Efficiency and information rents with supplier rationing: renewable energy auctions in India *†

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Abstract

This paper studies studies allocation efficiency and suggests ways of improving it without affecting total information rent in the auctions for solar and wind energy capacity creation conducted by Solar Energy Corporation of India (SECI). These auctions account for 54 gigawatts of renewable electricity capacity in India. SECI's auctions usually have large procurement targets, which are beyond the capacity of a single firm. The auctions are open descending bid format, where bidders publicly reveal their capacity constraint and bid on the selling price of their electricity. The market clearing price is the one with least excess demand, and a residual award is provided to the last exiting bidder. This rationing rule and asymmetric capacities of bidders lead to inefficient allocation. Using SECI's bidding data, I structurally estimate the cost distribution of the bidders, and to conduct counterfactual analysis. Switching to sealed bid reduces probability of inefficient selection by 20-33 percentage points without affecting payment by SECI.

JEL classification: C57;D44;D42;Q49;C72
Keywords: Auctions; econometrics of games; procurement; applied market design; renewable electricity.

*This is a preliminary draft. Please do not cite.

†This study has benefited heavily from guidance of Laurent Lamy (ENPC), Olivier Tercieux (PSE, ENS), and Philippe Gagnepain (PSE, UP1). I also thank Catherine Bobtcheff (PSE), Evan Friedman (PSE), Sylvie Lambert (PSE), Abhijit Tagade (LSE), Aviman Satpathy (PSE, CNRS) and Robin Ng (Mannheim). Financial support from the Agence Nationale pour la Recherche (ANR DACRERISK) is also gratefully aknowledged.

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1 Introduction

Efficient allocation is a central problem in the domain of market design. The knowledge of cost of a potential supplier of a good or a value for a buyer is often unknown to the designer, often a government. The governments use auctions to elicit this private information and allocate the good to a bidder. In order to find the most efficient bidder, many auction mechanisms allow bidders to have some markup, which is their information rent. Allowing for this rents can achieve with the targets of granting rights to natural resources to the most efficient users. However, this efficiency comes at the cost to public funds as governments either pay the information rent to a potential seller in case of procurement, or doesn't extract them from a buyer when it sells some object or resource.

In recent years, auctions have been utilized for the rights to harness solar and wind power on a large scale, with the aim of fostering the growth of renewables-based electricity market. The policy of establishing utility-scale solar and wind farms in developing countries is seen as a crucial step in achieving the global policy target of net carbon neutrality. Depending on their respective policy targets, governments across the world use varied and nuanced auction designs for renewable energy market (for examples, see IRENA, 2015, guide to design of auctions for renawables). The allocation rules within these auctions give rise to strategic interactions among competing potential suppliers. An inefficient supplier may strategically bid in the auction and out-compete a more efficient supplier, or vice versa. In this paper, I study the allocation inefficiency in the solar and wind auctions in India. I demonstrate the inefficient allocation by characterizing equilibrium bidding behavior. Building upon this theoretical framework, I conduct an empirical analysis using data provided by the Solar Energy Corporation of India (SECI) to demonstrate that efficiency can be improved with adjustments to the auction mechanism.

Studying inefficiency in these auctions is important due to their role in creating a large and economically significant renewable energy market in India. The combined production capacity of wind and solar farms in India is 109 gigawatts(GW), which is the 5$^{\text{th}}$ largest in the world. Solar and Wind power accounted for around 90% of the additional electricity capacity for India in 2022 (report by Ember-Climate, 2023). Besides helping India and the world achieve their climate goals, this capacity can also provide electricity to around 28 million urban Indian households per year. The auctions have seen high levels of participation from 158 diverse firms, around 80 of whom have been selected as suppliers. International players like Softbank, Sembcorp, and large Indian firms like Adani, Tata, Renew regularly participate. Global energy firms like Total Energy are also present in the market, through their partnership with local producers. Such high participation has led to reduction in the wholesale price of solar and wind electricity in some parts of India to around INR2.5 per kilowatt-hour (3 cents of USD).

These auctions are conducted by both state and federal government agencies, which sell long-term power purchase agreements to multiple bidders. Each agreement specifies the total capacity of the projects that each bidder is expected to develop and a fixed selling price per kWh of their production (tariff). Prior to the auctions, the auctioneers announce a procurement target. The auctions have a qualifier round and a final round. In the qualifier round, bidders bid their desired capacity (assumed to be equal to their capacity constraint) and tariff.[1] The target set by auctioneers is often too large to be fulfilled by any single bidder's capacity. Moreover, it is common that the total capacity of the bidders with most competitive bids is not exactly equal to the target. This creates a market clearing problem, which is resolved through the allocation rule of implemented in the final round auction.

The final round is same for almost all the agencies. It is an open descending bid auction with two non-standard features- supplier rationing and public revelation of heterogeneous bidder capacities. The award is decided by progressively eliminating the high-price bidders until the cumulative capacity of the remaining bidders falls

[1]Capacity bid need not necessarily be a capacity constraint, but a part of a bidder's production plan. However, in this paper, I abstract from bidders' dynamic optimisation problem of allocating it's production capacity over different auctions.

below procurement target. At this point, the market is cleared through a simple rationing rule which awards a positive residual quantity to the last eliminated bidder. The bid of this bidder is the price of electricity for the winning bidders too, making this a uniform price auction. This allocation rule is the same as in Holmberg and Wolak (2018), who assume that bidders are symmetric in their capacity. The rationale behind such a rule is to clear the market in a simple and transparent way, while fostering competition by bidders who want to avoid rationing.[2]

However, the presence of such rationing can induce to strategic behavior, where a bidder with lower cost may agree to accept the residual award at a higher price, thereby having a high payoff. This leads to an inefficient allocation. A preliminary analysis of the data provided by SECI regarding bids and awards from each bidder suggests the existence of such incentives.[3] In approximately half of the auctions, bidders cease to compete as long as they can receive a residual award. Notably, the bidders with high capacities tend to make such decision more often. This observation indicates a relatively low level of competition.

I further investigate these incentives through theoretical investigation of the final round, which is modelled as a descending clock auction with supplier rationing. The bidders have their costs drawn independently and identically from the same distribution. At the start of the auction, clock shows a reserve bid b^R, which decreases continuously during the auction. In case of two bidders, the auction ends when a bidder decides to exit the auction at a particular price. The exiting bidder gets residual award and the remaining bidder gets own capacity as the award.

In this game, the equilibrium is characterised by higher capacity bidder being less

[2]It could be argued that bidders may over-report their capacity in the qualifier and subsequently agree to a residual award at a higher price. However, in practice, bidders have to prove their capacity to the auctioneer before the auction. Thus, the over-reporting is not a problem. On the other hand, a bidder may under-report with an anticipation that the opponent may agree to residual at a higher price. This strategy allows the bidder to secure a higher price for a lower capacity. A formal investigation of this conjecture is a possible future work.

[3]A major reason to focus on SECI's auctions is that they have helped create half of the solar and wind capacity of India. Moreover it has been shown to be relatively low risk counterparty, which means that one can abstract from risk-premium considerations and focus solely on the strategic considerations in analysing bids (Ryan, 2021).

competitive. In other words, she exits at a higher bid for any given cost. Furthermore, if her cost is above a certain threshold, she exits at the reserve bid (bunching at the reserve).[4] As such, higher capacity bidder is more likely to be the one to exit earlier even if her cost is low. Such equilibrium properties can explain the observed less competition by large capacity bidder. It provides clear insights into the link between auction rules and inefficient market design.

Intuitively, the higher capacity bidder is less competitive due to the higher residual award they receive upon exiting the auction. Thus, the gain in terms of capacity award is not as high if she is more competitive. Moreover, she gets much lower price for the residual if she loses despite being competitive. Aggressively competing is not advantageous to her regardless of the her cost type. Thus, open bidding, rationing rules, and public information about opponents' capacities create strategic incentives, wherein a high-capacity bidder would prefer not to compete even if their cost is low. This leads to an inefficient market design.

This gives rise to an important policy question. Since the auctioneers may not want the market clearing rule to be more complicated and aim to fulfill all of their procurement demand, can these auctions be made more efficient with minor tweaks? I address this question econometrically in two steps: estimating the cost distribution and conducting simulations to compare the welfare properties of the existing mechanism with a sealed bid one.

To estimate the cost distribution, I utilize the bids of bidders who receive zero awards. In an open descending price auction, any bidder receiving a zero residual has a dominant strategy to reduce their bid to their cost. If a bidder refrains from competing, they miss the opportunity to receive a positive award at a bid higher than their cost, should another bidder exit. Conversely, if they compete and eventually exit at their cost, they still receive an award of zero. Hence, the observed bids of

[4]These are the properties of the semi-seperating bayes-nash equilibrium. There are other bayes-nash equilibria, where one of the bidders never exits while the other one exits immediately, like in a war of attrition. These pooling equilibria survive on a non-credible threat, and are not studied in detail here.

such bidders who receive zero awards are equivalent to their costs.[5] Their bids can, then, be used to estimate the cost distribution.

These cost observations are censored on the right tail, and the threshold of censoring is endogenous. Only a subset of bidders with bids below a certain threshold in the qualifier round of SECI auctions are selected for the final round. However, this qualification threshold bid is dependent on the cost distribution itself. The endogeneity thus created leads to a situation where observed costs are not drawn independently of each other.

In order to resolve this endogeneity, I use probability density of observing certain order statistics of costs; conditional on the observation pertaining to some higher order statistic. This conditional probability density resembles the density of order statistics of costs, when looking at a sub-sample of independently drawn costs, all of which are below some randomly drawn threshold. Since we also observe bidder identities, identification of the cost distribution from this subsample can be accomplished similarly to Dutch auctions with observed identities, for which identification results are well established. (see Athey and Haile, 2007, for example).

I estimate the distribution parametrically, considering the small data size (116 bids from 25 auctions). The conditional probability density provides the likelihood function, the maximization of which yields estimates of a parametric cost distribution. These estimates are adjusted for auction-specific characteristics. To control for bidder-specific characteristics, I categorize them into strong and weak bidders. As a simple proxy, I identify the top 7 renewable energy producers in the country, as per the report by BloombergNEF (2022), as strong bidders. Among the 25 auctions, three of these seven bidders account for 33% of the bids that receive a positive award, whereas they represent only 18% of the bids awarded zero. This suggests that these bidders possess a significant cost advantage over others, potentially contributing to their high market share.

[5] An important caveat here is that bidders are assumed to not engage in jump bidding. The inference in presence of jump bidding is addressed in more details in Haile and Tamer (2003). However, in this paper, I assume that such bidders don't engage in such bidding.

Using these estimates, I compare the selection inefficiency in the India's renewable energy auction with a sealed bid discriminatory price auction, with 2 asymmetric players. I am able to show that using discriminatory pricing instead of uniform can lead to reduction in inefficient selection by 20-34% depending on the level of asymmetry, when $M = 300$. This translates to significant welfare improvements to the tune of USD 600,000-1,000,000, depending on the level of asymmetry and bidder capacity. The payment made by the government doesn't change significantly through this switch. As such, the auction can be lead to a more efficient market with a minor change in the procedure, which don't affect the procurement target and transparency. An important future avenue is to conduct a proper counterfactual analysis to get more insights into welfare implications of switching to some alternative allocation mechanism.

The rest of the paper is as follows. Section 2 compares the paper with related papers in the literature. Section 3 provides institutional background. Section 4 provides stylized facts regarding bidding behavior using SECI data. Section 5 provides the model and equilibrium for simpler version of the auction. Section 6 is on identification and estimation of the cost distribution. Comparison of the extent of inefficiency can be found in Section 7. Section 8 concludes the paper.

2 Related literature

This paper contributes to the literature on auctions, both theoretical and applied, and to market design for renewable energy.

The auction studied in this paper is a version of asymmetric all-pay auctions, where loser is defined as the player who gets a smaller positive award. It is closely related to Holmberg and Wolak (2018), who studied symmetric version of the same problem. Their aim was to compare the properties of uniform versus discriminatory price mechanisms, something done in this paper too. Another closely related paper is Betto and Thomas (2024) which studies asymmetric two player all pay auctions with spillovers, under complete information. This paper aims to provide a general model for R&D race, and show that it's possible for a high cost player to have a higher payoff when there are spillovers.

The proof of equilibrium existence and uniqueness is inspired by the exposition of theoretical results in Lebrun (2006) and Lizzeri and Persico (2000). The former provides the conditions for existence and uniqueness of pure strategy monotonic equilibrium in sealed bid first price auction, which have a problem of singularity. The same problem arises in the analysis of SECI auctions. The intuition behind uniqueness of equilibrium is same as that used in Lizzeri and Persico (2000) for second price all-pay auctions. Furthermore, uniqueness result of this paper adds to that of Lizzeri and Persico (2000), as one of the assumptions used by them is invalid in SECI's auctions. Thus, the techniques used in theoretical results of this paper add to the literature on equilibrium existence and uniqueness.

In the empirical part of the paper, I solve for an endogeneity problem arising out of the context specificity of SECI auctions. The identification technique is similar to that in Song (2006), which uses conditional density to estimate the distribution of bidder types when the number of bidders is unknown. Identifying the cost distribution from observed order statistics of the cost of bidders is similar to identification of bid distribution in dutch auctions where we observe winning bid and bidders' identities. This result can be found in Athey and Haile (2007) and Paarsch, Hong, et al. (2006). In the dutch auctions, there is an additional step where the bid distribution is used to estimate cost distribution. Such a step is not needed in my case.

Finally, the paper adds to the literature on market design in renewable sector. A closely related problem was studied by Fabra and Llobet (2019), where the bidders' cost is considered common information, but their capacities are private information. Allowing bidders to control their production enables them to make high markups in such a scenario. Some other papers have looked at specific electricity markets. Regarding India, a formal study is conducted by Ryan (2021). This paper showed that the participation and competition was higher in the auctions conducted by

SECI in comparison to auctions conducted by other agencies. The explanation for the same is that other agencies are more likely to default on their payments to the bidders, which makes them risky. As a result bidders charge risk premium through their bids in the auction, which holds up investments in their auctions. The paper, however, abstracts from certain strategically important nuances of the auction procedure used by SECI. Besides this, Probst et al. (2020) provide reduced form results on the impact of local content requirement on the price discovered in SECI auctions. Such a requirement was discontinued in 2017. Besides India, Hara (2023) studies the importance of risk premiums for bidding in Brazilian renewable energy auctions. A case study comparing auction designs in Brazil and Mexico is presented in Hochberg and Poudineh (2018), which talks about competition and price discovery in the two countries. My paper adds to this literature by providing a formal analysis of SECI's auction nuances, specially of supplier rationing, which lead to inefficiencies in market design.

To conclude this section, I can say that this paper belongs to the literature on market design for renewable energy, and contributes to empirical and theoretical literature on auctions.

3 Institutional background

At 180GW, India has 4^{th} largest installed capacity of electricity production from renewable sources. Of this 109GW is based on solar and wind. A huge proportion of this capacity is concentrated in large utility-scale grid connected solar and wind farms. The farms are built through auctions of power purchase agreements (PPA), which are signed between the auctioneer and bidders. PPAs mention the size of projects which a single bidder has to construct, and the price at which they sell their eletricity to the auctioneer for 25 years.

Many agencies at state and central level conduct these auctions. Around 50% of the solar and wind capacity is created by SECI and National Thermal Power Corporation

(NTPC)(joint report by JMK and IEEFA, 2023). Some other important agencies are the state level energy development corporations, among which just the 2 states Gujarat and Maharashtra auctions account for 15% of the auctioned capacity.

In this section, I describe SECI's allocation procedure in detail. Similar auction procedure is also used by NTPC. I focus on these two because of their combined size, and also because they are considered relatively risk-free of counterparties (Ryan, 2021). In this case, the analysis of bids can abstract from risk-premium considerations. Moreover, many other agencies also share the allocation rules used by SECI in the final round. While the paper investigates efficiencies arising from the final round, I provide econometrically relevant details of qualification round as well.

Before the 2 rounds, the auctioneer releases a Request for Submission (RfS) document, which specifies auction specific details. It mentions if the project has to be solar or wind or hybrid, if it has be located in a particular place in India or if it's location neutral. RfSs state that it's bidders responsibility to find the land (unless the auction is for solar park) and connect their project to the grid. It provides the incentives and penalties for good and bad post-auction performance, respectively. RfS mentions procurement target (M) and reserve tariff for the qualifier round ($\bar{p}$).

In the qualifier round, each bidder submits two envelops. The first envelop shows the financial and technical competence of the bidder. The second envelope contains the bids on price and bidder's capacity. SECI doesn't open second envelope until it has ascertained the veracity of the first envelope. If allowed, the price could be Viability Gap Funding (VGF), which is the minimum amount required by the bidder to make her project feasible, while selling electricity it would produce at $\bar{p}$. Bidders bid VGF per MW of their capacity bid. Otherwise, price is the tariff at which she would sell each unit (Kilowatt-hour) of electricity. This price is valid only for the capacity and project started as a result of winning the PPA in a particular auction. VGF bidding was discontinued after 2017, except for very specific cases, as more and more winners were bidding only on tariffs.

Table 1: Example allocation rule with target of 500

Bidder	Qualifier		Final	Award
	q_i	p_i^I	p_i^{II}	
B_1	100	1.5	1.5	100
B_2	50	2.6	2.1	50
B_3	200	2.8	2.1	200
B_4	450	3.0	2.1	150
B_5	150	3.2	3.0	0
B_6	100	3.4	2.5	0
B_7	300	3.5	NQ	-

The allocation rule is exhibited through the example in Table 1. The total number of bidders in the qualifier round can be denoted by N_1, and the mechanism is low price sealed bid. SECI ranks these N_1 bidders according to the price bid, with lowest (best) rank for lowest price. If VGF is allowed, the bidders asking for VGF are ranked higher than the ones bidding tariff. In SECI auctions, the auctioneer selects top m bidders such that their cumulative capacity just exceeds M for the final round. Additionally, top half of the remaining bidders also qualify.[6] In the example table, $N_1 = 7, m = 4$ and selected bidders would be $B_1 - B_6$. If the total of bidders' capacities is less than M, the auctioneer reduces the value of M in second round in a pre-defined manner and all the bidders would qualify. In NTPC auctions, all but last ranked bidder qualify. I denote the number of bidders in final round by N.

The N bidders compete online in an open descending bid auction.[7] Each bidder is able to see opponents' price and capacity throughout the auction. The starting bid of each bidder is their bid from the previous round, and they can only reduce it. The minimum reduction allowed is 0.01 INR (0,00012 USD). The bidders can't change their capacity, which allows me to treat them as exogenous during the auction. The auction lasts for at least one hour and it ends when there has been no change in bids for 8 minutes. At the end of the auction, top W bidders by price, whose cumulative

[6]Assume that the quantity bid by i^th ranked bidder is $q(i)$. The auctioneer would select top m ranking bidders, such that $m = \min_m \sum_{j=1}^{m} q(j) \geq M$, and half of the remaining bidders, for the second round.

[7]which is strategically equivalent to uniform price auction. See Krishna (2009)

capacity just falls below M are awarded a contract to build their desired capacity. In table 1, these are B_1, B_2, B_3. The lowest price bidder among the remaining (or, the marginal winner) is awarded the residual amount for capacity creation. Both NTPC and SECI have followed this rationing rule. In Table 1, this rationed bidder is B_4. [8]

4 Data and stylized facts

In this section, I analyse the bidding data in order to understand the bidding behaviour. Data is obtained from 2 sets of documents provided on the website of Solar Energy Corporation of India, SECI. The first set of documents are requests for submission (RfSs), which are issued by the auctioneer to invited bidders. This document provides auction specific characteristics like technology specifications, location restrictions, procurement target etc. They also provide the details of auction mechanism, allocation, and transfer rule among other things. The second set of the documents are the ones containing the result of the auction. These documents provide the first and second round bids of all the bidders, and the capacity awarded to each bidder at the conclusion of the auction. Similar auctions are also conducted by National Thermal Power Corporation (NTPC) and I append that to the SECI data. For NTPC auctions, however, I rely on public reports provided by Mercom India, which is a market tracker. Thus, data is not available for all the NTPC auctions, and the one available is only for round 2.

In total, there is data from 62 auctions conducted by SECI and NTPC. Two of these auctions restrict participation to Public Sector Enterprises, two have very small procurement target. One of them is for Round-the-Clock supply and another one has two part tariff. Data is not complete for 2 of the auctions. I remove these

[8] In some SECI auctions during and after 2019, if the bidder was awarded capacity less than 50% of its quantity bid, they could reject the offer and this capacity would lapse. However, I am abstracting from this rejection option, as it was exercised only in 4 auctions, which I will exclude from my empirical analysis.

eight auctions from the analysis because they are not directly comparable to other auctions. This leaves us with 54 auctions, having a total of 374 bids for the second round.

I convert the VGF bids, which are per Megawatt (MW) of capacity, into equivalent tariff bids, which are per Kilowatt-hour (KWh), using following calculation.

$$tariff = \frac{VGF * capacity}{capacity * CUF * 24 * 365 * 25 * 1000}$$

Here numerator captures the amount of money which the bidder has asked for, and denominator captures the number of KWhs of energy they will produce over 25 years. CUF is the expected capacity utilisation factor, which is set at 0.2 for this exercise.[9] In the conversion formula above, I am assuming that bidders value the present and the future production equally.

Table 2 shows the average number of participants to the qualifier round across large SECI auctions.[10] The table shows that the participation varies over the years. In total, 138 firms have participated in the large auctions. The high participation in 2017 is driven by auctions for projects in Bhadla Solar Park in the state of Rajasthan, which is now the largest Solar Park in Asia. Among these potential suppliers for SECI, 66 have never won any positive award. 27% of the capacity constructed via large SECI's auctions is concentrated with just 3 bidders- ReNew (10%), Adani (9%), and Softbank Energy (7%).[11] Top 10 bidders have 53% of the capacity. These bidders include Singapore-based Sembcorp, home-grown firms like ACME, Azure power. As a result of this competition, the average market clearing price declined from Rs. 4.5/unit in 2015 auctions to Rs. 2.4/unit in 2021 auctions. This price is the bid of the bidder who received the residual award in the final round.

[9] As per the Saur-News-Bureau (2023) report, average CUF for solar is 15-19% and as per the WindInsider (2023) report, for wind it's 25%.

[10] This excludes 1 auction for Round the clock supply, 1 auction with two part tariff, and 2 auctions which restricted entry to public sector firms.

[11] In 2023, Adani acquired all the renewable projects of Softbank in India for USD 3.5 Billion, making them the largest renewable producer for SECI.

Table 2: Average participation in large auctions

Year	**Ind**	**N**	**S**	**SP**	**Aggregate**
2015			8.5		8.50
2016		3	8.5	6	4.40
2017	12			21.67	17.80
2018	9.5			7	9.14
2019	4.25	9			5.20
2020	9.6	13			10.57
2021	15	22			17.33
2022	12.75		10		11.83
2023	7				7
Aggregate	8.96	7	9	15.6	9.22

Among the tariff auctions with large procurement targets (above 200MW), I observe that in 24 out of 40 auctions, the bidder who gets residual award has bid within Rupee 0.02 of the lowest of bids of all the losing bidders or within 0.01 of her first round bid, if there is no losing bidder. Such a bidder is said to have conceded or not competed. In 17 of such 24 auctions, it is the bidder with highest quantity bid who concedes and accepts to be rationed. In two cases, all bidders have same capacity. In three of the auctions, no bidder gets rationed in the outcome. In 5 auctions, the bidders exercise the option to reject the residual capacity allotted to them. This right is provided to them only after 2019, if the award is less than half of capacity bid. In these auctions, we do not observe any competition.

Among the 14 VGF auctions analysed (with $M \geq 25$MW), 6 auctions have no residual award as each bidder's quantity bid equals M. Among the remaining, the residual winner doesn't compete in 3 auctions. In all of them, this bidder is also the one with highest capacity report. In 2 auctions, competition is observed. In 3 auctions, the winner had a very low first round bid and capacity bid equal to M, which led to absence of competition in second round. Overall, I can say that in the auctions where there was a positive residual award, there was no competition in almost half of the occurrences by the highest quantity bidder.

Presence of both competitive and non-competitive behaviour in bidding rules out any collusive scheme where the bidders collude to get a high tariff. However, 67.5%

of all the auctions observe bidder agreeing to be rationed without any competition, and in most cases it's the bidder with highest capacity report. Thus, there seems to a relationship between capacity report and competitive behavior. To further explore this relation, I estimate a simple linear probability model and a probit model. For this purpose, I restrict attention to SECI auctions.

In this model, the decision to not compete is captured by indicator variable $concede_{ia}$. For tariff auctions, $concede_{ia} = 1$ if in auction a, B_i gets residual award and bid same as or within INR 0.02 of the lowest bid among all the bidders who got zero award. If no bidder gets award of zero, I compare the final bids to qualifier bids. $concede_{ia} = 1$ if B_i gets a positive residual award and her second round bid is within INR 0.02 of her qualifier round bid. For VGF auctions, same procedure is followed with a threshold of INR 100,000.

I remove the auctions where the bidders who are awarded their capacity report without changing bids between the 2 rounds from this analysis. I don't use the bids from auctions where none of the bidders were awarded a positive residual capacity. I also exclude auctions where some bidder exercised the right to reject the residual award. Whenever a bidder decides to concede and gets zero award, the auction doesn't end, as the allocation is not yet decided. It continues with lesser number of bidders, any subset of whom (including empty subset) might get a positive award if they decide to not compete further. Thus, a subgame is created among remaining bidders. If I observe such a situation in a particular auction, I consider the subgame generated by exit of a bidder as a separate auction a. In each such subgame a where the bidder i decides to compete and not agree to a residual capacity, $concede_{ia} = 0$. In the terminal subgame, $concede_{ia} = 1$ if the bidder who gets positive residual doesn't compete further. Treating these subgames as independent of each other imposes limitation on utility of the linear probability model. As such, the model here measures just a correlation, and not the causal effect of rationing on decision to exit immediately. The bidders getting zero award are not considered for this analysis because their decision to concede is not based on strategic choice regarding agreeing to residual at a higher bid, but on their individual rationality.

To capture the extent of rationing, I calculate a potential residual award for all the bidders who were awarded their capacity report. This is the capacity award they would have obtained if they had chosen to concede. To this end, I subtract the quantity bids of winners whose bids are same for both the rounds (if any) from M. This gives me adjusted M. I also remove these bidders from analysis. The potential residual award is then difference between adjusted M and capacity of all other bidders. The potential residual is then floored at 0. I take its ratio with respect to the capacity report in order to measure the extent of rationing. I use number of fully rationed competitors (one who would have gotten zero award if they had conceded at a bid higher than the market clearing bid), and the number of partially rationed competitors as additional regressors, which can account for impact of competition.

I model following regression specification:

$$concede_{ia} = \nu_0 + \nu_1(residual_{ia}/capacity_{ia}) + \nu_3 nPR_a + \nu_4 nFR_a + \chi X_a + \epsilon_{ia}$$

where X_a are auction specific controls, nPR_a are the number of partially rationed bidders and nFR is the number of fully rationed bidders. The results are provided in Table 3 This econometric specification doesn't use any measure for cost of bidders, which is important for exit decisions. However, this should not be a problem because the aim here is not to claim any causality, but find some correlation between a bidder's decision to concede and her capacity.

We can notice that the measure of rationing is an important determinant of probability of immediate exit by a bidder. Moreover, there is a positive relation between both of the variables, which implies that the bidder is more likely to concede immediately if she is not being rationed a lot. The relation carries on to Probit specification. Another important determinant is the number of partially rationed bidders, the bidders whose decision to concede ends the game. More the number of such players, more is the chance that at least one of them can be outcompeted without reducing

Table 3: Relation between residual award size and immediate exit decision

	Dependent variable:					
	ImmediateExit					
	OLS			*probit*		
	(1)	(2)	(3)	(4)	(5)	(6)
Constant	0.271***	0.326***	0.346***	−0.565	-0.345	-0.314
	(0.095)	(0.108)	(0.126)	(0.428)	(0.487)	(0.527)
residual/capacity	0.374***	0.397***	0.404***	1.231***	1.457***	1.483***
	(0.101)	(0.103)	(0.105)	(0.475)	(0.486)	(0.494)
nFR	−0.029	−0.037*	−0.032	−0.140	−0.173*	−0.159
	(0.019)	(0.020)	(0.023)	(0.094)	(0.099)	(0.111)
nPR	−0.047***	−0.057***	−0.059***	−0.181**	−0.237***	−0.245**
	(0.016)	(0.019)	(0.022)	(0.077)	(0.091)	(0.104)
Region controls		✓	✓		✓	✓
Type Controls			✓			✓
Observations	158	158	158	158	158	158
R^2	0.154	0.167	0.169			
Adjusted R^2	0.138	0.134	0.124			
Log Likelihood				-59.852	-58.411	-58.345
Akaike Inf. Crit.				127.704	130.823	134.690
Note:						*p<0.1; **p<0.05; ***p<0.01

own bid by much. Thus, the bidders are less likely to concede when nPR rises.

The histogram of ratio of residual and quantity bid in Figure 1 can help visualise this relationship between the extent rationing and decision to not compete. For this histogram, I filtered out the observations where rationed quantity was zero. The graph shows that proportion of red for lower values to residual to capacity bid ratio is higher.

These reduced form empirical models present correlation between residual quantity and the competitiveness of bidders. While the models here are not at all causal, the

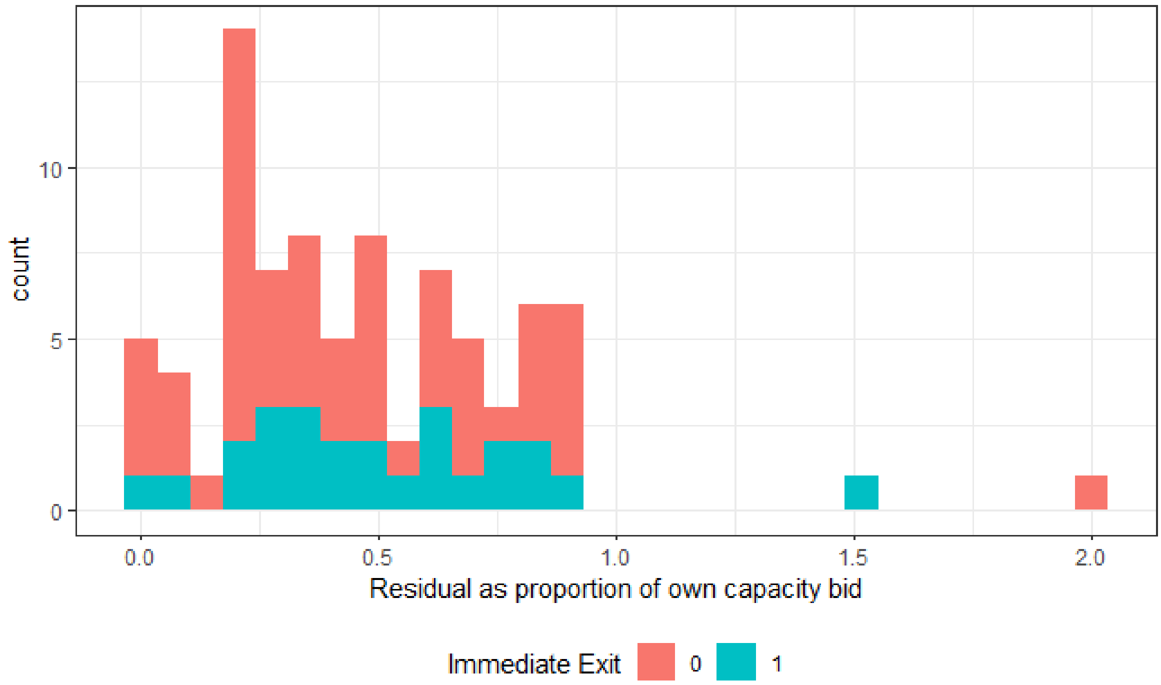


Figure 1: Histogram of ratio of potential residual award

presence of such correlations and other stylized facts presented in this section rule out a collusion based explanation for such behavior. The observed patterns warrant a more theoretical analysis.

5 Theoretical modelling of supplier rationing

This section models the final round as a descending clock auction with residual award. The aim of this and next section is to provide a simple game theoretical explanation of the stylized facts presented in Section 4. I make assumptions on game timing, and bidder and auctioneer information which incorporate relevant information from qualifier round. Although seemingly strong, such assumptions help this paper remain focused on incentives for bidders to compete or not, when facing rationing.

Before the auction, government announces the procurement target M for that auction. Each bidder, i announces her capacity $q_i \leq M$, which is the capacity they can create and provide to the government. I assume that this quantity is reported truthfully. Set of all the bidders is denoted by $\mathcal{N}$. The bidders are assumed to be risk-neutral. In the procedure described in section 2, the reserve bid is individualised as it depends on their first round bid. However, I abstract from this and assume that the announces the reserve price ($=b^R$) which is same for all the bidders.

The abstraction on reserve price doesn't lead to much loss of generality as can be explained in the following example. Let's suppose $M = 100$ and we start the final round with 5 bidders with capacities $\{30, 40, 50, 35, 10\}$ and price $\{3, 4, 5, 6, 7\}$. Then, the highest possible bid for any bidder is 7. However, the 4^{th} and 5^{th} bidder get 0 if they bid 7. Thus, they would gradually reduce their bid from their starting bid, with the hope of out-competing some other bidder while respecting individual rationality constraints. Suppose they reduce their bid to 5 and then last bidder exits the auction as she doesn't find it profitable to provide any amount at price below 5. If the auction were to end at this bid, first and second bidder would get 70 in total as their award. This means that third and fourth bidder have to compete for remaining 30 if the auction continues. The situation is similar to an auction where bidders bid for an award of 30, and have the same reserve bid of 5. Moreover, the game can continue in such a way that the common reserve becomes 4 and the total award size is 70. Thus, assuming a common reserve bid, instead of individualised reserve (as in reality) doesn't affect the theoretical understanding of the bidding strategies in this auction, and this is essentially due to open nature of bidding.

Each bidder is assumed to have a constant marginal cost of supplying the product, denoted by c_i. For each bidder i, c_i is private information, revealed to her before the auction. $c_i \overset{i.i.d}{\sim} F_i(c)$ and $c_i \in [0, \bar{c}]$. Suppose that there is a very small atom at 0. For the baseline model, $F_i(c) = F(c), \forall i$. I denote the reversed hazard rate of this distribution, $f(c)/F(c)$ by $\sigma(c)$ and assume that $\sigma'(c) < 0, \forall c > 0$. It is possible that there might be some learning among bidders from the qualification bids of their opponents. Any such learning can be captured by assuming heterogeneous priors over opponents' costs. As I show through a extensions in appendix B, heterogenous priors can be easily accommodated in the baseline model.

M is allotted via an open descending price auction, modelled as descending clock auctions where bidders bid the per unit price they would ask the government for providing the good.[12] Right before the start of the auction, each bidder's capacity, q_i, is made public. The auction process can be thought of as a descending clock auction auction. At the start of the auction, auctioneer displays bid b^R on a screen and all the bidders enter an arena. As auction proceeds, the displayed bid reduces in a continuous manner. If a bidder wishes to exit at a bid b, she leaves the arena when screen displays $b \leq b^R$. When she leaves, she gets a residual quantity award of $Max\{0, M - \sum_i q_i \mathbb{1}_{B_i \in \mathcal{I}(b)}\}$, where $\mathcal{I}(b)$ is the set of bidders in the arena at bid b. The auction stops when a bidder gets a positive award when she exits, or if $M - \sum_i q_i \mathbb{1}_{B_i \in \mathcal{I}(b)} = 0$. The bidders who are still in the arena at the end of auction are awarded their quantity at the bid displayed on the screen at that time. Thus at any point, the bidders who would get a positive residual on exiting the arena decide to either accept the residual at the current bid, or to wait for the bid to drop so that another opponent exits. If they decide the former, they get higher price but lower quantity, and vice-versa if they decide the latter.

In such a game, any bidder who would get a zero award on exiting, would not exit until the displayed price is same as her cost. If they exit at a higher bid, they still get a payoff of zero. However, if the don't exit, there is a chance that some other players will exit and this bidder may get a positive award. Thus, it's beneficial for her to not exit at a bid above cost. This characteristic of equilibrium bids of zero award bidders plays crucial role in identification of the cost distribution from SECI data.

The descending clock auction is essentially a dynamic game, where the bidders have 2 options (exit and continue) at any given instant. However, one can also think of

[12]This is an abstraction from the idea of price bids being the tariff on produced electricity and not the price of constructed capacity. The price bids in this model can be thought to be the per unit markup these bidders desire added to the Lifetime average Cost of Electricity they expect to produce. Any adjustments made for this equivalence don't harm the equilibrium results as long as capacity utilisation factors and discount rates are assumed same across agents.

this as a stage game. At the start of the game, each bidder chooses a cutoff bid at which she would exit, if none of her opponents would have exited by that bid. If a bidder exits and gets an award of zero, a subgame starts, and each bidder in this subgame finds a new cutoff bid. If in any subgame, the exiting bidder gets a positive residual award, the game ends. Thus, bidders have cutoff strategies in this stage game, where the cutoff bid depends on the set of quantities of all the players in the subgame. Bidder i's strategy is to choose her cutoff bid (or simply, bid) b_i in each subgame. The analysis amounts to finding Bayes Nash Equilibria (BNE) in pure strategies of this game. To keep the results simple and tractable, I focus on games with just 2 bidders.

5.1 Pure strategy equilibrium with 2 players

This section provides the results on characteristics and existence of pure strategy equilibria for auctions with 2 players and 3 players. In general, opponent of B_i is denoted by B_{-i}, her bid by b_i, and her equilibrium bid function by $\beta_i(c)$. A bidder is said to be large if their capacity is larger than the procurement target. The simplest case with 2 bidders would be when $M < q_i$ for both i, i.e., both are large. This case reduces the auction to a simple english auction, where $\beta_i(c) = c$ for both i. The other cases are a bit more involved.

5.1.1 A large bidder and a small bidder

Assume $M = q_1 > q_2$ without loss of generality. In this case B_2 gets 0 if her bid is higher. On the other hand, B_1 gets her capacity in all the cases. B_i's ex-post payoff, conditional on winning and losing respectively, are:

$$\pi_i^W(b_i; c_i, \mathbf{q}, b_{-i}) = q_i(p - c_i)$$
$$\pi_i^L(b_i; c_i, \mathbf{q}, b_{-i}) = Max\{0, M - q_{-i}\}(p - c_i)$$

where $p = Max\{b_1, b_2\}$

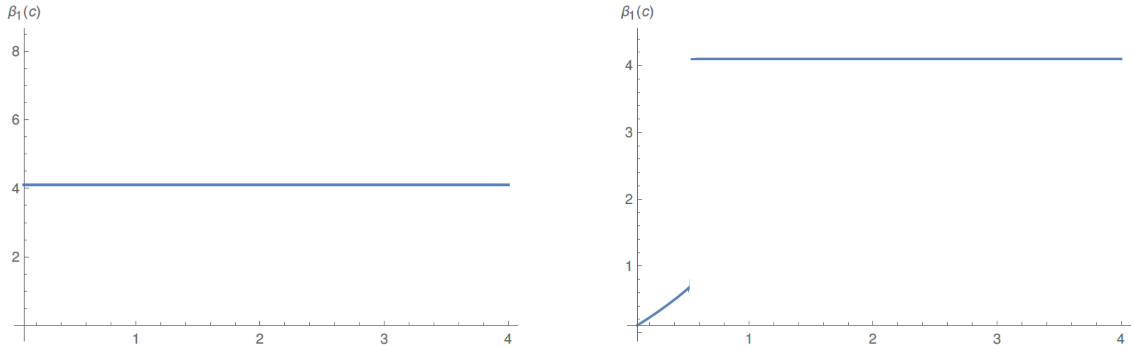
B_2 would find it weakly dominant to bid her cost. If she bids above and loses, she gets 0. If she wins with this bid, she pays price equal to opponent's bid, which would higher than her cost. Thus, she isn't really better off by bidding above her cost. Bidding lower than cost is dominated as that gives negative payoff. Thus, her equilibrium bid function, $\beta_2(c) = c$.

$\beta_1(c)$ is obtained as B_1's best response to $\beta_2(c) = c$. This is obtained by maximisation of her expected payoff, which is given by:

$$\pi_1(b_1; c_1, \beta_2(c)) = (M - q_2)(b_1 - c_1)F(b_1) + q_1 \int_{b_1}^{b^R} (x - c_1) dF(x)$$

For B_1, this situation reduces, analytically, to a decision problem, rather than a game. $\beta_1(c_1)$ is attained by finding $b_1 \in \underset{b \leq b^R}{ArgMax} \pi_1(b; c_1, \beta_2(c))$ for each c_1. If $\beta_1(c_1) < b^R$, then $\sigma(\beta_1(c_1))(\beta_1(c_1) - c_1) = \frac{M-q_2}{q_2}$ which is the first order condition of optimisation at an interior point. If for some c_1 this equality doesn't hold $\forall b < b^R$, $\beta_1(c_1) = b^R$, i.e., B_1 exits immediately at b^R. Strategies $\beta_1(c), \beta_2(c)$ constitute the equilibrium of this case.

To have an illustration of equilibrium, suppose $c_i \overset{iid}{\sim} U(0, 1)$ without an atom. This implies that if there is an internal optimum for some cost type, she bids according to function $\beta_1(c) = \frac{q_2}{2q_2 - M} c$. Note that if $q_2 < M/2$, this yields negative bids, which are dominated. Thus, if $q_2 < M/2$, there is no internal optimum and B_1 bids b^R regardless of her cost ($\beta_1(c) = b^R$), which implies complete pooling. Otherwise, she would be pooling partially. For example, when $M = q_1 = 3, q_2 = 2$, she would bid b^R for $c > 0.2\sqrt{31} - 0.8 \approx 0.313$. For other values of c, $\beta_1(c) = 2c$. Notice that the bidding function is discontinuous. This discontinuity is further illustrated in Figure 2b where a truncated lognormal distribution is assumed. Since it is dominant for B_2 to bid her type c_2, and the computed $\beta_1(c)$ is the unique maximiser of B_1's payoff, the equilibrium described here is unique BNE.



(a) Complete pooling: $q_1 = 100$, $q_2 = 40$ (b) Partial pooling: $q_1 = 100$, $q_2 = 80$

Figure 2: Equilibrium bid function of B_1

Equilibrium bid function for B_1 when $M = 100$, $b^R = 4.1$, and $F : [0, 4] \to [0, 1]$ is constrained Log-Normal with $\mu = 1$, $\sigma = 1$. Note that the scales on x-axis and y-axis are different.

B_1 bids b^R for a type c_1 if $\sigma(b)(b - c_1) < \frac{M-q_2}{q_2}$ $, \forall b < b^R$. If M or q_1 rise, and/or q_2 declines, this inequality is likely to be satisfied for a wider range of c_1. Thus, the extent of bunching would increase. Intuitively, rise in M and decline in q_2 reduces the extent of rationing faced by B_1. This makes her reluctant to compete when her cost isn't low enough to defeat B_2 who bids truthfully.

5.1.2 2 small bidders

In this case, $M > q_1 > q_2$, and $q_1 + q_2 > M$. In this case, both bidders would get a positive reward in case their bids are higher. B_i's ex-post win and loss payoffs can be written as:

$$\pi_i^W(b_i; c_i, \mathbf{q}, b_{-i}) = q_i(p - c_i)$$
$$\pi_i^L(b_i; c_i, \mathbf{q}, b_{-i}) = (M - q_{-i})(p - c_i)$$

where $p = \max\{b_1, b_2\}$.

Any ties are broken in favour B_2.[13] Unlike, the previous case and second price

[13]This tie-breaking rule is not without loss of generality. In fact, it is set in this way in order to have equilibrium existence. This is similar to the idea in Simon and Zame (1990) on endogenising the tie-breaking rule. They prove that in the game where indeterminacy can arise due to unspecified tie-breaking rule, one can always find a tie-breaking rule consistent with equilibrium existence.

auction, none of the players would bid truthfully in this case. B_i's expected payoff from the auction when she bids b_i, conditional on opponent's bid, b_{-i} and capacities q_1, q_2, M is:

$$\pi_i(b_i; b_{-i}, c_i, \mathbf{q}, M) = (M - q_{-i})(b_i - c_i)Pr(b_i > b_{-i}) + q_i \mathbb{E}_F(b_i - c_i | b_i < b_{-i})Pr(b_i < b_{-i})$$

There are 2 complete pooling Bayes Nash Equilibria, where either B_1 or B_2 never exit the arena, and their opponent exits immediately. In other words, one of the bidder commits to bid lower than the other bidder, who in turn bids b^R. Such BNE are sustained by some crazy type, and lead to completely inefficient screening.[14] Thus, it is natural to look for any other possible BNE, where screening is better. Following lemma characterises such a BNE:

Lemma 1. *For each B_i, $\beta_i(c)$ constitute a semi-separating Bayes Nash Equilibrium of the 2 player clock auction with rationing if and only if it satisfies following properties:*

(i) $\beta_i(c)$ *is non-decreasing in* c.

(ii) $\beta_i(c)$ *is continuous and atomless for* $b < b^R$ *for both* i.

(iii) $\beta_i(0) = 0$

(iv) For each player B_i, $\beta_i(c)$ *solves:*

$$\sigma(\beta_{-i}^{-1}(\beta_i(c)))\beta_{-i}^{-1\prime}(\beta_i(c))(\beta_i(c) - c)(q_1 + q_2 - M) = (M - q_{-i}) \tag{1}$$

for $c > 0$

(v) $\beta_2(\bar{c}) = b^R$, *and* $\exists c^*$ *such that* $\beta_1(c) = b^R$ $, \forall c \in [c^*, \bar{c}]$.

Proof. See Appendix A.1 ■

[14] If we look at the descending auction in dynamic version explained earlier, such an equilibrium will not be a perfect bayesian equilibrium.

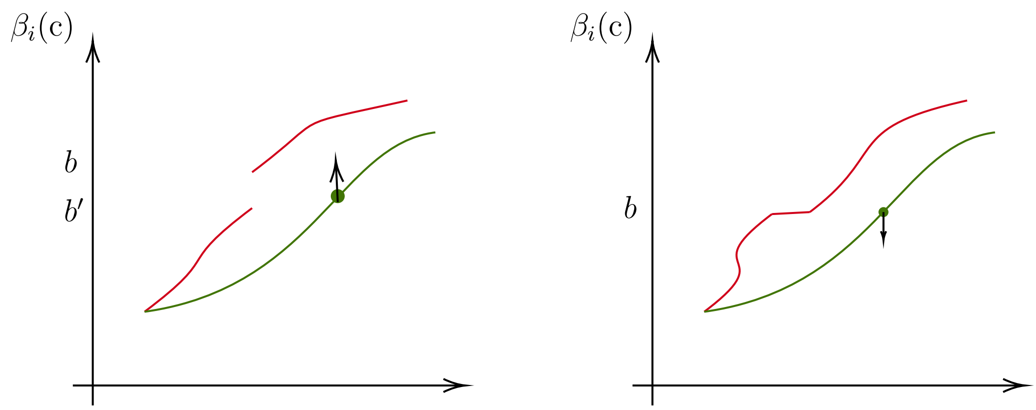


Figure 3: Possible deviations in case of discontinuity and presence of atom

Characteristic (i) can be shown by exhibiting that payoff function satisfies increasing differences property. (ii) can be shown through standard arguments for continuity and atomlessness. If there is an atom at some bid b, the opponent's type which bids b will deviate to a bid slightly lower than b, if latter's strategy is continuous. If there is a discontinuity in strategies, such that the type $\beta(c) = b$ and type $\beta(c^-) = b' < b$, than the opponent types bidding between b' and b would prefer to bid b. These deviations are shown in Figure 3. Characteristic (iii) can be shown through arguments similar to Bertrand competition.Characteristic (iv) can be obtained through first order conditions for optimum at an interior point. It requires invertibility of bid function, which is ensured by conditions (i) and (ii).

Property (v) is the key characteristic of interest. It implies that a positive mass of high cost types of B_1 bid b^R, i.e., B_1 bunches at b^R. It relies on the relative marginal payoffs of two players at any point of intersection of the solution curves, which are such that $\frac{\beta_2'(c)}{\beta_1'(c)} = \frac{M-q_1}{M-q_2} < 1$ if $\beta_i(c)$s intersect at the cost c. The marginal payoffs are such that their solution curves intersect just once. Then, by continuity, strict monotonicity at $b < b^R$, and property (iii) and (iv), I show that even in the immediate neighbourhood of 0, $\beta_1(c) > \beta_2(c)$. Thus, the point of intersection can only be at 0. Therefore, the solution curves don't intersect at $b > 0$. Combined with the property that highest types of both players should bid b^R, it implies that $\beta_1(c) = b^R, \forall c \in [c^*, \bar{c}]$, while $\beta_2(\bar{c}) = b^R$. This property also shows the importance of tie breaking rule in favor of B_2. In absence of this rule, whenever the two players bid b^R, B_2 has an incentive to reduce the bid slightly below b^R and avoid rationing with positive probability because B_1 is bunching at b^R. This tie-breaking rule makes B_2 indifferent between bidding b^R or slightly below b^R. Such an incentive doesn't exist for B_1 as possibility of tie for her is 0 because B_2 doesn't bunch.

Intuitively, B_1 is less aggressive and bunches because she has a higher marginal cost of competing (or reducing her bid) for any given cost type because she has a higher residual award. The gain in quantity conditional on winning is same for both the bidders ($=q_1 + q_2 - M$). Residual award is higher for B_1, which implies that competing is costlier for her. Thus, she is less aggressive, which gives her a higher markup ($= \beta_1(c) - c$) so that her overall marginal cost of competing is not as high. Thus, B_1's bid function is above B_2's until both of them have types in the immediate neighbourhood of 0. This also implies that for high cost types, B_1 has no incentive to compete at all, which leads to bunching. An important implication of the property (v) of the Lemma is that we can rule out existence of any completely seperating equilibrium in this auction as long as the capacities of the two bidders are different. Figure 4 shows the equilibrium as characterised in Lemma 1.

This figure also exhibits the selection inefficiency in these auctions. If B_2 has cost c_2 and B_1 has cost $c_1 < c_2$ as in the figure, B_2 will be bidding lower. As such, she will be awarded q_2 and B_1 gets $M - q_2$. Total cost of production in this scenario is $c_2 q_2 + c_1(M - q_2) = c_1 M + (c_2 - c_1)q_2$. On the other hand, if B_2 was rationed, the cost would have been $c_1 q_1 + c_2(M - q_1) = c_2 M - (c_2 - c_1)q_1 < c_1 M + (c_2 - c_1)q_2$. Thus, the allocation is not cost efficient.

So far, I haven't analysed the existence and uniqueness of equilibrium described in the lemma. This is important because in absence of such an equilibrium, the game only has the complete pooling equilibria. In order to show this, I first need to define functions, $\phi_1(b), \phi_2(b)$ as follows:

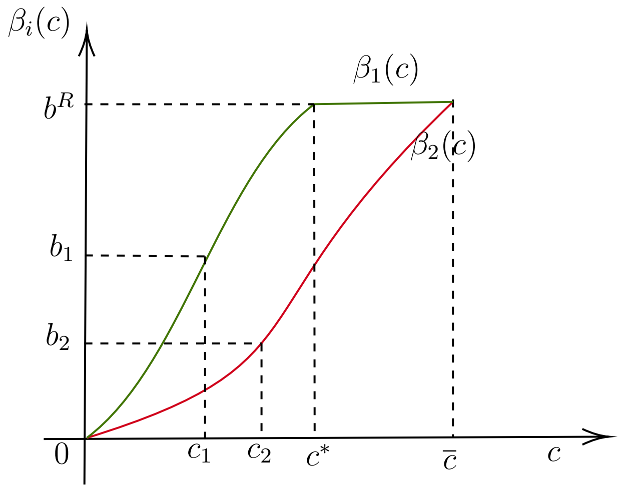


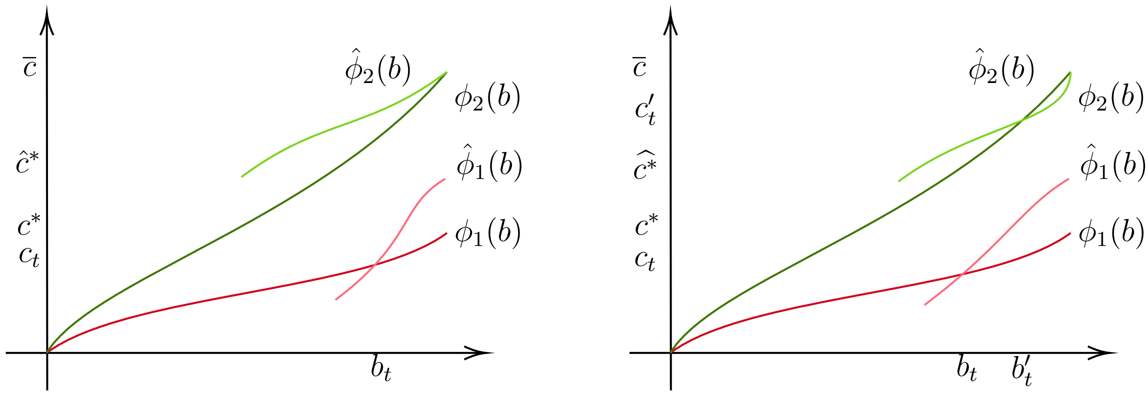
Figure 4: Asymmetric equilibrium with 2 players

$$\phi_i(b) := \begin{cases} 0 & for b = 0 \\ \beta_i^{-1}(b) & for\ 0 < b < b^R \\ Inf\{c : \beta_i(c) = b^R\} & for\ b = b^R \end{cases}$$

Hereafter, these functions are called solution curves. Since Lemma 1 that $\beta_1(c) > \beta_2(c), \forall c \in (0, \bar{c}$, it also implies that $\phi_1(b) < \phi_2(b), \forall b > 0$.

Any equilibrium is attained from the solution to Boundary value problem (BVP) given by FOCs (equations 1) and boundary conditions given by $\phi_2(b^R) = \bar{c}$, $\phi_1(b^R) = c^* < \bar{c}$ such that $\phi_1(0) = \phi_2(0) = 0$. The differential equations of this BVP have a division by 0 at the left boundary and hence, cauchy-lipschitz theorem is not applicable at $(0, 0)$. Thus, right boundary has to be used to establish existence, which is endogenously determined for $\phi_1(b)$. This is similar to the problem of existence and uniqueness in first price auction, as studied in (Lebrun, 2006). Using the FOCs, I can show existence of a c^* such that $\phi_1(b^R) = c^*$ and $\phi_1(0) = \phi_2(0) = 0$. Theorem 1 is formal statement of existence and uniqueness of equilibrium in Lemma 1, which I prove in the appendix.

Theorem 1. *The BNE as described in Lemma 1, exists and is unique.*



(a) (b)

Figure 5: Intersecting solution curves

Proof. See Appendix A.2 ■

Uniqueness can be understood through the argument similar to that of relative toughness in Lizzeri and Persico (2000). Consider two sets of solution curves $\phi_i(b)$ and $\hat{\phi}_i(b)$ such that $\phi_2(b^R) = \hat{\phi}_2(b^R) = \bar{c}$ and $\phi_1(b^R) = c^* < \hat{\phi}_1(b^R) = \hat{c}^*$, pertaining to "$\phi$" and "$\hat{\phi}$" situations respectively. As I show formally in appendix, this would imply that $\hat{\phi}_1(b) > \phi_1(b)$ and $\hat{\phi}_2(b) < \phi_2(b)$ for all $b > 0$. To understand this intuitively, consider the situation in the Figure 5a. At b^R, B_2 is bidding same in both equilibria, but is "marginally" more aggressive at b^R in $\hat{\phi}$ equilibrium (i.e., $\hat{\phi}_2'(b^R) < \phi_2'(b^R)$). As such, the probability of B_2's exit when B_1 bids in the immediate neighbourhood of b^R is lower. Thus, B_1 of type c^* should be less aggressive in $\hat{\phi}$ in order to compensate for this lower probability through a higher markup, as indicted by FOCs too. However, the figure suggests otherwise, and hence that situation can't happen. Thus, if $\hat{\phi}_1(b) > \phi_1(b), \hat{\phi}_2(b) < \phi_2(b)$ in the neighbourhood of b^R.

Next, lets consider the points of intersection of $\hat{\phi}_1(b)$ and $\phi_1(b)$ and take the one with highest bid. Denote it by (b_t, c_t). B_1 is bidding same in both ϕ and $\hat{\phi}$ equilibria, but is less aggressive at the margin in the latter. As before, this will imply that B_2 should be more aggressive when her cost is c_t. This will suggest a situation show in the figure 5b. Finally, such an intersection in B_2's solution curves, by similar logic would imply that B_1 should be less aggressive when her type is the one who

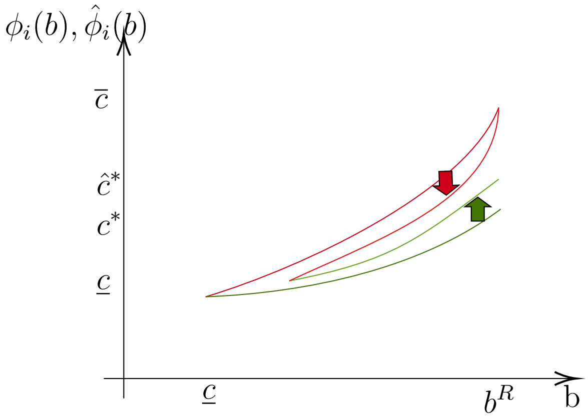


Figure 6: Co-movement of $\phi_1(b)$ and $\phi_2(b)$ in response to change in c^*

bids b'_t in ϕ, which is not in accordance to what we see in the figure. Thus, the solution curves in ϕ and $\hat{\phi}$ equilibrium should not intersect for all $b > 0$. Therefore, if $\hat{\phi}_1(b) > \phi_1(b), \hat{\phi}_2(b) < \phi_2(b)$.

At 0, given that the slope of the solution curves is infinite, we can't use the same logic. However, first order conditions require that the relative marginal payoff in the neighbourhood of a point of intersection should be such that $\phi'_2(0^+)/\phi'_2(0^+) = (M - q_2)/(M - q_1)$, which is a constant. This would imply that $\hat{\phi}_i(b) > \phi_i(b)$ for both i, which is not possible, as we already saw. Therefore, if $\hat{\phi}_1(b) > \phi_1(b), \hat{\phi}_2(b) < \phi_2(b), \forall b$. Therefore, only one possibility remains, in which either $\phi_1(0) = \phi_2(0) = 0$ or $\hat{\phi}_1(0) = \hat{\phi}_2(0) = 0$, but not both, as shown in Figure 6. Under certain regularity conditions, which I verify in the appendix, this implies uniqueness and existence of equilibrium.

While the result on existence and uniqueness is in line with the results on all-pay auctions without any residual reward for the losing bidder, there are some subtle differences. For example, results in Lizzeri and Persico (2000) required loss payoff to be nonpositive. The result I have is attained even when the "loss" payoff is positive. Moreover, my result is in contrast with result on 2 player asymmetric war of attrition in Nalebuff and Riley (1985), which had a continuum of equilibria. In their case, many possible solutions to the FOCs satisfy the condition that player with highest type will wait for infinite time.

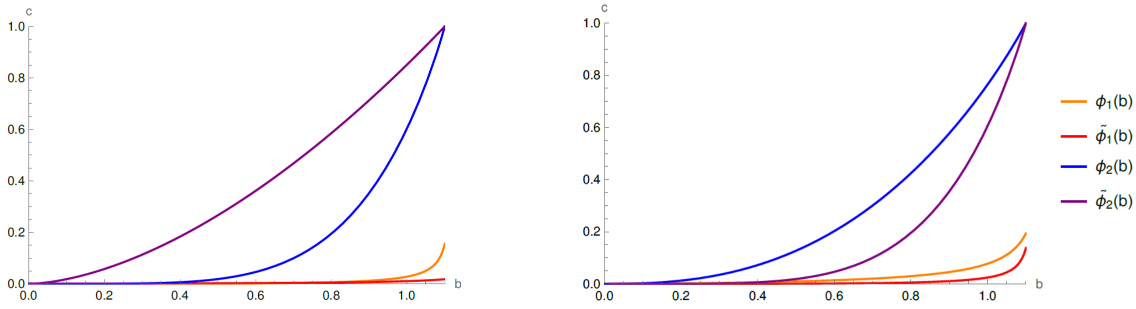
The equilibrium characteristic that B_1 bunches depends crucially on the finite reserve bid and assumption that ex-post payoff are the only source of ex-ante asymmetry. So far in the paper, this asymmetry has been imposed by capacity differences and the cost distribution is same for both bidders. However, ex-ante asymmetry can arise from differences in cost distributions too. Till now, I have focused only on the former in order to clearly understand the effect of such an asymmetry. The insights developed here on the effect of quantity award heterogeneity also carry on to the situations where both sources of asymmetry are considered. However, the identity of bunching bidder depends on the net effect of dominance of cost distribution and ex-post award. I show this in Appendix B, where I provide a formal characterisation of the equilibrium and proofs for following 2 cases of heterogeneity in cost distribution of the two players:

1. $c_i \in [0, \bar{c}_i]$, such that $\bar{c}_1 < \bar{c}_2$ and $c_i \overset{i.i.d}{\sim} F_i(c)$ such that $\sigma_1(c) = \sigma_2(c), \forall c \in [0, min\{\bar{c}_1, \bar{c}_2\}]$. Intuitively speaking, B_2 is likely to have larger costs than B_1.

2. $c_i \overset{i.i.d}{\sim} F_i(.)$ where each F_i has same support, $[0, \bar{c}]$. Denote by $\sigma_i(c)$ the reversed hazard rate (RHR) of $F_i(c)$; $\sigma_i'(c) < 0$. Suppose that the distribution F_1 RHR dominates F_2, i.e., $\sigma_1(c) \geq \sigma_2(c) \forall c \in [0, \bar{c}]$. Dominance can imply having higher probability of higher costs.

Through these cases, I can show that the intuition regarding the effect of differences in ex-post quantity award in the case of same cost distributions for each bidder case is robust to differences in cost distributions, even though the net effect is different. What matters for the equilibrium structure, and specially for the identity of bunching bidder is the net effect of cost distribution dominance and quantity bids.

To conclude the analysis, I provide the comparative statics with respect to M and q_i. The simulations show that any effect of increase in $q_1/(M - q_2)$, depends on its value, and the extent of change in it. This is shown in Figure 7. In Figure 7a, q_1 rises from 60 to 90, and that leads to B_1 being very less aggressive ($\tilde{\phi}_1(b) < \phi_1(b)$), while



(a) (b)

Figure 7: Change in $\phi_i(b)$s in response to quantity changes

In the left figure, $\phi_i(b)$s are defined for $q_1 = 60, q_2 = 50, m = 100$ and $\tilde{\phi}_i(b)$ are defined for $q_1 = 80, q_2 = 50, m = 100$. In the right figure, $\phi_i(b)$s are defined for $q_1 = 60, q_2 = 50, m = 100$ and $\tilde{\phi}_i(b)$ are defined for $q_1 = 70, q_2 = 50, m = 100$. The costs are drawn in i.i.d manner from $U[0, 1]$ with a $b^R = 1.1$.

B_2 becomes more aggressive ($\tilde{\phi}_2(b) > \phi_2(b)$). In Figure 7b, q_1 rises from 60 to 70, which makes both the players less aggressive. Thus, changes in bidding behavior in response to change in q_1 and extent of rationing are not obvious and not monotonic.

The theoretical exercises of this section and the appendix C on 3 players, show that the descending clock auction with rationing allocates inefficiently. While such an auction design is attractive because of the simplicity of allocation rules and transparency, the market of renewable electricity created by it is not cost-efficient. Thus a question arises regarding possibility of making this market more cost-efficient without using more complicated methods. I take an empirical approach to answer this question. This not only helps me quantify the cost-inefficiency in the auctions, but also tells the extent to which auctions can be made more efficient by slightly different mechanisms. The first step of this approach is to identify the cost distribution of the bidders from the observables in the data. The second step is to estimate the cost distribution, and final step is to conduct a simulation based study of various counterfactual mechanisms.

6 Identification and estimation of the cost distribution

In this section, I explain the identification methodology for distribution of costs of bidders using the data provided publicly by SECI. As mentioned earlier, this data contains bidders' identities, bids and awards. In particular, it provides bids of the bidders who are awarded no capacity. Identification, then, relies on the equilibrium characteristic that bidders who get an award of zero find it dominant to bid their cost in descending price open auction. Their bids can be ranked and provide the order statistics of costs.[15] The distribution of order statistics can identify the parent distribution of the costs.

6.1 Identification

For each auction $t \in \mathcal{T}$, SECI provides us data of the final round bids of the bidders with zero award, which also pertain to the order statistics of costs. These bids are from a self-selected set of bidders with cost below a certain threshold. The selection threshold is decided by their bids in the qualification round. The bids in the qualification round depend on the distribution of costs. In this section, such dependence of threshold on the distribution prevents us from using distribution of a single order statistic to estimate the parent distribution as in English auctions. Thereafter, we will see that the structure of distribution of order statistics can be used to resolve this problem, and identify underlying cost distribution if at least 2 order statistics are observed.

Suppose, for some auction t, N_t bidders draw their costs independently and identically from the distribution $F_t \in \mathcal{F}$, where $\mathcal{F}$ is an ordered family of distributions.

[15] Suppose $X_1, X_2, ..., X_N$ are N independent draws from a distribution F. If we arrange them in increasing order and represent this arrangement by $Y^{(1:N)}, Y^{(2:N)}, ..., Y^{(N:N)}$, then $Y^{(1:N)}$ is the first order statistic, $Y^{(2:N)}$ is the second order statistic, and so on. We can further find order statistics distribution $F^{(i:N)}$ for $Y^{(i:N)}$ using the independence of draws. David and Nagaraja (2004) textbook provides more details on order statistics.

To commence, suppose that the econometrician observes the second lowest cost, c, among N_t players, all of whom have cost below some randomly chosen selection threshold, $\bar{b}_t$. The probability density function pertaining to the distribution of the second lowest statistic in an arbitrary auction t is:

$$f_t^{2:N_t}(c|c \leq \bar{b}_t; N_t) = \frac{N_t(N_t-1)(F_t(\bar{b}_t) - F_t(c))^{N_t-2} F_t(c) f_t(c)}{F_t(\bar{b}_t)^{N_t}}$$

The CDF is given by:

$$F_t^{2:N_t}(c|c \leq \bar{b}_t; N_t) = N_t(N_t-1) \int_0^{F_t(c|c \leq \bar{b}_t)} u^{(N_t-2)}(1-u)du$$

These functions can be found in David and Nagaraja (2004). From the expression above, it can be noted that $F_t^{2:N_t}(c|c \leq \bar{b}_t)$ is an increasing function of $F_t(c|c \leq \bar{b}_t)$. Thus, the latter can be identified by observing the former in the data (See Paarsch, Hong, et al., 2006; Athey and Haile, 2007, for similar results).

Now suppose, as in SECI auctions, the threshold ($\bar{b}_t$) is not randomly chosen, but depends on the cost distribution, F_t, i.e., $\bar{b}_t = \bar{b}(F_t)$. As before, the probability density function of observing the second lowest order statistic is:

$$f_t^{2:N_t}(c|c \leq \bar{b}(F_t); N_t) = \frac{N_t(N_t-1)(F_t(\bar{b}(F_t) - F(c))F_t(c)^{N_t-2} f_t(c)}{F(\bar{b}(F_t))^{N_t}}$$

and the CDF is:

$$\begin{aligned}
F_t^{2:N_t}(c|c \leq \bar{b}(F_t); N_t) =& N_t(N_t-1) \int_0^{F_t(c|c \leq \bar{b}(F_t))} u^{(N_t-2)}(1-u)du \\
\implies F_t^{2:N_t}(c|c \leq \bar{b}(F_t); N_t) =& N_t(N_t-1) \int_0^{F_t(c)/F_t(\bar{b}(F_t))} u^{(N-2)}(1-u)du = \mu(F_t, \bar{b}(F_t)) \\
\implies \mu'(F_t, \bar{b}(F_t)) =& N_t(N_t-1)(F_t(c|c \leq \bar{b}(F_t)))^{N_t-2}(1 - F_t(c|c \leq \bar{b}(F_t))) \\
& \left(\frac{1}{F_t(\bar{b}(F_t))} - \frac{F_t(c)}{F_t^2(\bar{b}(F_t))} \frac{\partial F_t(\bar{b}(F_t))}{\partial(\bar{b}(F_t))} \frac{\partial \bar{b}(F_t)}{\partial F_t} \right)
\end{aligned}$$

It can be noted from above that if $\bar{b}$ is not dependent on F_t, then $\frac{\partial \bar{b}(F_t)}{\partial F_t} = 0 \implies$

$\mu'(F_t, \bar{b}(F_t)) > 0$, i.e., the mapping between $F_t^{2:N_t}(c|c \leq \bar{b}(F_t))$ and $F_t(c|c \leq \bar{b}(F_t))$ is monotonic. In SECI's case, we can't say if $\mu'(F_t, \bar{b}(F_t)) > 0$ or $\mu'(F_t, \bar{b}(F_t)) < 0$ for all $F_t \in \mathcal{F}$ unless we know $\bar{b}(F_t)$. In the absence of monotonicity of $\mu(F_t, \bar{b}(F_t))$, the mapping between $F_t^{2:N_t}(c|c \leq \bar{b}(F_t))$ and $F_t(c|c \leq \bar{b}(F_t))$ is not one-one, which is a problem for identification. Even if $\bar{b}(F_t)$ is known, it can have an irregular shape, in which case $F_t^{2:N_t}(c|c \leq \bar{b}(F_t))$ may not be strictly monotonic in $F_t(c|c \leq \bar{b}(F_t))$.

Thus, we can't use distribution of a single order statistic for identification. Now, suppose that data provides at least 2 order statistics for each auction, say $c^{(k_1:N)} = x, c^{(k_2:N)} = y$, where $k_1 < k_2, x < y$. Define a truncated distribution $H(x; \bar{b}(F)) = F(x)/F(\bar{b}(F))$ and denote its density by $h(x; \bar{b}(F))$.[16] The costs observed in the second round are drawn from $H(x; \bar{b}(F))$.

The PDF corresponding to the probability of observing $c^{(k_1:N)} = x$ conditional on $c^{(k_2:N)} = y$ is:

$$\begin{aligned}
p_{k_1|k_2}(x|y) &= \frac{(k_2-1)! H(x; \bar{b}(F(c)))^{k_1-1} h(x; \bar{b}(F(c)))(H(y; \bar{b}(F)) - H(x; \bar{b}(F)))^{k_2-k_1-1}}{(k_1-1)!(k_2-k_1-1)! H(y; \bar{b}(F))^{k_2-1}} \\
&= \frac{(k_2-1)! F(x)^{k_1-1} f(x)(F(y) - F(x))^{k_2-k_1-1}}{(k_1-1)!(k_2-k_1-1)! F(y)^{k_2-1}} \\
&= \frac{(k_2-1)!}{(k_1-1)!(k_2-k_1-1)!} \left(\frac{F(x)}{F(y)}\right)^{k_1-1} \frac{f(x)}{F(y)} \left(1 - \frac{F(x)}{F(y)}\right)^{k_2-k_1-1} \\
&= f^{k_1:k_2-1}(x|y), \ \textit{where } f(x|y) = f(x)/F(y)
\end{aligned} \tag{2}$$

The conditional density is same as the density of k_1^{th} order statistic from a sample of size $k_2 - 1$, which is truncated at y. Since y is drawn independently (unlike $\bar{b}(F(c))$), $F^{k_1:k_2-1}(x|y)$ is increasing function of $F(x|y)$. Thus, the latter can be identified by the observing former in the data (as in the case with exogenous threshold). $F(x)$ is then identified upto a threshold ($= \max_t c^{k_{t2}:N_t}$), by computing $F(x|y) \forall y$ observed over all the auctions. This methodology is same as the one proposed in Song (2006) for estimating type distribution with unknown number of bidders.

[16] I drop the subscript t denoting an auction for easy notation.

The identification here exploits the structure imposed by the order statistics. Intuitively, when we use the conditional distribution of order statistics (2), we are using the order statistics from a subset of observations lower than an exogenous threshold. Conditional on being below this threshold, the costs are drawn independently and can be used for identification of the distribution.

So far, I have assumed that bidders draw costs from same distribution, which is seldom the case. The estimates should account for any heterogeneity in the bidders' cost distributions. In order to so, econometrician needs to observe identity of all the bidders and the bidder corresponding to each observed bid. This is, fortunately, the case with SECI data.

6.1.1 Heterogeneity in bidders' cost distributions

Identification of cost distributions with asymmetric bidders in SECI auctions is similar to that of bid distributions in asymmetric dutch auctions with observations of winner's bid and identity and identities of all the other bidders. A dutch procurement auction is an open auction where the buyer would start from a low price and increase it until someone agrees to sell. Observing the winning bid (w) and winner identity (B_i for some i) amounts to observing the lowest order statistic of the bids and the bidder who bid it.[17] From Berman (1963), and subsequent work synthesized in Athey and Haile (2007) and Paarsch, Hong, et al. (2006), we know that the distributions of bids $G_i(b|b \leq \bar{b})$ can be identified for asymmetric dutch auctions with exogenous reserve $\bar{b}$, if we observe identities of all the bidders, winning bid, and the winning bidder. Since bids are same as cost in our case, these results imply that the cost distributions of each bidder can also be identified, as long as $\bar{b}$ is independent of these distributions. If it is not, identification is possible as long as each auction provides at least 2 order statistics.

I illustrate the identification with 3 bidders. For this purpose, I define some no-

[17] In contrast, in an English auction, observing the winning bid and winner identity tells the second lowest bid but doesn't tell the corresponding bidder identity.

tations. Denote the set of all the bidders by $\mathcal{I}$. $c^{k:\mathcal{I}}_{B_i}$ denotes the k^{th} lowest cost, which is bid by the bidder B_i. $B^{k:\mathcal{I}}$ denotes the bidder bidding the k^{th} lowest cost. Econometrician observes the following:

- 2 adjacent order statistics and their bidders, viz.
  - $c^{2:\mathcal{I}}_{B_j} = c_2$ for some arbitrary B_j.
  - $c^{3:\mathcal{I}}_{B_k} = c_3$ for some arbitrary B_k.
- B_i bids $b_1 \leq c_2$ for some arbitrary B_i

The PDF for joint probability of observing above is given as:

$$f^{(2,3:\mathcal{I})}_{(B_i,B_j,B_k)}(c_2, c_3) = H_{B_i}(c_2)h_{B_j}(c_2)h_{B_k}(c_3) \tag{3}$$

where (B_i, B_j, B_k) is an ordered tuple with bidder identities ordered by their bids; $H_{B_i}(x) = F_{B_i}(x)/F_{B_i}(\bar{b}(F_{B_i}, F_{B_j}, F_{B_k}))$.[18] The PDF corresponding to observing $c^{2:\mathcal{I}}_{B_j} = c_2$ conditional on $c^{3:\mathcal{I}}_{B_k} = c_3$ is given by:

$$\begin{aligned} p^{2|3:\mathcal{I}}_{(B_i,B_j,B_k)}(c_2|c_3) &= \frac{H_{B_i}(c_2)h_{B_j}(c_2)}{H_{B_i}(c_3)H_{B_j}(c_3)} \\ \implies p^{2|3:\mathcal{I}}_{(B_i,B_j,B_k)}(c_2|c_3) &= \frac{F_{B_i}(c_2)}{F_{B_i}(c_3)}\frac{f_{B_j}(c_2)}{F_{B_j}(c_3)} \\ &= f^{2:\{B_i,B_j\}}_{(B_i,B_j)}(c_2|c_3) \end{aligned} \tag{4}$$

Above is the PDF corresponding to the joint distribution of events $c^{2:\{B_i,B_j\}}_{B_j} = c_2$ and $c^{1:\{B_i,B_j\}}_{B_i} \leq c_2$, conditional on $c_2 \leq c_3$. Denote the corresponding CDF by $F^{2:\{B_i,B_j\}}_{(B_i,B_j)}(c_2|c_3)$. Then:

$$F^{2:\{B_i,B_j\}}_{(B_i,B_j)}(c_2|c_3) = \int_{\underline{c}}^{c_2} F_{B_i}(x|c_3)dF_{B_j}(x|c_3)$$

[18] As in dutch auctions, (4) can identify $F_i(c; c \leq \bar{b}(\mathcal{F}))\forall i$, which doesn't account for the endogeneity problem.

Argument for identification of $F_{B_j}(x|c_3)$ from here is same as that for asymmetric Dutch auctions with exogenous reserve with observation of winning bid, winning bidder, and other bidder identities (Paarsch, 1997; Athey and Haile, 2007). Just like in i.i.d case, we can identify $F_{B_j}(x)$ up to a threshold determined by $(max_{t\in\mathcal{T}}\{c^{3:\mathcal{I}_t}_{B_k t}\})$. Note that the mapping of bidder identity to bids is arbitrary and B_j is any arbitrary bidder, and hence the distribution is identified for each bidder.

The identification here is illustrated for the simplest possible case with 2 adjacent order statistics for the ease of exposition. This argument can be easily extended to more bidders case and to the situations where order statistics are not adjacent as long as we observe at least 2 of them.

6.2 Estimating cost distribution

In this section, I provide parametric estimates for the cost distribution. While this imposes additional non-testable structure, the limited amount of data prevents using non-parametric estimation. Moreover, parametric estimation enables us to find a non-truncated distribution, which may be helpful.[19]

The parametric estimation helps account for auction and bidder specific observed heterogeneity regarding cost distribution in a parsimonious fashion. In total, we can observe 103 costs across 23 SECI auctions, which can be used for estimating the distribution. Participation varies a lot across these auctions, with some of them having only 2 order statistics, and some others have 10. These 23 auctions are different in three aspects:

1. Temporal: There is a trend of decline in costs of renewable technology, which affects the cost distribution.

2. Technology: Some auctions are specifically for solar power, some for wind, and some are technologically neutral (hybrid).

[19]An analysis of tradeoff between the two methods can be found in Paarsch (1997) 3.4.

3. Geography: Some auctions specify the state where the project must be constructed, some are specific to certain solar parks, and some others are location-neutral. This affects the average cost of production because different parts of India have different solar irradiance. Moreover, a part of cost is connecting the project to the grid, which can be slightly low if the project is specific to a solar park.

I provide average of costs across all these different dimensions in Table 4. For the temporal part, I divide the sample into upto and after 2018, instead of looking at each year separately because the data is not balanced across years when segregated by technology and location. It can be noted that the average is lower for later years, as expected. But this lower average is driven mostly by solar auctions, specially in southern India. Looking at post 2018, we can also notice that the costs are lowest for solar auctions, followed by hybrid and then wind. In the data, we can observe that the wind and hybrid projects are location neutral. Location is mentioned only for solar projects, and it can be noticed prices are much higher in southern states before 2019, and slightly higher for later years. While the data mentions the exact state, I club the states or solar parks as south Indian if they are to the south of tropic of cancer. This grouping of states is done in order to have enough data across different categories along the geography dimension.

A major part of bidder specific heterogeneity emerges from financing costs. According to BloombergNEF (2022) report, lenders ascribe high importance to producer's track record. Other bidder specific factors include equipment suppliers and downstream contractors, chiefly. Report also mentions the difference in debt financing rates between wind and solar projects and bidders prefer to bid more in solar park as that reduces costs due delay in land acquisition. Minor importance is ascribed to project location. The report is based not just on SECI auctions, but on overall the renewable electricity sector in India.

Since I don't obtain detailed data of each firm's track record, downstream contractors or supplier networks, the bidder specific effects are captured through an indicator variable. This variable, X_i, equates 1 when the bidder B_i is one of the important bidders. A bidder is said to be important, if she is one of the seven largest power producers in the country as per the aforementioned bloomberg report. In particular, it includes Adani group, Greenko, ReNew, NTPC, Azure, Tata, and ACME.[20] To be precise, these are the seven largest producers based on all the contracts acquired by them, not just the ones obtained via SECI and NTPC auctions. For example, Tata Power has not been very successful in SECI auctions, but is a major producer at national level. Greenko doesn't participate in them. An important assumption here is that the large bidders were able to contribute so much to the capacity because they could access financing at cheaper rates, which can be due to any of their characteristics.

This assumption may not be unfounded if we look at some statistics from 23 SECI auctions used for estimation. We can notice that just four bidders- Adani, Renew, NTPC, and ACME- make up 33% of 106 bids with positive award and 38% of the overall capacity award of 24GW in the 23 auctions. These large firms account for 18.4% of the bids which were awarded zero. It seems that these firms have a bit of cost advantage over the others. Thus, being a large producer can be a proxy for the bidder specific characteristics which affect costs.

In particular, I assume that the cost of bidder i in an auction t is given by:

$$c_{it} \sim \mathcal{N}(\mu_{it}, var)$$
$$where\ \mu_{it} = \alpha_0 + \alpha_1 X_i + \alpha_2 X_t$$

where X_t are the auction specific controls, and $X_i = 1$ if the bidder B_i is important bidder.

The parameters $\alpha_0, \alpha_1, \alpha_2, var$ are estimated through maximum likelihood estimation. For each auction $t \in \mathcal{T}$, I can write a log likelihood function of observing the

[20] Note that NTPC is an auctioneer as well as a supplier. This may lead to some specific strategic considerations on their part, and may make them better informed too.

Table 4: Average of observed costs and number of auctions for each technology type before and after 2018 in SECI auctions

Average cost	Upto 2018	After 2018	Aggregate
Hybrid		2.507(16)	2.507(16)
Solar	3.080(36)	2.396(21)	2.828(57)
No location	2.774(5)	2.535(6)	2.644(11)
North	2.855(26)	2.328(13)	2.679(39)
South	4.558(5)	2.42(2)	3.947(7)
Wind	2.985(17)	2.897(13)	2.947(30)
Aggregate	3.050(53)	2.562(50)	2.813(103)

certain cost order statistics and corresponding bidder identities as:

$$\begin{aligned}&Log\mathcal{L}_t(\alpha_0,\alpha_1,\alpha_2,var;(\mathbf{c_t},\mathbf{b_t}),\mathcal{I}_t,X_t,\mathbf{X_i})\\ =&\Bigg(\sum_{i\in\mathcal{I}_{lt}}(\Phi(c^{l_t:\mathcal{I}_t};\alpha_0+\alpha_1X_{B_i}+\alpha_2X_t,var)-\Phi(c^{h_t:\mathcal{I}_t};\alpha_0+\alpha_1X_{B_i}+\alpha_2X_t,var)\\ &+\sum_{k=l_t}^{h_t-1}(\phi(c^{k:\mathcal{I}_t};\alpha_0+\alpha_1X_{B^{l_t:\mathcal{I}_t}}+\alpha_2X_t,var)-\Phi(c^{h_t:\mathcal{I}_t};\alpha_0+\alpha_1X_{B^{l_t:\mathcal{I}_t}}+\alpha_2X_t,var))\Bigg)\end{aligned}$$

where $(\mathbf{c_t},\mathbf{b_t})$ is the vector containing ordered pairs of cost order statistics and bidder identities pertaining to each of these statistics in auction t, l_t is the lowest ordered statistic in the auction t, h_t is the lowest ordered statistic, $\mathcal{I}_t$ is the set of bidders in auction t, $\mathcal{I}_{lt}$ is the set of bidders with bid less than $c^{l:\mathcal{I}_t}$, $B^{k:\mathcal{I}_t}$ is the bidder bidding k^{th} order statistic, where $k \geq l_t$.

Using this, I can write the log likelihood function for all the observations as:

$$Log\mathcal{L}(\alpha_0,\alpha_1,\alpha_2,var;Info_{\mathcal{T}})=\sum_{t\in\mathcal{T}}Log\mathcal{L}_t(\alpha_0,\alpha_1,\alpha_2,var;(\mathbf{c_t},\mathbf{b_t}),\mathcal{I}_t)$$

where $Info_{\mathcal{T}}$ is the set containing $(\mathbf{c_t},\mathbf{b_t}),\mathcal{I}_t,l_t,h_t,X_t,\mathbf{X}_i$ for all the auctions.

Due to requirement that we observe at least 2 order statistics of costs in each auction,

Table 5: Estimates of parameters of cost Distribution

Constant	Solar	Post 2018	Solar×Post2018	Important bidder	var
3.3338	0.8161	0.6539	−1.0877	−0.7450	0.8662
(0.2374)	(0.2624)	(0.2501)	(0.3085)	(0.1621)	(0.0557)

only 25 auctions (23 SECI, 2 NTPC) could be used for estimation. These auctions revealed 116 draws of different order statistics. Table 5 provides bootstrapped estimates of parameters of cost distribution. Notice that the bidders classified as important are likely to have lower costs on average. This implies that such bidders have stronger cost distributions in the sense of stochastic dominance. Hereafter, I refer to them as strong bidders, instead of important, in line with the literature in auction theory.

7 Alternatives to reduce inefficiency

In this section, I provide insights into the impact of making small changes in the allocation mechanism on the efficiency and payoffs of SECI auctions, given our estimates. In the literature on comparing different auction designs, there is no consensus on which method is better for the auctioneer and/or the society. As per Holmberg and Wolak (2018), the suitability of low/high price sealed bid vs open ascending/descending bid auction depends on the application at hand, and hence warrants a separate investigation for each setting. Moreover, sealed bid auctions are also regularly used in other sectors of Indian economy, most important of them being the telecom spectrum allocation.[21] As such, it may be interesting to see what can be gained from replacing open bidding with sealed bidding in the final round, without changing other parts like complete information on bidder capacities.

I compare the two methods regarding their allocation efficiency and the payoffs for

[21] These auctions were marred by corruption scandal accusations (most notably 2G scam of 2008), which may have been a motivation behind having open auctions in renewable energy.

the auctioneer. Efficiency of each mechanism is measured by following 2 outcomes:

- %Inefficient: This is the probability that the lower cost bidder was rationed (i.e., got smaller capacity award).
- Cost inefficiency: This is the difference in cost incurred in creation of capacity when award is made as per a particular auction mechanism, with the cost incurred if the high cost bidder is rationed (welfare benchmark).

Besides efficiency, I compare these auctions with regards to the payments made by auctioneer to the suppliers.

The focus of this exercise is to gain insights from the case of 2 small bidders, i.e., $M > q_i \forall i$ and $q_1 + q_2 > M$. The theoretical results are well established for this case in the section 5.1.2. While such a restriction may hamper the direct applicability of the results, the aim here is to provide an evidence that welfare can be improved if we switch from uniform pricing to sealed bid discriminatory pricing, without changing any other feature of the auction.

While the bidders are asymmetric in cost distribution, they don't know if their opponent is strong or weak when in the auction. Thus, their belief about opponents cost can be thought of as mixture of the two distributions, weak and strong. In the data, it's observed that in 33% of the auctions, one of the bidders who got some positive award was the stronger bidder. In a very simplistic manner, I interpret it as 33% chance that one of the bidders who can get a residual award is a strong one. As such, bidders are assumed believe that their opponent is strong with a probability of 0.33 and weak with remaining probability. In any case, the exact mixture doesn't affect the qualitative result on welfare improvement.

Table 6 summarises the bootstrapped estimates for welfare outcomes and payments made by the auctioneer in USD over 25 year period, when $M = 300$ for different values of q_1, q_2. We can notice that the savings in social cost are significant across the mechanisms regardless of the level of asymmetries. As the asymmetry in quantities

Table 6: Performance of uniform price (UP) and discriminatory price (DP) auctions

q_1/q_2	%Inefficient	Avg diff	Cost ineff.	Avg diff	Payment	Avg diff
200/170						
UP	29.01		954,157		48,355,454	
	(3.032)		(124,567)		(3,974,700)	
DP	5.18	23.83	186,213	1,123,430	47,058,238	1,297,217
	(1.576)	(3.030)	(52,936)	(144,046)	(3,855,957)	(223,792)
220/170						
UP	28.38		1,002,057		33,401,929	
	(4.54)		(135,079)		(616,881)	
DP	8.03	20.35	301,693	1,271,372	32,285,294	1,116,634
	(2.181)	(3.694)	(120,706)	(135,753)	(744,330)	(509,935)
200/130						
UP	47.15		588,679		35,156,792	
	(4.039)		(58,338)		(630,855)	
DP	13.21	33.93	158,448	745,699	34,317,544	839,248
	(2.737)	(3.96)	(36,686)	(87,593)	(1,243,385)	(1,035,954)

The table reports values of average and standard deviation (in brackets) for 200 cost draws from 200 distributions, whose parameters are estimated from bootstrapped samples. This is a total of 200 observations for percentage of inefficiency, and 40,000 for others.
The draws are made from normal distribution truncated at 0.4 and 5.5. Reserve bid, $b^R = 5.6$, procurement demand $M = 300$.
Differences are calculated as UP-DP.
For the difference in cost inefficiency, the differences between UP and DP are averaged over the number of instances with different allocation.

increases, we can notice that the probability of inefficient selection increases in both uniform and discriminatory price auctions. The increase is much higher for uniform price vis-a-vis discriminatory. Notice from the column on social cost that at such level of asymmetry, the social cost itself is not very high. At this level of asymmetry, the cost saving is not as high in magnitude either, but it is significant. The payment made by auctioneer is also found to be lower when using discriminatory pricing. However, the differences in payments may not always be significant. This is because the markup earned by bidders in discriminatory price auction is much higher than in the uniform price auction.

From the figure 8, we can notice that bidder's bidding functions (and hence, bid distributions) are closer to each other in sealed bid discriminatory price auction,

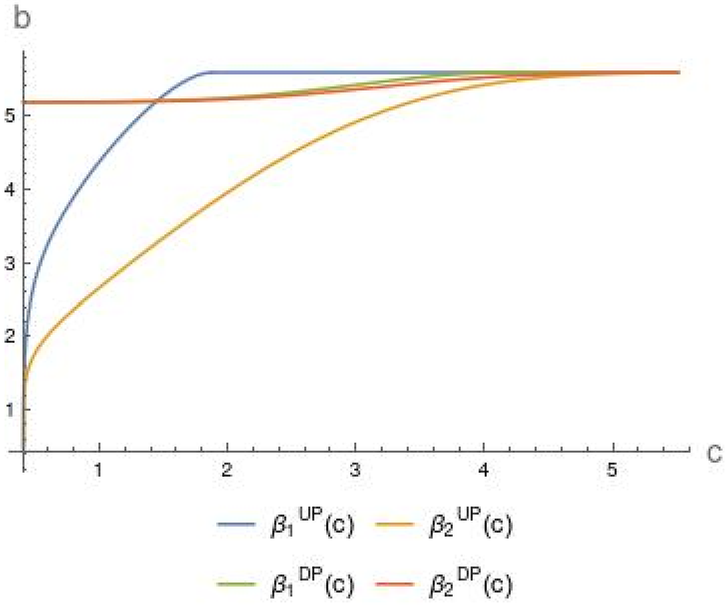


Figure 8: Bidding behavior: uniform versus discriminatory price

vis-a-vis open uniform price auction. The reason being that in the latter, bidders receive what they bid when they win, not what their opponent bids. As such, they are inclined to make bids with higher markup. Given that the highest cost type bids reserve, higher markup implies that the bidding functions are at higher levels for all costs in sealed bid compared to open bidding. This would imply that their bidding functions and the bid distributions are close. Such difference in bidding behavior across auctions should be seen as long as bidders are asymmetric. This leads me to hypothesize that discriminatory pricing will reduce social costs even if there are more bidders, as long as they are asymmetric in their capacities. If the bidders are symmetric, we don't have inefficient selection.

8 Conclusion

In this paper, I analyse the final round of auction mechanism used by Solar Energy Corporation of India (SECI), theoretically as well as empirically. This round is open descending price auction (uniform pricing). In particular, I analyse bidding behaviour and resultant inefficiencies arising from two key features of these auctions-public reporting of asymmetric bidders capacities, finite reserve, and rationing of suppliers. I show that switching from a uniform price to discriminatory price auction, while maintaining these features, leads to significant cost savings without affecting payment by auctioneer.

A preliminary analysis of SECI data tells us that in more than half of the auctions, a bidder agrees to getting rationed without competing much. It also tells us that the decision to compete or concede (and agree to be rationed) is positively correlated with the size of residual award vis-a-vis own capacity report. In order to explain such behavior, I characterise the semi-separating equilibrium of a descending clock version of this auction. I show that a finite reserve bid, bidder capacity asymmetries and rationing lead to bunching at the reserve by the supplier with higher capacity. The results also show that the bidder with higher capacity is likely to be less aggressive in the auction in such an equilibrium. Another feature of this equilibrium is that the bid at which bidder would exit, unless their opponent exits at a higher bid, is found to be related to the ratio of residual award and own capacity, as in the data. I show that this equilibrium necessarily exists and is unique semi-separating equilibrium for 2 player game. I further extend the results to the case of 3 bidders, where third bidder has very low capacity. As such, the findings from the data are corroborated by the theoretical results.

The theoretical results also show that these auctions are selecting inefficiently. In order to see if there exist alternative allocation mechanisms with lesser inefficiency, I need to know the distribution of costs of the bidders. For this purpose, I first estimate the bidders' cost distribution parametrically. Here the identification of parameters is enabled by observation of bids of bidders with zero award, who bid their cost. These costs provide us order statistics, albeit only of the bidders whose costs is below a threshold, which depends on the cost distribution parameter itself. I show that such an endogeneity can be dealt with, by using the density of probability of observing a low order statistic, conditional on a higher order statistic, when we can observe identities and bids of all the bidders. The identification then follows from the literature on dutch auctions with observed bidder identity.

Using these parameters, I use simulations to show that moving from uniform pricing of open auction to a sealed bid discriminatory pricing can help save social costs. This is because latter method lets suppliers have a higher markup, and reduces the asymmetry in their bidding functions. However, one should keep in mind that such savings are quantified using simulations, and with 2 bidders because of absence of complete theoretical results for more than two bidders. As I argue in the section 7, the result on attainment of cost savings on switching to discriminatory pricing should hold, the magnitude of cost saved would change. I also need to compare the existing method with some other counterfactual methods, in order to have more policy implications

As a future direction of work, a proper counterfactual exercise should be conducted to properly quantify the extent of benefits of moving from open to sealed bidding. At the same time, it is possible that some firms have easier access to financing because of their proximity to the government, which implies that their cost advantage is not based on efficiency. As such, it might be important to look at the effect of such proximity and conduct a counterfactual exercise to measure the extent of selection inefficiency created by selecting such bidders. Besides these immediate avenues of research, we can look into modelling the qualification round and for generalising theoretical results for final round.

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A Proofs for section 5

I continue with the assumption that $q_1 > q_2$. Throughout the proofs I denote $\underset{x \to x^-}{lim} u(x)$ by $u(x^-)$ and $\underset{x \to x^+}{lim} u(x)$ by $u(x^+)$ for any function $u(x)$.

A.1 Proof of Lemma 1

Proof. First I prove that the equilibrium should satisfy the specified conditions. Then, I show that there is no unilateral deviation from a bid suggested by these properties, for any type of any bidder.

Only if direction:

To prove condition (i), it is sufficient to show that payoff of a player satisfies Single crossing property of incremental returns (SCP IR). Consider any 2 arbitrary cost types of B_i, c_i and c_i' such that $c_i < c_i'$ and 2 bids b_i, b_i' such that $b_i < b_i'$. Then the property is satisfied if $\pi_i(b_i', c_i) - \pi_i(b_i, c_i) > 0$ implies $\pi_i(b_i', c_i') - \pi_i(b_i, c_i') > 0$ when B_{-i} bids with a non-decreasing strategy. Without loss of generality, assume $i = 1$.

$$\begin{aligned}\pi_1(b_1', c_1; b_2) &= (M - q_2)(b_1' - c_1)Pr(b_2 < b_1') + q_1\mathbb{E}(b_2 - c_1|b_2 > b_1')Pr(b_2 > b_1')\\ \pi_1(b_1, c_1; b_2) &= (M - q_2)(b_1 - c_1)Pr(b_2 < b_1) + q_1\mathbb{E}(b_2 - c_1|b_2 > b_1)Pr(b_2 > b_1)\end{aligned} \tag{5}$$

where b_2 is the random variable denoting B_2's bid.

$$\begin{aligned}&\therefore A(b_1', b_1, c_1, b_2) \equiv \pi_1(b_1', c_1; b_2) - \pi_1(b_1, c_1; b_2)\\ =&(M - q_2)[(b_1' - c_1)Pr(b_2 < b_1') - (b_1 - c_1)Pr(b_2 < b_1)]\\ &+ q_1[\mathbb{E}(b_2 - c_1|b_2 > b_1')Pr(b_2 > b_1') - \mathbb{E}(b_2 - c_1|b_2 > b_1)Pr(b_2 > b_1)]\end{aligned} \tag{6}$$

Suppose $A(b_1', b_1, c_1, b_2) > 0.$,

$$
\begin{aligned}
&\pi_1(b_1', c_1'; b_2) - \pi_1(b_1, c_1'; b_2)\\
=&(M - q_2)[(b_1' - c_1')Pr(b_2 < b_1') - (b_1 - c_1')Pr(b_2 < b_1)]\\
&+ q_1[\mathbb{E}(b_2 - c_1'|b_2 > b_1')Pr(b_2 > b_1') - \mathbb{E}(b_2 - c_1'|b_2 > b_1)Pr(b_2 > b_1)]\\
=&(M - q_2)[(b_1' - c_1 + c_1 - c_1')Pr(b_2 < b_1') - (b_1 - c_1 + c_1 - c_1')Pr(b_2 < b_1)]\\
&+ q_1[\mathbb{E}(b_2 - c_1 + c_1 - c_1'|b_2 > b_1')Pr(b_2 > b_1') - \mathbb{E}(b_2 - c_1 + c_1 - c_1'|b_2 > b_1)Pr(b_2 > b_1)]\\
=&A(b_1', b_1, c_1, b_2) + (M - q_2)(c_1 - c_1')[Pr(b_2 < b_1') - Pr(b_2 < b_1)] + q_1(c_1 - c_1')[Pr(b_2 > b_1') - Pr(b_2 > b_1)]\\
=&A(b_1', b_1, c_1, b_2) + (M - q_2)(c_1 - c_1')[Pr(b_2 < b_1') - Pr(b_2 < b_1)] + q_1(c_1 - c_1')[-Pr(b_2 < b_1') + Pr(b_2 < b_1)]\\
=&\underbrace{A(b_1', b_1, c_1, b_2)}_{>0} + \underbrace{(M - q_2 - q_1)}_{<0}\underbrace{(c_1 - c_1')}_{<0}\underbrace{[Pr(b_2 < b_1') - Pr(b_2 < b_1)]}_{>0}
\end{aligned}
\tag{7}
$$

As $b_1' > b_1$, $Pr(b_1' = max\{b_1', b_2\}) - Pr(b_1 = max\{b_1, b_2\}) > 0$ because the event $b_2 < b_1$ is a subset of the event $b_2 < b_1'$. This along with $A(b', b, c_1, b_2) > 0$, $c_1 < c_1'$, $M < q_1 + q_2$, ensures that above expression above is positive. Thus, $\pi_1(b_1', c_1'; b_2) - \pi_1(b_1, c_1'; b_2) > 0$, which proves the SCP-IR. Thus, equilibrium is monotonic.

(ii) property establishes atomlessness at $b < b^R$ and continuity of bidding strategies.

Continuity: For this, I proceed in two steps. First I show that the bids have common support, and then I show that they have full support. Given the monotonicity of equilibrium, the only type of discontinuity is the one where for some type c_1 of B_1, $\beta_i(c_i^-) = b' < \beta_i(c_i) = b$. Suppose they don't have common support. Then, as shown in the figure 9a, $\exists \tilde{c}_2$ s.t. $\beta_2(\tilde{c}_2) \in [b', b]$. The payoff to this type of B_2 is $\pi_2(\beta_2(\tilde{c}_2), \tilde{c}_2) = (\beta_2(\tilde{c}_2) - \tilde{c}_2)(M - q_1)Pr(b_1 < \beta_2(\tilde{c}_2)) + q_2\mathbb{E}(b_1 - \tilde{c}_2|b_1 > \beta_2(\tilde{c}_2))Pr(b_1 > \beta_2(\tilde{c}_2))$.

If she bids b, her payoff is $\pi_2(b, \tilde{c}_2) = (b - \tilde{c}_2)(M - q_1)Pr(b_1 < b) + q_2\mathbb{E}(b_1 - \tilde{c}_2|b_1 > b)Pr(b_1 > b)$. The monotonicity of B_1's strategy and a hole in her bid distribution on (b', b), and atomlessness of cost distribution, $Pr(b_1 > b) = Pr(b_1 > \beta_2(\tilde{c}_2))$ and $Pr(b_1 < b) = Pr(b_1 < \beta_2(\tilde{c}_2))$. Thus, $\pi_2(b, \tilde{c}_2) - \pi_2(\beta_2(\tilde{c}_2), \tilde{c}_2) = (b - \beta_2(\tilde{c}_2))(M - q_1)Pr(b_1 < b) + q_1\mathbb{E}(b_1 - \tilde{c}_2|b_1 > b)Pr(b_1 > b) > 0$.

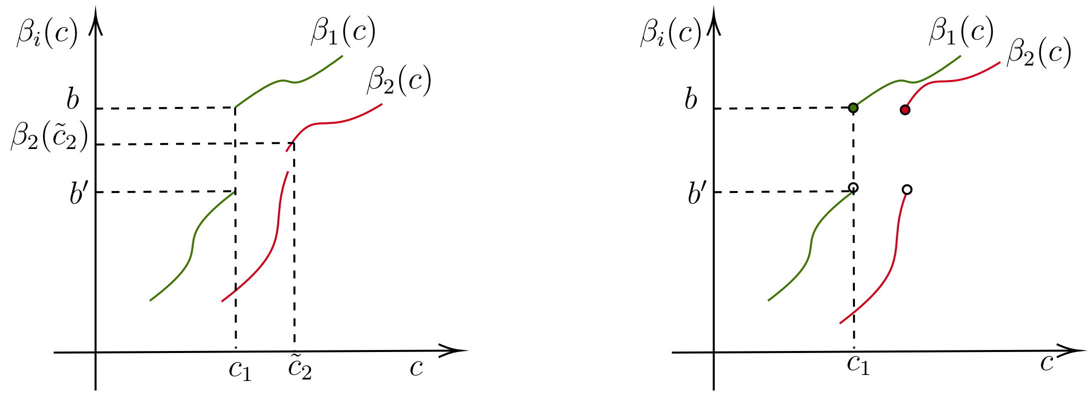
Now suppose that there is a range of bids $[b', b)$, $b' < b$ which are bid by none of the bidder and all bids below b' are being bid by some type of each bidder. This is shown in Figure 9b. Consider a type $c_1 - \epsilon$, $\epsilon \to 0$ of B_1 such that her type c_1 bids b. Given the result on full support and monotonicity, this type would bid $b' - \delta(\epsilon)$, $\delta(\epsilon) \to 0$ when $\epsilon \to 0$. Her payoff is:

$$
\begin{aligned}
&\pi_1(b' - \delta(\epsilon); c_1 - \epsilon, b_2) \\
=&(M - q_2)(b' - \delta(\epsilon) - c_1 + \epsilon)Pr(b_2 < b' - \delta(\epsilon)) + q_1\mathbb{E}(b_2 - c_1 - \epsilon(\epsilon)|b_2 > b' - \delta(\epsilon))Pr(b_2 > b' - \delta(\epsilon))
\end{aligned}
$$

If she instead bids b, her payoff is:

$$
\begin{aligned}
&\pi_1(b; c_1 - \epsilon, b_2) \\
=&(M - q_2)(b - c_1 + \epsilon)Pr(b_2 < b) + q_1\mathbb{E}(b_2 - c_1 + \epsilon|b_2 > b)Pr(b_2 > b) \\
=&(M - q_2)(b - c_1 + \epsilon)Pr(b_2 < b') + q_1\mathbb{E}(b_2 - c_1 + \epsilon|b_2 > b')Pr(b_2 > b') \\
=&(M - q_2)(b - c_1 + \epsilon)(Pr(b_2 < b' - \delta(\epsilon)) + Pr(b' - \delta(\epsilon) < b_2 < b')) \\
&+ q_1(\mathbb{E}(b_2 - c_1 - \epsilon|b_2 > b' - \delta(\epsilon))Pr(b_2 > b' - \delta(\epsilon)) - \mathbb{E}(b_2 - c_1 - \epsilon|b' - \delta < b_2 < b')Pr(b' - \delta(\epsilon) < b_2 < b'))
\end{aligned}
$$

where the last expression follows from non-degeneracy of cost distribution. Using above two expressions, I can infer the following:



(a) (b)

Figure 9: Discontinuity of bidding functions

$$\begin{aligned}
&\pi_1(b; c_1-\epsilon, b_2) - \pi_1(b'-\delta(\epsilon); c_1-\epsilon, b_2) \\
=&(b-b'+\delta(\epsilon))(M-q_2)Pr(b_2 < b'-\delta(\epsilon)) + ((M-q_2)(b-c_1+\epsilon) \\
&- q_1\mathbb{E}(b_2-c_1-\epsilon|b'-\delta(\epsilon) < b_2 < b'))Pr(b'-\delta(\epsilon) < b_2 < b') \\
\therefore &\lim_{\epsilon\to 0} \pi_1(b; c_1-\epsilon, b_2) - \pi_1(b'-\delta(\epsilon); c_1-\epsilon, b_2) = (b-b')(M-q_2)Pr(b_2 < b') > 0
\end{aligned}$$

Thus, there is a strictly positive deviation for B_1 when the bids do not have full support. Similar deviation can be shown for B_2 too. Thus, the result on common and full support for bids of both players tells us that their strategies are continuous.

No atom at bids below b^R: In any equilibrium, a cost type of a bidder has to be locally indifferent between the bid suggested by PBE and a bid slightly lower or higher. Suppose that in equilibrium, B_1 has an atom of probability mass $\varepsilon > 0$ at some bid $b_1 < b^R$. If opponent bids continuously. Then B_2 has a type $c_2 + \delta$, where $\delta \to 0$ and type c_2 bids b_1. This is exhibited in Figure 10. If this type decides to reduce her bid to b_1^-, then her marginal cost is almost zero, but marginal benefit is $(q_1 + q_2 - M)\varepsilon(b_1 - c_2)$. Thus, B_2 of this type (c_2^+) can profit by bidding slightly lower than b_1. Thus, there is no equilibrium where there is an atom for $b < b^R$.

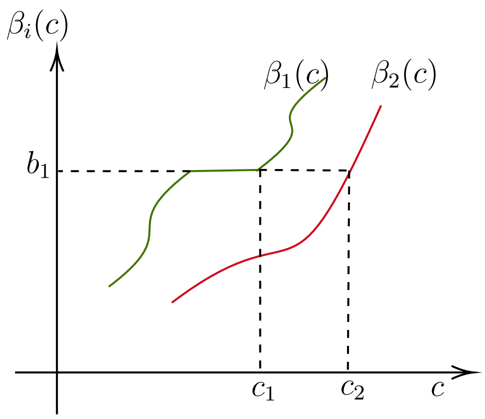


Figure 10: Deviation if there is an atom in bids

From (i) and (ii), we know that $\beta_i(c)$ is invertible for all c as long as $\beta_i(c) \neq b^R$.

Thus, for each i, I can define the functions $\phi_i(b), \forall i$ as follows:

$$\phi_i(b) := \begin{cases} \beta_i^{-1}(b) & \text{for } b < b^R \\ Inf\{c : \beta_i(c) = b^R\} & \text{for } b = b^R \end{cases}$$

$\phi_i(b)$ gives the cost type of B_i who would some bid $b < b^R$ in equilibrium. If the bidder bids b^R, then $\phi_i(b)$ gives the smallest cost type of B_i who would bid b^R. Since the equilibrium bids are continuous monotonic, the inverse is also continuous and monotonic.

Condition (iii) can be argued as follows. Suppose wlog that in equilibrium $\beta_1(0) = \underline{b}$ but $\beta_2(c_*) = \underline{b}$ for some $c_* > 0$ and $\underline{b} > 0$. Given the strict monotonicity of $\phi_i(b)$, the type $c_* + \epsilon, \epsilon \to 0$ of B_2 would bid some $\underline{b} + \delta(\epsilon)$, $\delta(\epsilon) \to 0$. It's payoff is:

$$\begin{aligned}
&\pi_2(\underline{b} + \delta(\epsilon), c_* + \epsilon) \\
=&(M - q_1)F(\phi_1(\underline{b} + \delta(\epsilon)))(\underline{b} + \delta(\epsilon) - c_* - \epsilon) + q_2 \int_{\underline{b}+\delta(\epsilon)}^{b^R} (x - c_* - \epsilon) dF(\phi_1(x)) \\
=&(M - q_1)(F(\phi_1(0)) + f(\phi_1(\underline{b}))\phi_1'(\underline{b}))(\underline{b} + \delta(\epsilon) - c_* - \epsilon) + q_2 \int_{\underline{b}+\delta(\epsilon)}^{b^R} (x - c_* - \epsilon) dF(\phi_1(x)) \\
\approx& q_2 \int_{\underline{b}}^{b^R} (x - c_* - \epsilon) dF(\phi_1(x)) - \underbrace{\delta(\epsilon)(q_1 + q_2 - M) f(\phi_1(\underline{b}))\phi_1'(\underline{b})(\underline{b} - c_* - \epsilon)}_{>0} \\
&+ (M - q_1) \underbrace{F(\phi_1(\underline{b}))}_{=0}(\underline{b} + \delta(\epsilon) - c_* - \epsilon) \\
<& q_2 \int_{\underline{b}}^{b^R} (x - c_* - \epsilon) dF(\phi_1(x)) \\
<& q_2 \int_{\underline{b}}^{b^R} (x - c_* - \epsilon) dF(\phi_1(x)) \\
=&\pi_2(\underline{b} - \gamma, c_* + \epsilon) \ , \forall \gamma > 0
\end{aligned}$$

Thus, there is a strictly profitable deviation for the type $c_* + \epsilon$. This deviation doesn't exist if $\underline{b} = 0$. Similar deviation can be shown if $\beta_2(0) = \underline{b} > \beta_1(0) = 0$. Therefore, in equilibrium $\beta_i(0) = 0$ for both i.

To see (iv), suppose that B_{-i} is playing as per ϕ_{-i} which satisfies equation 1 (when replacing $\beta_i^{-1}(c)$ with $\phi_i(b)$. Then, the payoff of B_i of type c_i when she bids b_i is:

$$\pi_i(b_i; c_i, \phi_{-i}(b)) = F(\phi_{-i}(b_i))(b_i - c_i)(M - \sum_{j \neq i} q_j) + q_i \int_{b_i}^{b^R} (x - c_i) dF(\phi_{-i}(x)) \quad (8)$$

Any interior optimum of this payoff will satisfy the first order condition of optimisation, which is:

$$f(\phi_{-i}(b_i))\phi'_{-i}(b_i)(b_i - c_i)(M - q_{-i} - q_i) + F(\phi_{-i}(b_i))(M - q_{-i}) = 0$$

Replacing c_i by $\phi_i(b)$, one can attain (1) for B_i.

Finally I prove (v), which states that B_1 partially pools at b^R in equilibrium. For this, I first prove that there can be at most one intersection between $\phi_2(b)$ and $\phi_1(b)$ and that intersection should be as in Figure 11. Then I show that even in the immediate right neighbourhood of 0, $\phi_2(b) > \phi_1(b)$, which shows that any intersection as shown in the figure is not possible. These two together imply that $\phi_2(b) > \phi_1(b)$ for $b > 0$.

For first step, note that at any point of intersection of $\phi_1(b)$ and $\phi_2(b)$, one can see from (1) that $\dfrac{\phi'_2(b)}{\phi'_1(b)} = \dfrac{M - q_2}{M - q_1} > 1$. This would imply that $\phi_2(b))$ should intersect that $\phi_1(b)$ just once and from below and left of it, as show in figure 11. This is because the inequality $\phi'_2(b) > \phi'_1(b)$ will not be satisfied at the second point of intersection. Note that if $\phi_1(b) < \phi_2(b)$ for some $b < b^R$, there will be no intersection between the two functions for bids above this b. Suppose that $\exists b_t \leq b^R$, such that $\phi_1(b) \geq \phi_2(b), \forall b \leq b_t$ with equality only at $b = b_t$ (as shown in Figure 11). Since $\phi_2(b)$ can intersect $\phi_1(b)$ only from left and below, all other cases are ruled out.

From (iii), we know that as $c \to 0^+$, $\beta_1(c) \to 0^+, \beta_2(c) \to 0^+$. This implies that $\beta_1(c) \to \beta_2(c)$ as $c \to 0^+$. From (i) and (ii), $\beta_i(c)$ is continuous and strictly monotonic when $c \to 0^+$, which implies that $\phi_i(b)$ is defined for all $b > 0$, and that $\lim_{b \to 0^+} \phi_i(b) = 0$.

For the second step, consider some $\delta > 0, \delta \to 0$ and suppose $\phi_i(\delta/n) = 0 + \epsilon_i(\delta/n)$

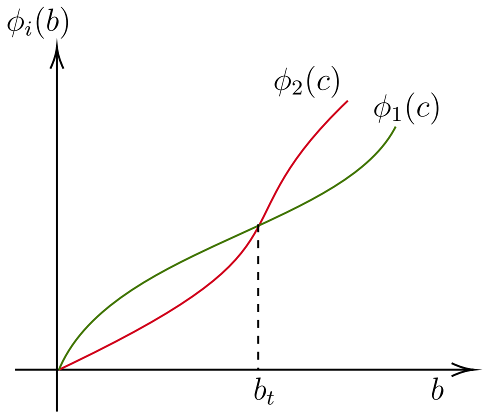


Figure 11: Possible intersection between $\phi_1(b)$ and $\phi_2(b)$

for some natural number $n \geq 1$. Then $\phi_i(\delta) - \phi_i(\delta/n) = \frac{n-1}{n}\delta\phi_i'(\delta) + \delta^2\kappa_i(\delta, \delta/n)$ for each i, where $\kappa_i(.)$ is a bounded function. Therefore,

$$\frac{\phi_2'(\delta)}{\phi_1'(\delta)} = \frac{\phi_2(\delta) - \phi_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n)}{\phi_1(\delta) - \phi_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)} = \frac{\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n)}{\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)} \tag{9}$$

From FOCs (equations 1), $\frac{\phi_2'(\delta)}{\phi_1'(\delta)} = \frac{M - q_2}{M - q_1}\frac{\sigma(0) + \epsilon_1(\delta)\sigma'(0)}{\sigma(0) + \epsilon_2(\delta)\sigma'(0)}\frac{\delta - \epsilon_2(\delta)}{\delta - \epsilon_1(\delta)}$. $\frac{\sigma(c)}{\sigma'(c)} = \frac{f(c)}{f'(c) - f^2(c)/F(c)}$. This implies that $\frac{\sigma(0)}{\sigma'(0)} \approx 0$ because $F(0)$ is almost 0 under the assumption of a very small atom at 0 in the cost distribution.

Thus $\frac{\phi_2'(\delta)}{\phi_1'(\delta)} = \frac{M - q_2}{M - q_1}\frac{\sigma'(0)(\sigma(0)/\sigma'(0) + \epsilon_1(\delta))}{\sigma'(0)(\sigma(0)/\sigma'(0) + \epsilon_2(\delta))}\frac{\delta - \epsilon_2(\delta)}{\delta - \epsilon_1(\delta)} = \frac{M - q_2}{M - q_1}\frac{\epsilon_1(\delta)}{\epsilon_2(\delta)}\frac{\delta - \epsilon_2(\delta)}{\delta - \epsilon_1(\delta)}$.

Alongwith Equation (9), this implies:

$$\frac{\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n)}{\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)} = \frac{M - q_2}{M - q_1}\frac{\epsilon_1(\delta)}{\epsilon_2(\delta)}\frac{\delta - \epsilon_2(\delta)}{\delta - \epsilon_1(\delta)}$$

$$\implies \underbrace{\frac{M - q_2}{M - q_1}}_{>1} = \underbrace{\frac{\epsilon_2(\delta)(\delta - \epsilon_1(\delta))}{\epsilon_1(\delta)(\delta - \epsilon_2(\delta))}}_{>1, \text{ if } \epsilon_2(\delta) > \epsilon_1(\delta)} \underbrace{\frac{\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n)}{\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)}}_{>1, \text{ if } \epsilon_2(\delta) > \epsilon_1(\delta)}$$

The relation above should hold for all $n > 1$. LHS>1 in above. Since $\delta \to 0$ and $\kappa_i(\delta, \delta/n)$ is bounded function, $\delta^2\kappa_i(\delta, \delta/n) \approx 0$. As $n \to \infty$, $\epsilon_2(\delta/n) \approx \epsilon_1(\delta/n)$ as the values of $\phi_i(b)$s approach 0, as $b \to 0$. Therefore, as $n \to \infty$, if $\epsilon_2(\delta) \leq \epsilon_1(\delta)$, then RHS$\leq$1 while LHS>1, which violates the equation above. Therefore, $\epsilon_2(\delta) > \epsilon_1(\delta)$.

This implies that $\phi_2(b) > \phi_1(b)$ in the immediate right neighbourhood of 0. Hence, there is no point of intersection between $\phi_2(b)$ and $\phi_1(b)$ for $b > 0$. Thus, for any $b \in (0, b^R]$, $\phi_2(b) > \phi_1(b)$ and, in particular, $\exists c^* < \bar{c},\ s.t.\ \phi_1(b^R) = c^*$. To obtain the bid function, one can then invert $\phi_i(b)$ for both i, and for $c > c^*$, assign $\beta_1(c) = b^R$, which is implied by the non-negative monotonicity of β_is.

<u>If direction:</u>

The conditions give equilibrium, if there is no deviation for any type c_i of any player B_i, from the bid recommended by $\beta_i(c_i)$. While the calculations here are for $i = 1$, the proof for $i = 2$ is the same. Suppose $\phi_1(b_1) = c_1$, where $0 < b_1 < b^R$. Define $\Pi_1(b_1', b_1, c_1; \phi_2(b)) := \pi_1(b_1', c_1; \phi_2(b)) - \pi_1(b_1, c_1; \phi_2(b))$ as the change in payoff of B_1 if she bids $b_1' \in [0, b^R]$ instead of b_1. Given the continuity, monotonicity and full support of bids, $\exists$ a type c_1' such that $\phi_1(b_1') = c_1'$. Since $\phi_1(b)$ satisfies 1,

$$
\begin{aligned}
&\frac{\partial}{\partial b_1'}\pi_1(b_1', b_1, c_1; \phi_2(b))\\
=&\frac{\partial}{\partial b_1'}\pi_1(b_1', c_1; \phi_2(b))\\
=&(M - q_2 - q_1)(b_1' - c_1)f(\phi_2(b_1'))\phi_2'(b_1') + (M - q_2)F(\phi_2(b_1'))\\
=&(M - q_2 - q_1)(b_1' - c_1' + c_1' - c_1)f(\phi_2(b_1'))\phi_2'(b_1') + (M - q_2)F(\phi_2(b_1'))\\
=&(c_1' - c_1)(M - q_2 - q_1)f(\phi_2(b_1'))\phi_2'(b_1')\\
=&(\phi_1(b_1') - \phi_1(b_1))(M - q_2 - q_1)f(\phi_2(b_1'))\phi_2'(b_1')
\end{aligned}
$$

where the second last equation is arrived by using the FOC 1 for type c_1'. Given the monotonicity of $\phi_1(b)$, $\phi_1(b_1') - \phi_1(b_1) > 0$ if $b_1' > b_1$, which implies that $\frac{\partial}{\partial b_1'}\Pi_1(b_1', b_1, c_1; \phi_2(b)) < 0$. Since $\Pi_1(b_1, b_1, c_1; \phi_2(b)) = 0$, this implies that any deviation from b_1 to a higher bid would lead to reduction in expected payoff. Similarly, when $b_1' < b_1$, $\phi_1(b_1') - \phi_1(b_1) < 0$, which would ultimately imply that any deviation from b_1 to a lower bid will lead to reduction in expected payoff. Thus, there is no strictly positive deviation for type c_1 of B_1 from the strategy recommended by conditions of Lemma 1. Since c_1 was chosen arbitrarily, I can infer that no such deviation can be found for any other type. Similar calculations can be done for B_2.

The absence of any unilateral deviation implies that a function $\beta_i(c)$ whcih satisfies the conditions in the Lemma indeed gives a bayes nash equilibrium. ■

A.2 Proof of Theorem 1

To show that equilibrium exists and is unique amounts to showing that there is exactly one pair of two functions $\beta_1(c)$ and $\beta_2(c)$ such that the conditions of Lemma 1 are satisfied. To do so, I proceed in following steps:

1. Consider an Initial Value Problem $\mathcal{P}$ as follows:

$$\begin{aligned} \phi_2'(b) &= \frac{M-q_2}{q_1+q_2-M}\frac{1}{\sigma(\phi_2(b))(b-\phi_1(b))} \\ \phi_1'(b) &= \frac{M-q_1}{q_1+q_2-M}\frac{1}{\sigma(\phi_1(b))(b-\phi_2(b))} \end{aligned} \tag{10}$$

   $\phi_2(b^R)=\bar{c}$, and $\phi_1(b^R)=c^*\leq\bar{c}$. Cauchy Lipschitz theorem implies that $\exists a$ such that a unique solution to $\mathcal{P}$ exists for interval $[b^R-a, b^R+a]$ because $b^R>\bar{c}$.

2. Show that this solution is monotonic and extend the local solution to the interval $(0, b^R]$.

3. Show that there is at most one IVP $\mathcal{P}$ whose solution $\phi_1(b), \phi_2(b)$ are such that $\underset{b\to 0}{lim}\phi_1(b) = \underset{b\to 0}{lim}\phi_2(b) = 0$.

4. Show that there is exactly one value of c^* such that $\underset{b\to 0}{lim}\phi_1(b) = \underset{b\to 0}{lim}\phi_2(b) = 0$, where $\phi_i(b)$ solve $\mathcal{P}$.

5. Extend the functions to include 0 in their domain, by assuming that $\phi_1(0) = \phi_2(0) = 0$.

6. Invert $\phi_i(b)$s. Note that the domain of $\phi_1^{-1}(c)$ is $[0, c^*]$. Thus, $\beta_1(c)$ is defined

as:

$$\beta_1(c) = \begin{cases} \phi_1^{-1}(c) & 0 \le c \le c^* \\ b^R & c^* < c \le \bar{c}. \end{cases}$$

append $\beta_1(c)$ with a constant function which takes value b^R for $c \in (c_1^*, \bar{c}]$. $\beta_2(c) = \phi_2^{-1}(c)$.

Step 1 is obvious from Cauchy Lipschitz theorem. I now prove steps 2,3, and 4. Steps 5 and 6 do not require any proof.

Proof of step 2:

Proof. The solution to $\mathcal{P}$ is monotonic. This can be seen through a contradiction argument. Since $\phi_1(b)$ and $\phi_2(b)$ are solutions to ODEs on some interval containing b^R, they are continuous and differentiable in that interval. Thus, if the solution was not monotonic, then $\exists b_i$ such that $\phi_i'(b_i) = 0$. Under the assumption that there is a small atom at 0, $F(c) > 0 \forall c \in [0, \bar{c}]$. Thus, $\phi_i'(b_i) = 0$ only if $|\phi_{-i}(b_i)| = \infty$. This violates the boundedness theorem. Thus, the solutions $\phi_1(b)$ and $\phi_2(b)$ are monotonic. Moreover, this monotonicity is positive. To see this, note that $b^R > \bar{c} \implies b^R - \phi_{2n}(b^R) > 0 \implies \phi_{1n}'(b^R) > 0$. A negative monotonicity would contradict this. Given this positive monotonicity, the IVP remains well defined and it's RHS is lipschitz continuous for any $b \ge b^R - a$. Thus, one can extend this local solution to any closed interval in $(0, b^R]$. ■

Proof of step 3:

Proof. Consider another IVP, $\hat{\mathcal{P}}$, which is same as $\mathcal{P}$ except that $\hat{\phi}_1(b^R) = \hat{c}^*$ for some $\hat{c}^* \in (c^*, \bar{c})$. Denote its solution by $(\hat{\phi}_1(b), \hat{\phi}_2(b))$.

Since $\phi_2(b^R) = \bar{c} = \hat{\phi}_2(b^R)$, $\sigma(\hat{\phi}_2(b^R)) = \sigma(\phi_2(b^R))$. Using FOCs, it can be inferred that $\hat{\phi}_2'(b^R)(b^R - \hat{\phi}_1(b^R)) = \phi_2'(b^R)(b^R - \phi_1(b^R)) = \frac{M - q_2}{(q_1 + q_2 - M)\sigma(\phi_2(b^R))}$. This further implies $\hat{\phi}_2'(b^R)(b^R - \hat{c}^*) = \phi_{2n}'(b^R)(b^R - c^*)$. Since $\hat{c}^* > c^*$, $b^R - \hat{c}^* < b^R - c^*$, which implies that $\hat{\phi}_2'(b^R) > \phi_2'(b^R)$. This implies that for any b in the immediate left-neighbourhood of b^R, $\hat{\phi}_2(b) < \phi_2(b)$.

Suppose for any $b_{2t} \in (0, b^R]$, $\hat{\phi}_2(b)$ and $\phi_2(b)$ intersect as shown in the figure 5b. Then, $\hat{\phi}_2(b_{2t}) = \phi_2(b_{2t})$ and $\hat{\phi}_2'(b_{2t}) < \phi_2'(b_{2t})$, which imply that $\sigma(\hat{\phi}_2(2t))\hat{\phi}_2'(2t) < \sigma(\phi_2(2t))\phi_2'(2t)$. From the FOCs, it can then be inferred that $b_{2t} - \hat{\phi}_{1n}(b_{2t}) > b_{2t} - \phi_1(b_{2t})$, which implies that $\hat{\phi}_1(b_{2t}) < \phi_1(b_{2t})$. This requires an intersection between $\hat{\phi}_1(b)$ and $\phi_1(b)$ at some point $b_{1t} > b_{2t}$. Thus, there are two solutions to the IVP defined by ODEs 10, and boundary at points b_{1t} and b_{2t}, which violates the cauchy-lipschitz theorem of uniquenss of IVP solution. Thus, $\forall b \in (0, b^R)$, $\hat{\phi}_2(b) < \phi_2(b)$ and $\hat{\phi}_2(b^R) = \phi_2(b^R)$, and $\hat{\phi}_1(b) > \phi_1(b) \forall b \in (0, b^R]$.

If solutions to both $\mathcal{P}$ and $\hat{\mathcal{P}}$ satisfy condition (ii) and (iii) of Lemma 1, then following needs to be satisfied.

$$\lim_{b\to 0}\phi_1(b) = \lim_{b\to 0}\phi_2(b) = \lim_{b\to 0}\hat{\phi}_1(b) = \lim_{b\to 0}\hat{\phi}_2(b) = 0 \tag{11}$$

As in proof of the lemma, following can be written for some $\delta > 0$,$\delta \to 0$ and a natural number $n \geq 1$:

$$\begin{aligned} \frac{M - q_2}{M - q_1} &= \frac{\epsilon_2(\delta)(\delta - \epsilon_1(\delta))}{\epsilon_1(\delta)(\delta - \epsilon_2(\delta))} \frac{\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n)}{\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)} \\ &= \frac{\hat{\epsilon}_2(\delta)(\delta - \hat{\epsilon}_1(\delta))}{\hat{\epsilon}_1(\delta)(\delta - \hat{\epsilon}_2(\delta))} \frac{\hat{\epsilon}_2(\delta) - \hat{\epsilon}_2(\delta/n) - \delta^2\hat{\kappa}_2(\delta, \delta/n)}{\hat{\epsilon}_1(\delta) - \hat{\epsilon}_1(\delta/n) - \delta^2\hat{\kappa}_1(\delta, \delta/n)} \end{aligned}$$

where $\phi_i(\delta/n) = \epsilon_i(\delta/n)$ and $\hat{\phi}_i(\delta/n) = \hat{\epsilon}_i(\delta/n)$. Above can be rewritten as:

$$\frac{\epsilon_2(\delta)(\delta - \hat{\epsilon}_2(\delta))}{\hat{\epsilon}_2(\delta)(\delta - \epsilon_2(\delta))} \frac{\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n)}{\hat{\epsilon}_2(\delta) - \hat{\epsilon}_2(\delta/n) - \delta^2\hat{\kappa}_2(\delta, \delta/n)} = \frac{\epsilon_1(\delta)(\delta - \hat{\epsilon}_1(\delta))}{\hat{\epsilon}_1(\delta)(\delta - \epsilon_1(\delta))} \frac{\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)}{\hat{\epsilon}_1(\delta) - \hat{\epsilon}_1(\delta/n) - \delta^2\hat{\kappa}_1(\delta, \delta/n)}$$

In order to satisfy (14), it is necessary that as $n \to \infty$, $\hat{\epsilon}_i(\delta/n) \to 0$ and $\epsilon_i(\delta/n) \to 0$. Furthermore as $\delta \to 0$ and κ_i and $\hat{\kappa}_i$ are bounded functions, one can further infer that in the limit of $n \to \infty$ and $\delta \to 0$:

$$\frac{\epsilon_2^2(\delta)(\delta - \hat{\epsilon}_2(\delta))}{\hat{\epsilon}_2^2(\delta)(\delta - \epsilon_2(\delta))} = \frac{\epsilon_1^2(\delta)(\delta - \hat{\epsilon}_1(\delta))}{\hat{\epsilon}_1^2(\delta)(\delta - \epsilon_1(\delta))}$$

From above, it can be noticed that if $\hat{\epsilon}_2(\delta) < \epsilon_2(\delta)$, then LHS>1. Thus, $\hat{\epsilon}_1(\delta) < \epsilon_1(\delta)$ for RHS>1. This contradicts the result that $\forall b \in (0, b^R)$, $\hat{\phi}_2(b) < \phi_2(b)$ and $\hat{\phi}_2(b^R) = \phi_2(b^R)$, and $\hat{\phi}_1(b) > \phi_1(b) \forall b \in (0, b^R]$. Thus, we can't have more than one IVP which satisfies all the conditions of Lemma 1.

To complete the proof, I use monotonicity result from step 2. In order to have $\phi_i'(b) > 0$, it is required that $b > \phi_{-i}(b)$ for each i. Thus, as $b \to 0$, we have $\phi_i(b) \to 0$ in order to satisfy the monotonicity. ■

Proof of Step 4:

Proof. Since the only difference in IVPs is at the initial value $\phi_1(b^R) = c^*$, I need to show that there is exactly one value of c^* such that the solution $\phi_i(b)$ to resultant IVP satisfies $\underset{b \to 0}{lim} \phi_i(b) = 0$. To see this, note that the condition $\hat{\phi}_2(b) < \phi_2(b)$ and $\hat{\phi}_2(b^R) = \phi_2(b^R)$, and $\hat{\phi}_1(b) > \phi_1(b) \forall b \in (0, b^R]$ implies that $\hat{\phi}_2(b) - \hat{\phi}_1(b) < \phi_2(b) - \phi_1(b) \forall b > 0$. Alongwith result of step 3, this further implies that if $\phi_2(b_t) = \phi_1(b_t)$ for some $b_t > 0$, then $\hat{\phi}_2(\hat{b}_t) = \hat{\phi}_1(\hat{b}_t)$ for some $\hat{b}_t > b_t$, where $\hat{\phi}_i(b)$ are solutions to $\hat{\mathcal{P}}$. Since the choice of c^* and $\hat{c}^*$ was arbitrary, this amounts to saying that the x-coordinate of point of intersection of ϕ_1 and ϕ_2 is strictly increasing in c^*.

Now, consider a function $H(c) : [\iota, \bar{c}] \to [\iota, \bar{c}]$ which maps c^* to b_t where $\phi_2(b_t) = \phi_1(b_t)$ for IVP with $\phi_1(b^R) = c^*$. This mapping is strictly monotonically increasing. Since the RHS of the differential equiations (10) is continuous, the solution to these equations is also continuous in the initial value c^*.[22] If $H(c_t^*) = b_t$ for some c_t^*, then $\phi_2(b) - \phi_1(b) < 0$ for $b < b_t$, where $\phi_1(b_t) = \phi_2(b_t)$ when $\phi_1(b^R) = c_t^*$. Given the strict monotonicity of $H(c)$, the continuity of IVP solution with respect to initial

[22]See Hirsch, Smale, and Devaney, 2012 chapters 7 and 17 for results on sensitivity analysis of IVP.

value implies that if $\phi_1(b^R) = c_t^* - \omega, \omega \to 0$, then $\phi_2(b) - \phi_1(b) < 0$ for $b < b_t - \delta(\omega)$ for some $\delta(\omega) \to 0$, and $\phi_1(b_t - \delta(\omega)) = \phi_2(b_t - \delta(\omega))$.Thus, $H(c_t - \omega) = b_t - \delta(\omega)$ for some $\delta(\omega) \to 0$, thereby establishing continuity of $H(c)$.

So far, we have established continuity and strictly positive monotonicity of $H(c)$. Notice further that $H(\bar{c}) = \bar{c}$ because one can always set $\phi_1(b^R) = \phi_2(b^R) = \bar{c}$ for the IVP. Therefore, using Extreme Value Theorem we can say that $H(c)$ will attain it's minimum, which is equal to ι, for exactly one value of c. This result holds $\forall \iota > 0$, and in particular for $\iota \to 0$. Thus, the solution to IVP given by equations (10), $\phi_1(b^R) = c^*$, $\phi_2(b^R) = \bar{c}$, is such that $\underset{b \to 0}{lim} \phi_2(b) = \underset{b \to 0}{lim} \phi_1(b) = 0$. ∎

B 2 player extensions

In this section, I present two extensions with asymmetric cost information. In the first extension the 2 bidders have cost distributions which can ordered as per their Reversed Hazard Rates. In the second extension, I assume that the distribution of one of the bidders is truncated version of that of another bidder. While both cases enable me to extend the equilibrium result for the case with same cost distribution, the second is important for the formalisation of 2P1F equilibrium characterisation.

B.1 Different reversed hazard rates

Suppose $c_i \overset{i.i.d}{\sim} F_i(c)$, and $c_i \in [0, \bar{c}]$ for each i. Denote reversed hazard of $F_i(c)$ by $\sigma_i(c)$. Suppose that they can be ordered in terms of their reversed hazard rate, i.e $\sigma_i(c) < \sigma_{-i}(c)$. Furthermore assume that $\underset{c \to 0^+}{lim} \sigma_i'(c) = \underset{c \to 0^+}{lim} \sigma_{-i}'(c)$. Then, as before, I can characterise the equilibrium in following lemma:

Lemma 2. *For each B_i, $\beta_i(c)$ constitutes a non-trivial BNE of the asymmetric 2 player button auction with rationing if and only if it satisfies following properties:*

(i) $\beta_i(c)$ is non-decreasing in c.

(ii) $\beta_i(c)$ *is continuous and atomless for* $b < b^R$ *for both* i.

(iii) $\beta_i(0) = 0$ $,\forall i$.

(iv) For each player B_i, $\beta_i(c)$ *solves:*

$$\sigma_{-i}(\beta_{-i}^{-1}(\beta_i(c)))\beta_{-i}^{-1'}(\beta_i(c))(\beta_i(c) - c)(q_1 + q_2 - M) = (M - q_{-i}) \tag{12}$$

(v) If $\frac{\sigma_1(c)}{\sigma_2(c)} > \frac{M-q_1}{M-q_2}, \forall c$, $\exists c_1^*$ *such that* $\beta_1(c_1^*) = b^R$ $,\forall c \in [c_1^*, \bar{c}]$, *and* $\beta_2(\bar{c}) = b^R$. *If* $\frac{\sigma_1(c)}{\sigma_2(c)} < \frac{M-q_1}{M-q_2}, \forall c$, $\exists c_2^*$ *such that* $\beta_2(c_2^*) = b^R$ $,\forall c \in [c_2^*, \bar{c}]$, *and* $\beta_1(\bar{c}) = b^R$.

Proof. Proof of $(i), (ii), (iii), (iv)$, are same as in case with same cost distributions for each bidder. For (v), I can proceed in the same way as before. Define ϕ_i as

$$\phi_i(b) := \begin{cases} \beta_i^{-1}(b) & for\ b < b^R \\ Inf\{c : \beta_i(c) = b^R\} & for\ b = b^R \end{cases}$$

At any point of intersection of $\phi_1(b)$ and $\phi_2(b)$, I can write $\dfrac{\phi_2'(b)}{\phi_1'(b)} = \dfrac{(M - q_2)\sigma_1(\phi_1(b))}{(M - q_1)\sigma_2(\phi_2(b))}$. If $\frac{\sigma_1(c)}{\sigma_2(c)} > \frac{M-q_1}{M-q_2}, \forall c$, $\phi_2'(b) > \phi_1'(b)$ at point of intersection. Given the assumption $\underset{c\to 0^+}{lim}\, \sigma_i'(c) = \underset{c\to 0^+}{lim}\, \sigma_{-i}(c)$, I can use same arguments as in proof of Lemma 1 to show that B_1 will bunch.

However, if $\frac{\sigma_1(c)}{\sigma_2(c)} < \frac{M-q_1}{M-q_2}, \forall c$, then B_2 bunches at b^R. ■

The result here implies that B_2 will bunch only if the likelihood that she has higher cost than B_1 is large. This provides a larger marginal benefit of reducing the bid, as there is now a higher probability of B_2's exit. If it is large enough, B_1 would be more aggressive as it offsets the effect of having a larger residual, which leads to higher cost of competition.

Existence and uniqueness can be proved with steps similar to the case of same distribution for both bidders.

B.2 Asymmetric support, same RHR

For each B_i, $c_i \in [0, \bar{c}_i]$. $\sigma(c)$ is same for both i for $c \in [0, min_i\{\bar{c}_i\}]$. If other words, cost distribution of one of the bidders is truncation of that of the other. Equilibrium is characterised by the lemma below:

Lemma 3. *For each B_i, $\beta_i(c)$ constitutes a non-trivial BNE of the 2 player asymmetric button auction with rationing if only if it satisfies following properties:*

(i) $\beta_i(c)$ *is non-decreasing in* c.

(ii) $\beta_i(c)$ *is continuous and atomless for* $b < b^R$ *for both* i.

(iii) $\beta_i(0) = 0$ $,\forall i$.

(iv) For each player B_i, $\beta_i(c)$ *solves:*

$$\sigma_{-i}(\beta_{-i}^{-1}(\beta_i(c)))\beta_{-i}^{-1'}(\beta_i(c))(\beta_i(c) - c)(q_1 + q_2 - M) = (M - q_{-i}) \tag{13}$$

(v) $\exists\Delta$ *such that if* $\bar{c}_2 - \bar{c}_1 < \Delta$, $\exists c_1^*$ *such that* $\beta_1(c) = b^R$ $,\forall c \in [c_1^*, \bar{c}_1]$ *and* $\beta_2(\bar{c}_2) = b^R$*, else,* $\exists c_2^*$ *such that* $\beta_2(c) = b^R$ $,\forall c \in [c_2^*, \bar{c}_2]$ *and* $\beta_1(\bar{c}_1) = b^R$

Proof. Proof of $(i), (ii), (iii), (iv)$ are same as in case with same cost distributions for each bidder. As before, define $\phi_i(b)$ as inverse of $\beta_i(c)$. For (v), it can be seen in the same way as in proof of Lemma 1 that $\phi_2(b) > \phi_1(b), \forall b > 0$ for a given set of least upper bounds (LUBs) of support of cost distribution, $\{\bar{c}_1, \bar{c}_2\}$. Consider a bid δ/n, where $\delta \to 0^+$ and $n \geq 1$ is some natural number. Then $\phi_i(\delta/n) = \epsilon_i(\delta/n)$ such that $\epsilon_i(\delta/n) \to 0$. Therefore, as in 2P0F, I can write

$$\frac{\phi_2'(\delta)}{\phi_1'(\delta)} = \frac{\phi_2(\delta) - \phi_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n)}{\phi_1(\delta) - \phi_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)} = \frac{\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n)}{\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)}$$

where $\kappa_i(.)$ is a bounded function. From the FOCs, I can further infer that:

$$\begin{aligned}
&\frac{\phi_2'(\delta)}{\phi_1'(\delta)} = \frac{M-q_2}{M-q_1}\frac{\sigma(\phi_1(\delta)}{\sigma(\phi_2(\delta)}\frac{\delta-\epsilon_2(\delta)}{\delta-\epsilon_1(\delta)} \\
\Longrightarrow\ &\frac{\epsilon_2(\delta)-\epsilon_2(\delta/n)-\delta^2\kappa_2(\delta,\delta/n)}{\epsilon_1(\delta)-\epsilon_1(\delta/n)-\delta^2\kappa_1(\delta,\delta/n)} = \frac{M-q_2}{M-q_1}\frac{\sigma(\phi_1(\delta)}{\sigma(\phi_2(\delta)}\frac{\delta-\epsilon_2(\delta)}{\delta-\epsilon_1(\delta)} \\
\Longrightarrow\ &\frac{M-q_2}{M-q_1} = \frac{\epsilon_2(\delta)}{\epsilon_1(\delta)}\frac{\delta-\epsilon_1(\delta)}{\delta-\epsilon_2(\delta)}\frac{\epsilon_2(\delta)-\epsilon_2(\delta/n)-\delta^2\kappa_2(\delta,\delta/n)}{\epsilon_1(\delta)-\epsilon_1(\delta/n)-\delta^2\kappa_1(\delta,\delta/n)}
\end{aligned} \tag{14}$$

Using the same reasoning as in Appendix A.1, I can conclude that $\epsilon_2(\delta) > \epsilon_1(\delta)$

If $\bar{c}_1 > \bar{c}_2$, B_1 would bunch because $\phi_2(b^R) = \bar{c}_2$ which needs to be higher than $\phi_1(b^R)$. This would imply that $\phi_1(b^R) < \bar{c}_2 < \bar{c}_1$.

Consider the case where $\bar{c}_1 \leq \bar{c}_2$. Consider two pairs of supremum of support of (c_1, c_2), $(\bar{c}_1, \bar{c}_1)$ and $(\bar{c}_1, \hat{\bar{c}}_2)$ such that $\hat{\bar{c}}_2 > \bar{c}_1$. Denote the corresponding equilibrium inverse bid functions generated from these suprema as $\phi_i(b)$ and $\hat{\phi}_i(b)$ respectively. From Lemma 1, we know that $\phi_1(b^R) = c^* < \bar{c}_1$ and $\phi_2(b^R) = \bar{c}_1$ and that $\underset{b\to 0^+}{lim}\phi_i(c) = 0$ for both i.

With regards to $\hat{\phi}_i(b)$, there are 2 possibilities- either $\hat{\phi}_2(b^R) > \phi_2(b^R) = \bar{c}_1$ or $\hat{\phi}_2(b^R) = \hat{c}_2^* < \phi_2(b^R) = \bar{c}_1$.

Let's consider the first case. Suppose $\exists b_t\ s.t.\ \hat{\phi}_2(b_t) = \phi_2(b_t)$, then $\hat{\phi}_2'(b_t) > \phi_2'(b_t)$. This implies that $\sigma(\hat{\phi}_2(b_t))\hat{\phi}_2'(b_t) > \sigma(\phi_2(b_t))\phi_2'(b_t)$, which implies that $\hat{\phi}_1(b_t) > \phi_1(b_t)$. This, further implies that $\hat{\phi}_1(b) > \phi_1(b),\ \forall b > 0$. Otherwise there are two solutions to IVP characterised by ODEs given by 13, and boundary values given by point of intersection of $\phi_i(b), \hat{\phi}_i(b)$ for each i, defined over any compact interval in $(0, b^R]$ containing the point of intersection. This violates the Cauchy-Lipschitz theorem.

Next, let's look at $\phi_i(b)$ and $\hat{\phi}_i(b)$ in the immediate neighbourhood of 0. For this, I can write following, as in (11),

$$\frac{M-q_2}{M-q_1} = \frac{\hat{\epsilon}_2(\delta)}{\hat{\epsilon}_1(\delta)}\frac{\delta-\hat{\epsilon}_1(\delta)}{\delta-\hat{\epsilon}_2(\delta)}\frac{\hat{\epsilon}_2(\delta)-\hat{\epsilon}_2(\delta/n)-\delta^2\hat{\kappa}_2(\delta,\delta/n)}{\hat{\epsilon}_1(\delta)-\hat{\epsilon}_1(\delta/n)-\delta^2\hat{\kappa}_1(\delta,\delta/n)},$$

where $\hat{\kappa}_i(.)$ is a bounded function. Above implies that $\hat{\phi}_2(b) > \hat{\phi}_1(b)$. I can further infer that:

$$\frac{\epsilon_2(\delta)}{\epsilon_1(\delta)}\frac{\delta-\epsilon_1(\delta)}{\delta-\epsilon_2(\delta)}\frac{\epsilon_2(\delta)-\epsilon_2(\delta/n)-\delta^2\kappa_2(\delta,\delta/n)}{\epsilon_1(\delta)-\epsilon_1(\delta/n)-\delta^2\kappa_1(\delta,\delta/n)} = \frac{\hat{\epsilon}_2(\delta)}{\hat{\epsilon}_1(\delta)}\frac{\delta-\hat{\epsilon}_1(\delta)}{\delta-\hat{\epsilon}_2(\delta)}\frac{\hat{\epsilon}_2(\delta)-\hat{\epsilon}_2(\delta/n)-\delta^2\hat{\kappa}_2(\delta,\delta/n)}{\hat{\epsilon}_1(\delta)-\hat{\epsilon}_1(\delta/n)-\delta^2\hat{\kappa}_1(\delta,\delta/n)}$$
$$\implies \frac{\epsilon_2(\delta)}{\hat{\epsilon}_2(\delta)}\frac{\delta-\hat{\epsilon}_2(\delta)}{\delta-\epsilon_2(\delta)}\frac{\epsilon_2(\delta)-\epsilon_2(\delta/n)-\delta^2\kappa_2(\delta,\delta/n)}{\hat{\epsilon}_2(\delta)-\hat{\epsilon}_2(\delta/n)-\delta^2\hat{\kappa}_2(\delta,\delta/n)} = \frac{\epsilon_1(\delta)}{\hat{\epsilon}_1(\delta)}\frac{\delta-\hat{\epsilon}_1(\delta)}{\delta-\epsilon_1(\delta)}\frac{\epsilon_1(\delta)-\epsilon_1(\delta/n)-\delta^2\kappa_1(\delta,\delta/n)}{\hat{\epsilon}_1(\delta)-\hat{\epsilon}_1(\delta/n)-\delta^2\hat{\kappa}_1(\delta,\delta/n)}$$

Above relation should hold for all n. As $\delta \to 0$ and $\kappa_i(.)$ and $\hat{\kappa}_i(.)$ are bounded functions $\delta^2\kappa_i(\delta,\delta/n) \approx 0$ and $\delta^2\hat{\kappa}_i(\delta,\delta/n) \approx 0$ for both i. Since both $\phi_i(b)$ and $\hat{\phi}_i(b)$ converge to 0^+ as $b \to 0^+$, I can further say that $\epsilon_i(\delta/n) \approx \hat{\epsilon}_i(\delta/n)$ as $n \to \infty$. If $\hat{\epsilon}_2(\delta) > (<)\epsilon_2(\delta)$, then LHS is above (below) 1. Thus, RHS will be above (below) 1 only if $\hat{\epsilon}_1(\delta) > (<)\epsilon_1(\delta)$.

Now, if $\hat{\epsilon}_2(\delta) < \epsilon_2(\delta)$, then $\hat{\epsilon}_1(\delta) < \epsilon_1(\delta)$. Since $\hat{\phi}_1(b_t) > \phi_1(b_t)$, where b_t is the point of intersection of $\hat{\phi}_i(b)$ and $\phi_i(b)$, this implies that $\hat{\phi}_i(b)$ intersects $\phi_i(b)$ for both i because $\hat{\phi}_2(b^R) = \hat{\bar{c}}_2 > \phi_2(b^R) = \bar{c}_1$. This situation is depicted in Figure 12. As explained in appendix A2, such intersections violate the Cauchy-Lipschitz theorem of unique solution. Thus, if $\hat{\phi}_2(b^R) = \hat{\bar{c}}_2 > \phi_2(b^R) = \bar{c}_1$, then $\hat{\phi}_2(b) > \phi_2(b)\forall b > 0$ which implies $\hat{\phi}_1(b) > \phi_1(b)\forall b > 0 \implies \hat{\phi}_1(b^R) = \hat{c}_1^* > \phi_1(b^R) = c^*$.

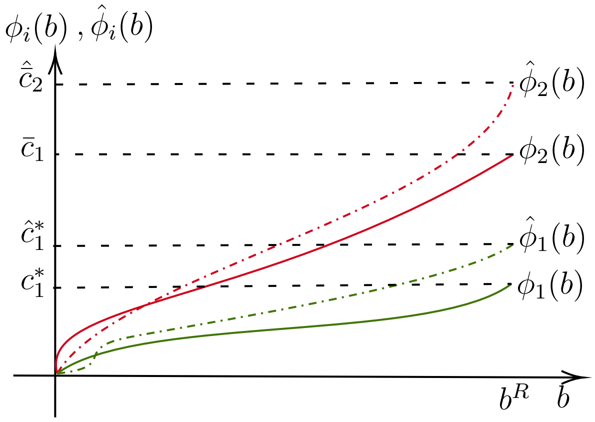


Figure 12: Intersecting solution curves

The second case is where $\hat{\phi}_2(b^R) = \hat{c}_2^* < \phi_2(b^R) = \bar{c}_1$. In this case, $\hat{\phi}_1(b^R) = \bar{c}_1$, else both players will have an atom, which is not possible in equilibrium. Thus, here, $\hat{\phi}_1(b^R) > \phi_1(b^R)$. As before, I can show that any intersection between $\hat{\phi}_2(b)$ and $\phi_2(b)$ would imply intersection between $\hat{\phi}_1(b)$ and $\phi_1(b)$. Hence, $\hat{\phi}_2(b) < \phi_2(b)$, and

$\hat{\phi}_1(b) > \phi_1(b)$, $\forall b > 0$. However, as shown above, this inequality wouldn't hold for the bids close to 0. Thus, this case leads to contradictions and hence, is not possible.

Therefore, when if the supremum of support of c_2 is higher, i.e., $\hat{\bar{c}}_2 > \bar{c}_2$, $\hat{\phi}_2(b) > \phi_2(b)$, and $\hat{\phi}_1(b) > \phi_1(b)$, $\forall b > 0$.

Define a function $M(\bar{c}_2) : [\bar{c}_1, \infty) \to \mathbb{R}^+$ such that $M(\bar{c}_2)$ maps LUB of support of c_2 to $\phi_1(b^R)$, where $\bar{c}_1$ is LUB of an arbitrary support of c_1. Since the choice of $\hat{\bar{c}}_2$ above is arbitrary, we can say that $M(\bar{c}_2) > 0$ is an increasing function. Continuity can be argued in the same way as in proof of Theorem 1 in Appendix A.2. Thus, for a given $\bar{c}_1$, as $\bar{c}_2$ increases from $\bar{c}_1$, c^* increases, and the size of B_1's atom at b^R reduces. The maximum value of c^* can be $\bar{c}_1$, which corresponds to atom size of 0. Due to monotonicity and continuity of $M(\bar{c}_2)$, $\exists \bar{c}_2^T$ such that $M(\bar{c}_2^T) = \bar{c}_1$. Then for $\bar{c}_2 \in [\bar{c}_1, \bar{c}_2^T)$, B_1 bunches at b^R and for $\bar{c}_2 > \bar{c}_2^T$, B_2 would bunch. This holds true regardless of the value of $\underline{c}_1$. I can thus define $\Delta \equiv \underline{c}_2^T - \underline{c}_1$, such that $B_1(B_2)$ bunches if $\underline{c}_2 < (>) \underline{c}_1 + \Delta$. This proves (v). ■

This result here has similar intuition as in previous extension. B_2 would bunch at b^R only if it is likely to have costs much higher than that of B_1. This extension is important not only for robustness checks, but also for formalising equilibrium in case with 2 small and 1 very small player.

Finally, I establish existence and uniqueness of this PBE in order to have characterisation of equilibrium of 2P1F case.

Theorem 2. *Equilibrium defined by Lemma 3 exists and is unique.*

Proof. From Lemma 3, it can be inferred that for some given values of $\bar{c}_1, \bar{c}_2$, only one of the bidders, B_1 or B_2 will be bunching.

The boundary value problem which gives equilibrium bid function is characterised by the differential equation 13, and boundaries given by $\phi_1(0) = \phi_2(0)$, and $\phi_2(b^R) = \bar{c}_2$ when $\bar{c}_2 > \bar{c}_1 + \Delta$, and $\phi_1(b^R) = \bar{c}_1$ otherwise. Comparing to the boundary value problem for 2P0F case, it can be noticed that the differential equation and left boundary are the same, while right boundary can be different.

From the proof of Theorem 1, we already know that equilibrium exists and is unique if the right boundary is $\phi_2(b^R) = \bar{c}_2$. Moreover, same arguments can be applied to the case where the right boundary is $\phi_1(b^R) = \bar{c}_1$. ■

C 3 player extension: 2 small and 1 very small bidder

Suppose 3 bidders B_1, B_2, and B_3 have quantities q_1, q_2, and q_3 respectively, such that, $q_1 > q_2 > q_3$, $q_1 + q_2 > M$ but $q_1 + q_3 < M$ and $q_2 + q_3 < M$. Thus, B_1 and B_2 can together cover the whole demand. For B_3, it is dominant to bid her cost, for the reasons same as in section 5.1.1. In this game, exit of B_1 or B_2 will end the game, but exit of B_3 will start a new subgame between the other two. As before, there are equilibria which require crazy types but the analysis here will focus on the semi-separating equilibrium which don't require such types. This equilibrium is also the perfect bayesian equilibrium of this game.

Denote the set of all players by $\mathcal{N}$, and set $\{B_1, B_2\}$ by $\mathcal{A}2$. In this section B_i refers to the elements of $\mathcal{A}2$ and B_{-i} is the element of $\mathcal{A}2 \setminus B_i$. For $i \in \{1, 2\}$, denote the equilibrium bid function of B_i by $\beta_{i,\mathcal{N}}(c)$ in the subgame with all players, and $\beta_{i,\mathcal{A}2}(c)$ in the subgame started by B_3's exit. $\mathbf{b}$ denotes the vector of bids of all the players. If a bidder in $\mathcal{A}2$ exits at any bid, she gets a strictly positive quantity award. As such, these bidders can be called partially rationed as opposed to fully rationed bidder, B_3. A partially rationed bidder B_i bids b_i, and the other partially rationed bidder bids b_{-i}, and B_3 bids b_3, her payoff when her type is c_i is:

$$\begin{aligned}\pi_i(b_i; c_i, \mathbf{b}) =& (M - q_{-i} - q_3)(b_i - c_i)Pr(b_i = max_j\{b_j\}) \\ &+ q_i\mathbb{E}(b_{-i} - c_i|b_{-i} > b_3, b_{-i} > b_i)Pr(b_{-i} = max_j\{b_j\}) \\ &+ \mathbb{E}(\pi^*_{i,\mathcal{A}2}(b_3)|b_i < b_3, b_{-i} < b_3)Pr(b_3 = max_j\{b_j\})\end{aligned}$$

where $\pi^*_{i,\mathcal{A}2}(b_3)$ is the payoff for B_i in the subgame started by B_3's exit.

$\beta_{3,\mathcal{N}}(c) = c$. B_1 and B_2 best respond to that and to each other in equilibrium, which is characterised in the following lemma:

Lemma 4. *$\beta_{3,\mathcal{N}}(c) = c$. $\beta_{i,\mathcal{N}}(c)$ and $\beta_{i,\mathcal{A}2}(c)$ for $i \in \{1,2\}$, give a PBE if and only if:*

(i) $\beta_{i,\mathcal{N}}(c)$ *is non-decreasing in c.*

(ii) $\beta_{i,\mathcal{N}}(c)$ *is continuous and atomless for* $b < b^R$ *for both i.*

(iii) $\beta_{i,\mathcal{N}}(0) = 0$ *,* $\forall i$.

(iv) $\forall i, \beta_{i,\mathcal{A}2}(c_i)$*, solve following differential equations:*

$$\begin{aligned}&(\pi^*_{i,\mathcal{A}2}(b; c_i) - (M - q_{-i} - q_3)(\beta_{i,\mathcal{N}}(c_i) - c_i))\frac{f(\beta_{i,\mathcal{N}}(c_i))}{F(\beta_{i,\mathcal{N}}(c_i))}\mathbb{1}_{b\leq\bar{c}} \\ &+ (\beta_{i,\mathcal{N}}(c_i) - c_i)(\sum_j q_j - M)\frac{f(\beta^{-1}_{-i,\mathcal{N}}(\beta_{i,\mathcal{N}}(c_i)))\beta^{-1'}_{-i,\mathcal{N}}(\beta_{i,\mathcal{N}}(c_i))}{F(\beta^{-1}_{-i,\mathcal{N}}(\beta_{i,\mathcal{N}}(c_i)))} = M - q_{-i} - q_3\end{aligned} \tag{15}$$

where $\pi^*_{i,\mathcal{A}2}(b; c_i)$ *is the payoff of* B_i *in the subgame started with exit of* B_3*.*

(v) $\exists c^*_1 \leq \bar{c}$ *such that* $\beta_{1,\mathcal{N}}(c) = b^R, \forall c \in [c^*_1, \bar{c}]$. $\beta_{2,\mathcal{N}}(\bar{c}) = b^R$ *if* $b^R > \bar{c}$ *and* $\lim_{c\to\bar{c}^-} \beta_{2,\mathcal{N}}(c) = b^R$ *if* $b^R = \bar{c}$.

(vi) $\beta_{i,\mathcal{A}2}(c)$ *for* $i \in \{1,2\}$ *are given by semi-seperating equilibrium in the subgame started by* B_3*'s exit at a bid b, which is characterised in Lemma 3 in Appendix B.2.*

Proof. See Appendix C.1. ■

PBE described here looks the same that of section 5.1.2, except that there is a kink at $b = \bar{c}$. The intuition behind a similar equilibrium as in case with 2 small bidders is that B_3's presence affects both B_1 and B_2 in the same way. It reduces their residual capacity by the same amount and the marginal probability of B_3's exit at any bid is same for both the bidders. Thus, B_1 is still less reluctant to compete vis-a-vis B_2.

The proof is also similar, except for some additional steps for (i) and (v). For (v), I show that there will be at most one point of intersection between $\beta_1(c)$ and $\beta_2(c)$. At any point of intersection, $\frac{\beta'_{1,\mathcal{N}}(c)}{\beta'_{2,\mathcal{N}}(c)} = \frac{M-q_2-q_3-(\pi^*_{1,\mathcal{A}2}(b,c)-(M-q_2-q_3)(b-c))\sigma(b)}{M-q_1-q_3-(\pi^*_{2,\mathcal{A}2}(b,c)-(M-q_1-q_3)(b-c))\sigma(b)}$ for $b \leq \bar{c}$. If B_3 were to exit at bid b pertaining to the point of intersection, then a subgame same as 2P0F starts with b as reserve. As we know from Lemma 1(v), B_1 of type c pertaining to this bid, will also exit at b in this subgame. This gives us the values for $\pi^*_{i,\mathcal{A}2}(b,c)$ for each i, which are such that the aforementioned slope ratio is above 1. Thus, there is only one possible point of intersection between $\beta_{1,\mathcal{N}}$ and $\beta_{2,\mathcal{N}}$, and that point is $(0,0)$ for reasons same as in section 5.1.2.

Furthermore, as I show in appendix, the PBE is such that in the subgame, B_2 would be bunching. This result eases the analysis for existence and uniqueness, as it gives explicit expressions of continuation values.

Looking at the equilibrium characteristics, it can be noticed that apart from FOC, every other property is same that of equilibrium in 2 players. FOC here is such that LHS is not continuous, unlike previous case. The key condition leading to uniqueness and existence in that case was that the solution to the boundary value problem for different boundaries is such that $\phi_2(b)$ is lower if $\phi_1(b)$ is higher for a given boundary (as in Figure 6). Although, this condition still holds, the lack of continuity leads to negative result on existence of pure strategy PBE.

If $b^R > \bar{c}$, there is a kink in the bidding function at $\bar{c}$. In this case, B_2 becomes more aggressive on the margin at $\bar{c}$. The best response for B_1 is, then, to be less aggressive in absolute manner, unless the quantities have some very specific values. This creates discontinuity in B_1's bidding function, by a logic similar to 2P0F. This violates property (i) of BNE described in Lemma 4. Thus, in such a case, we can only have trivial BNE. However, such a problem doesn't exist when $b^R = \bar{c}$. Thus, the result on existence and uniqueness of equilibrium doesn't extend to this case when $b^R > \bar{c}$.

Theorem 3. *If $b^R > \bar{c}$, equilibrium described by Lemma 4 may not always exist, but when it exists, it is unique. If $b^R \leq \bar{c}$, the equilibrium exists and is unique.*

Proof. See Appendix C.2. ∎

C.1 Proof of Lemma 4

Proof. For the very small bidder B_3, it is weakly dominant to bid her cost. The reason is same as for 1P1F case. The proof proceeds in the way similar to that in 2P0F (Appendix A2). However, there are some additional nuances involved in proving property (i) and (v).

As in Section A.1, I show (i) condition by proving that a player's expected payoff satisfies SCP-IR property, when opponent is playing as per an increasing strategy. As before, I will show it for B_1. Consider any two types c_1, c_1' of B_1, such that $c_1 < c_1'$, and any two arbitrary bids b_1, b_1', where $b_1 < b_1'$. To show monotonicity, all I need to show is that when B_2 follows a non-decreasing strategy, if $\pi_1(b_1', c_1; b_2, c_3) - \pi_1(b_1, c_1; b_2, c_3) > 0$, then $\pi_1(b_1', c_1'; b_2, c_3) - \pi_1(b_1, c_1'; b_2, c_3) > 0$, where b_2 is random variable (RV) denoting B_2's bid, and c_3 is RV for B_3's cost type (and equivalently, her bid).

$$
\begin{aligned}
\pi_1(b_1', c_1; b_2, c_3) =& (M - q_2 - q_3)(b_1' - c_1)Pr(b_1' = max\{b_1', b_2, c_3\}) \\
&+ q_1 \mathbb{E}(b_2 - c_1 | b_2 = max\{b_1', b_2, c_3\}) Pr(b_2 = max\{b_1', b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1) | c_3 = max\{b_1', b_2, c_3\}) Pr(c_3 = max\{b_1', b_2, c_3\}) \\
\pi_1(b_1, c_1; b_2, c_3) =& (M - q_2 - q_3)(b_1 - c_1)Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1 \mathbb{E}(b_2 - c_1 | b_2 = max\{b_1, b_2, c_3\}) Pr(b_2 = max\{b_1, b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1) | c_3 = max\{b_1, b_2, c_3\}) Pr(c_3 = max\{b_1, b_2, c_3\})
\end{aligned} \tag{16}
$$

Denote $\pi_1(b_1', c_1; b_2, c_3) - \pi_1(b_1, c_1; b_2, c_3)$ by $A(b_1', b_1, c_1, b_2, c_3)$, or simply, A. Suppose that $A > 0$ always. Furthermore,

$$
\begin{aligned}
\pi_1(b_1', c_1'; b_2, c_3) =& (M - q_2 - q_3)(b_1' - c_1')Pr(b_1' = max\{b_1', b_2, c_3\}) \\
&+ q_1 \mathbb{E}(b_2 - c_1' | b_2 = max\{b_1', b_2, c_3\}) Pr(b_2 = max\{b_1', b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1') | c_3 = max\{b_1', b_2, c_3\}) Pr(c_3 = max\{b_1', b_2, c_3\}) \\
\pi_1(b_1, c_1'; b_2, c_3) =& (M - q_2 - q_3)(b_1 - c_1')Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1 \mathbb{E}(b_2 - c_1' | b_2 = max\{b_1, b_2, c_3\}) Pr(b_2 = max\{b_1, b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1') | c_3 = max\{b_1, b_2, c_3\}) Pr(c_3 = max\{b_1, b_2, c_3\})
\end{aligned} \tag{17}
$$

which implies,

$$
\begin{aligned}
\pi_1(b_1', c_1'; b_2, c_3) =& (M - q_2 - q_3)(b_1' - c_1' + c_1 - c_1)Pr(b_1' = max\{b_1', b_2, c_3\}) \\
&+ q_1\mathbb{E}(b_2 - c_1' + c_1 - c_1 | b_2 = max\{b_1', b_2, c_3\})Pr(b_2 = max\{b_1', b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1')|c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1)|c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
&- \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1)|c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
\pi_1(b_1, c_1'; b_2, c_3) =& (M - q_2 - q_3)(b_1 - c_1' + c_1 - c_1)Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&+ q_1\mathbb{E}(b_2 - c_1' + c_1 - c_1 | b_2 = max\{b_1, b_2, c_3\})Pr(b_2 = max\{b_1, b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1')|c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\}) \\
&+ \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1)|c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\}) \\
&- \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1)|c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\})
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&\therefore \pi_1(b_1', c_1'; b_2, c_3) - \pi_1(b_1, c_1'; b_2, c_3) \\
&= A + (M - q_2 - q_3)(c_1 - c_1')Pr(b_1' = max\{b_1', b_2, c_3\}) - (M - q_2 - q_3)(c_1 - c_1')Pr(b_1 = max\{b_1, b_2, c_3\}) \\
&\quad + q_1\mathbb{E}(c_1 - c_1'|b_2 = max\{b_1', b_2, c_3\})Pr(b_2 = max\{b_1', b_2, c_3\}) - q_1\mathbb{E}(c_1 - c_1'|b_2 = max\{b_1, b_2, c_3\})Pr(b_2 = max\{b_1, b_2, c_3\}) \\
&\quad + \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1')|c_3 = max\{b_1', b_2, c_3\} - \pi^*_{1,\mathcal{A}2}(c_3, c_1)|c_3 = max\{b_1', b_2, c_3\})Pr(c_3 = max\{b_1', b_2, c_3\}) \\
&\quad - \mathbb{E}(\pi^*_{1,\mathcal{A}2}(c_3, c_1')|c_3 = max\{b_1, b_2, c_3\} - \pi^*_{1,\mathcal{A}2}(c_3, c_1)|c_3 = max\{b_1, b_2, c_3\})Pr(c_3 = max\{b_1, b_2, c_3\})
\end{aligned}
\tag{19}
$$

From Lemma 3, I can write continuation value in the subgame following B_3's exit, $\pi^*_{1,\mathcal{A}2}(c_3, c_1)$, as:

$$
\pi^*_{1,\mathcal{A}2}(c_3, c_1) = \underset{b_1'' \leq c_3}{Max}\left[(M - q_2)(b_1'' - c_1)\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)} + q_1\int_{b_1''}^{c_3}(x - c_1)\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\right]
$$

where $\phi_2^{sg}(b)$ is given by Lemma 3 in Appendix A.3.2 and $a(c_3)$ denotes the probability that B_2's cost type is from that subset of $[0, \bar{c}]$ which bids less than c_3 in the subgame with preceding B_3's exit. I can further write,

$$
\begin{aligned}
\pi^*_{1,\mathcal{A}2}(c_3,c_1) =& \underset{b_1''\leq c_3}{Max}\Big[(M-q_2)(b_1''-c_1+c_1'-c_1')\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)}+q_1\int_{b_1''}^{c_3}(x-c_1+c_1'-c_1')\frac{dF^{sg}(\phi_2(x))}{a(c_3)}\Big] \\
\implies \pi^*_{1,\mathcal{A}2}(c_3,c_1) \leq& \underset{b_1''\leq c_3}{Max}\Big[(M-q_2)(x-c_1')\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)}+q_1\int_{b_1''}^{c_3}(x-c_1')\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\Big] \\
&+\underset{b_1''\leq c_3}{Max}\Big[(M-q_2)(c_1'-c_1)\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)}+q_1\int_{b_1''}^{c_3}(c_1'-c_1)\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\Big] \\
\implies \pi_1(c_3,c_1')-\pi_1(c_3,c_1) \geq& -\underset{b_1''\leq c_3}{Max}\Big[(M-q_2)(c_1'-c_1)\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)}+q_1\int_{b_1''}^{c_3}(c_1'-c_1)\frac{dF(\phi_2^{sg}(x))}{a(c_3)}\Big]
\end{aligned}
\tag{20}
$$

Since we have supposed that B_2 has non-decreasing strategies in the subgame before B_3's exit, and Lemma 3(i) states that $\phi_2^{sg}(x)$ is an increasing function, (20) implies

$$
\begin{aligned}
\pi_1(c_3,c_1')-\pi_1(c_3,c_1) \geq& -\underset{b_1''\leq c_3}{Max}\Big[(M-q_2)(c_1'-c_1)\frac{F(\phi_2^{sg}(b_1''))}{a(c_3)}+q_1(c_1'-c_1)\frac{a(c_3)-\phi_2^{sg}(b_1'')}{a(c_3)}\Big] \\
\implies \pi_1(c_3,c_1')-\pi_1(c_3,c_1) \geq& -q_1(c_1'-c_1)
\end{aligned}
\tag{21}
$$

where the last line follows from the idea that this objective function will be maximised when $b_1''=0$.

$$
\begin{aligned}
&\pi_1(b_1',c_1';b_2,c_3)-\pi_1(b_1,c_1';b_2,c_3) \\
&\geq A+(M-q_2-q_3)(c_1-c_1')Pr(b_1'=max\{b_1',b_2,c_3\})-(M-q_2-q_3)(c_1-c_1')Pr(b_1=max\{b_1,b_2,c_3\}) \\
&+q_1(c_1-c_1')Pr(b_2=max\{b_1',b_2,c_3\})-q_1(c-c_1)Pr(b_2=max\{b_1,b_2,c_3\}) \\
&+q_1(c_1-c_1')Pr(c_3=max\{b_1',b_2,c_3\})-q_1(c-c_1)Pr(c_3=max\{b_1,b_2,c_3\}) \\
&=A+(M-q_2-q_3)(c_1-c_1')Pr(b_1'=max\{b_1',b_2,c_3\})-(M-q_2-q_3)(c_1-c_1')Pr(b_1=max\{b_1,b_2,c_3\}) \\
&+q_1(c_1-c_1')Pr(b_1'\neq max\{b_1',b_2,c_3\})-q_1(c_1-c_1')Pr(b_1\neq max\{b_1,b_2,c_3\}) \\
&=A+(M-q_2-q_3)(c_1-c_1')Pr(b_1'=max\{b_1',b_2,c_3\})-(M-q_2-q_3)(c_1-c_1')Pr(b_1=max\{b_1,b_2,c_3\}) \\
&+q_1(c_1-c_1')(1-Pr(b_1'=max\{b_1',b_2,c_3\}))-q_1(c_1-c_1')(1-Pr(b_1=max\{b_1,b_2,c_3\})) \\
&=\underbrace{A}_{>0}+\underbrace{(M-q_2-q_3-q_1)}_{<0}\underbrace{(c_1-c_1')}_{<0}\underbrace{(Pr(b_1'=max\{b_1',b_2,c_3\})-Pr(b_1=max\{b_1,b_2,c_3\}))}_{>0}
\end{aligned}
\tag{22}
$$

$Pr(b_1'=max\{b_1',b_2,c_3\})-Pr(b_1=max\{b_1,b_2,c_3\})>0$ because $b_1'>b_1$ and event that b_1 is greater than both b_2 and c_3 is subset of the event that b_1' is greater than both b_2 and c_3. This along with $A>0$, $c_1<c_1'$, $M<q_1+q_2+q_3$, $b_1'>b_1$, ensures that above expression is positive. This proves condition (i).

Proof of $(ii),(iii)$ is same as 2P0F. (iv) can be shown from first order conditions of optimisation of B_i's payoff.

For (v), consider a point of intersection (b_t, c_t) of $\phi_{1,\mathcal{N}}$ and $\phi_{2,\mathcal{N}}$ where $b_t < \bar{c}$. At this point,

$$\frac{\phi'_{2,\mathcal{N}}(b_t)}{\phi'_{1,\mathcal{N}}(b_t)} = \frac{M - q_2 - q_3 - (\pi^*_{1,\mathcal{A}2}(b_t, c_t) - (M - q_2 - q_3)(b_t - c_t))\sigma(b_t)}{M - q_1 - q_3 - (\pi^*_{2,\mathcal{A}2}(b_t, c_t) - (M - q_1 - q_3)(b_t - c_t))\sigma(b_t)} \tag{23}$$

Note that $\pi^*_{1,\mathcal{A}2}(b_t, c_t)$ is the payoff if B_3 exits at b_t. Since this is also a point of intersection, the subgame started by B_3's exit is same as 2P0F, with $c_i \in [0, c_t]$. Moreover, at this point, both players have type c and the reserve bid for 2P0F is b_t. Thus, from Lemma 1, B_1 of type c_t bids b_t, but is bunching and hence, gets residual. B_2 of type c_t will also bid b_t, but is not bunching. Consequently, their continuation value at this point are $\pi^*_{1,\mathcal{A}2}(b_t, c_t) = (M - q_2)(b_t - c_t)$, $\pi^*_{2,\mathcal{A}2}(b_t, c_t) = q_1(b_t - c_t)$. Thus, we can write

$$\frac{\phi'_{2,\mathcal{N}}(b_t)}{\phi'_{1,\mathcal{N}}(b_t)} = \frac{(M - q_2 - q_3) - q_3(b_t - c_t)\sigma(b_t)}{(M - q_1 - q_3) - (\sum_{j=1}^{3} q_j - M)(b_t - c_t)\sigma(b_t)} > 1$$

where inequality arises because $M - q_1 - q_3 < M - q_2 - q_3$ while $\sum_j q_j - M > q_3$. This implies that $\phi_1(b)$ intersects at most once with $\phi_2(b)$ for $b > 0$.

The exit of B_3 starts a subgame which is same as the extension in Appendix A.3.2. In this subgame, either B_1 or B_2 is bunching. This further means that at any given bid b, if B_3 exits, then Lemma 3 tells us that either B_1 or B_2 of the type $\phi_i(b)$ would also exit at b and get a residual.

Consider a bid δ/n, where $\delta \to 0$ and $n \geq 1$ is some natural number. Then, $\phi_{i,\mathcal{N}}(\delta/n) = \epsilon_i(\delta/n)$, where $\epsilon_i(\delta) \to 0$ by continuity. Suppose that B_1 is bunching in the subgame started by B_3's exit at (δ). Then, in the same way as in other cases, I can write the following from the FOCs of case 2P1F:

$$(\delta - \epsilon_1(\delta))(q_3\sigma(\delta) + (q_1 + q_2 + q_3 - M)\sigma(\epsilon_2(\delta))(\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n))) = M - q_2 - q_3$$

$$(\delta - \epsilon_2(\delta))(q_1 + q_2 + q_3 - M)(\sigma(\delta) + \sigma(\epsilon_1(\delta))(\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n))) = M - q_1 - q_3$$

Using the fact that $\sigma(0)/\sigma'(0) = 0$ and that $\sigma'(0) = \infty$, I can infer the following from above:

$$\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{(q_3\delta + \epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n))(q_1 + q_2 + q_3 - M))}{(q_1 + q_2 + q_3 - M)(\delta + \epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)))} = \frac{M - q_2 - q_3}{M - q_1 - q_3} \tag{24}$$

$$\frac{\epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n))} \approx \frac{q_3}{q_1 + q_2 + q_3 - M} < 1 \tag{25}$$

Inputting (21) in (20), I obtain

$$\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{\epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n))} = \frac{M - q_2 - q_3}{M - q_1 - q_3} > 1$$

As in 2P0F, above implies that $\epsilon_2(\delta) > \epsilon_1(\delta)$. However that is a contradiction because (21) implies otherwise. Thus, B_1 can't be bunching.

Now, consider the case where B_2 is bunching in the subgame started by B_3's exit at the bid δ, $\delta \to 0$. From the FOCs for 2P1F, I can infer following using facts that $\sigma(0)/\sigma'(0) = 0$ and $\sigma'(0) = \infty$:

$$\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{(q_1 + q_2 + q_3 - M)(\delta + \epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n))))}{q_3\delta + \epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n)))(q_1 + q_2 + q_3 - M)} = \frac{M - q_2 - q_3}{M - q_1 - q_3} \tag{26}$$

$$\frac{\epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n))} = \frac{q_1 + q_2 + q_3 - M}{q_3} \tag{27}$$

Inputting (23) in (22) gives:

$$\frac{\delta - \epsilon_1(\delta)}{\delta - \epsilon_2(\delta)} \frac{\epsilon_2(\delta)(\epsilon_2(\delta) - \epsilon_2(\delta/n) - \delta^2\kappa_2(\delta, \delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta) - \epsilon_1(\delta/n) - \delta^2\kappa_1(\delta, \delta/n))} = \frac{M - q_2 - q_3}{M - q_1 - q_3} \tag{28}$$

As argued before, above requires $\epsilon_2(\delta) > \epsilon_1(\delta)$ (which, unlike the previous case, is not in contradiction with (23)).

Finally, I need to check if the necessary and sufficient condition for B_2's bunching in the subgame are also satisfied. The FOCs of 2P0F with asymmetric support (Appendix A.3.2) imply that when B_2 bunches $\exists\ \tilde{\epsilon}_2(\delta) < \epsilon_2(\delta)$ such that B_2 pools for costs between $\tilde{\epsilon}_2(\delta)$ and $\epsilon_2(\delta)$. Therefore,

$$\frac{\sigma(\tilde{\epsilon}_2(\delta))}{\sigma(\epsilon_1(\delta))}\frac{\phi'_{2,\mathcal{A}2}(\delta)}{\phi'_{1,\mathcal{A}2}(\delta)}\frac{\delta-\epsilon_1(\delta)}{\delta-\tilde{\epsilon}_2(\delta)}=\frac{M-q_2}{M-q_1}$$

which implies that $\dfrac{\delta-\epsilon_1(\delta)}{\delta-\tilde{\epsilon}_2(\delta)}\dfrac{\tilde{\epsilon}_2(\delta)(\tilde{\epsilon}_2(\delta)-\tilde{\epsilon}_2(\delta/n)-\delta^2\tilde{\kappa}_2(\delta,\delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta)-\epsilon_1(\delta/n)-\delta^2\kappa_1(\delta,\delta/n))}=\dfrac{M-q_2}{M-q_1}$, where $\tilde{\kappa}_2(.)$ is a bounded function. Since $\tilde{\epsilon}_2(\delta)<\epsilon_2(\delta)$, this further implies

$$\frac{\delta-\epsilon_1(\delta)}{\delta-\epsilon_2(\delta)}\frac{\tilde{\epsilon}_2(\delta)(\epsilon_2(\delta)-\epsilon_2(\delta/n)-\delta^2\tilde{\kappa}_2(\delta,\delta/n))}{\epsilon_1(\delta)(\epsilon_1(\delta)-\epsilon_1(\delta/n)-\delta^2\kappa_1(\delta,\delta/n))}>\frac{M-q_2}{M-q_1}$$

Given the convergence of $\phi_i(b)$ to 0 as $b\to 0$ and its continuity, $\epsilon_i(\delta/n)\to 0$ as $n\to\infty$. Since, $\tilde{\epsilon}_2(\delta/n)<\epsilon_2(\delta/n)$, $\tilde{\epsilon}_2(\delta/n)\to 0$ too as $n\to\infty$. Thus, I can infer that inequality $\dfrac{\delta-\epsilon_1(\delta)}{\delta-\epsilon_2(\delta)}\dfrac{\epsilon_2(\delta)}{\epsilon_1(\delta)}\dfrac{(\epsilon_2(\delta)-\epsilon_2(\delta/n)-\delta^2\tilde{\kappa}_2(\delta,\delta/n)))}{(\epsilon_1(\delta)-\epsilon_1(\delta/n)-\delta^2\kappa_1(\delta,\delta/n)))}>\dfrac{M-q_2}{M-q_1}$ should hold when B_2 is bunching in the subgame.

As Lemma 3 lists all the necessary and sufficient conditions for the equilibrium, and this inequality is derived from the conditions listed in that lemma, it is a necessary and sufficient condition for B_2 to bunch in the subgame started by exit of B_3. Since $\frac{M-q_2-q_3}{M-q_1-q_3}>\frac{M-q_2}{M-q_1}$ when $q_1>q_2$ and $\delta^2\approx 0$ when $\delta\to 0$, equation (28) implies that the condition is satisfied.

Therefore, $\epsilon_2(\delta)>\epsilon_1(\delta)$ and given that at the point of intersection, solution curve of B_2 needs to have higher slope than that of B_1; the curves will not intersect. Thus, $\phi_{2,\mathcal{N}}(b)>\phi_{1,\mathcal{N}}(b)\ \forall b>0$. This would imply that $\phi_{2,\mathcal{N}}(b^R)=\bar{c}>\phi_{1,\mathcal{N}}(b^R)=c_1^*$.

Finally, notice that if B_2 is bunching in subgame started by B_3's exit at any bid b, she is bunching in such a subgame for all b. Else, there exists a bid b_T such that for $b<b_T$, B_2 bunches and above that, B_1 bunches in the subgame. Thus, B_1's payoff in the subgame, $\pi^*_{1,\mathcal{A}2}(b;c_i)$ would fall discontinuously at b_T. As such, the FOC is satisfied only if $\phi'_{2,\mathcal{N}}(b_T^-)<\phi'_{2,\mathcal{N}}(b_T^+)$. Similarly, $\phi'_{1,\mathcal{N}}(b_T^-)>\phi'_{1,\mathcal{N}}(b_T^+)$. The distance between $\phi_1(b)$ and $\phi_2(b)$ would increase which, as per Lemma 3, implies that B_2 should bunching in the subgame started by B_3's exit at bids above b_T, which is a contradiction. As such, there is no such b_T. Thus, if B_2 is bunching in subgame started by B_3's exit at any bid b, she is bunching in such a subgame for all b. ■

C.2 Proof of Theorem 3

Proof. The proof is similar to that of Theorem 1 (Appendix A.2). To see this, notice that the proof of Lemma 4 tells us that B_2 is bunching in the subgame started by B_3's exit. Thus, I can rewrite the FOCs as:

$$\begin{aligned}
&(q_1+q_2+q_3-M)(b-\phi_{1,\mathcal{N}}(b)))\sigma(b)\mathbb{1}_{b\le\bar{c}} \\
&+(b-\phi_{1,\mathcal{N}}(b))(q_1+q_2+q_3-M)\sigma(\phi_{2,\mathcal{N}}(b))\phi'_{2,\mathcal{N}}(b)=M-q_2-q_3 \\
&q_3(b-\phi_{2,\mathcal{N}}(b))\sigma(b)\mathbb{1}_{b\le\bar{c}} \\
&+(b-\phi_{2,\mathcal{N}}(b))(q_1+q_2+q_3-M)\sigma(\phi_{1,\mathcal{N}}(b))\phi'_{1,\mathcal{N}}(b)=M-q_1-q_3
\end{aligned} \tag{29}$$

Suppose first that $b^R>\bar{c}$. For any $b\in(\bar{c},b^R]$, the FOCs are similar to that of 2P0F. The solution to any IVP given by those FOCs, and boundary conditions $\phi_{2,\mathcal{N}}(b^R)=\bar{c}$, and $\phi_{1,\mathcal{N}}(b^R)=c^*$ exists for all possible c^* and is unique. Furthermore, a structure similar to that of 2P1F also implies that if $\hat{\phi}_{2,\mathcal{N}}(b)<\phi_{2,\mathcal{N}}(b)$, then $\hat{\phi}_{1,\mathcal{N}}(b)>\phi_{1,\mathcal{N}}(b)$ for solutions to any two IVPs which are same except for the initial value $\phi_{1,\mathcal{N}}(b^R)$.

Thus, for any 2 such IVPs, if $\hat{\phi}_{2,\mathcal{N}}(b)<\phi_{2,\mathcal{N}}(b)$, then $\hat{\phi}_{2,\mathcal{N}}(b)<\phi_{2,\mathcal{N}}(\bar{c})$ and $\hat{\phi}_{1,\mathcal{N}}(b)>\phi_{1,\mathcal{N}}(\bar{c})$.

For any bids less than $\bar{c}$, the equations 29 can be rewritten as:

$$\begin{aligned}
(b-\phi_{1,\mathcal{N}}(b)))(\sigma(b)+\sigma(\phi_{2,\mathcal{N}}(b))\phi'_{2,\mathcal{N}}(b)) &= \frac{M-q_2-q_3}{q_1+q_2+q_3-M} \\
(b-\phi_{2,\mathcal{N}}(b))\left(\frac{q_3}{(q_1+q_2+q_3-M)}\sigma(b)+\sigma(\phi_{1,\mathcal{N}}(b))\phi'_{1,\mathcal{N}}(b)\right) &= \frac{M-q_1-q_3}{q_1+q_2+q_3-M}
\end{aligned} \tag{30}$$

Consider a sequence $\{\frac{\delta}{2^n}\}_{n\in\mathbb{N}}$. For each n, consider two initial value problems $\mathcal{P}_n$ and $\hat{\mathcal{P}}_n$ defined on $[\frac{\delta}{2^n},\bar{c}]$. The problems have same ODEs as (25) except that I replace function $\phi_{i,\mathcal{N}}$ by $\phi_{in,\mathcal{N}}$. The initial values are $\phi_{2n,\mathcal{N}}(\bar{c})=c^*_{2n}$, $\phi_{1n,\mathcal{N}}(\bar{c})=c^*_{1n}$, and $\hat{\phi}_{2n,\mathcal{N}}(\bar{c}) = \hat{c}^*_{2n}$, $\hat{\phi}_{1n,\mathcal{N}}(\bar{c}) = \hat{c}^*_{1n}$, where $c^*_{2n} > \hat{c}^*_{2n}$ and $c^*_{1n} < \hat{c}^*_{1n}$.[23]

Now, I can proceed as in 2P0F to show that for each n there is a unique pair of boundary conditions $\phi_{1n,\mathcal{N}}(\bar{c}) = c^*_{1n}$ and $\phi_{2n,\mathcal{N}}(\bar{c}) = c^*_{2n}$, such that solution to $\mathcal{P}_n$ is such that $\phi_{1n,\mathcal{N}}(0) = \phi_{2n,\mathcal{N}}(0)$. Furthermore, it can be shown from arguments similar to 2P0F that the solution is positively monotonic function. Thus, I can argue that as $n \to \infty$, we will get solution such that $\phi_{in,\mathcal{N}}(0) \to 0$. The rest of the argument is same as before to show that there is a unique pair of initial values (c^*_1, c^*_2) such that $\underset{c \to 0^+}{lim} \phi_{i,\mathcal{N}}(c) = 0$.

Now consider the IVP below, defined on $[\bar{c}, b^R]$:

$$(b - \phi_{1,\mathcal{N}}(b))(q_1 + q_2 + q_3 - M)\sigma(\phi_{2,\mathcal{N}}(b))\phi'_{2,\mathcal{N}}(b) = M - q_2 - q_3$$
$$(b - \phi_{2,\mathcal{N}}(b))(q_1 + q_2 + q_3 - M)\sigma(\phi_{1,\mathcal{N}}(b))\phi'_{1,\mathcal{N}}(b) = M - q_1 - q_3$$

$\phi_{i,\mathcal{N}}(\bar{c}) = c^*_i$

This IVP has a unique solution. However, there is exactly one value of b^R where the solution is such that $\phi_{2,\mathcal{N}}(b^R) = \bar{c}$. Thus, there is no guarantee that the equilibrium exists. However, parameters are such that it does, it is unique.

Note however that when $b^R = \bar{c}$, there is a singularity on the right boundary also. However, I can still proceed as in 2P0F barring some changes.The sequence of BVPs with ODEs as in 25, would be defined on $[\frac{\delta}{2^n}, \bar{c} - \frac{\delta}{2^n}]$ with boundaries $\phi_{2n,\mathcal{N}}(\bar{c} - \frac{\delta}{2^n}) = \bar{c} - \frac{\delta}{2^n} + \frac{\delta^2}{4^n}$ and $\phi_{1n,\mathcal{N}}(\frac{\delta}{2^n}) = \phi_{2n,\mathcal{N}}(\frac{\delta}{2^n})$. The solution to BVPs will generate a sequence of non-decreasing functions $\phi_{in,\mathcal{N}}(b)$. This can be used to generate another sequence of functions $w_{in}(b)_{n \in \mathbb{N}}$ defined as:

$$w_{in}(b) = \begin{cases} \phi_{in,\mathcal{N}}(\bar{c} - \frac{\delta}{2^n}), \ b \in [\bar{c} - \frac{\delta}{2^n}, \bar{c}] \\ \phi_{in,\mathcal{N}}(b), \ b \in [\frac{\delta}{2^n}, \bar{c} - \frac{\delta}{2^n}] \\ \phi_{in,\mathcal{N}}(\frac{\delta}{2^n}), \ b \in [0, \frac{\delta}{2^n}] \end{cases}$$

[23] Case where $c^*_{2n} > \hat{c}^*_{2n}$ and $c^*_{1n} > \hat{c}^*_{1n}$ is of no interest because it violates the condition that if $\hat{\phi}_{2,\mathcal{N}}(b) < \phi_{2,\mathcal{N}}(\bar{c})$, then $\hat{\phi}_{1,\mathcal{N}}(b) > \phi_{1,\mathcal{N}}(\bar{c})$.

The rest of the argument leverages the monotone convergence theorem as in 2P0F, to show that the $\underset{n\to\infty}{lim}\, w_{in}$ converges. Define $\phi_{i,\mathcal{N}}(b)$ as $\underset{n\to\infty}{lim}\, w_{in}$, which then implies that $\underset{c\to 0^+}{lim}\, \phi_{i,\mathcal{N}}(c) = 0$ for each i, and $\underset{c\to \bar{c}^-}{lim}\, \phi_{2,\mathcal{N}}(c) = \bar{c}$.

■