T, \lambda)$. This solves (18) holding with equality. Two cases are possible.

Case 1: $T \leq \bar{S}^{NI} \iff T \leq -\frac{1}{\lambda}\mathcal{W}_{-1}(-\kappa)$. This inequality is satisfied when either $\frac{1}{\lambda} > T$ or $\begin{cases} \frac{1}{\lambda} & \leq T \\ \kappa & \leq e^{-\lambda T}\lambda T. \end{cases}$ Given $T \leq \bar{S}^{NI}$, (18) holding with equality becomes

$$p_2(T)\, v - Tc = 0.$$

Solving it for κ yields $\kappa = e^{-\lambda T}\left(\frac{e^{\lambda T} - 1}{\lambda T} - 1\right)$.

Case 2: $T > \bar{S}^{NI}$. This inequality is satisfied when $\frac{1}{\lambda} \leq T$ and $\kappa > e^{-\lambda T}\lambda T$. Given $T > \bar{S}^{NI}$, (18) holding with equality becomes

$$vp_2\left(\bar{S}^{NI}\right) - c\bar{S}^{NI} = 0 \iff v\left(1 - p_0\left(\bar{S}^{NI}\right) - p_1\left(\bar{S}^{NI}\right)\right) = c\bar{S}^{NI},$$

where (recall that $\dot{p}_2\left(\bar{S}^{NI}\right) = \frac{c}{v}$)

$$p_0\left(\bar{S}^{NI}\right) = \frac{1}{\lambda^2 \bar{S}^{NI}}\dot{p}_2\left(\bar{S}^{NI}\right) = \frac{c}{\lambda^2 \bar{S}^{NI} v} = \frac{\kappa}{\lambda \bar{S}^{NI}}$$

and

$$p_1\left(\bar{S}^{NI}\right) = \frac{1}{\lambda}\dot{p}_2\left(\bar{S}^{NI}\right) = \frac{c}{\lambda v} = \kappa.$$

Consequently,

$$vp_2\left(\bar{S}^{NI}\right) - c\bar{S}^{NI} = v - v \cdot \kappa\left(1 + \lambda \bar{S}^{NI} + \frac{1}{\lambda \bar{S}^{NI}}\right).$$

Let $y := \lambda \bar{S}^{NI}$. Note that, by definition, $y > 1$. Then $\kappa = ye^{-y}$, and so

$$\left(vp_2\left(\bar{S}^{NI}\right) - c\bar{S}^{NI}\right)/v = 1 - e^{-y}\left(1 + y + y^2\right).$$

It follows that $V^{NI}(0)$ is nonnegative whenever $\lambda \bar{S}^{NI} \geq y_0 \doteq 1.79328$, which is equivalent to

$$\kappa \leq \kappa_0 \doteq 0.298426.$$

Finally, putting the two cases together yields

$$\kappa^{NI}(T, \lambda) = \begin{cases} \kappa_0 \doteq 0.298426, & \text{if } \frac{1}{\lambda} \leq T \text{ and } \kappa \geq e^{-\lambda T}\lambda T \\ e^{-\lambda T}\left(\frac{e^{\lambda T}-1}{\lambda T} - 1\right), & \text{otherwise.} \end{cases} \tag{19}$$

□

Proof of Lemma 3. The principal chooses $a_t \in \{0, 1\}$ sequentially given the observed realizations of $x_t \in \{0, 1, 2\}$. Whenever the principal observes $t = \tau_2$, he immediately chooses $a_t = 0$ and gets v.

Consider the case $x_t = 1, t < T$. If it is optimal to continue the project over the internal $[t, t + \Delta_t)$, then

$$V^{FI}_{t|1} = -c\Delta_t + \lambda\Delta_t \cdot v + (1 - \lambda\Delta_t) \cdot V^{FI}_{t+\Delta_t|1}, \tag{20}$$

where $V^{FI}_{t|n}$ stands for continuation value of the principal at date t given full information and n completed stages of the project. Consider a candidate policy given by $\tau = \tau_2 \wedge T$. $V^{FI}_{t|1}(\tau_2 \wedge T)$ is given by

$$V^{FI}_{t|1}(\tau_2 \wedge T) = v\,\mathrm{P}(\tau_2 \leq T|x_t = 1) - c\,\mathrm{E}[\tau_2 \wedge T - t|x_t = 1].$$

$\tau_2|x_t = 1$ corresponds to the time between two consecutive Poisson arrivals, and thus has exponential distribution. First, consider $\mathrm{P}(\tau_2 \leq T|x_t = 1)$:

$$\mathrm{P}(\tau_2 \leq T|x_t = 1) = 1 - e^{-\lambda(T-t)}.$$

Next, consider $\mathrm{E}[\tau_2 \wedge T - t|x_t = 1]$:

$$\begin{aligned} &\mathrm{E}[\tau_2 \wedge T|x_t = 1] - t \\ =&\mathrm{P}(\tau_2 \leq T|x_t = 1)\int_t^T z \cdot \frac{\lambda e^{-\lambda(z-t)}}{\mathrm{P}(\tau_2 \leq T|x_t = 1)}dz + \mathrm{P}(\tau_2 > T|x_t = 1)T - t \\ =&\frac{1}{\lambda}\left(1 - e^{-\lambda(T-t)}\right) + t - e^{-\lambda(T-t)}T + \mathrm{P}(\tau_2 > T|x_t = 1)T - t \\ =&\frac{1}{\lambda}\left(1 - e^{-\lambda(T-t)}\right). \end{aligned} \tag{21}$$

Thus,

$$\begin{aligned} V^{FI}_{t|1}(\tau_2 \wedge T) &= v\left(1 - e^{-\lambda(T-t)}\right) - c\frac{1}{\lambda}\left(1 - e^{-\lambda(T-t)}\right) \\ &= \left(v - \frac{c}{\lambda}\right)\left(1 - e^{-\lambda(T-t)}\right). \end{aligned} \tag{22}$$

First, consider the case $v > \frac{c}{\lambda}$. From (22) one observes that if $v > \frac{c}{\lambda}$, then $V^{FI}_{t|1}(\tau_2 \wedge T) > 0, \forall t \in [0, T)$. $V^{FI}_{t|1}(\tau_2 \wedge T)$ for this parametric case is illustrated in the Figure 6. As, by optimality, $V^{FI}_{t|1}$ in (20) is weakly higher than $V^{FI}_{t|1}(\tau_2 \wedge T)$, it holds that $V^{FI}_{t|1} > 0, \forall t \in [0, T)$. Thus, the principal invests until $\tau_2 \wedge T$, which verifies that the candidate policy is optimal and the optimal continuation value given full information and one completed stage of the project is given by

$$V^{FI}_{t|1} = \left(v - \frac{c}{\lambda}\right)\left(1 - e^{-\lambda(T-t)}\right). \tag{23}$$

Second, consider the case $v = \frac{c}{\lambda}$. In this case, the principal is indifferent between continuing and stopping at any date. Third, consider $v < \frac{c}{\lambda}$. In this case, from (22), $V^{FI}_{t|1}(\tau_2 \wedge T) < 0, \forall t \in [0, T]$. It can be shown that this implies that $V^{FI}_{t|1}$ can not be strictly positive at any date t. Thus, $v < \frac{c}{\lambda}$ leads to the trivial case in which the principal does not start investing at $t = 0$ in the full information benchmark. Thus, we assume $v \geq \frac{c}{\lambda}$, or, equivalently $\kappa \leq 1$.

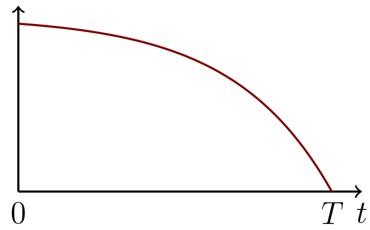

Figure 6: $V^{FI}_{t|1}(\tau_2 \wedge T)$, the continuation value of the principal under full information, $\tau = \tau_2 \wedge T$ policy, and conditional on one stage of the project being completed.

Consider now the case of $x_t = 0, t < T$, i.e., no stages of the project have yet been completed. If it is optimal to continue the project over the internal $[t, t + \Delta_t)$, then

$$V^{FI}_{t|0} = -c\Delta_t + \lambda\Delta_t V^{FI}_{t|1} + (1 - \lambda\Delta_t)V^{FI}_{t+\Delta_t|0}.$$

By Taylor expansion ($V^{FI}_{t+\Delta_t|0} = V^{FI}_{t|0} + \dot{V}^{FI}_{t|0}\Delta_t$), we obtain the Hamilton-Jacobi-Bellman equation

$$c = \lambda(V^{FI}_{t|1} - V^{FI}_{t|0}) + \dot{V}^{FI}_{t|0}.$$

After plugging $V^{FI}_{t|1}$ given by (23) into the HJB equation, one can solve this ODE for $V^{FI}_{t|0}$. A generic solution is

$$z_0 = v - \frac{2c}{\lambda} + t(\lambda v - c)e^{-(T-t)\lambda} + C_0 e^{t\lambda}, \tag{24}$$

where C_0 is an integration constant.

It can be shown that the stopping boundary for the principal's optimal stopping problem is a regular boundary, i.e., the smooth pasting condition holds at the stopping boundary. Applying value matching ($V^{FI}_{S_0|0} = 0$) and smooth pasting ($\dot{V}^{FI}_{S_0|0} = 0$) conditions to the HJB equation, we can get the stopping boundary S_0. It is implicitly given by

$$c = \lambda V^{FI}_{S_0|1}, \tag{25}$$

where $V^{FI}_{S_0|1}$ is given by (23).

The equation (25) can be solved for S_0. We denote the solution by S_0^P:

$$S_0^P = T + \frac{1}{\lambda}\log\left(\frac{1-2\kappa}{1-\kappa}\right). \tag{26}$$

Finally, (26) pins down C_0, which is implicitly given by

$$v - \frac{2c}{\lambda} + S_0^P(\lambda v - c)\frac{1-2\kappa}{1-\kappa} + C_0 e^{S_0^P\lambda} = 0, \tag{27}$$

and by the standard verification theorem, it can be demonstrated that the value function $V_{t|0}^{FI}$ given by (24) with C_0 given by (27) and S_0^P given by (26) is optimal.

The principal is willing to start investing at $t = 0$ iff $S_0^P \geq 0$. We denote the upper bound on the cost-benefit ratio κ such that the principal chooses to start investing in $t = 0$ under full information by $\kappa^{FI}(T, \lambda)$, we solve $S_0^P = 0$ for κ and obtain

$$\kappa^{FI}(T, \lambda) = \frac{1 - e^{-\lambda T}}{2 - e^{-\lambda T}}. \tag{28}$$

In summary, under full information, if $\kappa \leq \kappa^{FI}(T, \lambda)$, then the principal starts investing at $t = 0$. Further, he stops at S_0^P if the first stage of the project has not been completed by that time. Otherwise, he proceeds to invest until $\tau_2 \wedge T$.

□

Proof of Lemma 1. Any information policy σ^μ induces an action process, which is a stopping time with respect to a filtration of the probability space. Thus, an information policy σ^μ can be represented as a stopping time τ. A stopping time τ is the principal's best response to at least one information policy σ^μ if and only if

$$V_t(\tau) \geq 0, \forall t \geq 0 \text{ and } V_\tau^{NI} < 0, \tag{29}$$

where $V_t(\tau)$ is the principal's continuation value given by (2) and V_t^{NI} is the principal's optimal continuation value in the absence of any additional information from the agent starting from the date t. We proceed with proving this claim.

Necessity. Assume $V_t(\tau) < 0$ for some t. In that case, it is optimal for the principal to deviate to stopping at $t < \tau$. Thus, there is no information policy σ^μ, for which this τ is the principal's best reply. Assume $V_\tau^{NI} \geq 0$. Thus, the principal deviates to stopping at $t > \tau$, and there is no σ^μ, for which this τ is the best reply.

Sufficiency. Assume (29) holds. $V_t(\tau) \geq 0$ for all $t < \tau$ implies that the principal prefers to continue rather than to stop the funding for all $t < \tau$. Thus, it can not be that case that the principal stops before τ. Further, $V_\tau^{NI} < 0$ implies that, conditional on reaching the date of stopping τ, it is better for the principal to stop immediately rather than to stop at $t > \tau$. Finally, given τ, there exists σ implementing it: consider a direct recommendation mechanism σ with $M = \{0, 1\}$ such that whenever, based on the realizations of the state process and randomization devices, the considered stopping time τ suggests stopping the funding, the direct recommendation mechanism sends the message $m = 0$ to the principal. As it is optimal for the principal to stop at τ, τ is the principal's best reply to σ.

As the agent chooses the distribution of the stopping time τ to maximize her long-run payoff, the constraint $V_{\tau}^{NI} < 0$ is inactive at optimum. Otherwise, the agent can prolong the expected funding by choosing a different τ. Thus, without loss of generality, we omit this constraint from the agent's problem, and the problem that the agent solves at $t = 0$ is given by (1). $\square$

Discussion of Assumption 2. $\kappa^{ND}(T, \lambda)$ is defined as follows: for any $\kappa \leq \kappa^{ND}(T, \lambda)$, the principal invests until T in the no-information benchmark. From Lemma 2, if the principal is willing to start investing, i.e., $\kappa \leq \kappa^{NI}(T, \lambda)$, then

$$S^{NI} = \overline{S}^{NI} \wedge T.$$

For the sake of instruction, below we consider relaxing the Assumption 2 and demonstrate how the relation between $\kappa^{ND}(T, \lambda)$ and $\kappa^{NI}(T, \lambda)$ changes between Case a (assumption alternative to the Assumption 2) and Case b (Assumption 2 holds).

Case a. $e^{\lambda T} \leq \lambda T(\lambda T + 1) + 1$. In this case, whenever the principal is willing to start investing in the no-information benchmark, she invests until T, i.e., $\kappa^{ND}(T, \lambda) = \kappa^{NI}(T, \lambda)$, where $\kappa^{NI}(T, \lambda)$ is given by (19). To see that, first, consider the extreme sub-case in which $T < \frac{1}{\lambda}$. As $-\lambda\overline{S}^{NI}$ must belong to -1 axis of Lambert W function, it has a lower bound corresponding to $\frac{1}{\lambda}$. Thus, $T < \overline{S}^{NI}$ for any $\kappa(T, \lambda)$. Second, consider $\lambda T \in \left[1, \tilde{\lambda T}\right]$, where $\tilde{\lambda T}$ solves $e^{\lambda T} = \lambda T(\lambda T + 1) + 1$. In this case, from (16), if $\kappa(T, \lambda) \leq e^{-\lambda T}\lambda T$ ($\kappa(T, \lambda) \geq e^{-\lambda T}\lambda T$, respectively), then $T \leq \overline{S}^{NI}$ ($T \geq \overline{S}^{NI}$, respectively). However, $\kappa^{NI}(T, \lambda) \leq e^{-\lambda T}\lambda T$. Thus, $\kappa^{ND}(T, \lambda) = \kappa^{NI}(T, \lambda)$.

Case b. $e^{\lambda T} > \lambda T(\lambda T + 1) + 1$. As before, it holds that if $\kappa(T, \lambda) \leq e^{-\lambda T}\lambda T$ ($\kappa(T, \lambda) \geq e^{-\lambda T}\lambda T$), then $T \leq \overline{S}^{NI}$ ($T \geq \overline{S}^{NI}$, respectively). Denote

$$\kappa^{ND}(T, \lambda) := e^{-\lambda T}\lambda T.$$

As $\kappa^{NI}(T, \lambda) > \kappa^{ND}(T, \lambda)$, two cases emerge. If $0 < \kappa \leq \kappa^{ND}(T, \lambda)$, then $T \leq \overline{S}^{NI}$, and from $\kappa \leq \kappa^{NI}(T, \lambda)$, it holds that $S^{NI} = T$ and as the agent does not strictly benefit from disclosing any information, she chooses non-disclosure. If $\kappa > \kappa^{ND}(T, \lambda)$, then $T > \overline{S}^{NI}$ and the agent can potentially benefit from information disclosure. $\square$

Proof of Lemma 4. Consider an information policy such that stopping of funding happens immediately at the completion of the second stage of the project; it is given by $\tau = \tau_2 \wedge T$. There exists such $\tilde{\kappa}(T, \lambda)$ that solves the principal's binding $t = 0$ individual rationality constraint when $\tau = \tau_2 \wedge T$:

$$V(\tau_2) = 0, \tag{30}$$

where

$$\begin{aligned} V(\tau_2) &= p_2(T)\,v - \mathrm{E}[\tau_2 \wedge T]\,c \\ &= v\left(1 - e^{-\lambda T} - \lambda T e^{-\lambda T}\right) - c\frac{1}{\lambda}\left(2 - 2e^{-\lambda T} - \lambda T e^{-\lambda T}\right). \end{aligned} \tag{31}$$

The solution to equation (30) is given by

$$\tilde{\kappa}(T,\lambda) = \frac{1 - e^{\lambda T} + \lambda T}{2 - 2e^{\lambda T} + \lambda T}. \tag{32}$$

Further, $\kappa > \tilde{\kappa}(T,\lambda) \Rightarrow V(\tau_2) < 0$ and $\kappa \leq \tilde{\kappa}(T,\lambda) \Rightarrow V(\tau_2) \geq 0$. □

Proof of Lemma 5. Consider the case of $\kappa \in (\kappa^{ND}(T,\lambda), \kappa^{NI}(T,\lambda)]$. The agent's relaxed problem for this case has the individual rationality constraints only for $t \in [0, \bar{S}^{NI}]$, and it is given by

$$\begin{aligned} &\max_{F_0,F_1,F_2} \{c \cdot \mathrm{E}[\tau]\} \\ &\text{s.t. } V_t(\tau) \geq 0, \forall t \in \left[0, \bar{S}^{NI}\right], \end{aligned} \tag{33}$$

where $V_t(\tau)$ is given by (2).

Consider the candidate information policy represented by τ such that $\tau \geq \bar{S}^{NI} \vee (\tau_2 \wedge T)$ and $V(\tau) = V^{NI}$, where V^{NI} is given by (5). We start with arguing that the candidate τ satisfies the system of individual rationality constraints. From Lemma 2, given candidate τ, the principal invests until $\bar{S}^{NI}$ with certainty and the constraints in (33) are satisfied for all $t \in [0, \bar{S}^{NI})$. Further, τ implies that $V_{\bar{S}^{NI}}(\tau) = 0$, i.e., the individual rationality constraint at $t = \bar{S}^{NI}$ is binding.

We proceed with arguing that the candidate τ maximizes the agent's objective function in (33). The agent's objective can be WLOG written out as:

$$W(\tau) = \underbrace{\mathrm{P}(x_\tau = 2)\, v}_{\text{total surplus}} - \underbrace{V(\tau).}_{\text{principal's surplus}} \tag{34}$$

By Lemma 4, a stopping time τ that assigns probability one to $\tau \geq \tau_2 \wedge T$ satisfies the individual rationality constraint at $t = 0$ in (33). Note that, given $\tau \geq \tau_2 \wedge T$, the total surplus in (34) is given by $\mathrm{P}(x_T = 2)\, v$, i.e., total surplus achieves its upper bound determined by the exogenously given project deadline T. The principal's surplus in (34) is given by $V(\tau) = V^{NI}$, i.e., principal's surplus achieves its lower bound specified by (5). This can be seen from the principal's decision problem, in which he best replies to an information policy σ^μ. As σ^μ allows the principal to condition his actions on the information regarding the evolution of the state process, the principal's equilibrium payoff can not be lower than V^{NI}, his equilibrium payoff when he is restricted to choosing actions without conditioning them on the information about the state process. Thus, τ solves the relaxed problem (33).

Consider the case of $\kappa \in (\kappa^{NI}(T,\lambda), \tilde{\kappa}(T,\lambda)]$. The agent's relaxed problem for this case has the individual rationality constraint only for the initial period, and it is given by

$$\begin{aligned} &\max_{F_0,F_1,F_2} \{c \cdot \mathrm{E}[\tau]\} \\ &\text{s.t. } V(\tau) \geq 0, \end{aligned} \tag{35}$$

where $V(\tau) = \mathrm{P}(x_\tau = 2)\, v - \mathrm{E}[\tau]\, c$.

Consider candidate information policy represented by τ such that $\tau \geq \tau_2 \wedge T$ and $V(\tau) = V^{NI}$. For such τ, agent's expected payoff (34) is given by $\mathrm{P}(x_T = 2)\, v - V^{NI}$. As discussed for the parametric case $\kappa \in (\kappa^{ND}(T,\lambda), \kappa^{NI}(T,\lambda)]$, the first term is at its upper bound. To see that the second term is at its lower bound, note that, from Lemma 2, $V^{NI} = 0$, and thus the individual rationality constraint in (35) is binding. Hence, τ solves the relaxed problem (35).

□

Proof of Proposition 1. The proof covers the case $\kappa \in (\kappa^{ND}(T,\lambda), \kappa^{NI}(T,\lambda)]$ and the case $\kappa \in (\kappa^{NI}(T,\lambda), \tilde{\kappa}(T,\lambda)]$ separately.

1. The case of $\kappa \in (\kappa^{ND}(T,\lambda), \kappa^{NI}(T,\lambda)]$.

We start with proving the existence of S^* such that $V(\tau) = V^{NI}$. Assume that $S^* > \bar{S}^{NI}$. For all $t \in [\bar{S}^{NI}, S^*)$, stopping never occurs, at $t = S^*$ it occurs if $x_{S^*} = 2$, and for all $t \in (S^*, \tau)$ it occurs at $t = \tau_2 \wedge T$. For $t \in [S^*, \tau)$, the absence of stopping induces posteriors $q_n(t)$. Further, for $t \in [S^*, \tau)$ the principal discounts future benefits from postponing stopping using the probability of the state being 2. Hence, the continuation value at $t = \bar{S}^{NI}$ can be written as

$$V_{\bar{S}^{NI}}(\tau) = v\lambda \left(\int_{\bar{S}^{NI}}^{S^*} p_1(z) - \kappa dz + \int_{S^*}^{T} (q_1(z) - \kappa)(1 - \mathrm{P}(x_z = 2))\, dz \right). \tag{36}$$

The principal's long-run payoff is given by

$$V(\tau) = \int_0^{\bar{S}^{NI}} (v \cdot p_1(s)\lambda - c)\, ds + V_{\bar{S}^{NI}}(\tau),$$

where $\int_0^{\bar{S}^{NI}} (v \cdot p_1(s)\lambda - c)\, ds = V^{NI}$. Thus, to ensure that S^* makes the individual rationality constraint bind at $t = \bar{S}^{NI}$, i.e., $V(\tau) = V^{NI}$, it is necessary and sufficient that $V_{\bar{S}^{NI}}(\tau) = 0$. Using (36), this equation can be written as

$$\int_{\bar{S}^{NI}}^{S^*} \kappa - p_1(z)\, dz = \int_{S^*}^{T} (q_1(z) - \kappa)(1 - \mathrm{P}(x_z = 2))\, dz.$$

Let $g(S) := \int_{\bar{S}^{NI}}^{S} \kappa - p_1(z)\, dz$ and $k(S) := \int_{S}^{T} (q_1(z) - \kappa)(1 - \mathrm{P}(x_z = 2))\, dz$, $S \in [\bar{S}^{NI}, \tau)$. $q_1(t) \geq \kappa$, for all $t \in [S^*, T)$. Thus, $g\left(\bar{S}^{NI}\right) = 0, k\left(\bar{S}^{NI}\right) > 0$. Further, $p_1(t) < \kappa$, for all $t \in (\bar{S}^{NI}, T]$. Hence, $g(T) > 0, k(T) = 0$. Finally, $p_1(t) \leq \kappa$, for all $t \in \left[\bar{S}^{NI}, T\right]$ implies that $g'(S) \geq 0$, for all $s \in \left[\bar{S}^{NI}, T\right]$, and $q_1(t) \geq \kappa$, for all $t \in [S^*, T]$ implies that $k'(S) \leq 0$, for all $s \in [S^*, T]$. Thus, by the intermediate value theorem, there exists S^* solving $V_{\bar{S}^{NI}}(\tau) = 0$. Thus, there exists $S^* > \bar{S}^{NI}$ such that principal's individual rationality constraint is binding at $t = \bar{S}^{NI}$.

We proceed with proving that the stopping time τ satisfies the conditions in Lemma 1 and thus it is obedient.

First, consider $t \leq \bar{S}^{NI}$. The principal's continuation value for all $t \in [0, \bar{S}^{NI}]$ can be written as

$$V_t(\tau) = \int_t^{\bar{S}^{NI}} v\lambda (p_1(s) - \kappa)\, ds + V_{\bar{S}^{NI}}(\tau).$$

Given the binding individual rationality constraint, it becomes

$$V_t(\tau) = \int_t^{\bar{S}^{NI}} v\lambda (p_1(s) - \kappa)\, ds, \text{ for all } t \in [0, \bar{S}^{NI}).$$

Finally, note that $V_t(\tau)$ above is equivalent to V_t^{NI} given by (17). Lemma 2 implies that given $\kappa \in (\kappa^{ND}(T,\lambda), \kappa^{NI}(T,\lambda)]$, $V^{NI}(0) = V(\tau) \geq 0$. Further, Lemma 8 implies that $V(\tau) \geq 0 \Rightarrow V_t(\tau) \geq 0, \forall t \in [0, \bar{S}^{NI})$.

Second, consider $t \in \left[\bar{S}^{NI}, S^*\right]$. Given $\kappa \in (\kappa^{ND}(T,\lambda), \kappa^{NI}(T,\lambda)]$, $p_1(t) \leq \kappa, \forall t \in \left[\bar{S}^{NI}, S^*\right]$. Thus, $V_t^{NI} = 0, \forall t \in \left[\bar{S}^{NI}, S^*\right]$. The principal's continuation value is given by

$$V_t(\tau) = \int_t^{S^*} v\lambda(p_1(s) - \kappa)\, ds + V_{S^*}(\tau). \tag{37}$$

As $p_1(t) \leq \kappa, \forall t \in \left[\bar{S}^{NI}, S^*\right]$, $\int_t^{S^*} v\lambda(p_1(s) - \kappa)\, ds \leq 0$ and it is increasing in t. As $V_{\bar{S}^{NI}}(\tau) = 0$, where $V_{\bar{S}^{NI}}(\tau)$ is given by (36), it follows that $V_t(\tau) \geq 0, \forall t \in \left[\bar{S}^{NI}, S^*\right]$.

Third, consider $t \in [S^*, \tau)$. The absence of stopping at $t \geq S^*$ reveals that $x_t \neq 2$. Thus, $q_1(t) = \frac{p_1(t)}{p_0(t)+p_1(t)} = \frac{\lambda t}{1+\lambda t}$, $\forall t \in [S^*, \tau)$, and, thus, $\dot{q}_1(t) > 0$. Further, $q_1(S^*) > \kappa$. The continuation value $\forall t \in [S^*, \tau)$ is given by

$$V_t(\tau) = \mathrm{E}\left[\textstyle\int_t^{\tau} v\lambda(q_1(z) - \kappa)\, dz \,|\, t < \tau\right].$$

Thus, $V_t(\tau) \geq 0$, $\forall t \in [S^*, \tau)$.

2. The case of $\kappa^{NI}(T,\lambda) < \kappa \leq \tilde{\kappa}(T,\lambda)$.

We start with proving the existence of S^* such that $V(\tau) = 0$. For all $t \in [0, S^*)$, stopping never occurs, at $t = S^*$ it occurs if $x_{S^*} = 2$, and for all $t \in (S^*, T]$ it occurs at $t = \tau_2 \wedge T$. The principal's long-run payoff can be written as

$$V(\tau) = v\lambda\left(\int_0^{S^*} p_1(z) - \kappa dz + \int_{S^*}^{T} (q_1(z) - \kappa)(1 - \mathrm{P}(x_z = 2))\, dz\right). \tag{38}$$

To ensure that S^* makes the individual rationality constraint bind at $t = 0$, it is necessary and sufficient that $V(\tau) = 0$. The next step of the proof consist of inspecting (38) to establish that there exists S^* ensuring that $V(\tau) = 0$. It follows the respective part from the proof for the parametric case $\kappa^{ND}(T,\lambda) < \kappa \leq \kappa^{NI}(T,\lambda)$, imposing $\bar{S}^{NI} = 0$ in it everywhere; thus, we omit it for the sake of space.

We proceed with proving that the stopping time τ satisfies the conditions in Lemma 1 and thus it is obedient. The principal's continuation value is given by (37). As $\kappa \in (\kappa^{NI}(T,\lambda), \tilde{\kappa}(T,\lambda)]$, it follows from Lemma 2 that $V_t^{NI} = 0, \forall t \in [0, S^*]$. *First, assume* $S^* \leq \bar{S}^{NI}$. From the proof of Lemma 2, it follows that $p_1(t) \leq \kappa, \forall t \in [0, \overline{S}]$, and $p_1(t) \geq \kappa, \forall t \in [\overline{S}, \bar{S}^{NI}]$. Thus,

$$\int_t^{\bar{S}^{NI}} v\lambda(p_1(s) - \kappa)\, ds \geq \int_0^{\bar{S}^{NI}} v\lambda(p_1(s) - \kappa)\, ds, \forall t[0, \bar{S}^{NI}]. \tag{39}$$

As $V_t(\tau)$ is given by (37), $V(\tau) = 0$ and (39) imply that $V_t(\tau) \geq 0, \forall t \in [0, S^*]$. *Second, assume* $S^* \geq \bar{S}^{NI}$. As $V(\tau) = 0$ and $\int_0^{\bar{S}^{NI}} v\lambda(p_1(s) - \kappa)\, ds < 0$, it must be that $V(\bar{S}^{NI}) > 0$. Further, $\int_t^{S^*} v\lambda(p_1(s) - \kappa)\, ds$ increases in t for $t \in [\bar{S}^{NI}, S^*]$. Thus, $V_t(\tau) \geq 0, \forall t \in [0, S^*]$.

Finally, the proof that $V_t(\tau) \geq 0, \forall t \in [S^*, \tau)$ follows the the respective part of the proof for the parametric case $\kappa \in (\kappa^{ND}(T,\lambda), \kappa^{NI}(T,\lambda)]$; thus, we omit it for the sake of space.

□

Proof of Lemma 6. We provide the proof for the parametric cases $\kappa^{ND}(T,\lambda) < \kappa \leq \kappa^{NI}(T,\lambda)$ and $\kappa^{NI}(T,\lambda) < \kappa \leq \tilde{\kappa}(T,\lambda)$ separately.

1. The case of $\kappa^{ND}(T,\lambda) < \kappa \leq \kappa^{NI}(T,\lambda)$.

Under any obedient optimal policy, the principal's individual rationality constraint is binding, thus, $V(\tau) = V^{NI}$, or equivalently $p_2(T)\, v - \mathrm{E}[\tau]\, c = p_2\left(\bar{S}^{NI}\right) v - \bar{S}^{NI} c$. Thus,

$$\mathrm{E}[\tau] = \frac{1}{\lambda\kappa}\left(p_2(T) - p_2\left(\bar{S}^{NI}\right)\right) + \bar{S}^{NI}.$$

Differentiating both sides with respect to κ yields

$$\frac{\partial\,\mathrm{E}[\tau]}{\partial\kappa} = \frac{e^{-T\lambda}(1+T\lambda) - e^{-\bar{S}^{NI}\lambda} - \kappa}{\kappa^2\lambda}.$$

The equation

$$e^{-T\lambda}(1+T\lambda) - e^{-\bar{S}^{NI}\lambda} - \kappa = 0$$

can be equivalently rewritten as

$$e^{-T\lambda} - e^{-\bar{S}^{NI}\lambda} = \kappa - e^{-T\lambda}T\lambda.$$

It has a unique solution corresponding to $\kappa = \kappa^{ND}(T,\lambda) := e^{-T\lambda}T\lambda$. As $\kappa > \kappa^{ND}(T,\lambda)$, it holds that $\partial\,\mathrm{E}[\tau]/\partial\kappa < 0$.

2. The case of $\kappa^{NI}(T,\lambda) < \kappa \leq \tilde{\kappa}(T,\lambda)$.

The principal's long-run payoff under any obedient optimal policy is given by

$$\mathrm{E}[\tau]\, c = p_2(T)\, v.$$

Rewriting it equivalently, $\mathrm{E}[\tau] = \frac{1}{\lambda}\frac{1}{\kappa}p_2(T) \Rightarrow \partial\,\mathrm{E}[\tau]/\partial\kappa < 0$.

□

Proof of Lemma 7. Lemma 4 implies that if the distribution of the stopping time τ assigns zero probability to stopping in states 0 and 1 then $V(\tau) < 0$ and the individual rationality constraint is violated. Thus, the necessary condition for a information policy represented by τ to be individually rational under $\kappa \in (\tilde{\kappa}(T,\lambda), \kappa^{FI}(T,\lambda))$ is that it assigns a positive probability to stopping not only in state 2, but also to stopping in either state 0 or state 1.

First, consider a stopping time τ that assigns a positive probability to stopping in state 1, i.e. $F_1(t|t_1) > 0$ for some $t \in [t_1, T)$. A pure information policy σ induces a stopping time τ^π defined on the probability space $(\mathcal{H}, \mathcal{F}, P)$, where $\mathcal{H}$ is the space of histories and $\mathcal{F}$ is the natural filtration of the state process x_t. Assume there is a positive mass of histories $H_1 \subseteq \mathcal{H}$ for some given stopping time τ^π:

$$\begin{aligned} H_1 := &\left(H^A := \{h|\tau_1(h) \leq \tau^\pi(h) < \tau_2(h) \leq T\}\right) \\ &\cup \left(H^B := \{h|\tau_1(h) \leq \tau^\pi(h) < T < \tau_2(h)\}\right) \\ &\cup \left(H^C := \{h|\tau_1(h) \leq \tau^\pi(h) = T < \tau_2(h)\}\right). \end{aligned}$$

We proceed with showing that at optimum, $P(H^A \cup H^B) = 0$, i.e., stopping in state 1 is possible only at T. Assume $P(H^A \cup H^B) > 0$ and consider a new stopping time $\tilde{\tau}^\pi$, which differs from τ^π as follows: $\forall h \in H_1$, let $\tilde{\tau}^\pi(h) = \tau_2(h) \wedge T$ (so that under $\tilde{\tau}^\pi$, $\forall h \in H^A$ stopping occurs in state 2), and $\forall h \in H \backslash H_1$, nothing is changed. Hereafter, for the sake of conciseness, we drop the argument h of the stopping time $\tau^\pi(h)$. Assessing the change in the principal's payoff yields:

$$\begin{aligned} &V(\tilde{\tau}^\pi) - V(\tau^\pi) \\ &\qquad = \int_{H^A} v dP(h) - \int_{H^A \cup H^B} c \cdot \tau_2 \wedge T dP(h) - \left(\int_{H^A \cup H^B} (0 - c \cdot \tau^\pi) \, dP(h) \right) \\ &\qquad = v \, \mathrm{P}(H^A) - c \int_{H^A \cup H^B} (\tau_2 \wedge T - \tau^\pi) \, dP(h). \end{aligned} \tag{40}$$

Let $H^\cup := H^A \cup H^B, H^A \cup H^B = \{h | \tau_1 \leq \tau^\pi < \tau_2 \wedge T\}$. Further, note that $H^A = H^\cup \cap \{\tau_2 \leq T\}$. Given this, the expression (40) becomes

$$\begin{aligned} &V(\tilde{\tau}^\pi) - V(\tau^\pi) \\ &\qquad = v \, \mathrm{P}\left(H^\cup \cap \{\tau_2 \leq T\}\right) - c \, \mathrm{E}\left(\tau_2 \wedge T - \tau^\pi | H^\cup\right) \mathrm{P}(H^\cup) \\ &\qquad = v \, \mathrm{P}\left(\tau_2 \leq T | H^\cup\right) \mathrm{P}\left(H^\cup\right) - c \, \mathrm{E}\left(\tau_2 \wedge T - \tau^\pi | H^\cup\right) \mathrm{P}(H^\cup) \\ &\qquad = \mathrm{P}\left(H^\cup\right) \Big(v \, \mathrm{P}\left(\tau_2 \leq T | H^\cup\right) - c \, \mathrm{E}\left(\tau_2 \wedge T - \tau^\pi | H^\cup\right) \Big). \end{aligned} \tag{41}$$

Further, inspecting the expression in the brackets in the last line of (41) yields:

$$\begin{aligned} & v \, \mathrm{P}\left(\tau_2 \leq T | H^\cup\right) - c \, \mathrm{E}\left(\tau_2 \wedge T - \tau^\pi | H^\cup\right) \\ = & \int_{\hat{H}} v \, \mathrm{P}\left(\tau_2 \leq T | H^\cup \cap \{\tau^\pi = S \pm \varepsilon\} \cap \{\tau_1 = t_1 \pm \varepsilon\}\right) \\ & - c \, \mathrm{E}\left(\tau_2 \wedge T - S | H^\cup \cap \{\tau^\pi = S \pm \varepsilon\} \cap \{\tau_1 = t_1 \pm \varepsilon\}\right) dP\left(h\right), \end{aligned} \tag{42}$$

where $\{\tau^\pi = t \pm \varepsilon\}$ is a shorthand for $\tau^\pi \in [t - \varepsilon, t + \varepsilon]$ and

$$\hat{H} = \{h | \{\tau^\pi = S \pm \varepsilon\} \cap \{\tau_1 = t_1 \pm \varepsilon\} S \in [0, T], t_1 \in [0, T]\}.$$

It can be show that

$$\{H^\cup \cap \{\tau^\pi = S\}\} \;\; = \;\; \{x_S = 1\}. \tag{43}$$

Further, it can be shown that for all $0 < t_1 \leq S \leq T$,

$$\begin{aligned} \mathrm{P}\left(\tau_2 \leq T | \{x_S = 1\} \cap \{\tau_1 = t_1\}\right) &= \mathrm{P}\left(\tau_2 \leq T | x_S = 1\right), \\ \mathrm{E}\left(\tau_2 \wedge T - S | \{x_S = 1\} \cap \{\tau_1 = t_1\}\right) &= \mathrm{E}\left(\tau_2 \wedge T - S | x_S = 1\right). \end{aligned} \tag{44}$$

Given (43) and (44), consider (42), where we integrate over the set $\hat{H}$. Consider subset of $\hat{H}$ such that $t_1 > S$, taking $\lim_{\varepsilon \to 0}$ of the integral over this subset yields 0 as the bounded convergence theorem applies. Now consider taking $\lim_{\varepsilon \to 0}$ of integral over the complement subset such that $t_1 \leq S$ (and applying the bounded convergence theorem): given (43), $\lim_{\varepsilon \to 0}$ of the expression under the integral in (42) becomes

$$v \, \mathrm{P}\left(\tau_2 \leq T | \{x_S = 1\} \cap \{\tau_1 = t_1\}\right) - c \, \mathrm{E}\left(\tau_2 \wedge T - S | \{x_S = 1\} \cap \{\tau_1 = t_1\}\right). \tag{45}$$

Further, given (44), for any t_1, S in the $t_1 \leq S$ subset of $\hat{H}$, (45) is given by

$$v\,\mathrm{P}\left(\tau_2 \leq T | x_S = 1\right) - c\,\mathrm{E}\left(\tau_2 \wedge T - S | x_S = 1\right) = \left(v - \frac{c}{\lambda}\right)\left(1 - e^{-\lambda(T-S)}\right) > 0, \tag{46}$$

where the sign of expression follows from $\kappa \leq \kappa^{FI} < 1$ and the expression for $\mathrm{E}\left(\tau_2 \wedge T - S | x_S = 1\right)$ is obtained in (21). Thus $\lim_{\varepsilon \to 0}$ of the integral over this subset yields a strictly positive value, and (42) is positive. Finally, as $P\left(H^{\cup}\right) > 0$ in (41), we get that $V(\tilde{\tau}^{\pi}) - V(\tau^{\pi}) > 0$. Given this, it is straightforward that $W(\tilde{\tau}^{\pi}) - W(\tau^{\pi}) > 0$. Thus, for a pure information policy σ to be optimal, it should not assign a positive probability to stopping in state 1. It can be shown that this necessary condition carries over to a mixed information policy σ^{μ}, and thus

$$F_1(t|t_1) = 0, \forall t \in [t_1, T). \tag{47}$$

Given (47), we can wlog restrict attention to τ^{π} which assigns a positive probability to stopping in states 0 and 2. Our goal here is to show that at optimum stopping in state 2 happens immediately. Given (47), wlog consider the following partition of $\mathcal{H}$ for some given τ^{π}:

(i). $H_0 := \{h | \tau^{\pi} < \tau_1 \wedge T\}$, i.e., such histories that stopping occurs in state 0,

(ii). $H_1 := \{h | \tau_1 \leq \tau^{\pi} = T < \tau_2\}$, i.e., stopping occurs in state 1,

(iii). $H_2 := \left(H_2^A := \{h | \tau_2 < \tau^{\pi} \leq T\}\right) \cup \left(H_2^B := \{h | \tau_2 = \tau^{\pi} \leq T\}\right)$, i.e., stopping occurs in state 2.

Showing that at optimum stopping in state 2 happens immediately boils down to showing that optimality requires that $\mathrm{P}(H_2^A) = 0$. Note that as histories are induced by pure information policy σ, while choosing τ^{π}, which occurs before τ_1, the principal does not distinguish between any of the histories, and thus

$$\tau^{\pi}(h) = S_0 \in [0, T], \forall h \in H_0,$$

where S_0 is deterministic. Given this, the partition becomes:

$$\begin{aligned} H_0 :=& \{h | S_0 < \tau_1 \wedge T\}, \\ H_1 :=& \{h | \tau_1 \leq S_0 \leq \tau^{\pi} = T < \tau_2\}, \\ H_2 :=& \left(H_2^A := \{h | \tau_1 \leq S_0\} \cap \{h | \tau_2 < \tau^{\pi} \leq T\}\right) \cup \\ & \cup \left(H_2^B := \{h | \tau_1 \leq S_0\} \cap \{h | \tau_2 = \tau^{\pi} \leq T\}\right). \end{aligned} \tag{48}$$

The goal is to show that optimality requires that $\mathrm{P}\left(H_2^A\right) = 0$. We proceed with constructing $\hat{\tau}^{\pi}$ which gives the principal a payoff higher than τ^{π}. $\hat{\tau}^{\pi}$ is constructed as follows. First, for all $h \in H_2^A$, at $t = \tau_2$, flip a coin with a distribution $\theta \in [0, 1]$. In the case of heads, stop right away, i.e., at τ_2. In the case of tails, proceed according to τ^{π}. Second, for all $h \in H$, add $\Delta S \in [0, T - S_0]$ to S_0 to ensure that

$$V(\tau^{\pi}) = V(\hat{\tau}^{\pi}) \geq 0, \tag{49}$$

where the inequality is implied by the $t = 0$ IR constraint for the agent. We proceed with showing that such $\Delta S \geq 0$ exists. Denote for each h:

$$\Delta\tau^\pi(h) := \begin{cases} \tau^\pi(h) - \tau_2(h), & \text{if } \tau_2(h) < \tau^\pi(h) \\ 0, & \text{otherwise.} \end{cases}$$

Writing out (49):

$$\begin{aligned} & v\,\mathrm{P}(H_2) - c\left(\int_{H_2} \tau_2 + \Delta\tau^\pi dP + S_0\,\mathrm{P}(H_0) + T\,\mathrm{P}(H_1)\right) \\ =& v\,\mathrm{P}(\hat{H}_2) - c\left(\int_{\hat{H}_2} \tau_2 + \Delta\tau^\pi dP - \theta \cdot \int_{\hat{H}_2^A} \Delta\tau^\pi dP + (S_0 + \Delta S)\,\mathrm{P}(\hat{H}_0) + T\,\mathrm{P}(\hat{H}_1)\right), \end{aligned} \tag{50}$$

where the sets $\hat{H}_0, \hat{H}_1, \hat{H}_2$ are defined as follows:

$$\begin{aligned} \hat{H}_0 :=& \{h | S_0 + \Delta S < \tau_1 \wedge T\} \\ \hat{H}_1 :=& \{h | \tau_1 \leq S_0 + \Delta S \leq \hat{\tau}^\pi = T < \tau_2(h)\} \\ \hat{H}_2 :=& \left(H_2^A := \{h | \tau_1 \leq S_0 + \Delta S\} \cap \{h | \tau_1 \leq \tau_2 < \hat{\tau}^\pi \leq T\}\right) \cup \\ & \cup \left(H_2^B := \{h | \tau_1 \leq S_0 + \Delta S\} \cap \{h | \tau_1 \leq \tau_2 = \hat{\tau}^\pi \leq T\}\right). \end{aligned} \tag{51}$$

The goal is to prove that the equation (50) has a solution in ΔS. Rewriting equation (50) equivalently, while keeping $V(\tau^\pi) = 0$ in mind yields:

$$\begin{aligned} 0 =& v\,\mathrm{P}(\hat{H}_2) - c\left(\int_{\hat{H}_2} \tau_2 + \Delta\tau^\pi dP - \theta \cdot \int_{\hat{H}_2^A} \Delta\tau^\pi dP\right) \\ & - c\left((S_0 + \Delta S)\,\mathrm{P}(\hat{H}_0) + T\,\mathrm{P}(\hat{H}_1)\right), \\ & \iff \\ c&\left(\int_{\hat{H}_2} \Delta\tau^\pi dP - \theta \cdot \int_{\hat{H}_2^A} \Delta\tau^\pi dP\right) \\ =& v\,\mathrm{P}(\hat{H}_2) - c\left(\int_{\hat{H}_2} \tau_2 dP + (S_0 + \Delta S)\,\mathrm{P}(\hat{H}_0) + T\,\mathrm{P}(\hat{H}_1)\right). \end{aligned} \tag{52}$$

Consider stopping times $\tilde{\tau}^\pi$ and $\tilde{\tilde{\tau}}^\pi$ given by

$$\tilde{\tau}^\pi = \begin{cases} \tau_2 \wedge T, & \text{if } x_{S_0} > 0 \\ S_0, & \text{otherwise.} \end{cases}$$

$$\tilde{\tilde{\tau}}^\pi = \begin{cases} \tau_2 \wedge T, & \text{if } x_{S_0 + \Delta S} > 0 \\ S_0 + \Delta S, & \text{otherwise.} \end{cases}$$

The equation (52) can be written as

$$\begin{aligned} c\left(\int_{\hat{H}_2} \Delta\tau^\pi dP - \theta \cdot \int_{\hat{H}_2^A} \Delta\tau^\pi dP\right) =& v\,\mathrm{P}\left(x_{\tilde{\tilde{\tau}}^\pi} = 2\right) - c\,\mathrm{E}\left[\tilde{\tilde{\tau}}^\pi\right] \\ & \iff \\ c\left(\int_{\hat{H}_2} \Delta\tau^\pi dP - \theta \cdot \int_{\hat{H}_2^A} \Delta\tau^\pi dP\right) =& V\left(\tilde{\tilde{\tau}}^\pi\right). \end{aligned} \tag{53}$$

Let $\mu(\Delta S)$ denote the LHS and $\lambda(\Delta S)$ denote the RHS of (53). It can be shown that $\mu(\Delta S)$ and $\lambda(\Delta S)$ are continuous in ΔS. First, consider the left bound, $\Delta S = 0$, and the LHS:

$$\begin{aligned}\mu(0) &= c\left(\int_{H_2} \Delta\tau^\pi dP - \theta \cdot \int_{H_2^A} \Delta\tau^\pi dP\right) \\ &= c\left(\int_{H_2^B} \Delta\tau^\pi dP + \int_{H_2^A} \Delta\tau^\pi (1-\theta)\, dP\right) > 0.\end{aligned} \tag{54}$$

Next, consider the RHS.

$$\lambda(0) = v\,\mathrm{P}(H_2) - c\left(\int_{H_2} \tau_2 dP + S_0\,\mathrm{P}(H_0) + T\,\mathrm{P}(H_1)\right) = V(\tilde{\tau}^\pi). \tag{55}$$

Further, it can be shown from $V(\tau^\pi) = 0$ that $V(\tilde{\tau}^\pi) = c\int_{H_2} \Delta\tau^\pi dP$. To see this note that

$$\begin{aligned} V(\tau^\pi) &= 0 \\ \iff v\,\mathrm{P}(H_2) - c\left(\int_{H_2} \tau^\pi dP + S_0\,\mathrm{P}(H_0) + T\,\mathrm{P}(H_1)\right) &= 0 \\ \iff v\,\mathrm{P}(H_2) - c\left(\int_{H_2} \tau_2 dP + \int_{H_2} \Delta\tau^\pi dP + S_0\,\mathrm{P}(H_0) + T\,\mathrm{P}(H_1)\right) &= 0 \\ \iff v\,\mathrm{P}(H_2) - c\left(\int_{H_2} \tau_2 dP + S_0\,\mathrm{P}(H_0) + T\,\mathrm{P}(H_1)\right) &= c\int_{H_2} \Delta\tau^\pi dP \\ \iff V(\tilde{\tau}^\pi) &= c\int_{H_2} \Delta\tau^\pi dP. \end{aligned}$$

Given this, (55) yields

$$\lambda(0) = c\int_{H_2} \Delta\tau^\pi dP. \tag{56}$$

From (54) and (56),

$$\lambda(0) > \mu(0) > 0. \tag{57}$$

Next consider the right bound, $\Delta S = T - S_0$. First, consider LHS:

$$\mu(T - S_0) = c\left(\int_{h:\{\tau_2 \le T\}} \Delta\tau^\pi dP - \theta \cdot \int_{h:\{\tau_2 \le T\} \cap \{\tau^\pi - \tau_2 \ge \varepsilon\}} \Delta\tau^\pi dP\right) \ge 0. \tag{58}$$

Second, consider RHS:

$$\begin{aligned}\lambda(T - S_0) &= v\,\mathrm{P}(\tau_2 \le T) - c\left(\int_{h:\{\tau_2 \le T\}} \tau_2 dP + T\,\mathrm{P}(\tau_1 > T) + T\,\mathrm{P}(\tau_1 \le T < \tau_2)\right) \\ &= v\,\mathrm{P}(\tau_2 \le T) - c\left(\int_{h:\{\tau_2 \le T\}} \tau_2 dP + T\,\mathrm{P}(\tau_2 > T)\right) \\ &= v\,\mathrm{P}(\tau_2 \le T) - \mathrm{E}\left[\tau_2 \wedge T\right] = V\left(\tau_2 \wedge T\right).\end{aligned}$$

Further, as $\kappa > \tilde{\kappa}$, it holds that $V(\tau_2 \wedge T) < 0$. Thus, $\lambda(T - S_0) < 0$ and

$$\lambda(T - S_0) < 0 \le \mu(T - S_0). \tag{59}$$

Given (57) and (59), by the intermediate value theorem, there exists such $\hat{\tau}^{\pi}$ that (49) holds. Finally, as

$$V(\tau^{\pi}) = \mathrm{P}(H_A)v - c\,\mathrm{E}[\tau^{\pi}],$$
$$V(\hat{\tau}^{\pi}) = \mathrm{P}(\hat{H}_A)v - c\,\mathrm{E}[\hat{\tau}^{\pi}],$$

$V(\tau^{\pi}) = V(\hat{\tau}^{\pi})$ by construction, and $\mathrm{P}(\hat{H}_A) > \mathrm{P}(H_A)$ by $\Delta S > 0$, and thus it follows that $\mathrm{E}[\hat{\tau}^{\pi}] > \mathrm{E}[\tau^{\pi}]$. Thus, $W(\hat{\tau}^{\pi}) > W(\tau^{\pi})$ and $\hat{\tau}^{\pi}$ gives the agent a payoff higher than τ^{π}. Thus, if a pure information policy σ is optimal then $\mathrm{P}\left(H_2^A\right) = 0$, i.e., stopping in state 2 happens immediately. Finally, it can be shown that this necessary condition carries over to a mixed information policy σ^{μ}, and thus

$$F_2(t|t_1,t_2) = 1, \forall t \in [t_2, T].$$

□

Proof of Proposition 2. Given Lemma 7, the space of candidate optimal information policies under $\kappa \in (\tilde{\kappa}(T,\lambda), \kappa^{FI}(T,\lambda)]$ simplifies to information policies such that stopping in state 2 happens at τ_2, and also stopping in state 0 happens with positive probability. Thus, to characterize the information policy under $\kappa \in (\tilde{\kappa}(T,\lambda), \kappa^{FI}(T,\lambda)]$, We need to characterize the assignment of the probability mass of stopping in state 0 that is optimal for the agent given the principal's individual rationality constraints, i.e., choice of $F_0(t)$.

At $t = 0$, the agent chooses a distribution F_0 on $[0, T]$, observable to both the agent and the principal. ρ stands for the random date at which the stopping occurs if the state is 0 by that date. ρ is drawn at $t = 0$ according to F_0, which is independent from the state process x_t, and the draw privately observed by the agent.

We proceed to solving the agent's problem:

$$\begin{aligned} &\max_{F_0} \left\{ \mathrm{E}_{F_0}\left[cE\left[\tau\right]\right]\right\} \\ &\text{s.t. } \mathrm{E}_{F_0}\left[V_t(\tau)|t<\tau\right] \geq 0, \forall t \geq 0, \end{aligned} \tag{60}$$

where τ is given by (61).

We proceed in two steps: first, we formulate and *solve the relaxed version of* (60) with individual rationality constraint only for $t = 0$; second, we demonstrate that the solution to the relaxed problem *satisfies the full system of constraints in* (60).

The individual rationality constraint in the relaxed problem is given by

$$\mathrm{P}\left(x_{\tau} = 2\right)v - \mathrm{E}\left[\tau\right]c \geq 0.$$

We proceed with a useful lemma.

Lemma 9. *Given an information policy represented by*

$$\tau = \begin{cases} \rho, & \text{if } x_{\rho} = 0 \\ \tau_2 \wedge T, & \text{otherwise,} \end{cases} \tag{61}$$

where $\rho \in [0, T]$, it holds that

$$\mathrm{P}(x_\tau = 2) = \mathrm{P}(x_T = 2) - \mathrm{P}(x_\rho = 0)\,\mathrm{P}(x_T = 2 | x_\rho = 0)$$

and

$$E[\tau] = E[\tau_2 \wedge T] - \mathrm{P}(x_\rho = 0)\,E[\tau_2 \wedge T - \rho | x_\rho = 0].$$

Proof. $\mathrm{P}(x_\tau = 2)$ stands for the mass of events such that the principal gets v. Given (61), the principal gets v either if the second stage is completed not later than ρ or if the first stage is completed not later than ρ and the second stage is completed not later than T. Thus,

$$\mathrm{P}(x_\tau = 2) = \mathrm{P}\Big(\{x_\rho = 1\} \cap \{\tau_2 \leq T\}\Big) + \mathrm{P}\Big(x_\rho = 2\Big).$$

Further,

$$\mathrm{P}\Big(\{x_\rho = 1\} \cap \{\tau_2 \leq T\}\Big) = \mathrm{P}(x_\rho = 1)\,\mathrm{P}(\tau_2 \leq T | x_\rho = 1).$$

Thus,

$$\mathrm{P}(x_\tau = 2) = \mathrm{P}\Big(x_\rho = 1\Big)\,\mathrm{P}\Big(\tau_2 \leq T | x_\rho = 1\Big) + \mathrm{P}\Big(x_\rho = 2\Big). \tag{62}$$

Further, from the full probability formula,

$$\begin{aligned}\mathrm{P}(x_\rho = 1)\,\mathrm{P}(\tau_2 \leq T | x_\rho = 1) = \\ &\mathrm{P}(x_T = 2) \\ &- \mathrm{P}(x_\rho = 0)\,\mathrm{P}(\tau_2 \leq T | x_\rho = 0) \\ &- \mathrm{P}(x_\rho = 2)\,\mathrm{P}(\tau_2 \leq T | x_\rho = 2).\end{aligned}$$

Plugging this into (62) yields

$$\mathrm{P}(x_\tau = 2) = \mathrm{P}(x_T = 2) - \mathrm{P}(x_\rho = 0)\,\mathrm{P}(\tau_2 \leq T | x_\rho = 0).$$

We proceed with proving the second result of Lemma 9. Given (61), it holds that

$$\begin{aligned}E[\tau] &= \mathrm{P}(x_\rho = 0)\,E[\tau | x_\rho = 0] + \mathrm{P}(x_\rho > 0)\,E[\tau | x_\rho > 0] \\ &= \mathrm{P}(x_\rho = 0)\,\rho + \mathrm{P}(x_\rho > 0)\,E[\tau_2 \wedge T | x_\rho > 0].\end{aligned} \tag{63}$$

Further, from the full probability formula,

$$\begin{aligned}\mathrm{P}(x_\rho > 0)\,E[\tau_2 \wedge T | x_\rho > 0] =& E[\tau_2 \wedge T] \\ &- \mathrm{P}(x_\rho = 0)\,E[\tau_2 \wedge T | x_\rho = 0].\end{aligned}$$

Plugging this into (63) yields

$$E[\tau] = E[\tau_2 \wedge T] - \mathrm{P}(x_\rho = 0)\,E[\tau_2 \wedge T - \rho | x_\rho = 0].$$

□

Using Lemma 9, the agent's relaxed problem can be written out as:

$$\begin{aligned}&\min_{F_0}\left\{\mathrm{E}_{F_0}\left[\mathrm{P}\left(x_\rho=0\right)\mathrm{E}\left[\tau_2\wedge T-\rho|x_\rho=0\right]\right]\right\}\\&\text{s.t. }\mathrm{E}_{F_0}\left[\mathrm{P}\left(x_\rho=0\right)\left(c\,\mathrm{E}\left[\tau_2\wedge T-\rho|x_\rho=0\right]-v\,\mathrm{P}\left(\tau_2\le T|x_\rho=0\right)\right)\right]\ge -V\left(\tau_2\right).\end{aligned}\tag{64}$$

The Lagrangian function for the problem is

$$\begin{aligned}\mathcal{L}=&\mathrm{E}_{F_0}\left[\mathrm{P}\left(x_\rho=0\right)\mathrm{E}\left[\tau_2\wedge T-\rho|x_\rho=0\right]\right]\\&-\mu\left(\mathrm{E}_{F_0}\left[\mathrm{P}\left(x_\rho=0\right)\left(c\,\mathrm{E}\left[\tau_2\wedge T-\rho|x_\rho=0\right]-v\,\mathrm{P}\left(\tau_2\le T|x_\rho=0\right)\right)\right]+V\left(\tau_2\right)\right),\end{aligned}$$

where $\mathrm{P}\left(x_\rho=0\right)=e^{-\lambda\rho}$,

$$\begin{aligned}&\mathrm{E}\left[\tau_2\wedge T-\rho|x_\rho=0\right]\\=&\mathrm{P}\left(\tau_2\le T|x_\rho=0\right)\int_\rho^T z\cdot\frac{\lambda^2\left(z-\rho\right)e^{-\lambda(z-\rho)}}{\mathrm{P}\left(\tau_2\le T|x_\rho=0\right)}dz+\mathrm{P}\left(\tau_2>T|x_\rho=0\right)T-\rho\\=&\frac{2}{\lambda}-\frac{2}{\lambda}e^{-\lambda(T-\rho)}-e^{-\lambda(T-\rho)}\left(T-\rho\right)\end{aligned}\tag{65}$$

and

$$\mathrm{P}\left(\tau_2\le T|x_\rho=0\right)=1-e^{-\lambda(T-\rho)}-\lambda\left(T-\rho\right)e^{-\lambda(T-\rho)}.\tag{66}$$

We obtain the F.O.C., which needs to hold for each value of ρ that has a positive probability in F_0:

$$e^{-\lambda T}\left(c\left(2e^{-\lambda(T-\rho)}-1\right)\left(\mu-1\right)-\mu\lambda v\left(e^{-\lambda(T-\rho)}-1\right)\right)=0.\tag{67}$$

The derivative of the left-hand side of (67) w.r.t. ρ is given by $e^{-\lambda\rho}\lambda\left(2c+\mu\left(\lambda v-2c\right)\right)$. As $\kappa^{FI}\left(T,\lambda\right)<\frac{1}{2}$, the derivative is positive. Thus, there exists at most one ρ that satisfies the FOC (67). Thus, the optimal F_0 is degenerate. We denote it with S_0^A, the interim deadline.

We proceed with characterizing the optimal S_0^A:

$$\begin{aligned}&\min_{S\in[0,T]}\left\{\mathrm{P}\left(x_S=0\right)\mathrm{E}\left[\tau_2\wedge T-S|x_S=0\right]\right\}\\&\text{s.t. }\mathrm{P}\left(x_S=0\right)\left(c\,\mathrm{E}\left[\tau_2\wedge T-S|x_S=0\right]-v\,\mathrm{P}\left(\tau_2\le T|x_\rho=0\right)\right)\ge -V\left(\tau_2\right).\end{aligned}\tag{68}$$

The system of F.O.C. is given by

$$\begin{cases}e^{-\lambda T}c\left(2e^{-\lambda(T-S)}-1\right)\left(\mu-1\right) & \ge 0 \text{ if } S=0\\ -e^{-\lambda T}\mu\lambda v\left(e^{-\lambda(T-S)}-1\right) & =0 \text{ if } S\in(0,T)\\ & \le 0 \text{ if } S=T\\[2ex] \frac{c}{\lambda}e^{-\lambda T}\left(2\left(e^{-\lambda(T-S)}-1\right)-\lambda\left(T-S\right)\right) & \\ -ve^{-\lambda T}\left(\left(e^{-\lambda(T-S)}-1\right)-\lambda\left(T-S\right)\right)+V\left(\tau_2\right)\ge 0 & =0 \text{ if } \mu>0.\end{cases}$$

Assume $\mu=0$. In this case, the first F.O.C. wrt S yields $-ce^{-\lambda T}\left(2e^{-\lambda(T-S)}-1\right)$. The expression is negative for all $S\in(0,T)$. Thus, $\mu>0$, and optimal S solves the binding constraint. Thus, We proceed with inspecting the corresponding equation given by

$$\begin{aligned} &\frac{c}{\lambda}e^{-\lambda T}\left(2\left(e^{-\lambda(T-S)}-1\right)-\lambda\left(T-S\right)\right) \\ &-ve^{-\lambda T}\left(\left(e^{-\lambda(T-S)}-1\right)-\lambda\left(T-S\right)\right) \\ =&-V\left(\tau_2\right), \end{aligned} \tag{69}$$

where $V\left(\tau_2\right)$ is given by (31).

The solution to (69) is given by

$$S=\frac{1}{\lambda}\left[\gamma+\mathcal{W}\left(-\gamma e^{-\gamma}\right)\right], \tag{70}$$

where $\gamma=e^{\lambda T}\frac{1-2\kappa}{1-\kappa}$ and $\mathcal{W}(.)$ denotes the Lambert W function.

Denote the 0 and -1 branches of the Lambert W function by $\mathcal{W}_0(.)$ and $\mathcal{W}_{-1}(.)$. $\kappa\in\left(0,\frac{1}{2}\right)$, thus, $\gamma>0$. (70) depends on γ and for each $\gamma\neq 1$ corresponds to two points as the Lambert W function has two branches. The values of (70) as a function of γ are presented in Figure 7. They are given by

$$S=\begin{cases}\left(\frac{1}{\lambda}\left[\gamma+\mathcal{W}_{-1}\left(-\gamma e^{-\gamma}\right)\right],0\right), & \text{if } \gamma<1\\ \left(0,\frac{1}{\lambda}\left[\gamma+\mathcal{W}_{0}\left(-\gamma e^{-\gamma}\right)\right]\right), & \text{if } \gamma>1\\ 0, & \text{if } \gamma=1.\end{cases}$$

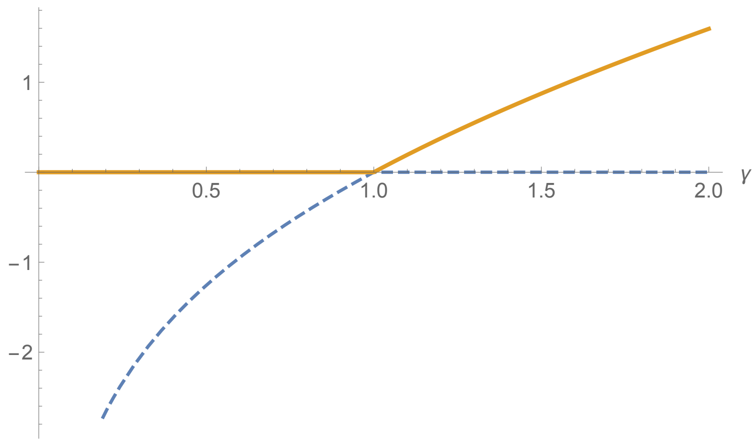

Figure 7: Roots of equation (69) as a function of the parameter γ:
root corresponding to branch 0 of the Lambert W function - **thick**;
root corresponding to branch -1 of the Lambert W function - **dashed**.

γ is decreasing in κ, and $\gamma_{|\kappa=\kappa^{FI}}=1$. As $\kappa\leq\kappa^{FI}$, which corresponds to $\gamma\geq 1$, the solution to (69) is given by

$$S_A=0,\qquad S_B=\tfrac{1}{\lambda}\left[\gamma+\mathcal{W}_0\left(-\gamma e^{-\gamma}\right)\right].$$

As the objective of (68) is decreasing in S and $S_B>S_A$, the solution to (68) is given by

$$S_0^A=\frac{1}{\lambda}\left[\gamma+\mathcal{W}_0\left(-\gamma e^{-\gamma}\right)\right],\gamma=e^{\lambda T}\frac{1-2\kappa}{1-\kappa}. \tag{71}$$

Finally, we can describe the solution to (64): τ is the stopping time such that stopping occurs either at the moment of completion of the second stage of the project or at S_0^A, conditional on the absence of the completion of the first stage of the project, i.e.

$$\tau = \begin{cases} S_0^A, & \text{if } x_{S_0^A} = 0 \\ \tau_2 \wedge T, & \text{otherwise,} \end{cases} \tag{72}$$

where S_0^A is given by (71).

We proceed with the second part of the proof: we demonstrate that (72) satisfies the full system of constraints in (60), and thus solves (60). To do this, we need to demonstrate that $V_t(\tau) \geq 0$, for all $t \in [0, \tau)$. If the recommendation mechanism τ is given by (72), then, for $t < S_0^A$ the absence of stopping at some t reveals that $x_t \neq 2$. Thus,

$$q_1(t) = \frac{p_1(t)}{p_1(t) + p_0(t)} = \frac{\lambda t}{1 + \lambda t}, \; \forall t < S_0^A.$$

Hence, $\dot{q}_1(t) > 0$, for all $t < S_0^A$. Further, for $t \geq S_0^A$, the absence of stopping reveals that $x_t = 1$. Thus, $q_1(t) = 1$, for all $t \geq S_0^A$.

We proceed with a useful lemma.

Lemma 10. *It holds that*

$$\dot{V}_t(\tau) = \lambda q_1(t) V_t(\tau) + v\lambda(\kappa - q_1(t)).$$

Proof. The continuation value of the principal at time t and given the information policy represented by τ is given by

$$\begin{aligned} V_t(\tau) &= (v\lambda q_1(t) - c)\Delta_t + (1 - \lambda q_1(t)\Delta_t) V_{t+\Delta_t}(\tau) \\ &= v\lambda(q_1(t) - \kappa)\Delta_t + (1 - \lambda q_1(t)\Delta_t) V_{t+\Delta_t}(\tau). \end{aligned}$$

Differentiating both sides w.r.t. Δ_t and considering $\lim_{\Delta_t \to 0}(.)$ yields

$$0 = v\lambda(q_1(t) - \kappa) - \lambda q_1(t) V_t(\tau) + \dot{V}_t(\tau),$$

which, after rearranging becomes

$$\dot{V}_t(\tau) = \lambda q_1(t) V_t(\tau) + v\lambda(\kappa - q_1(t)). \tag{73}$$

□

Writing out $V_t(\tau)$ based on Lemma 10 yields

$$\dot{V}_t(\tau) = \lambda q_1(t) V_t(\tau) + v\lambda(\kappa - q_1(t)). \tag{74}$$

$q_1(0) = 0$ and $\dot{q}_1(t) > 0$, for all $t < S_0^A$. we define $\tilde{t}$ as the solution of $\frac{\lambda t}{1+\lambda t} = \kappa$. $q_1(t) < \kappa$, for all $t \in \left[0, \tilde{t} \wedge S_0^A\right]$.

We argue that $V(\tau) \geq 0 \Rightarrow V_t(\tau) \geq 0$, for all $t \in \left(0, \tilde{t} \wedge S_0^A\right)$. Assume that this is not true, then $\exists \hat{t}$ such that $\hat{t} := \inf\left\{t \in \left(0, \tilde{t} \wedge S_0^A\right) : V_t(\tau) < 0\right\}$. As $V_t(\tau)$ is continuous in t, it follows that $V_{\hat{t}}(\tau) = 0$, and by the mean value theorem there must be $\bar{t} \in \left(0, \hat{t}\right)$ such that $\dot{V}_{\bar{t}}(\tau) \leq 0$. But this is in contradiction with the fact that $V_{\bar{t}}(\tau) \geq 0$ and 74.

Consider now $t \in [\tilde{t} \wedge S_0^A, \tau)$. The continuation value can be written as

$$V_t(\tau) = \mathrm{E}\left[\int_t^\tau v\lambda\left(q_1(z) - \kappa\right) dz \,|\, t < \tau\right]. \tag{75}$$

As $\kappa < \frac{1}{2}$ and $q_1(t) = 1$, for all $t \in [S_0^A, \tau)$, it holds that $q_1(t) \geq \kappa$, $\forall t \in [\tilde{t} \wedge S_0^A, \tau)$. Thus, it can be seen from (75) that $V_t(\tau) \geq 0$, $\forall t \in [\tilde{t} \wedge S_0^A, \tau)$.

□

Proof of Proposition 3. We assume it is not the case that $\alpha = 1$ and $\beta = 0$ as, otherwise, agent is indifferent and discloses no information. We start with proving existence of $\tilde{\kappa}$ and then proceed to proving that when the project is promising, an information policy, in which stopping never occurs in state 0, is optimal. Proving existence of $\tilde{\kappa}$ follows the steps of the proof of Lemma 4. The principal's expected payoff is given by

$$V(\tau) = \alpha \,\mathrm{P}(x_\tau = 2)\, v \,\mathrm{E}\left[e^{-r\tau} | \tau_2 \leq \tau\right] - \mathrm{E}\left[\int_0^\tau e^{-rs} ds\right] c.$$

$\tilde{\kappa}$ solves $V(\tau_2) = 0$, or, equivalently

$$\alpha \,\mathrm{P}(x_{\tau_2 \wedge T} = 2)\, v \,\mathrm{E}\left[e^{-r \cdot \tau_2 \wedge T} | \tau_2 \leq T\right] = \mathrm{E}\left[\int_0^{\tau_2 \wedge T} e^{-rs} ds\right] c, \tag{76}$$

where $\mathrm{P}(x_{\tau_2 \wedge T} = 2) = p_2(T)$. Solving (76) for κ yields

$$\tilde{\kappa}(T, \lambda, r, \alpha) = \frac{1}{\lambda\alpha} \frac{\mathrm{P}(x_{\tau_2 \wedge T} = 2)\,\mathrm{E}\left[e^{-r \cdot \tau_2 \wedge T} | \tau_2 \leq T\right]}{\mathrm{E}\left[\int_0^{\tau_2 \wedge T} e^{-rs} ds\right]}.$$

Finally, $V(\tau)$ decreases in κ. Thus, if $\kappa < \tilde{\kappa}(T, \lambda, r, \alpha)$, then a stopping time $\tau = \tau_2 \wedge T$ satisfies the principal's individual rationality constraint.

Consider now the agent's expected payoff $W(\tau)$ given by

$$W(\tau) = (1 - \alpha)\,\mathrm{P}(x_\tau = 2)\, v \,\mathrm{E}\left[e^{-r\tau} | \tau_2 \leq \tau\right] + \mathrm{E}\left[\int_0^\tau e^{-rs} ds\right] \beta c.$$

Consider the case $\kappa \leq \tilde{\kappa}(T, \lambda, r, \alpha)$. Consider a stopping time τ given by (61), i.e., such that stopping happens either immediately at the moment of the second stage completion, or in state 0 at a possibly random interim deadline. Further, consider an alternative stopping time $\hat{\tau} = \tau_2 \wedge T$. Given the two stopping times, $\mathrm{P}(x_{\hat{\tau}} = 2) > \mathrm{P}(x_\tau = 2)$. Further, $\mathrm{E}\left[e^{-r\hat{\tau}} | \tau_2 \leq \hat{\tau}\right] = \mathrm{E}\left[e^{-r\tau} | \tau_2 \leq \tau\right]$ and $\mathrm{E}\left[\int_0^{\hat{\tau}} e^{-rs} ds\right] > \mathrm{E}\left[\int_0^\tau e^{-rs} ds\right]$. As $W(\hat{\tau}) > W(\tau)$ and $\kappa < \tilde{\kappa}(T, \lambda, r, \alpha)$, the agent prefers to implement the stopping time $\hat{\tau}$ rather than τ.

Consider now the case $\kappa > \tilde{\kappa}(T, \lambda, r, \alpha)$. The application of the arguments from the proof of Lemma 7 establishes the result.

□

C The case of no project completion deadline

Importantly, the presence of a hard project deadline T serves as one of the necessary and sufficient conditions for the agent to commit to an interim reporting deadline. Without a hard deadline T, the principal's incentives under full information are different. Recall from Lemma 3 the principal's incentive to continue investing decreases in the length of absence of the first stage completion. In the case $T \to \infty$, the continuation value $V_{t|1}^{FI}$ is constant and given by $v(1-\kappa)$. As a result, the principal's incentive to continue investing given the absence of stage completion does not change over time. Thus, if the principal opts in, he never chooses to stop investing before the completion of the second stage occurs. As a result, setting an interim deadline stops serving as an agent's tool to incentivize the principal's investment. The agent's information policy in the case of no project deadline is given in Lemma 11.

Lemma 11. *Assume that $T \to \infty$. In that case, if $\kappa < \frac{1}{2}$, then the agent uses the information policy presented in Proposition 1.*

Proof of Lemma 11. Under full information and the absence of an exogenous deadline, the principal assigns value v_x to each state $x \in \{0, 1, 2\}$. Clearly, $v_2 = v$ as the principal stops immediately and gets v. In state 1, at each t the principal gets $v\Delta_t$ with probability $\lambda\Delta_t$, v_1 with probability $1 - \lambda\Delta_t$ and pays $c\Delta_t$. As the principal's problem is stationary, the principal's continuation value v_1 does not change with t. Assume that $\kappa < 1$, as otherwise $c \geq \lambda v$ and the principal chooses not to invest in state 1. As the principal's continuation value in state 1 does not change over time,

$$0 = \lambda \cdot (v_2 - v_1) - c,$$

and so

$$v_1 = v - \frac{c}{\lambda} = v(1-\kappa).$$

Thus, the principal wants to invest in state 0 if $c \leq \lambda v_1$, i.e., $\kappa \leq \frac{1}{2}$.

Finally, as the information regarding τ_1 is not decision-relevant for the principal, for $\kappa < \frac{1}{2}$, the agent chooses the information policy that discloses only the completion of the second stage of the project and optimally postpones the disclosure to make the principal's individual rationality constraint bind.

□