Sequential creation of Surplus and the Shapley Value*

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Abstract

We introduce a new family of games, the *Games with Intertemporal Externalities*, where the surplus is created by two sets of disjoint players who play sequentially and where the players in the first stage may affect the conditions for the creation of worth of the players in the second stage. In this class of games, we discuss some extensions of the Shapley properties, and we propose two sharing rules: the one-coalition externality value and the naive value. We first introduce them by computing the players' expected contribution for two random arrival processes and show that each corresponds to the Shapley value of an associated game in characteristic function form. Then, we characterize each of the two values through the basic Shapley axioms together with an additional axiom in the spirit of equal treatment.

1 Introduction

Our choices today may directly or indirectly affect the interests of future generations. This is especially true for decisions with long time horizons, such as the extraction of

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non-renewable resources, those aiming to reduce greenhouse gas (GHG) emissions, the ones concerning nuclear waste disposal, the construction of long-lived infrastructures, or the investment in technical innovation.

From a normative perspective, if today's choices shape future generations' conditions, then it is necessary to discuss how we take the future players (our children, our grandchildren, and those who will follow) into account when deciding the sharing of the surplus of these decisions.

Our paper considers this cooperative inter-generational situation by defining a new family of games, which we refer to as *Games with Intertemporal Externalities*, and proposes cooperative solutions that acknowledge that one generation may be making decisions for people who cannot speak for their interests at the time.

Take the example of global warming. This is a cooperative game with intertemporal externalities, where today's choices are represented by the coalitions formed by today's players. Today, players are aware of the future effects of their decisions on global warming; it is estimated that nowadays, a significant joint effort is needed to meet global warming below the 1.5ºC and 2ºC targets by the end of the XXI century. This is an externality for the future generation. But it is also true that these efforts to reduce greenhouse gas do not seem that urgent for the present generation since the consequences will be seen in the future, and today's generation may not internalize the externalities imposed on the next generation. In an intertemporal externalities game, any coalition formed by the present cohort generates worth today, and the partition of today's generation exerts an externality on the future cohort. The worth that future players create depends on the coalitions they form and the externality inherited from the past generation. We claim that in such a game, a cooperative sharing value needs to consider the two periods and the two sets of players.

We first adapt the classic Shapley axioms to intertemporal externalities games and study their implications. They do not suffice to single out a unique solution. Then, we use the common interpretation of the Shapley value as the players' expected contributions to coalitions to introduce two values: the *one-coalition externality value* and the *naive value*. We also show the relationship between these values and the Shapley value of two associated games.

Our main results characterize the two values by adding one additional property to the classic Shapley axioms, respectively. We show that a property of equal treatment of contributions leads to the characterization of the one-coalition externality value. In contrast, a property of equal treatment of externalities characterizes the naïve value.

The games with intertemporal externalities differ from other cooperative games.

However, they share similarities with the "games with externalities," also called "partition function form games" (Thrall and Lucas, 1963). In this class of games, there is a unique set of players, and the worth of each coalition depends on the partition containing it. Recent literature studies these games, where the worth of a coalition of players depends on the organization of the outside players (see, e.g., Macho-Stadler, Pérez-Castrillo, and Wettstein, 2007, De Clippel and Serrano, 2008, and McQuillin, 2009).[1] However, the family of intertemporal externalities games is not included and does not include the family of games with externalities.

The rest of the paper is organized as follows. Section 2 introduces the family of intertemporal externalities games. Section 3 adapts the Shapley axioms and shows properties of any value that satisfies them. Section 4 introduces the one-coalition externality and the naïve values in an intuitive manner. Sections 5 and 6 axiomatically characterize these values, respectively. Section 7 discusses the prescription of the values for games with intertemporal additive externalities. In particular, it addresses the question of whether the present generation should transfer resources to the future when today's decisions harm (or benefit) future players.

2 Framework

We introduce a new family of games, which we call "games with intertemporal externalities." A game with intertemporal externalities is played by two disjoint sets of players, N_1 and N_2, with $N_1 \cap N_2 = \emptyset$. We think of players in N_1 interacting at period $t = 1$, whereas players in N_2 interact at $t = 2$.[2] We denote generic players of N_1 by i, i', generic players of N_2 by j, j', and generic players of $N_1 \cup N_2$ by h, h'.

A coalition S_1 of N_1 is a group of players of that set, that is, a non-empty subset of N_1, $S_1 \subseteq N_1$. If a coalition S_1 forms, the players obtain jointly a surplus of $v_1(S_1) \in \mathbb{R}$. The worth $v_1(S_1)$ only depends on the coalition S_1 and not on how the other players in $N_1 \setminus S_1$ or N_2 are organized.

Similarly, a coalition S_2 of N_2 is a non-empty subset of N_2, $S_2 \subseteq N_2$. Contrary to what happens at $t = 1$, the worth obtained by a coalition of N_2 depends not only on the identity of the players in the coalition but also on the past organization of the players in N_1, that is, there are intertemporal externalities between $t = 1$ and $t = 2$.

[1] For a review of the literature on values for games with externalities see Macho-Stadler, Pérez-Castrillo, and Wettstein (2019).

[2] There are other environments with two sets of players where our model applies. For instance, the two groups of players may live at two completely separate locations along a river.

To formally express these externalities, denote by $\mathcal{P}(M)$ the set of partitions of a finite set M. Then, if the coalition S_2 forms and the players in N_1 were organized according to the partition $P_1 \in \mathcal{P}(N_1)$, the coalition S_2 generates a surplus $v_2(S_2; P_1) \in \mathbb{R}$.

The utility is transferable among all the players; that is, the cooperative game is a transferable utility (TU) game. In our two-period interpretation of the model, being a TU game requires the existence of a perfect credit market that allows transferring money at zero interest rate (or at zero cost) in any direction between $t = 1$ and $t = 2$.

Therefore, a *game with intertemporal externalities*, or simply a *game*, is a pair (N, v) with $N = (N_1, N_2)$ and $v = (v_1, v_2)$, where $v_1 : 2^{N_1} \to \mathbb{R}$ and $v_2 : 2^{N_2} \times \mathcal{P}(N_1) \to \mathbb{R}$, with $v_1(\emptyset) = 0$ and $v_2(\emptyset; P_1) = 0$ for any $P_1 \in \mathcal{P}(N_1)$. We denote by $\mathcal{G}$ the set of all games.

We look for proposals for the division of the surplus created in games with intertemporal externalities. A *value* is a mapping $\Phi : \mathcal{G} \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ that satisfies

$$\sum_{h \in N_1 \cup N_2} \Phi_h(N, v) = v_1(N_1) + v_2(N_2; \{N_1\}).$$

Note that we have in mind environments where it is efficient that the grand coalition forms in both periods. Hence, our definition of a value entails *efficiency*.

3 The "basic" axioms and first properties

In this section, we first introduce some reasonable requirements to impose on a value by extending those characterizing the Shapley value in TU games without externalities. These are the axioms of linearity, anonymity, and "dummy" player. We first define the operations of *addition* and *multiplication by a scalar*, and the notions of *permutation of games* and *dummy player*.

Definition 1. *(a) The addition of two games (N, v) and (N, v') is the game $(N, v + v')$ defined by $v + v' = (v_1 + v_1', v_2 + v_2')$, where $(v_1 + v_1')(S_1) \equiv v_1(S_1) + v_1'(S_1)$ for all $S_1 \subseteq N_1$ and $(v_2 + v_2')(S_2; P_1) \equiv v_2(S_2; P_1) + v_2'(S_2; P_1)$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.*

(b) Given a game (N, v) and a scalar $\lambda \in \mathbb{R}$, the game $(N, \lambda v)$ is defined by $\lambda v = (\lambda v_1, \lambda v_2)$, where $(\lambda v_1)(S_1) \equiv \lambda v_1(S_1)$ for all $S_1 \subseteq N_1$ and $(\lambda v_2)(S_2; P_1) \equiv \lambda v_2(S_2; P_1)$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.

The permutation of a game uses the notion of a permutation of $N = (N_1, N_2)$: A *permutation of* $N = (N_1, N_2)$ is a pair $\sigma = (\sigma_1, \sigma_2)$, where σ_1 is a permutation of N_1 and σ_2 is a permutation of N_2.

Definition 2. *Let* $(N, v) \in \mathcal{G}$ *and* σ *be a permutation of* N. *The permuted game* $(N, \sigma v)$ *is defined by* $\sigma v = (\sigma v_1, \sigma v_2)$, *where* $\sigma v_1(S_1) \equiv v_1(\sigma_1(S_1))$ *for all* $S_1 \subseteq N_1$, *and* $\sigma v_2(S_2; P_1) \equiv v_2(\sigma(S_2); \sigma(P_1))$ *for all* $S_2 \subseteq N_2$ *and* $P_1 \in \mathcal{P}(N_1)$.

A player in N_1 may influence the surplus generated at both periods. On the other hand, a player in N_2 only affects the surplus generated at $t = 2$, although her influence depends on the organization of the players at $t = 1$. This is why the definition of a dummy player is different for the players in N_1 and N_2.

For every finite set M, partition $P \in \mathcal{P}(M)$, and player $h \in M$, we define $P^{-h} = \{T \setminus \{h\} : T \in P\} \cup \{\{h\}\}$. Then:

Definition 3. *(a) Player* $i \in N_1$ *is a dummy player in the game* (N, v) *if*

$$v_1(S_1) = v_1(S_1 \backslash \{i\}) \qquad \text{for all } S_1 \subseteq N_1 \text{ and}$$
$$v_2(S_2; P_1) = v_2\left(S_2; P_1^{-i}\right) \quad \text{for all } S_2 \subseteq N_2 \text{ and all } P_1 \in \mathcal{P}(N_1).$$

(b) Player $j \in N_2$ *is a dummy player in the game* (N, v) *if* $v_2(S_2; P_1) = v_2(S_2 \backslash \{j\}; P_1)$ *for all* $S_2 \subseteq N_2$ *and* $P_1 \in \mathcal{P}(N_1)$.

We now adapt the three original Shapley (1953) value axioms to our environment:

1. *Linearity*: A value Φ is linear if
   1.1. $\Phi(N, v + v') = \Phi(N, v) + \Phi(N, v')$ for any $(N, v), (N, v') \in \mathcal{G}$, and
   1.2. $\Phi(N, \lambda v) = \lambda \Phi(N, v)$ for any $\lambda \in \mathbb{R}$ and $(N, v) \in \mathcal{G}$.
2. *Anonymity*: A value Φ satisfies anonymity if for any game $(N, v) \in \mathcal{G}$ and any permutation σ of N,
$$\Phi(N, \sigma v) = \sigma \Phi(N, v).$$
3. *Dummy player*: A value Φ satisfies the dummy player axiom if, for any game $(N, v) \in \mathcal{G}$, $\Phi_h(N, v) = 0$ if $h \in N_1 \cup N_2$ is a dummy player in the game (N, v).

Axioms 1 to 3 characterize a unique value (Shapley, 1953) in the set of games in characteristic function form, which we will refer to as *CFF games*. Let us denote $\mathcal{G}^{CFF}$ the set of CFF games and $(M, \hat{v}) \in \mathcal{G}^{CFF}$, i.e., M is the set of players and $\hat{v} : 2^M \to \mathbb{R}$ is the characteristic function.[3] The Shapley value Sh of a player $h \in M$ can be written as

$$Sh_h(M, \hat{v}) = \sum_{S \subseteq M} \beta_h(M, S)\, \hat{v}(S) = \sum_{S \subseteq M, S \ni h} \beta_h(M, S)\, MC_h(S),$$

[3] We will use characters with "hat," as $\hat{v}$, to easily identify when we refer to a characteristic function of a CFF game instead of a worth function in a game with intertemporal externalities.

where $MC_h(S)$ is the contribution of player $h \in M$ to a coalition S that includes her, that is, $MC_h(S) = \hat{v}(S) - \hat{v}(S \backslash \{h\})$, and for every $S \subseteq M$,[4]

$$\beta_h(M,S) = \begin{cases} \frac{(|S|-1)!(m-|S|)!)}{m!} & \text{if } h \in S \\ \frac{-(|S|!(m-|S|-1)!))}{m!} & \text{if } h \in M \backslash S. \end{cases}$$

Note that if $N_1 = \emptyset$ or $N_2 = \emptyset$, then the game with intertemporal externalities (N, v) is essentially a CFF game where the set of players is either N_2 or N_1, respectively. Therefore, any value that satisfies axioms 1 to 3 proposes the Shapley value for those games.

Moreover, consider a game (N, v) where both sets, N_1 and N_2, are non-empty, but there are no intertemporal externalities. That is, suppose that the surplus generated by any coalition of N_2 does not depend on the organization of the players in $t = 1$. Denote $(N_1, \hat{v}_1)$ the CFF game where $\hat{v}_1(S_1) = v_1(S_1)$ for all $S_1 \in N_1$.[5] Also, for a game without externalities, denote $\hat{v}_2(S_2) \equiv v(S_2; P_1)$ for any $S_2 \in N_2$ and $P_1 \in \mathcal{P}(N_1)$. Then, a value satisfying the three axioms allocates the Shapley value of $(N_1, \hat{v}_1)$ to the players of N_1 and the Shapley value of $(N_2, \hat{v}_2)$ to the players of N_2. We state and prove this result in Proposition 1.

Proposition 1. *Take a value Φ satisfying linearity, anonymity, and the dummy player axiom. Also, consider a game (N, v) without externalities, that is, $v_2(S_2; P_1) = v_2(S_2; Q_1)$ for all $S_2 \subseteq N_2$ and $P_1; Q_1 \in \mathcal{P}(N_1)$. Then, denoting $\hat{v}_2(S_2) \equiv v(S_2; P_1)$ for any $S_2 \in N_2$ and $P_1 \in \mathcal{P}(N_1)$, we have*

$$\Phi_i(N, v) = Sh_i(N_1, \hat{v}_1) \text{ for all } i \in N_1 \text{ and}$$
$$\Phi_j(N, v) = Sh_j(N_2, \hat{v}_2) \text{ for all } j \in N_2.$$

Proof. Define the games (N, v^a) and $\left(N, v^b\right)$ as follows:

$$v_1^a(S_1) = v_1(S_1) \text{ for all } S_1 \subseteq N_1,$$
$$v_2^a(S_2; P_1) = 0 \text{ for all } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1),$$
$$v_1^b(S_1) = 0 \text{ for all } S_1 \subseteq N_1,$$
$$v_2^b(S_2; P_1) = v_2(S_2; P_1) \text{ for all } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1).$$

[4] We denote $|M|$ the number of players in M, for any finite set M.

[5] We will use v_1 to refer to the first component of the vector v in a game with intertemporal externalities (N, v); whereas $\hat{v}_1$ refers to the characteristic function of CFF game without externalities, i.e., $(N_1, \hat{v}_1)$.

Note that $(N,v) = \left(N, v^a + v^b\right)$. Then, by linearity, $\Phi_h(N,v) = \Phi_h(N,v^a) + \Phi_h(N,v^b)$ for all $h \in N_1 \cup N_2$.

All the players in N_2 are dummy players in (N, v^a). Then, by the dummy player axiom $\Phi_j(N,v^a) = 0$ for every $j \in N_2$. Moreover, (N, v^a) is essentially a CFF game among the players in N_1 with a characteristic function $\hat{v}_1$, which is equal to the function v_1^a. Then, we can follow the same steps as in the original proof by Shapley (1953) and conclude that $\Phi_i(N,v^a) = Sh_i(N_1, \hat{v}_1)$ for every $i \in N_1$.

Similarly, all the players in N_1 are dummy players in $\left(N, v^b\right)$: A player $i \in N_1$ does not generate any value in v_1^b and her position in the partition formed at $t = 1$ does not affect the surplus of any coalition $S_2 \subseteq N_2$. Hence, by the dummy player property, $\Phi_i\left(N, v^b\right) = 0$, for every $i \in N_1$. Since the game $\left(N, v^b\right)$ is without externalities, v_2^b is equivalent to a CFF game with the player set N_2. Then, the classic characterization of the Shapley value implies that $\Phi_j\left(N, v^b\right) = Sh_j\left(N_2, \hat{v}_2^b\right)$, where $\hat{v}_2^b = \hat{v}_2$, for every $j \in N_2$.

We obtain the expressions in the proposition using $\Phi_h(N,v) = \Phi_h(N,v^a) + \Phi_h(N,v^b)$ for all $h \in N_1 \cup N_2$. □

Proposition 2 goes a step forward. It shows that because there are no externalities affecting the function v_1, the worth generated at $t = 1$ should always be split only among the players in N_1, and the sharing should be done according to the Shapley value. On the other hand, the function v_2 receives the influence of players in N_1 and N_2; hence, all the players may share the surplus obtained at $t = 2$.

Proposition 2. *Take a value Φ satisfying linearity, anonymity, and the dummy player axiom. Then, for every $(N,v) \in \mathcal{G}$,*

$$\Phi_i(N,v) = Sh_i(N_1, \hat{v}_1) + f_i(N_1, N_2, v_2) \text{ for all } i \in N_1 \text{ and}$$
$$\Phi_j(N,v) = f_i(N_1, N_2, v_2) \text{ for all } j \in N_2,$$

where the function f satisfies

$$\sum_{h \in N_1 \cup N_2} f_h(N_1, N_2, v_2) = v_2(N_2; \{N_1\}).$$

Proof. We define the games (N, v^a) and $\left(N, v^b\right)$ as in the proof of Proposition 1. As before, the players in N_2 are dummy players in (N, v^a) hence, $\Phi_j(N, v^a) = 0$ for every $j \in N_2$. By the same argument as in the previous proof, $\Phi_i(N, v^a) = Sh_i(N_1, \hat{v}_1)$ for every $i \in N_1$.

On the other hand, (N, v^b) is a game where players in N_1 do not generate value in $t = 1$ but they exert externalities in $t = 2$. The value obtained by the players in the game (N, v^b) can depend on the sets N_1 and N_2 and on the function v_2, but not on v_1. That is, $\Phi_h(N, v^b)$ corresponds to a function $f_h(N_1, N_2, v_2)$, for every $h \in N_1 \cup N_2$.

The linearity of the value implies $\Phi_h(N, v) = \Phi_h(N, v^a) + \Phi_h(N, v^b)$ for all $h \in N_1 \cup N_2$, which leads to the expressions of $\Phi_h(N, v)$ stated in the proposition.

Finally, $\sum_{h \in N_1 \cup N_2} f_h(N_1, N_2, v_2) = \sum_{h \in N_1 \cup N_2} \Phi_h(N, v^b) = v_1^b(N_1) + v_2^b(N_2; \{N_1\}) = v_2(N_2; \{N_1\})$ because of the efficiency of Φ. □

4 The players' expected contribution for two random arrival processes

A common interpretation of the Shapley value of a player in a CFF game $(M, \hat{v}) \in \mathcal{G}^{CFF}$ is that it corresponds to her expected contribution to coalitions, where the distribution of coalitions arises in a particular way. Specifically, suppose the players enter a room in some order and that all $|M|$ orderings of the players in M are equally likely. Then $Sh_h(M, \hat{v})$ is the h' expected contribution as she enters the room.

In the next two subsections, we propose two "natural" ways in which players can enter the room in a game with intertemporal externalities; each of them leads to a value for $\mathcal{G}$.

4.1 All orderings are feasible

Consider a situation where to compute the expected contribution of a player, we assume that the players can "arrive" in any order.

Take a game $(N, v) \in \mathcal{G}$. An *ordering* of $N_1 \cup N_2$ is an injective mapping $\omega : N_1 \cup N_2 \to \{1, \ldots, |N_1| + |N_2|\}$. Let $\Omega(N_1 \cup N_2)$ denote the set of orderings of $N_1 \cup N_2$. We divide in two the set of predecessors at a given step, $k \in \{1, \ldots, |N_1| + |N_2|\}$:

$$B_k^\omega(N_1) = \omega^{-1}(\{1, \ldots, k\}) \cap N_1$$
$$B_k^\omega(N_2) = \omega^{-1}(\{1, \ldots, k\}) \cap N_2$$

and $B_0^\omega(N_1) = B_0^\omega(N_1) = \emptyset$. That is, $B_k^\omega(N_1)$ (respectively, $B_k^\omega(N_2)$) is the set of predecessors of the player who arrives at step k who belong to N_1 (respectively, N_2).

We compute the contribution of a player given an ordering ω. Take the player who arrives in the k^{th} step, that is, player $\omega^{-1}(k)$. If she belongs to N_1, then she contributes to changing the surplus obtained according to v_1 since the worth of the coalition $B_k^\omega(N_1)$ may be different from that of $B_{k-1}^\omega(N_1)$ due to the addition of $\omega^{-1}(k)$. Hence, the first contribution of player $\omega^{-1}(k)$ is $v_1(B_k^\omega(N_1)) - v_1(B_{k-1}^\omega(N_1))$. Moreover, player $\omega^{-1}(k)$ may also contribute by changing the externality that players in N_1 exert over the coalition of N_2 formed at this step, that is, $B_k^\omega(N_2)$ (that coincides with $B_{k-1}^\omega(N_2)$). In this logic, we assume that the players in N_1 who did not arrive yet, that is, the players in $N_1 \setminus B_k^\omega(N_1)$, remain singleton. Hence, the contribution of player $\omega^{-1}(k)$ to the worth generated by the players in N_2 is $v_2(B_k^\omega(N_2); \{B_k^\omega(N_1), \{i\}_{i\in N_1\setminus B_k^\omega(N_1)}\}) - v_2(B_k^\omega(N_2); \{B_{k-1}^\omega(N_1), \{i\}_{i\in N_1\setminus B_{k-1}^\omega(N_1)}\})$.

If the player $\omega^{-1}(k)$ is in N_2, she may only change the surplus generated by the function v_2. This contribution depends on the set of players in N_1 who have already arrived. Following the same logic as before, the contribution of $\omega^{-1}(k)$, in this case, is $v_2\left(B_k^\omega(N_2); \{B_k^\omega(N_1), \{i\}_{i\in N_1\setminus B_k^\omega(N_1)}\}\right) - v_2(B_{k-1}^\omega(N_2); \{B_k^\omega(N_1), \{i\}_{i\in N_1\setminus B_k^\omega(N_1)}\})$.

Therefore, using that $B_k^\omega(N_2) = B_{k-1}^\omega(N_2)$ if $\omega^{-1}(k) \in N_1$ and $B_k^\omega(N_1) = B_{k-1}^\omega(N_1)$ if $\omega^{-1}(k) \in N_2$, we can write the contribution to (N, v) of the player who arrives at step $k \in \{1, \ldots, |N_1| + |N_2|\}$ of ω as:

$$\begin{aligned} m_k^\omega(N, v) &= v_1(B_k^\omega(N_1)) - v_1(B_{k-1}^\omega(N_1)) \\ &+ v_2(B_k^\omega(N_2); \{B_k^\omega(N_1), \{i\}_{i\in N_1\setminus B_k^\omega(N_1)}\}) - v_2(B_{k-1}^\omega(N_2); \{B_{k-1}^\omega(N_1), \{i\}_{i\in N_1\setminus B_{k-1}^\omega(N_1)}\}). \end{aligned}$$

The *one-coalition externality value* Φ^{1c} is defined for every (N, v) and $h \in N_1 \cup N_2$ as player h's expected contribution, that is,[6]

$$\Phi_h^{1c}(N, v) = \frac{1}{(|N_1| + |N_2|)!} \sum_{\omega\in\Omega(N_1\cup N_2)} m_{\omega(h)}^\omega(N, v). \tag{1}$$

Note that Φ^{1c} is a well-defined value because, for each order, the contributions of all the players in $N_1 \cup N_2$ add up to $v_1(N_1) + v_2(N_2; \{N_1\})$. That is, for any $\omega \in \Omega(N_1 \cup N_2)$,

$$\begin{aligned} \sum_{k=1}^{|N_1|+|N_2|} m_k^\omega(v_2) &= v_1(N_1) + v_2\left(N_2; \{N_1\}\right) - v_1(\emptyset) - v_2(\emptyset; \{\{i\}_{i\in N_1}\}) \\ &= v_1(N_1) + v_2\left(N_2; \{N_1\}\right). \end{aligned}$$

We now show that the one-coalition externality value corresponds to the Shapley value of the associated CFF game $(N_1 \cup N_2, \hat{v}^{1c})$, defined for every $S \subseteq N_1 \cup N_2$ by

$$\hat{v}^{1c}(S) = v_1\left(S \cap N_1\right) + v_2\left(S \cap N_2; \{S \cap N_1, \{i\}_{i\in N_1\setminus S}\}\right).$$

[6] We call it the one-coalition externality value because it only considers the externalities exerted when, at most, one coalition of N_1 is formed.

Proposition 3 states that the one-coalition externality value of a game with intertemporal externalities (N, v) and the Shapley value of the associated CFF game $(N_1 \cup N_2, \hat{v}^{1c})$ coincide.

Proposition 3. *For any game with intertemporal externalities* $(N, v) \in \mathcal{G}$*,*

$$\Phi^{1c}(N, v) = Sh(N_1 \cup N_2, \hat{v}^{1c}). \tag{2}$$

Proof. The set of the orderings that allow computing the Shapley value of the game $(N_1 \cup N_2, \hat{v}^{1c})$ is the same set that we have used to define the one-coalition externality value of (N, v). Moreover, it is immediate to check that, for any order, a player's contribution in both games is the same. Hence, the two values coincide. □

4.2 Players in N_1 go first

The existence of intertemporal externalities suggests that we may want to only consider orderings where the players in N_1 go before the players in N_2. We will call them "constrained orderings." For a game $(N, v) \in \mathcal{G}$, a *constrained ordering* of $N_1 \cup N_2$ is an injective mapping $\theta : N_1 \cup N_2 \to \{1, \ldots, |N_1| + |N_2|\}$ such that $\theta(i) < \theta(j)$, for all $i \in N_1$ and $j \in N_2$. We denote $\Theta(N_1 \cup N_2)$ the set of constrained orderings of $N_1 \cup N_2$. As above, $B_k^\theta(N_1)$ and $B_k^\theta(N_2)$ are the sets of predecessors of the player who arrives at step k who belong to N_1 and N_2.

A player's contribution given a constrained ordering θ is easy to compute. When a player $j \in N_2$ arrives, all the players in N_1 have already arrived; hence, N_2 has been formed. Therefore, the order of arrival does not change the externality that the players in N_1 generate on the worth of the coalitions in N_2. Therefore, the contribution to (N, v) of the player who arrives at step $k \in \{1, \ldots, |N_1| + |N_2|\}$ of θ is:

$$m_k^\theta(N, v) = \begin{cases} v_1(B_k^\theta(N_1)) - v_1(B_{k-1}^\theta(N_1)) & \text{if } \theta^{-1}(k) \in N_1 \\ v_2(B_k^\theta(N_2); \{N_1\}) - v_2(B_{k-1}^\theta(N_2); \{N_1\}) & \text{if } \theta^{-1}(k) \in N_2. \end{cases}$$

We define the *naive value* Φ^n as the players' expected contribution to constrained orderings, that is,

$$\Phi_h^n(N, v) = \frac{1}{|N_1|!\,|N_2|!} \sum_{\theta \in \Theta(N_1 \cup N_2)} m_{\theta(h)}^\theta(N, v),$$

for any $h \in N_1 \cup N_2$. It is easy to check that Φ^n is well-defined, that is, it is efficient.

It is easy to relate the naive value of a game (N, v) with the Shapley value of two CFF games, the first one involving the players of N_1 and the second involving the players in N_2. Define

$$\hat{v}_2^{N_1}(S_2) = v_2(S_2; \{N_1\}) \tag{3}$$

for every $S_2 \subseteq N_2$. Then,

Proposition 4. *For any game with intertemporal externalities $(N, v) \in \mathcal{G}$,*

$$\varPhi_h^n(N, v) = \begin{cases} Sh_h(N_1, \hat{v}_1) & \text{if } h \in N_1 \\ Sh_h\left(N_2, \hat{v}_2^{N_1}\right) & \text{if } h \in N_2. \end{cases} \tag{4}$$

Proof. Any ordering of N_1 appears $|N_2|$ times in the set $\varTheta(N_1 \cup N_2)$ of constrained orderings of $N_1 \cup N_2$ (that is, as many times as there are orderings of the players in N_2, who arrive later). Hence, each contribution of the players in N_1 (which determine their $Sh(N_1, \hat{v}_1)$) appears $|N_2|$ times in the computation of $\varPhi^n(N, v)$. Therefore, the equality 5 holds if $h \in N_1$. A similar argument applies if $h \in N_2$, taking into account that the contributions in both, $\varPhi_h^n(N, v)$ and $Sh_h\left(N_2, \hat{v}_2^{N_1}\right)$, are computed when $P_1 = N_1$. □

Proposition 4 shows that, according to the naive value, players in N_1 only receive the value they create at the first period. They do not enjoy or suffer the consequences of the externality that they generate in the second period by forming the grand coalition in the first period.

The naive value also corresponds to the Shapley value of a related CFF game involving the two sets of players, which we denote $(N_1 \cup N_2, \hat{v}^n)$, defined for every $S \subseteq N_1 \cup N_2$ by

$$\hat{v}^n(S) = v_1\left(S \cap N_1\right) + v_2\left(S \cap N_2; \{N_1\}\right).$$

Proposition 5 states the result.

Proposition 5. *For any game with intertemporal externalities $(N, v) \in \mathcal{G}$,*

$$\varPhi^n(N, v) = Sh(N_1 \cup N_2, \hat{v}^n). \tag{5}$$

Proof. The argument is similar to that in the proof of Proposition 4 since, for instance, for players in N_1, only their position with respect to the other players in N_1 matters to compute their contribution in $\hat{v}^n$. □

In this section, we have introduced the values $\varPhi^{1c}$ and $\varPhi^n$ for the set of games with intertemporal externalities $\mathcal{G}$. Each value is obtained as the expected marginal contributions to coalitions, for a particular coalition arrival process. We have also shown that each corresponds to the Shapley value of an associated CFF game. In the next two sections, we propose new properties to complement the basic axioms described in section 3 to characterize Φ^{1c} and Φ^{n}.

5 Characterization of the one-coalition externality value

In this section, we characterize the one-coalition externality value by adding an equal treatment property to the basic axioms introduced in Section 3. In order to present this axiom, we first define the notion of *equally relevant* players. As we discuss after the definition, it is a demanding notion.

Definition 4. *(a) Players $i, i' \in N_1$ are equally relevant in (N, v) if*

$$v_1(S_1) - v_1(S_1 \setminus \{i\}) = v_1(S_1) - v_1(S_1 \setminus \{i'\}) \qquad \text{for every } S_1 \subseteq N_1 \text{ , and}$$
$$v_2(S_2; P_1) - v_2(S_2; P_1^{-i}) = v_2(S_2; P_1) - v_2(S_2; P_1^{-i'}) \text{ for every } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1).$$

(b) Players $j, j' \in N_2$ are equally relevant in (N, v) if

$$v_2(S_2; P_1) - v_2(S_2 \setminus \{j\}; P_1) = v_2(S_2; P_1) - v_2(S_2 \setminus \{j'\}; P_1) \text{ for every } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1).$$

(c) Players $i \in N_1$ and $j \in N_2$ are equally relevant in (N, v) if

$$v_1(S_1) = v_1(S_1 \setminus \{i\}) \text{ for all } S_1 \subseteq N_1, \text{ and}$$
$$v_2(S_2; P_1) - v_2(S_2 \setminus \{j\}; P_1) = v_2(S_2; P_1) - v_2(S_2; P_1^{-i}) \text{ for every } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1).$$

We provide an intuition of Definition 4. Consider two players in N_1. To be equally relevant, the two players must contribute equally to *any* coalition of N_1 (not only to those coalitions containing both players, as in the classic definition of "equal players"). Moreover, the effect in any coalition of N_2 of having either of the two players isolated is also the same. Hence, considering two players in N_1 equally relevant requires satisfying a very demanding condition. The condition for two players in N_2 to be equally relevant is in the same spirit as the first part of the condition for players in N_1. Finally, we also propose a definition of equally relevant for one agent in N_1 and one agent in N_2. In this case, we require that the player in N_1 does not have an effect on v_1 and the contribution of the player in N_2 be the same as the effect on the externality of the player in N_1 being isolated.

The next value axiom requires that two equally relevant players obtain the same payoff in the value. As we have discussed above, being equally relevant is a very demanding condition; hence, the axiom is weak. In fact, for equally relevant players in N_2, the property is implied by the axiom of anonymity. We state this fact in Remark 1.

Remark 1. *Consider two equally relevant players $j, j' \in N_2$. Then, anonymity implies $\Phi_j(N,v) = \Phi_{j'}(N,v)$, for any $(N,v) \in \mathcal{G}$. Indeed, let Φ be an anonymous value and $\sigma = (\sigma_1, \sigma_2)$ a permutation of N, with σ_1 the identity on N_1 and σ_2 the permutation on N_2 such that $\sigma_2(j) = j'$, $\sigma_2(j') = j$, and $\sigma_2(j'') = j''$, for every $j'' \in N_2 \setminus \{j, j'\}$. Then, it is easy to see that $\sigma v = v$ and by anonymity $\Phi_j(N,v) = \Phi_{j'}(N,v)$.*

We now introduce the axiom of equal treatment.

4. *Equal treatment*: A value Φ satisfies equal treatment if, for any game $(N,v) \in \mathcal{G}$, $\Phi_h = \Phi'_h$, for any equally relevant players $h, h' \in N_1 \cup N_2$.

Theorem 1 states the characterization of the one-coalition externality value using the axiom of equal treatment.

Theorem 1. *Φ^{1c} is the only value satisfying the axioms of linearity, anonymity, dummy player, and equal treatment.*

Proof. We start by showing that Φ^{1c} satisfies all the properties. We use Proposition 3 and Shapley's original axioms for CFF games. By convenience, we refer to $(N_1 \cup N_2, \hat{v}^{1c})$ as "the associated CFF game" of (N,v).

The linearity of Φ^{1c} follows from (a) the associated CFF game of the sum of two games is the sum of the two corresponding associated CFF games, (b) the associated CFF game of the product of a game and a scalar is the product of the corresponding associated CFF game and the scalar, and (c) the linearity of the Shapley value.

Similarly, the anonymity of Φ^{1c} follows from the fact that the associated CFF game of a permuted game is a permuted game of the associated CFF game and the anonymity of the Shapley value.

For the dummy player property, let $i \in N_1$ be a dummy player in (N,v). Then, for every $S \subseteq N_1 \cup N_2$,

$$\begin{aligned}\hat{v}^{1c}(S) =& v_1(S \cap N_1) + v_2\left(S \cap N_2; \{S \cap N_1, \{i'\}_{i' \in N_1 \setminus S}\}\right)\\ =& v_1((S \setminus \{i\}) \cap N_1) + v_2\left(S \cap N_2; \{(S \setminus \{i\}) \cap N_1, \{i'\}_{i' \in N_1 \setminus (S \setminus \{i\})}\}\right)\\ =& v_1((S \setminus \{i\}) \cap N_1) + v_2\left((S \setminus \{i\}) \cap N_2; \{(S \setminus \{i\}) \cap N_1, \{i'\}_{i' \in N_1 \setminus (S \setminus \{i\})}\}\right)\\ =& \hat{v}^{1c}(S \setminus \{i\}),\end{aligned}$$

where the first and last equalities follow the definition of $\hat{v}^{1c}$; the second equality holds because $i \in N_1$ is a dummy player hence, her marginal contribution to $v_1((S \setminus \{i\}) \cap N_1)$ is zero, and the partitions $\{S \cap N_1, \{i'\}_{i' \in N_1 \setminus S}\}$ and $\{(S \setminus \{i\}) \cap N_1, \{i'\}_{i' \in N_1 \setminus (S \setminus \{i\})}\}$ only differ in the affiliation of the dummy player i, which does not affect the worth of the coalition $S \cap N_2$, and the third equality holds because $(S \setminus \{i\}) \cap N_2 = S \cap N_2$.

If $j \in N_2$ is a dummy player in (N, v) then, for every $S \subseteq N_1 \cup N_2$,

$$\begin{aligned}
\hat{v}^{1c}(S) =& v_1(S \cap N_1) + v_2\left(S \cap N_2; \{S \cap N_1, \{i\}_{i \in N_1 \setminus S}\}\right) \\
=& v_1(S \cap N_1) + v_2\left((S \setminus \{j\}) \cap N_2; \{S \cap N_1, \{i\}_{i \in N_1 \setminus S}\}\right) \\
=& v_1((S \setminus \{j\}) \cap N_1) + v_2\left((S \setminus \{j\}) \cap N_2; \{(S \setminus \{j\}) \cap N_1, \{i\}_{i \in N_1 \setminus (S \setminus \{j\})}\}\right) \\
=& \hat{v}^{1c}(S \setminus \{j\}),
\end{aligned}$$

where the second equality holds because $j \in N_2$ is a dummy player and the third because $(S \setminus \{j\}) \cap N_1 = S \cap N_1$ if $j \in N_2$. Then, the dummy player property of Φ^{1c} follows from the homonymous property of the Shapley value.

Finally, we prove that Φ^{1c} satisfies the equal treatment property. Let $(N, v) \in \mathcal{G}$ and $i, i' \in N_1$ be equally relevant players in (N, v). We show that the two players obtain the same payoff in Φ^{1c} if we prove that they are symmetric in the associated game $(N_1 \cup N_2, \hat{v}^{1c})$. Consider any $S \subseteq N_1 \cup N_2$ such that $i, i' \in S$. Then,

$$\begin{aligned}
\hat{v}^{1c}(S \setminus \{i\}) =& v_1\left((S \setminus \{i\}) \cap N_1\right) + v_2\left((S \setminus \{i\}) \cap N_2; \{(S \setminus \{i\}) \cap N_1, \{l\}_{l \in N_1 \setminus (S \setminus \{i\})}\}\right) \\
=& v_1\left((S \cap N_1) \setminus \{i\}\right) + v_2\left(S \cap N_2; \{(S \cap N_1) \setminus \{i\}, \{l\}_{l \in N_1 \setminus S}, \{i\}\}\right) \\
=& v_1\left((S \cap N_1) \setminus \{i\}\right) + v_2\left(S \cap N_2; P_1^{-i}\right),
\end{aligned}$$

where $P_1 \equiv \{(S \cap N_1), \{l\}_{l \in N_1 \setminus S}\}$. A similar equation holds for $\hat{v}^{1c}(S \setminus \{i'\})$. Since i and i' are equally relevant players, $v_1\left((S \cap N_1) \setminus \{i\}\right) = v_1\left((S \cap N_1) \setminus \{i'\}\right)$ and $v_2\left(S \cap N_2; P_1^{-i}\right) = v_2\left(S \cap N_2; P_1^{-i'}\right)$; hence, $\hat{v}^{1c}(S \setminus \{i\}) = \hat{v}^{1c}(S \setminus \{i'\})$, and the players are symmetric, as we wanted to prove.

Next, we show that if $i \in N_1$ and $j \in N_2$ are equally relevant in (N, v), then they are symmetric players in $(N_1 \cup N_2, \hat{v}^{1c})$. Following the same steps as above, we can check that $\hat{v}^{1c}(S \setminus \{j\}) = v_1\left(S \cap N_1\right) + v_2\left((S \cap N_2) \setminus \{j\}; P_1\right)$. Then, for any $S \subseteq N_1 \cup N_2$ such that $i, j \in S$, we have $\hat{v}^{1c}(S \setminus \{i\}) = \hat{v}^{1c}(S \setminus \{j\})$ because $v_1\left((S \cap N_1) \setminus \{i\}\right) = v_1\left(S \cap N_1\right)$ and $v_2\left(S \cap N_2; P_1^{-i}\right) = v_2\left((S \cap N_2) \setminus \{j\}; P_1\right)$. Therefore, i and j obtain the same payoff in $Sh(N, \hat{v}^{1c})$, and, hence, in $\Phi^{1c}(N, v)$.

For the uniqueness, let Φ be a value on $\mathcal{G}$ satisfying the properties. By Proposition 1, we only need to show that the value is uniquely determined for the games $(N, v^b) \in \mathcal{G}^b \equiv \{(N, v) \in \mathcal{G} : v_1(S_1) = 0 \text{ for all } S_1 \subseteq N_1\}$. To show it, we use a basis of the family of games $\mathcal{G}^b$. For any non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$, we define the unanimity game of $(S_2; P_1)$, $(N, v^{(S_2;P_1)}) \in \mathcal{G}^b$, by[7]

$$v_1^{(S_2;P_1)}(T_1) = 0 \text{ for all } T_1 \subseteq N_1 \tag{6}$$

$$v_2^{(S_2;P_1)}(T_2; Q_1) = \begin{cases} 1 & \text{if } S_2 \subseteq T_2 \text{ and } P_1 \preceq Q_1 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We claim that $\{(N, v^{(S_2;P_1)}) : \emptyset \neq S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1)\}$ is a basis of $\mathcal{G}^b$. Clearly, $\mathcal{G}^b$ is a vector space of dimension $(2^{|N_2|}-1)|\mathcal{P}(N_1)|$. Then, it is enough to check that the set of unanimity games is linearly independent. Let $\{\lambda_{(S_2;P_1)} : \emptyset \neq S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1)\}$ be a set of scalars such that $\sum_{\substack{\emptyset \neq S_2 \subseteq N_2 \\ P_1 \in \mathcal{P}(N_1)}} \lambda_{(S_2;P_1)} v^{(S_2;P_1)}$ is the null game. Suppose, by contradiction, that not all the scalars are equal to zero. Then, we choose one of them, $\lambda_{(T_2;Q_1)} \neq 0$, such that for every $T_2' \subseteq T_2$ and $Q_1' \preceq Q_1$, $\lambda_{(T_2';Q_1')} = 0$. The worth of $\sum_{\substack{\emptyset \neq S_2 \subseteq N_2 \\ P_1 \in \mathcal{P}(N_1)}} \lambda_{(S_2;P_1)} v^{(S_2;P_1)}$ evaluated in $(T_2; Q_1)$ is $\sum_{\substack{\emptyset \neq S_2 \subseteq N_2 \\ P_1 \in \mathcal{P}(N_1)}} \lambda_{(S_2;P_1)} v_2^{(S_2;P_1)}(T_2; Q_1) = \lambda_{(T_2;Q_1)} \neq 0$, which is a contradiction and proves the claim.

By linearity, we only need to show that Φ is uniquely determined for every element of the basis. Consider $(N, v^{(S_2;P_1)})$, for any non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. For convenience, we write the partition P_1 as $P_1 = \{A_1, \ldots A_k\} \cup \{\{l\} : l \in A_{k+1}\}$, where $A_1, \ldots A_k$ are non-singleton coalitions. That is, A_{k+1} includes all the players, if any, of the singleton coalitions of P_1. We show that $\Phi_h(N, v^{(S_2;P_1)})$ is uniquely determined for every $h \in N_1 \cup N_2$.

First, take $j \in N_2 \setminus S_2$. It is easy to check that j is a dummy player in $v^{(S_2;P_1)}$. Then, by the dummy player property, $\Phi_j(N, v^{(S_2;P_1)}) = 0$.

Second, consider $i \in A_{k+1}$. Then i is a dummy player in $v^{(S_2;P_1)}$. Indeed, $v_1^{(S_2;P_1)}(T_1) = v_1^{(S_2;P_1)}(T_1 \setminus \{i\}) = 0$ for all $T_1 \subseteq N_1$. Moreover, $P_1 \preceq Q_1$ if and only if $P_1 \preceq Q_1^{-i}$ for every $Q_1 \in \mathcal{P}(N_1)$; hence, $v_2^{(S_2;P_1)}(T_2; Q_1) = v_2^{(S_2;P_1)}(T_2; Q_1^{-i})$ for every $T_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. Then, by the dummy player property, $\Phi_i(N, v^{(S_2;P_1)}) = 0$.

We next show that the payoffs to the agents in $S_2 \cup A_1 \cup \cdots \cup A_k$ are also uniquely determined.

Let $j \in S_2$. Then, $v_2^{(S_2;P_1)}(T_2 \setminus \{j\}; Q_1) = 0$ for every $T_2 \subseteq N_2$ and every $Q_1 \in \mathcal{P}(N_1)$. Since $v_2^{(S_2;P_1)}(T_2 \setminus \{j\}; Q_1) = 0$ is the same for every $j \in S_2$, by anonymity, Φ allocates the same payoff to all the agents in S_2 (see Remark 1).

Consider now $i \in N_1 \setminus A_{k+1}$. Observe that $P_1 \not\preceq Q_1^{-i}$ for every $Q_1 \in \mathcal{P}(N_1)$, because player i forms a singleton coalition in Q_1^{-i} and belongs to a non-singleton coalition in

[7] Given $P, Q \in \mathcal{P}(M)$, we say that P is finer than Q and write $P \preceq Q$ if for every $S \in P$ there is a $T \in Q$ such that $S \subseteq T$.

P_1. Then $v_2^{(S_2;P_1)}(T_2;Q_1^{-i}) = 0$ for every $T_2 \subseteq N_2$ and every $Q_1 \in \mathcal{P}(N_1)$. Recall that, since $v^{(S_2;P_1)} \in \mathcal{G}^b$, players in N_1 do not generate value in the first period. Hence, all the players in $N_1 \setminus A_{k+1}$ are equally relevant. Therefore, by the equal treatment property, Φ allocates the same payoff to all of them.

Moreover, note that we have just seen that for every $i \in N_1 \setminus A_{k+1}$ and $j \in S_2$, $v_1^{(S_2;P_1)}(T_1) = v_1^{(S_2;P_1)}(T_1 \setminus \{i\}) = 0$ for all $T_1 \subseteq N_1$ and

$$\begin{aligned} v_2^{(S_2;P_1)}(T_2;Q_1) - v_2^{(S_2;P_1)}(T_2 \setminus \{j\};Q_1) =& v_2^{(S_2;P_1)}(T_2;Q_1) \\ =& v_2^{(S_2;P_1)}(T_2;Q_1) - v_2^{(S_2;P_1)}(T_2;Q_1^{-i}), \end{aligned}$$

for every $T_2 \subseteq N_2$ and every $Q_1 \in \mathcal{P}(N_1)$. Therefore, $i \in N_1 \setminus A_{k+1}$ and $j \in S_2$ are equally relevant players. Hence, by the equal treatment property, Φ allocates the same payoff to all players in $S_2 \cup A_1 \cup \cdots \cup A_k$. Then, the efficiency implicit in the definition of a value yields

$$\Phi_h\left(v^{(S_2;P_1)}\right) = \begin{cases} \frac{1}{|S_2|+|N_1 \setminus A_{k+1}|} & \text{if } h \in S_2 \cup A_1 \cup \cdots \cup A_k \\ 0 & \text{otherwise,} \end{cases}$$

which corresponds to the value $\Phi^{1c}(N, v^{(S_2;P_1)})$. □

6 Characterization of the naive value

For the characterization of the naive value, we are going to use ideas related to the equal treatment of the players in N_1 who generate similar externalities. We introduce these ideas in some simple games, denoted $u^{(S_2;P_1)}$, that are part of a basis for the set of games with intertemporal externalities $\mathcal{G}$.[8]

Consider a non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. We define the game $u^{(S_2;P_1)} = \left(u_1^{(S_2;P_1)}, u_2^{(S_2;P_1)}\right)$, where $u_1^{(S_2;P_1)} : 2^{N_1} \to \mathbb{R}$ and $u_2^{(S_2;P_1)} : 2^{N_2} \times \mathcal{P}(N_1) \to \mathbb{R}$, by:

$$u_1^{(S_2;P_1)}(R_1) = 0 \quad \text{for all} \quad R_1 \subseteq N_1$$

$$u_2^{(S_2;P_1)}(R_2;Q_1) = \begin{cases} 1 & \text{if } (R_2;Q_1) = (S_2;P_1) \\ 0 & \text{otherwise.} \end{cases}$$

[8] In the proof of Theorem 1, we use for convenience a different basis, which we denoted $\{v^{(S_2;P_1)}\}_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1)}$, for the same set of games.

The set $\{u^{(S_2;P_1)}\}_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)}$ is a basis for the set of games $\mathcal{G}^b \equiv \{(N,v)\in\mathcal{G} : v_1(S_1)=0 \text{ for all } S_1\subseteq N_1\}$.[9,10] Indeed, for any game $(N,v^b)\in\mathcal{G}^b$, we have:

$$v^b = \sum_{S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v^b(S_2;P_1)u^{(S_2;P_1)}. \tag{8}$$

In a game $u^{(S_2;P_1)}$, the role of all the players in N_1 is "similar": it is only when they form precisely the partition P_1 that they generate an externality on the coalition S_2. Our new axiom states that since the role of the players in N_1 in a game $u^{(S_2;P_1)}$ is similar, they should receive the same payoff in "compensation" of the externality that they generate. We call it the axiom of "equal treatment of externalities."

5. *Equal Treatment of Externalities*: A value Φ satisfies equal treatment of externalities if

$$\Phi_i(N,u^{(S_2;P_1)}) = \Phi_{i'}(N,u^{(S_2;P_1)}) \text{ for all } i,i'\in N_1, S_2\subseteq N_2, \text{ and } P_1\in\mathcal{P}(N_1). \tag{9}$$

Lemma 1 provides some information about the payoff obtained by the players in a value that satisfies equal treatment of externalities in addition to the basic axioms.

Lemma 1. *Consider a value Φ that satisfies linearity, anonymity, dummy player, and equal treatment of externalities. Then, there exists weights $\{\alpha(S_2;P_1)\}_{\emptyset\neq S_2\subseteq N_2; P_1\in\mathcal{P}(N_1)}$ satisfying $\sum_{P_1\in\mathcal{P}(N_1)}\alpha(S_2;P_1)=1$ for all $S_2\subseteq N_2$, such that*

$$\Phi_i(N,v) = Sh_i(N_1,\hat{v}_1) + \sum_{S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v_2(S_2;P_1)\Phi_k(N,u^{(S_2;P_1)}) \tag{10}$$

$$\Phi_j(N,v) = Sh_j(N_2,\hat{v}_2^\alpha) - \sum_{S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\not\supseteq\{j\}} \frac{|N_1|}{|N_2\setminus S_2|} v_2(S_2;P_1)\Phi_k(N,u^{(S_2;P_1)}), \tag{11}$$

[9] In the game $u^{(S_2;P_1)}$, forming the grand coalition in both periods is not efficient unless $(S_2;P_1) = (\{N_2\},\{N_1\})$. We use these functions for convenience. However, the same analysis can be done if we define a basis using the functions $w^{(S_2;P_1)}$, which are identical to $u^{(S_2;P_1)}$ except that $w^{(S_2;P_1)}(R_2;Q_1) = 1$ if either $(R_2;Q_1) = (S_2;P_1)$ or $(R_2;Q_1) = (\{N_2\};\{N_1\})$.

[10] Consider the game (N,u^{S_1}), where $u^{S_1} = \left(u_1^{S_1}, u_2^{S_1}\right)$ is defined by:

$$u_1^{S_1}(R_1) = \begin{cases} 1 & \text{if } R_1 = S_1 \\ 0 & \text{otherwise.} \end{cases}$$
$$u_2^{S_1}(R_2;Q_1) = 0 \quad \text{for all} \quad (R_2;Q_1)\in 2^{N_2}\times\mathcal{P}(N_1).$$

Then, the set $\{(N,u^{S_1})\}_{\emptyset\neq S_1\subseteq N_1} \cup \{(N,u^{(S_2;P_1)})\}_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)}$ constitutes a basis for the set of games with intertemporal externalities $\mathcal{G}$.

for any $i \in N_1$ and $j \in N_2$, where $\Phi_k(N, u^{(S_2;P_1)})$ is the value obtained by any $k \in N_1$ in the basis game $(N, u^{(S_2;P_1)})$ and $(N_2, \hat{v}_2^{\alpha})$ is a CFF game defined by

$$\hat{v}_2^{\alpha}(S_2) \equiv \sum_{P_1 \in \mathcal{P}(N_1)} \alpha(S_2; P_1) v_2(S_2; P_1) \tag{12}$$

for any $S_2 \subseteq N_2$.

Proof. We decompose the game (N, v) in the games (N, v^a) and $\left(N, v^b\right)$, as in the proof of Proposition 1. We know that $\Phi(N, u^a)$ assigns $Sh(N_1, \hat{v}_1)$ to the players in N_1 and 0 to the players of N_2. We now focus on $\Phi(N, u^b)$.

Since Φ satisfies *linearity*, then

$$\Phi_h(N, v^b) = \sum_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1)} v^b(S_2; P_1) \Phi_h(N, u^{(S_2;P_1)}) \quad \text{for all} \quad h \in N_1 \cup N_2. \tag{13}$$

The *anonymity* of Φ implies that

$$\Phi_j(N, u^{(S_2;P_1)}) = \Phi_{j'}(N, u^{(S_2;P_1)}) \;\; \text{if} \;\; j, j' \in S_2, \;\text{or}\; j, j' \in N_2 \setminus S_2, \tag{14}$$

and its *efficiency* implies

$$\sum_{h \in N_1 \cup N_2} \Phi_h(N, u^{(S_2;P_1)}) = 0 \;\; \text{if} \;\; (S_2; P_1) \neq (N_2, \{N_1\}), \tag{15}$$

$$\sum_{h \in N_1 \cup N_2} \Phi_h(N, u^{(N_2,\{N_1\})}) = 1. \tag{16}$$

Moreover, because Φ satisfies *linearity*, *anonymity*, and *dummy player*, then

$$\sum_{P_1 \in \mathcal{P}(N_1)} \Phi_h(N, u^{(S_2;P_1)}) = \begin{cases} 0 & \text{if} \;\; h \in N_1 \\ \beta_h(N_2, S_2) & \text{if} \;\; h \in N_2 \end{cases} \tag{17}$$

where $\beta_h(N_2, S_2)$ corresponds to the Shapley number. Equation (17) follows Proposition 1 because $\sum_{P_1 \in \mathcal{P}(N_1)}(N, u^{(S_2;P_1)})$ is a game without externalities; hence, the worth $\sum_{P_1 \in \mathcal{P}(N_1)}(N, u^{(S_2;P_1)})(N_2; \{N_1\})$ (which is equal to 0 unless $(S_2; P_1) = (N_2; \{N_1\})$, in which case the worth is 1) is shared among the players in N_2 according to their Shapley value.

Using (9) and (14), we can express equations (15) and (16) as follows:

$$|N_1| \, \Phi_k(N, u^{(S_2;P_1)}) + |S_2| \, \Phi_j(N, u^{(S_2;P_1)}) + |N_2 \setminus S_2| \, \Phi_{j'}(N, u^{(S_2;P_1)}) = 0 \tag{18}$$

for any $k \in N_1$, $j \in S_2$, and $j' \in N_2 \setminus S_2$, and

$$|N_1| \, \Phi_k(N, u^{(N_2,\{N_1\})}) + |N_2| \, \Phi_j(N, u^{(N_2,\{N_1\})}) = 1, \tag{19}$$

for any $k \in N_1$ and $j \in N_2$.

We write equation (18) as:

$$\Phi_{j'}(N, u^{(S_2;P_1)}) = -\frac{|S_2|}{|N_2 \setminus S_2|}\Phi_j(N, u^{(S_2;P_1)}) - \frac{|N_1|}{|N_2 \setminus S_2|}\Phi_k(N, u^{(S_2;P_1)}), \quad (20)$$

for any $k \in N_1$, $j \in S_2$, and $j' \in N_2 \setminus S_2$, and we notice that the Shapley numbers satisfy the following relation:

$$|S_2|\, \beta_j(N_2, S_2) + |N_2 \setminus S_2|\, \beta_{j'}(N_2, S_2) = 0 \quad (21)$$

for all $j \in S_2$ and $j' \in N_2 \setminus S_2$.

Using (21), we substitute $|N_2 \setminus S_2|$ in equation (20) to obtain:

$$\Phi_{j'}(N, u^{(S_2;P_1)}) = \beta_{j'}(N_2, S_2)\frac{1}{\beta_j(N_2, S_2)}\Phi_j(N, u^{(S_2;P_1)}) - \frac{|N_1|}{|N_2 \setminus S_2|}\Phi_k(N, u^{(S_2;P_1)}), \quad (22)$$

for any $k \in N_1$, $j \in S_2$, and $j' \in N_2 \setminus S_2$.

Define the "weights" $\alpha(S_2; P_1)$ as follows:

$$\alpha(S_2; P_1) \equiv \frac{1}{\beta_j(N_2, S_2)}\Phi_j(N, u^{(S_2;P_1)}), \quad (23)$$

where j is any player in S_2.

Notice that, using (17), $\sum_{P_1 \in \mathcal{P}(N_1)} \alpha(S_2; P_1) = \frac{1}{\beta_j(N_2,S_2)} \sum_{P_1 \in \mathcal{P}(N_1)} \Phi_j(N, u^{(S_2;P_1)}) = 1$ (where j is any player in S_2), for all $S_2 \subseteq N_2$.

Then, equations (23) and (22) lead to

$$\Phi_j(N, u^{(S_2;P_1)}) = \beta_j(N_2, S_2)\alpha(S_2; P_1), \quad (24)$$

$$\Phi_{j'}(N, u^{(S_2;P_1)}) = \beta_{j'}(N_2, S_2)\alpha(S_2; P_1) - \frac{|N_1|}{|N_2 \setminus S_2|}\Phi_k(N, u^{(S_2;P_1)}) \quad (25)$$

for any $j \in S_2$, $j' \in N_2 \setminus S_2$, and $k \in N_1$.

Using (13), (24), and (25), we can express the worth of any player $j \in N_2$ in a game (N, v^b) according to a value Φ that satisfies linearity, anonymity, and equal treatment of externalities as follows:

$$
\begin{aligned}
\Phi_j(N,v^b) =& \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v^b(S_2;P_1)\Phi_j(N,u^{(S_2;P_1)})\\
=& \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\supseteq\{j\}} v^b(S_2;P_1)\beta_j(N_2,S_2)\alpha(S_2;P_1)\\
+& \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\not\supseteq\{j\}} v^b(S_2;P_1)\left(\beta_j(N_2,S_2)\alpha(S_2;P_1) - \frac{|N_1|}{|N_2\setminus S_2|}\Phi_k(N,u^{(S_2;P_1)})\right)\\
=& \sum_{\emptyset\neq S_2\subseteq N_2} \beta_j(N_2,S_2) \sum_{P_1\in\mathcal{P}(N_1)} \alpha(S_2;P_1)v^b(S_2;P_1)\\
-& \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\not\supseteq\{j\}} v^b(S_2;P_1)\frac{|N_1|}{|N_2\setminus S_2|}\Phi_k(N,u^{(S_2;P_1)})\\
=& Sh_j(N_2,\hat{v}_2^\alpha) - \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\not\supseteq\{j\}} \frac{|N_1|}{|N_2\setminus S_2|}\Phi_k(N,u^{(S_2;P_1)})v^b(S_2;P_1),
\end{aligned}
$$

where k is any player in N_1.

Similarly, using (13), we can express the worth of any player $i\in N_1$ as follows:

$$\Phi_i(N,v^b) = \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v^b(S_2;P_1)\Phi_i(N,u^{(S_2;P_1)}).$$

Given that $\Phi_k(N,v^{(S_2;P_1)})$ is the same for every $k\in N_1$, linearity and equal treatment of externalities imply that all the players in N_1 obtain the same payoff in a game (N,v^b).

Finally, the expression in the lemma follows from the linearity of Φ and $v = v^a + v^b$. □

Lemma 1 states that the axiom of equal treatment of externalities, together with linearity, anonymity, and the dummy player axiom, restricts the set of values. However, it does not allow singling out one value. Next, we strengthen this axiom in a "natural" way.

Equal treatment of externalities advocates that the players in N_1 should receive the same payoff in a basis game $u^{(S_2;P_1)}$ because their role in creating the externality is similar. "Strong equal treatment of externalities" requires that, since the role of the players in N_1 in the games $u^{(S_2;P_1)}$ and $u^{(S_2;P_1')}$ are similar, for any $P_1,P_1'\in\mathcal{P}(N_1)$, their payoffs in these game should also be the same.

5'. *Strong Equal Treatment of Externalities*: A value Φ satisfies strong equal treatment of externalities if

$$\Phi_i(N,u^{(S_2;P_1)}) = \Phi_{i'}(N,u^{(S_2;P_1')}) \text{ for all } i,i'\in N_1, S_2\subseteq N_2, \text{ and } P_1,P_1'\in\mathcal{P}(N_1). \tag{26}$$

Theorem 2 uses Lemma 1 to characterize the naive value through our basic axioms plus the strong equal treatment of externalities axiom.

Theorem 2. *Φ^n is the only value satisfying the axioms of linearity, anonymity, dummy player, and strong equal treatment of externalities.*

Proof. We first show that Φ^n satisfies the four axioms. Given the characterization of Φ^n provided in Proposition 4, it is immediate to check that it satisfies linearity, anonymity, and dummy player. It also satisfies strong treatment of externalities because $\Phi_i^n(N, u^{(S_2;P_1)}) = 0$ for all $i \in N_1$, $S_2 \subseteq N_2$, and $P_1 \in \mathcal{P}_1$.

Notice that Φ^n corresponds to the value identified in Lemma 1 when the weights are $\alpha^n(S_2; P_1) \equiv 0$ and $\alpha^n(S_2; \{N_1\}) \equiv 1$, for all $S_2 \subseteq N_2$ and $P_1 \neq \{N_1\}$. For these weights, $\hat{v}_2^{N_1} = \hat{v}_2^{\alpha}$ (see equations (3) and (12)).

We now prove that Φ^n is the only value that satisfies all the axioms. Take Φ satisfying the axioms. We show that $\Phi(N, v) = \Phi^n(N, v)$ for all $(N, v) \in \mathcal{G}$.

First, take $i \in N_1$. Strong equal treatment of externalities requires that, for any $S_2 \subseteq N_2$, $\Phi_i(N, u^{(S_2;P_1)})$ is the same for all $P_1 \in \mathcal{P}(N_1)$. Equation (17) implies that $\sum_{P_1 \in \mathcal{P}(N_1)} \Phi_i(N, u^{(S_2;P_1)}) = 0$. Therefore, $\Phi_i(N, u^{(S_2;P_1)}) = 0$ for all $i \in N_1$, $S_2 \subseteq N_2$, and $P_1 \in \mathcal{P}(N_1)$. Then, using equation (10), $\Phi_i(N, v) = Sh_i(N_1, \hat{v}_1) = \Phi_i^n(N, v)$ for any $i \in N_1$.

Take now $j \in N_2$. Equation (11), together with $\Phi_k(N, u^{(S_2;P_1)}) = 0$ for all $k \in N_1$, implies that

$$\Phi_j(N, v) = Sh_j(N_2, \hat{v}_2^{\alpha}), \tag{27}$$

where $\hat{v}_2^{\alpha}$ is defined in (12), for some weight system α. We prove that it is necessarily the case that $\alpha = \alpha^n$ by induction on the size of the coalition S_2. If $S_2 = N_2$, efficiency requires $\alpha(N_2; P_1) = 0$, for any $P_1 \neq \{N_1\}$. Otherwise, suppose $\alpha(N_2; P_1) \neq 0$ for some $P_1 \neq \{N_1\}$, and consider the game $v = u^{(N_2;P_1)}$. For this game, $\hat{v}_2^{\alpha}(N_2) = \alpha(N_2; P_1)$. Therefore, $Sh(N_2, \hat{v}_2^{\alpha})$ shares $\alpha(N_2; P_1) \neq 0$ among the players in N_2, whereas the efficiency of Φ requires that the sum of the players' payoff be $v(N_2; \{N_1\}) = 0$. Moreover, $\alpha(N_2; P_1) = 0$ for any $P_1 \neq \{N_1\}$ implies $\alpha(N_2; \{N_1\}) = 1$. Hence, $\alpha(N_2; P_1) = \alpha^n(N_2; P_1)$ for all $P_1 \in \mathcal{P}_1$.

Assume now that $\alpha(S_2; P_1) = \alpha^n(S_2; P_1)$ for all $P_1 \in \mathcal{P}_1$ holds for all $S_2 \subseteq N_2$ with $|S_2| \geq m$, for $1 < m \leq |N_2|$.

Consider $S_2 \subseteq N_2$ with $|S_2| = m - 1$, $j \in N_2 \setminus S_2$, and $P_1 \in \mathcal{P}_1$. Define the game (N, w) by $w = u^{(S_2 \cup \{j\};P_1)} + u^{(S_2;P_1)}$. That is, the worth of the coalitions $S_2 \cup \{j\}$ and S_2 is 1 if the partition P_1 has been formed; in any other case, the worth of a coalition is zero. The agent j is a dummy player in (N, w); hence, the dummy player axiom implies $\Phi_j(N, w) = 0$. Moreover, given the worth of the coalitions in w, the CFF game $(N_2, \hat{w}_2^\alpha)$ satisfies

$$\begin{aligned} w_2^\alpha(S_2 \cup \{j\}) =& \alpha(S_2 \cup \{j\}; P_1) \\ w_2^\alpha(S_2) =& \alpha(S_2; P_1) \\ w_2^\alpha(T_2) =& 0 \text{ for all } T_2 \neq S_2, T_2 \neq S_2 \cup \{j\}\}. \end{aligned}$$

The contribution of j to any coalition in the game $(N_2, \hat{w}_2^\alpha)$ is zero, except possibly to S_2. Her contribution to S_2 is $\alpha(S_2 \cup \{j\}; P_1) - \alpha(S_2; P_1)$. Then, $0 = \Phi_j(N, w) = Sh_j(N_2, w_2^\alpha)$ implies that this contribution must be zero; hence, $\alpha(S_2; P_1) = \alpha(S_2 \cup \{j\}; P_1)$ for all $P_1 \in \mathcal{P}_1$. Since $|S_2 \cup \{j\}| = m$, we use the induction argument and obtain $\alpha(S_2; P_1) = \alpha(S_2 \cup \{j\}; P_1) = \alpha^n(S_2 \cup \{j\}; P_1) = 0$ for all $P_1 \neq \{N_1\}$ and $\alpha(S_2; \{N_1\}) = \alpha(S_2 \cup \{j\}; \{N_1\}) = \alpha^n(S_2 \cup \{j\}; \{N_1\}) = 1$.

This completes the induction argument. We have shown that $\alpha = \alpha^n$; hence, Φ^n is the only value satisfying the four axioms. □

7 Games with intertemporal additive externalities

In this section, we introduce a particular class of games, which we call *games with intertemporal additive externalities*. They are games where the intertemporal externality does not vary across the different coalitions that can be formed in the second period. We first illustrate in this class of games the form that any sharing rules satisfying the basic axioms has, and then we illustrate the differences in the distribution of the surplus between the one-coalition externality and the naive values in this family of games.

Formally, a game with intertemporal additive externalities $(N, v) \in \mathcal{G}$ satisfies $v_2(\emptyset; P_1) = 0$ for any $P_1 \in \mathcal{P}(N_1)$ and, for every non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$,

$$v_2(S_2; P_1) = \hat{v}_2(S_2) + e(P_1).$$

The function $\hat{v}_2 : 2^{N_2} \setminus \emptyset \to \mathbb{R}$ provides the worth generated by any non-empty coalition of players in N_2, and the function $e : \mathcal{P}(N_1) \to \mathbb{R}$ measures the externality generated in any coalition by the partition formed among the players in N_1. We normalize the function such that $e(\{\{i\} : i \in N_1\}) = 0$. This assumption is without loss of generality as we could subtract the worth of the partitions of singletons from all the externalities and add it to the game v_1. We denote $\mathcal{G}^A \subset \mathcal{G}$ the family of games with intertemporal additive externalities.

Let us consider a value Φ satisfying linearity, anonymity, and dummy player. We decompose any $(N, v) \in \mathcal{G}^A$ as the sum of two games (N, v') and (N, v''). The game (N, v') satisfies $v_1' = v_1$ and $v_2'(\emptyset; P_1) = 0$ and $v_2'(S_2; P_1) = \hat{v}_2(S_2)$ for any $P_1 \in \mathcal{P}(N_1)$ and every non-empty $S_2 \subseteq N_2$. The game (N, v'') is defined by $v_1'' = 0$ and $v_2''(\emptyset; P_1) = 0$ and $v_2''(S_2; P_1) = e(P_1)$ for any $P_1 \in \mathcal{P}(N_1)$ and every non-empty $S_2 \subseteq N_2$.

Note that (N, v') is a game without externalities. Then, by Proposition 1, $\Phi_i(N, v') = Sh_i(N_1, \hat{v}_1)$ for all $i \in N_1$ and $\Phi_j(N, v') = Sh_j(N_2, \hat{v}_2)$ for all $j \in N_2$. Therefore, any difference between two values in the sharing of the surplus of (N, v) is due to $\Phi(N, v'')$, that is, in the way they share the surplus $e(N_1)$ among the players in $N_1 \cup N_2$.

Concerning the sharing of (N, v''), the only general feature that all the values satisfy is that, by anonymity, all the players in N_2 obtain the same payoff, hence $\Phi_j^n(N, v'') = \Phi_{j'}^n(N, v'')$.

We now present the sharing proposed by Φ^n and Φ^{1c} for games with intertemporal additive externalities.

Consider the naive value, that is, $\Phi = \Phi^n$. Proposition 4 implies that $\Phi_i^n(N, v'') = Sh_i(N, \hat{v}_1'') = 0$ for all $i \in N_1$. Moreover, since all the players in N_2 must obtain the same payoff, $\Phi_j^n(N, v'') = \frac{e(N_1)}{|N_2|}$ for all $j \in N_2$. That is, the naive value allocates equally the surplus (positive or negative) generated by the formation of the grand coalition N_1 to the players in N_2.

Consider now the one-coalition externality value, that is $\Phi = \Phi^{1c}$. For this value, the externality generated by the formation of N_1 is shared among the players in $N_1 \cup N_2$ and not only among the players in N_2. Using equation (1), we compute the equal value assigned by Φ^{1c} to the players in N_2:

$$\Phi_j^{1c}(N, v'') = \sum_{\substack{S_1 \subseteq N_1 \\ |S_1| \geq 2}} \frac{|S_1|!(|N_1 \cup N_2| - |S_1| - 1)!}{|N_1 \cup N_2|!} e(S_1, \{l\}_{l \in N_1 \setminus S_1}),$$

for all $j \in N_2$. On the other hand, the players in N_1 are not symmetric. Following also equation (1), the marginal contributions of a player in N_1 determine the value that Φ^{1c} assigns to her:

$$\Phi_i^{1c}(N, v'') = \sum_{\substack{S_1 \subseteq N_1 \\ S_1 \supseteq \{i\}}} \left(\frac{(|S_1| - 1)!(|N_1| - |S_1|)!}{|N_1|!} - \frac{(|S_1| - 1)!(|N_1 \cup N_2| - |S_1|)!}{|N_1 \cup N_2|!} \right) \times \left(e(S_1, \{l\}_{l \in N_1 \setminus S_1}) - e(S_1 \setminus \{i\}, \{l\}_{l \in N_1 \setminus (S_1 \setminus \{i\})}) \right),$$

for all $i \in N_1$.

To illustrate the previous results on how the externality is shared among the players in N_1 and N_2, consider two games (N, v'') with $N_1 = \{1, 2, 3\}$, $N_2 = \{4\}$, one with positive externalities (that is, forming the grand coalition at $t = 1$ generates the maximum surplus at $t = 2$) and another with negative externalities (that is, forming the grand coalition at $t = 1$ generates the minimum surplus at $t = 2$):

P_1	$e^+(P_1)$	$e^-(P_1)$
$e(\{1\}, \{2\}, \{3\})$	-12	9
$e(\{1, 2\}, \{3\})$	-12	6
$e(\{1, 3\}, \{2\})$	-9	6
$e(\{2, 3\}, \{1\})$	-6	6
$e(\{1, 2, 3\})$	-3	0

For these numerical examples, we have: Using $\Phi = \Phi^n$,

- for e^+: $\Phi_1^n = \Phi_2^n = \Phi_3^n = 0$, $\Phi_4^n = -3$
- for e^-: $\Phi_1^n = \Phi_2^n = \Phi_3^n = 0$, $\Phi_4^n = 0$

Using $\Phi = \Phi^{1c}$,

- for e^+: $\Phi_1^{1c} = 0$, $\Phi_2^{1c} = 1$, $\Phi_3^{1c} = 2$, $\Phi_4^{1c} = -6$
- for e^-: $\Phi_1^{1c} = \Phi_2^{1c} = \Phi_3^{1c} = -1$, $\Phi_4^{1c} = 3$

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