Nash implementation of supermajority rules*

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Abstract

A committee of n experts from a university department must choose whom to hire from a set of m candidates. Their honest judgments about the best candidate must be aggregated to determine the socially optimal candidates. However, experts' judgments are not verifiable. Furthermore, the judgment of each expert does not necessarily determine his preferences over candidates. To solve this problem, a mechanism that implements the socially optimal aggregation rule must be designed. We show that the smallest quota q compatible with the existence of a q-supermajoritarian and Nash implementable aggregation rule is $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor$. Moreover, for such a rule to exist, there must be at least $m \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ impartial experts with respect to each pair of candidates.

Key Words: Aggregation of experts' judgments; supermajority rules; Nash implementation.
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1 Introduction

A committee of n experts from a university department must choose whom to hire from a set of m candidates. Although all experts have the same information about the candidates, their honest judgments about who is the best do not necessarily coincide (for example, the experts may differ in the importance they assign to different characteristics of the candidates). Therefore, experts' judgments must be aggregated to decide the winning candidates. The problem is that judgments are not verifiable. Furthermore, the judgment of each expert does not necessarily determine his preferences over candidates. For example, an expert might be interested in hiring a candidate who is his friend, even if he does not think that candidate is the best. To solve this problem, we should design a mechanism (or voting system) that provides the right incentives for the experts to choose the candidates prescribed by the judgment aggregation rule. The aggregation rule is said to be implementable when this can be done.

As usual in implementation problems, whether a judgment aggregation rule is implementable may depend on the characteristics of that rule. However, in this setting, an additional element is decisive: how experts' judgments and preferences are related. For example, an aggregation rule might be implementable if all experts prefer the candidates they consider to be the best to win, but not if all experts have the same friend whom they want to favor.

Concerning the characteristics of the aggregation rule, we focus on supermajority rules. An aggregation rule is q-supermajoritarian (with $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq q \leq n$) if, whenever at least q experts have the same judgment about the best candidate, that is the only candidate selected by the rule. Note that the smaller q, the more stringent the q-supermajority criterion.

Regarding the relationship between judgments and preferences, following Amorós (2020), we say that an expert is impartial with respect to two candidates if the planner knows that, whenever the expert honestly believes that one of the two candidates is the best, he prefers that candidate to the other.

Our goal is to study the existence of q-supermajoritarian aggregation rules that are Nash implementable. Specifically, we are interested in studying (1) what is the smallest quota q compatible with the existence of a q-supermajoritarian and Nash implementable aggregation rule and (2) what requirements this imposes on the impartiality of the group of experts.

Concerning the first point, we show that $n - \left\lfloor \frac{n-1}{m} \right\rfloor$ is a lower bound on

q for the existence of a q-supermajoritarian aggregation rule that is Nash implementable (Proposition 1). This lower bound holds even in the most favorable situation where all experts are impartial with respect to all pairs of candidates.

About the second point, we show that, for a Nash implementable and $(n-\lfloor\frac{n-1}{m}\rfloor)$-supermajoritarian aggregation rule to exist, for each pair of candidates, there must be at least $m\lfloor\frac{n-1}{m}\rfloor+1$ experts who are impartial with respect to them (Proposition 2). In particular, if for at least one pair of candidates, there are precisely $m\lfloor\frac{n-1}{m}\rfloor+1$ impartial experts, those experts must be impartial with respect to all other pairs of candidates (Proposition 2). Moreover, in this case, the existence of a Nash implementable and $(n-\lfloor\frac{n-1}{m}\rfloor)$-supermajoritarian aggregation rule is guaranteed (Proposition 3).

Related literature

Amorós (2020, 2021) are the closest papers to ours. They analyze the same setting as our paper and study necessary conditions for implementation in an ordinal equilibrium concept.[1] Amorós (2020) demonstrates that implementing a majoritarian aggregation rule in an ordinal equilibrium concept requires all experts to be impartial with respect to all pairs of candidates.[2] Amorós (2021) generalizes this result and shows that implementing a q-supermajoritarian aggregation rule in an ordinal equilibrium concept requires that, for each pair of candidates, there are at least $2(n-q)+1$ experts who are impartial with respect to them. In particular, this condition implies that implementing a $(n-\lfloor\frac{n-1}{m}\rfloor)$-supermajoritarian aggregation rule requires at least $2\lfloor\frac{n-1}{m}\rfloor+1$ impartial experts for each pair of candidates. However, our paper shows that these necessary conditions for implementation are not sufficient when the ordinal equilibrium concept is Nash equilibrium. Firstly, a corollary of our Proposition 1 is that no majoritarian aggregation rule is implementable in Nash equilibrium, even if all experts are impartial with respect to all pairs of candidates. Secondly, our Proposition 2 shows that the necessary condition of impartiality for implementing a $(n-\lfloor\frac{n-1}{m}\rfloor)$-supermajoritarian aggregation rule is stronger than stated by

[1] An equilibrium concept is ordinal if it only depends on the ordinal preferences of the agents, not on the cardinal utility. For example, dominant strategy and Nash equilibria are ordinal, but Bayesian equilibrium is not.

[2] A majoritarian aggregation rule is a q-supermajoritarian rule for the smallest possible q (*i.e.*, $q=\lfloor\frac{n}{2}\rfloor+1$).

Amorós (2021) when the ordinal equilibrium concept is Nash equilibrium, as $m\left\lfloor\frac{n-1}{m}\right\rfloor+1>2\left\lfloor\frac{n-1}{m}\right\rfloor+1$ if $m>2$.

Some papers study a simpler model where all experts have the same judgment (*e.g.*, Amorós, 2013; Yadav, 2016). In this case, the only reasonable rule selects the candidate that all experts judge to be the best. The condition over the impartiality of the experts for this rule to be implementable only requires that, for each pair of candidates, there is at least one expert who is impartial with respect to them. Another series of papers analyze the problem of selecting a ranking of candidates instead of a subset of winners (*e.g.*, Amorós, 2009b; Adachi, 2014). The definitions of judgment, aggregation rule, or impartiality are different in this problem, and then the conditions for implementation are not comparable with our results.

Amorós (2009a) studies the problem of selecting alternatives based on agents' preferences. In this setting, the unequivocal majority of a rule is the number of agents such that, whenever at least this many experts agree on the top alternative, only this alternative is chosen. He shows that $n-\left\lfloor\frac{n-1}{m}\right\rfloor$ is a lower bound for the unequivocal majority of any Maskin-monotonic rule. Although this result closely resembles our Proposition 1, they are independent results. The reason is that, while in Amorós (2009a) a rule chooses alternatives based on preferences, in our work, a rule chooses candidates based on judgments (and judgments do not determine preferences).[3]

Mackenzie (2020) studies how the pope is elected in the Roman Catholic Church. This problem is a particular case of our model where the cardinals are both the experts and the candidates. Holzman and Moulin (2013) study the problem of choosing one winner when the experts are the candidates themselves and each expert only cares about winning and is indifferent among everyone else so that his preferences do not depend on his judgment. Mackenzie (2015) analyzes a stochastic version of the Holzman and Moulin (2013) model. Tamura (2016) establishes a characterization result in the context of impartial nomination rules that satisfy anonymity, symmetry, and monotonicity.

The rest of the paper is organized as follows. In Section 2, we describe the model and notation. In Section 3, we state and prove the results. In Section 4, we offer concluding remarks.

[3]Moreover, Amorós (2009a) only considers strict preferences, while our model allows for indifferences.

2 Setting

Let E be a set of $n \geq 2$ experts and C a set of $m \geq 2$ candidates. Each expert i has an (honest) *judgment* about the best candidate, $J_i \in C$. The experts' judgments must be aggregated to determine the deserving winner. The aggregation procedure is represented by a *social choice rule* (SCR), namely a correspondence $F : C^n \rightarrow 2^C \backslash \{\emptyset\}$ that associates each possible profile of experts' judgments with a non-empty subset of candidates.

Our focus in this paper is on supermajoritarian SCRs. For each $J \in C^n$ and $x \in C$, let $E_J^x = \{i \in E \mid J_i = x\}$.

DEFINITION Let $q \in \mathbb{N}$ be such that $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq q \leq n$. An SCR F is q-*supermajoritarian* if, whenever $J \in C^n$ is such that $|E_J^x| \geq q$ for some $x \in C$, then $F(J) = x$.

Roughly speaking, q-supermajoritarianism requires that whenever a candidate is judged as best by at least q experts, the SCR selects only that candidate. Note that the higher q, the less demanding the q-supermajoritarian condition.

Experts have preferences over candidates that may depend on their judgments. Let $\Re$ denote the class of all complete, reflexive, and transitive preference relations over C. A *preference function* for an expert i is a mapping $R_i : C \rightarrow \Re$ that associates with each possible judgment J_i a preference relation $R_i(J_i)$ (the strict part is denoted $P_i(J_i)$).

Let $[C]^2$ denote the collection of pairs of candidates. Following Amorós (2020), we say that an expert is *impartial* with respect to a pair of candidates if the planner knows that whenever the expert believes one of the two candidates is the best, he prefers that candidate to the other. Each expert i is characterized by a set of pairs of candidates with respect to whom the planner knows that i is impartial, $I_i \subset [C]^2$. A preference function $R_i : C \longrightarrow \Re$ is *admissible* for i at I_i if, for every $J_i, x, y \in C$ such that $J_i = x$ and $xy \in I_i$, we have $x \ P_i(J_i) \ y$. Let $\mathcal{R}(I_i)$ be the class of all preference functions that are admissible for i at I_i.

A *jury configuration* is a profile $I = (I_i)_{i \in E}$. A profile $R \equiv (R_i)_{i \in E}$ is admissible at I if $R_i \in \mathcal{R}(I_i)$ for every $i \in E$. Let $\mathcal{R}(I)$ denote the set of admissible profiles of preference functions at I. Given a jury configuration I, a *state* is a profile $(J, R) \in C^n \times \mathcal{R}(I)$. A *mechanism* is a pair $\Gamma = (M, g)$, where $M \equiv \times_{i \in E} M_i$, M_i is a message space for expert i, and g :

$M \to C$ is an outcome function. A profile $m \in M$ is a *Nash equilibrium* of Γ at state (J, R) if, for every $i \in E$ and $\hat{m}_i \in M_i$, $g(m_i, m_{-i})\ R_i(J_i)\ g(\hat{m}_i, m_{-i})$. Let $N(\Gamma, J, R) \subset M$ denote the set of Nash equilibria of Γ at (J, R). The corresponding candidates selected by the mechanism are denoted $g(N(\Gamma, J, R))$.

Given a jury configuration I, a mechanism $\Gamma = (M, g)$ *implements* an SCR F in Nash equilibrium if, for each state $(J, R) \in C^n \times \mathcal{R}(I)$, $g(N(\Gamma, J, R))$ $= F(J)$.

3 Results

A well-known result in the literature on mechanism design states that every Nash implementable SCR is *Maskin-monotonic*: no outcome can be dropped from being chosen unless its desirability deteriorates for at least one agent (Maskin, 1999). Amorós (2020) showed that, in our setting, Maskin-monotonicity is equivalent to the following condition: if some candidate x is socially considered to be a deserving winner when the profile of judgments is J but not when the profile is $\hat{J}$, then there must be some expert i who judges x as the best candidate at J but not at $\hat{J}$ and who is impartial with respect to the pair $J_i\hat{J}_i$.

DEFINITION Given a jury configuration I, an SCR F satisfies *impartiality of relevant experts* (IRE) if, for every $J, \hat{J} \in C^n$ and $x \in C$, if $x \in F(J)$ and $x \notin F(\hat{J})$, then there exists $i \in E$ with $J_i = x \neq \hat{J}_i$ and $J_i\hat{J}_i \in I_i$.

LEMMA 1 *Given any jury configuration I, if an SCR F is Nash implementable, it satisfies IRE.*

Although Lemma 1 can be obtained as a corollary of Maskin (1999; Theorem 2) and Amorós (2020; Proposition 1), we include a new proof in the Appendix for completeness.

Whether an SCR satisfies IRE depends on the following two elements: (1) the properties of the SCR itself and (2) the jury configuration. Regarding the properties of the SCR, in this paper, we are interested in SCRs that are q-supermajoritarian for some $q \in \left[\left\lfloor \frac{n}{2} \right\rfloor + 1, n\right]$. Note that the smaller q, the more demanding the q-supermajoritarian requirement, and therefore the more difficult it will be to find a q-supermajoritarian SCR that satisfies IRE. Regarding the jury configuration, the most favorable situation for an

SCR to satisfy IRE is that all experts be impartial with respect to all pairs of candidates, *i.e.*, $I_i = [C]^2$ for every $i \in E$ (if an SCR does not satisfy IRE for this jury configuration, it does not satisfy it for any other).

Next, we establish some conditions on the two previous elements for a q-supermajoritarian SCR to be implementable in Nash equilibrium. First, we show that $n - \left\lfloor \frac{n-1}{m} \right\rfloor$ is a lower bound on q for the existence of a q-supermajoritarian SCR that satisfies IRE. This lower bound holds even in the most favorable situation where all experts are impartial with respect to all pairs of candidates.

PROPOSITION 1. *Given any jury configuration I, no Nash implementable SCR is q-supermajoritarian with $q < n - \left\lfloor \frac{n-1}{m} \right\rfloor$.*

Proof. Suppose by contradiction and w.l.o.g. that a Nash implementable SCR F is q-supermajoritarian with $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor - 1$ (if F is $\hat{q}$-supermajoritarian with $\hat{q} < n - \left\lfloor \frac{n-1}{m} \right\rfloor - 1$, it is q-supermajoritarian with $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor - 1$). From Lemma 1, because F is implementable in Nash equilibrium, it satisfies IRE.

Case 1. $n \leq m$.

Because $n \leq m$, then $q = n - 1$. Let $J \in C^n$ be such that $J_i \neq J_j$ for every $i, j \in E$ (because $n \leq m$, such a profile exists). Let $x \in F(J)$. Let $y \in C \backslash \{x\}$ and $\hat{J} \in C^n$ be such that, for every $i \in E$, (i) if $J_i \neq x$ then $\hat{J}_i = y$ and (ii) if $J_i = x$ then $\hat{J}_i = J_i$. Because $J_i \neq J_j$ for every $i, j \in E$, there is at most one expert i with $J_i = x$. Therefore, $\left| E^y_{\hat{J}} \right| \geq n - 1$. Hence, because F is q-supermajoritarian for $q = n - 1$, $F(\hat{J}) = y$. Then, $x \in F(J)$ and $x \notin F(\hat{J})$. However, there is no $i \in E$ with $J_i = x \neq \hat{J}_i$, which contradicts that F satisfies IRE, regardless of the jury configuration I.

Case 2. $m < n$.

Suppose now that $m < n$. Let $C^1, C^2 \subset C$ be such that $C^1 \cap C^2 = \emptyset$, $C^1 \cup C^2 = C$, $|C^1| = n - m \left\lfloor \frac{n}{m} \right\rfloor$, and $|C^2| = m - n + m \left\lfloor \frac{n}{m} \right\rfloor$. Let $J \in C^n$ be such that, (i) for each $x \in C^1$, $|E^x_J| = \left\lfloor \frac{n}{m} \right\rfloor + 1$, and (ii) for each $x \in C^2$, $|E^x_J| = \left\lfloor \frac{n}{m} \right\rfloor$. Let $x \in F(J)$. Let $y \in C \backslash \{x\}$ and $\hat{J} \in C^n$ be such that, for every $i \in E$, (i) if $J_i \neq x$ then $\hat{J}_i = y$ and (ii) if $J_i = x$ then $\hat{J}_i = J_i$. Note that there are at most $\left\lfloor \frac{n}{m} \right\rfloor + 1$ experts with $J_i = x$. Therefore, $\left| E^y_{\hat{J}} \right| \geq n - \left\lfloor \frac{n-1}{m} \right\rfloor - 1$. Because F is q-supermajoritarian for $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor - 1$, $F(\hat{J}) = y$. Then, $x \in F(J)$ and $x \notin F(\hat{J})$. However, there is no $i \in E$ with

$J_i = x \neq \hat{J}_i$, which contradicts that F satisfies IRE, regardless of the jury configuration I. ■

Proposition 1 implies that if we are interested in q-supermajoritarian SCRs that are implementable in Nash equilibrium, we must discard those whose quota q is less than $n - \left\lfloor \frac{n-1}{m} \right\rfloor$, regardless of the jury configuration.

An SCR is considered majoritarian if it is $(\left\lfloor \frac{n}{2} \right\rfloor + 1)$-supermajoritarian. Amorós (2020) demonstrated that implementing a majoritarian aggregation rule in an ordinal equilibrium concept requires all experts to be impartial with respect to all pairs of candidates. Notably, unless $m = 2$, $n = 2$, or $m = 3$ and $n = 4$, we have $n - \left\lfloor \frac{n-1}{m} \right\rfloor > \left\lfloor \frac{n}{2} \right\rfloor + 1$. Thus, an immediate consequence of Proposition 1 is that, even under the total impartiality requirement stated by Amorós (2020), no majoritarian SCR is implementable when the equilibrium concept is Nash equilibrium.

From Proposition 1, a natural question arises: is $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor$ the smallest supermajoritarian quota compatible with Nash implementation? In other words, is there any q-supermajoritarian SCR with $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor$ that is implementable in Nash equilibrium?

To answer this question, we first study what conditions the jury configuration must satisfy for such an SCR to exist. Our following result shows that, for a q-supermajoritarian SCR with $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor$ to satisfy IRE, the jury configuration has to be such that, for each pair of candidates, there are at least $m \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ experts who are impartial with respect to them. The jury configuration can satisfy this condition in many different ways. The most obvious of these is that, for every pair of candidates, all experts are impartial with respect to them. Suppose on the contrary that, for at least one pair of candidates, there are precisely $m \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ experts who are impartial with respect to them. It turns out that, in this case, those same experts must be totally impartial in that they are impartial with respect to all pairs of candidates. For each jury configuration I and each pair of candidates $xy \in [C]^2$, let E^I_{xy} be the group of experts that are impartial with respect to xy, *i.e.*, $E^I_{xy} = \{i \in E \mid xy \in I_i\}$. Let E^I be the group of experts who are impartial with respect to every pair of candidates, *i.e.*, $E^I = \{i \in E \mid xy \in I_i$ for every $xy \in [C]^2\}$.

PROPOSITION 2. *Given a jury configuration I, suppose a Nash implementable SCR F exists that is q-supermajoritarian with $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor$. Then:*

(1) $\left|E^I_{xy}\right| \geq m\left\lfloor\frac{n-1}{m}\right\rfloor + 1$ *for every* $xy \in [C]^2$*, and*

(2) *if* $\left|E^I_{xy}\right| = m\left\lfloor\frac{n-1}{m}\right\rfloor + 1$ *for some* $xy \in [C]^2$*, then* $\left|E^I\right| = m\left\lfloor\frac{n-1}{m}\right\rfloor + 1$.[4]

Proof. From Lemma 1, because F is Nash implementable, it satisfies IRE.

Step 1. If $J \in C^n$ *and* $x \in C$ *are such that* $|E^x_J| \leq \left\lfloor\frac{n-1}{m}\right\rfloor$*, then* $x \notin F(J)$.

Suppose by contradiction that $x \in F(J)$. Let $\hat{J} \in C^n$ and $y \in C$ be such that (i) $\hat{J}_i = J_i$ for every $i \in E^x_J$, and (ii) $\hat{J}_i = y$ for every $i \notin E^x_J$. Then $\left|E^y_{\hat{J}}\right| \geq n - \left\lfloor\frac{n-1}{m}\right\rfloor$. Because F is q-supermajoritarian with $q = n - \left\lfloor\frac{n-1}{m}\right\rfloor$, $F(\hat{J}) = y$. Then, $x \in F(J)$ and $x \notin F(\hat{J})$. However, there is no $i \in E$ with $J_i = x \neq \hat{J}_i$, which contradicts that F satisfies IRE.

Step 2. $\left|E^I_{xy}\right| \geq m\left\lfloor\frac{n-1}{m}\right\rfloor + 1$ *for every* $xy \in [C]^2$.

Suppose by contradiction that $\left|E^I_{xy}\right| \leq m\left\lfloor\frac{n-1}{m}\right\rfloor$ for some $xy \in [C]^2$. Then, there are at least $n - m\left\lfloor\frac{n-1}{m}\right\rfloor$ experts who are not impartial with respect to xy, *i.e.*, $\left|E/E^I_{xy}\right| \geq n - m\left\lfloor\frac{n-1}{m}\right\rfloor$. Let $J \in C^n$ be such that (i) $|E^z_J| = \left\lfloor\frac{n-1}{m}\right\rfloor$ for every $z \in C\backslash\{x\}$, (ii) $|E^x_J| = n - (m-1)\left\lfloor\frac{n-1}{m}\right\rfloor$, and (iii) $J_i = x$ for $n - m\left\lfloor\frac{n-1}{m}\right\rfloor$ of the experts who are not impartial with respect to xy.[5] Because $|E^z_J| = \left\lfloor\frac{n-1}{m}\right\rfloor$ for every $z \in C\backslash\{x\}$, by Step 1, we have $F(J) = x$. Let $\hat{J} \in C^n$ be such that (i) $\hat{J}_i = y$ for every $i \notin E^x_J$, (ii) $\hat{J}_i = y$ for every $i \notin E^I_{xy}$, and (iii) $\hat{J}_i = J_i$ for every $i \in E^x_J \cap E^I_{xy}$. Then, $\left|E^y_{\hat{J}}\right| = (m-1)\left\lfloor\frac{n-1}{m}\right\rfloor + n - m\left\lfloor\frac{n-1}{m}\right\rfloor = n - \left\lfloor\frac{n-1}{m}\right\rfloor$. Because F is q-supermajoritarian with $q = n - \left\lfloor\frac{n-1}{m}\right\rfloor$, $F(\hat{J}) = y$. Then, $x \in F(J)$ and $x \notin F(\hat{J})$. However, there is no $i \in E$ with $J_i = x \neq J_i$ and $J_i\hat{J}_i \notin I_i$, which contradicts that F satisfies IRE.

Step 3. If $J \in C^n$ *and* $x \in C$ *are such that* $x \in F(J)$ *and* $|E^x_J| = \left\lfloor\frac{n-1}{m}\right\rfloor + \alpha$ *for some* $\alpha \geq 1$ *then, for every* $y \in C\backslash\{x\}$*, we have* $\left|E^x_J \cap E^I_{xy}\right| \geq \left\lfloor\frac{n-1}{m}\right\rfloor + 1$.

Suppose by contradiction that, for some $y \in C\backslash\{x\}$, $\left|E^x_J \cap E^I_{xy}\right| \leq \left\lfloor\frac{n-1}{m}\right\rfloor$. Let $\hat{J} \in C^n$ be such that (i) $\hat{J}_i = x$ for every $i \in E^x_J \cap E^I_{xy}$ and (ii) $\hat{J}_i = y$ for every $i \notin E^x_J \cap E^I_{xy}$. Note that $\left|E^y_{\hat{J}}\right| \geq n - \left\lfloor\frac{n-1}{m}\right\rfloor$. Because F is q-supermajoritarian with $q = n - \left\lfloor\frac{n-1}{m}\right\rfloor$, $F(\hat{J}) = y$. Then $x \in F(J)$ and

[4] If $n \leq m$, the condition stated in point (1) of Proposition 2 only requires that, for each pair of candidates, there is at least one expert who is impartial with respect to them. If $n > m$, the condition is more stringent. In particular, if $n-1$ is a multiple of m, the condition requires that all experts be impartial with respect to all pairs of candidates.

[5] Note that then $\left|E^x_J \cap E/E^I_{xy}\right| = n - m\left\lfloor\frac{n-1}{m}\right\rfloor$.

$x \notin F(\hat{J})$. Note that, for every $i \in E_J^x$ with $\hat{J}_i \neq x$, we have $\hat{J}_i = y$ and $i \notin E_{xy}^I$, which contradicts that F satisfies IRE.

Step 4. If $\left|E_{xy}^I\right| = m \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ *for some* $xy \in [C]^2$*, then* $E_{xy}^I \subset E^I$.

Suppose by contradiction that there is some $i \in E_{xy}^I$ such that $i \notin E^I$. Then, there is $\hat{x}\hat{y} \in [C]^2 \setminus \{xy\}$ such that $\hat{x}\hat{y} \notin I_i$. Because $\hat{x}\hat{y} \neq xy$, either $\hat{x} \notin \{x, y\}$ or $\hat{y} \notin \{x, y\}$ (or both). Suppose w.l.o.g. that $\hat{x} \notin \{x, y\}$. Let $J \in C^n$ be such that (i) $\hat{J}_i = \hat{x}$, (ii) $\hat{J}_j = x$ for every $j \notin E_{xy}^I$, (iii) $\left|E_J^{\hat{x}} \cap E_{xy}^I\right| = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$, and (iv) $\left|E_J^z \cap E_{xy}^I\right| = \left\lfloor \frac{n-1}{m} \right\rfloor$ for every $z \in C \setminus \{\hat{x}\}$. Note that $|E_J^x| = \left\lfloor \frac{n-1}{m} \right\rfloor + n - m \left\lfloor \frac{n-1}{m} \right\rfloor - 1$, $\left|E_J^{\hat{x}}\right| = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$, and $|E_J^z| = \left\lfloor \frac{n-1}{m} \right\rfloor$ for every $z \in C \setminus \{x, \hat{x}\}$.

Claim 4.1. $x \notin F(J)$.

Note that $n - m \left\lfloor \frac{n-1}{m} \right\rfloor - 1$ is a non-negative integer. If $n - m \left\lfloor \frac{n-1}{m} \right\rfloor - 1 \geq 1$, then $|E_J^x| = \left\lfloor \frac{n-1}{m} \right\rfloor + \alpha$ for some $\alpha \geq 1$. In this case, by Step 1 and since $\left|E_J^x \cap E_{xy}^I\right| = \left\lfloor \frac{n-1}{m} \right\rfloor$, we have $x \notin F(J)$. If $n - m \left\lfloor \frac{n-1}{m} \right\rfloor - 1 = 0$, then $|E_J^x| = \left\lfloor \frac{n-1}{m} \right\rfloor$ and, by Step 1, $x \notin F(J)$.

Claim 4.2. $\hat{x} \notin F(J)$.

Because $\left|E_J^{\hat{x}}\right| = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$, $\hat{J}_i = \hat{x}$, and $\hat{x}\hat{y} \notin I_i$, then $\left|E_J^{\hat{x}} \cap E_{\hat{x}\hat{y}}^I\right| \leq \left\lfloor \frac{n-1}{m} \right\rfloor$. Therefore, $\left|E_J^{\hat{x}}\right| = \left\lfloor \frac{n-1}{m} \right\rfloor + \alpha$ for some $\alpha \geq 1$ but $\left|E_J^x \cap E_{xy}^I\right| < \left\lfloor \frac{n-1}{m} \right\rfloor + 1$. Hence, by Step 3, $\hat{x} \notin F(J)$.

Claim 4.3. $z \notin F(J)$ *for every* $z \in C \setminus \{x, \hat{x}\}$.

Let $z \in C \setminus \{x, \hat{x}\}$. Because $|E_J^z| = \left\lfloor \frac{n-1}{m} \right\rfloor$, by Step 1, we have $z \notin F(J)$.

From Claims 4.1, 4.2, and 4.3, we have $F(J) = \emptyset$, which contradicts that F is an SCR. ■

Amorós (2021) demonstrated that if an aggregation rule is q-supermajoritarian and implementable in an ordinal equilibrium concept, then, for each pair of candidates, there are at least $2(n - q) + 1$ experts who are impartial with respect to them. As Nash equilibrium is an ordinal equilibrium concept, a corollary of the previous result is that, if an aggregation rule is q-supermajoritarian with $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor$ and Nash implementable, then, for each pair of candidates, there are at least $2 \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ experts who are impartial with respect to them. However, since $m \left\lfloor \frac{n-1}{m} \right\rfloor + 1 > 2 \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ if $m > 2$, our Proposition 2 demonstrates that, in general, the necessary condition of impartiality is indeed stronger.

Let us then assume that there are $m \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ experts who are impartial with respect to every pair of candidates and return to the question at hand: is there any q-supermajoritarian SCR with $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor$ that is implementable in Nash equilibrium? The following result shows that the answer to this question is positive.

PROPOSITION 3. *Suppose that $n \geq 3$. Let I be a jury configuration such that $\left|E^I\right| \geq m \left\lfloor \frac{n-1}{m} \right\rfloor + 1$. Then, a Nash implementable and q-supermajoritarian SCR with $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor$ exists.*

Proof. Let F^* be an SCR such that, for each $J \in C^n$:

$$F^*(J) = \{x \in C : \left|E_J^x \cap E^I\right| \geq \left\lfloor \frac{n-1}{m} \right\rfloor + 1\}$$

First, note that, because $\left|E^I\right| \geq m \left\lfloor \frac{n-1}{m} \right\rfloor + 1$, for every $J \in C^n$, there is at least one $x \in C$ such that $\left|E_J^x \cap E^I\right| \geq \left\lfloor \frac{n-1}{m} \right\rfloor + 1$, and then $F^*(J) \neq \emptyset$.

Claim 1. F^* is q-supermajoritarian with $q = n - \left\lfloor \frac{n-1}{m} \right\rfloor$.

Let $J \in C^n$ be such that $|E_J^x| \geq n - \left\lfloor \frac{n-1}{m} \right\rfloor$ for some $x \in C$. Note that $\left|E \backslash E^I\right| \leq n - m \left\lfloor \frac{n-1}{m} \right\rfloor - 1$. Moreover, $n - \left\lfloor \frac{n-1}{m} \right\rfloor \geq n - m \left\lfloor \frac{n-1}{m} \right\rfloor - 1 + \left\lfloor \frac{n-1}{m} \right\rfloor + 1$. Then $\left|E_J^x \cap E^I\right| \geq \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ and, by definition of F^*, $x \in F^*(J)$.

Claim 2. F^* is implementable in Nash equilibrium.

Case 2.1. $m < n$.

Maskin (1999) showed that if there are at least three agents, any SCR satisfying Maskin monotonicity and no veto power is implementable in Nash equilibrium. In our setting, Maskin monotonicity is equivalent to IRE (Amorós, 2020; Proposition 1). No veto power requires an alternative being F-optimal whenever it is the most preferred for at least $n - 1$ agents. Next, we show that F^* satisfies both conditions.

Step 2.1.1 F^* satisfies IRE.

Let $J, \hat{J} \in C^n$ and $x \in F^*(J)$ be such that $x \notin F^*(\hat{J})$. Then, $\left|E_J^x \cap E^I\right| \geq \left\lfloor \frac{n-1}{m} \right\rfloor + 1$ and $\left|E_{\hat{J}}^x \cap E^I\right| < \left\lfloor \frac{n-1}{m} \right\rfloor + 1$. Therefore, there is at least one expert $i \in E_J^x \cap E^I$ such that $i \notin E_{\hat{J}}^x$. Hence, $J_i = x \neq \hat{J}_i$ and, because $i \in E^I$, $J_i \hat{J}_i \in I_i$.

Step 2.2.2 F^* satisfies no veto power.

Note that, for every $i \in E^I$, $R_i \in \mathcal{R}(I_i)$, $J_i \in C$, and $x \in C \backslash \{J_i\}$, we have $J_i \ P_i(J_i) \ x$; *i.e.*, the most preferred candidate for each expert $i \in E^I$ is J_i. Let $(J, R) \in C^n \times \mathcal{R}(I)$ be such that some candidate x is the most preferred for at least $n-1$ experts. Then $\left|E_J^x \cap E^I\right| \geq \left|E^I\right| - 1$. Hence, because $\left|E^I\right| \geq m\left\lfloor\frac{n-1}{m}\right\rfloor + 1$, $\left|E_J^x \cap E^I\right| \geq m\left\lfloor\frac{n-1}{m}\right\rfloor$. Moreover, because $m \geq 2$ and $m < n$, $m\left\lfloor\frac{n-1}{m}\right\rfloor \geq \left\lfloor\frac{n-1}{m}\right\rfloor + 1$. Then, $\left|E_J^x \cap E^I\right| \geq \left\lfloor\frac{n-1}{m}\right\rfloor + 1$. Therefore, $x \in F^*(J)$.

Case 2.2. $n \leq m$.

In this case $\left\lfloor\frac{n-1}{m}\right\rfloor = 0$, and then $\left|E^I\right| \geq 1$ and, for each $J \in C^n$, $F^*(J) = \{x \in C : \left|E_J^x \cap E^I\right| \geq 1\}$.

Subcase 2.2.1. $\left|E^I\right| > 1$.

The proof that F^* is implementable in Nash equilibrium is almost identical to that of Case 2.1, except for the argument that F^* satisfies no veto power. Let $(J, R) \in C^n \times \mathcal{R}(I)$ be such that some candidate x is the most preferred for at least $n-1$ experts. Then, because $\left|E^I\right| > 1$, x is the most preferred candidate for at least one expert in E^I. Hence, since the most preferred candidate for each expert $i \in E^I$ is J_i, $\left|E_J^x \cap E^I\right| \geq 1$. Therefore, $x \in F^*(J)$.

Subcase 2.2.2. $\left|E^I\right| = 1$.

Let i be the only expert in E^I. Then, for each $J \in C^n$, $F^*(J) = J_i$. Because $i \in E^I$, the most preferred candidate for i is J_i. Therefore, F^* is implementable in Nash equilibrium through the simple mechanism $\Gamma = (M, g)$ where $M_j = C$ for every $j \in E$ and $g(m) = m_i$ for every $m \in M$. ■

4 Concluding remarks

We have studied the problem of the existence of Nash implementable supermajority rules to aggregate the judgments of a group of possibly biased experts. We have stated conditions on the supermajority quota and the experts' impartiality for these rules to exist.

Here are some suggestions for promising lines of extensions.

a. The general conditions for subgame perfect implementation are less demanding than those for Nash implementation (see Moore and Repullo, 1988). It would be interesting to study what results can be obtained using a stage mechanism in which experts make choices sequentially.

b. One of the most significant difficulties when implementing a rule in Nash equilibrium is ensuring that the mechanism does not have "bad" equilibria that result in candidates other than the socially optimal. Knowing that some experts have friends or enemies among the candidates may help to eliminate these bad equilibria. It would be interesting to extend our work to this case.

Appendix

PROOF OF LEMMA 1

Suppose that F is implementable in Nash equilibrium.

Claim 1. For every $J, \hat{J} \in C^n$, every $x \in F(J)$ with $x \notin F(\hat{J})$, and every $R, \hat{R} \in R(I)$, there exist $i \in E$ and $y \in C$ such that $x\ R_i(J_i)\ y$ and $y\ \hat{P}_i(\hat{J}_i)\ x$.

Let $\Gamma = (M, g)$ be a mechanism implementing F in Nash equilibrium. Suppose by contradiction that there exist $J, \hat{J} \in C^n$, $x \in F(J)$ with $x \notin F(\hat{J})$, and $R, \hat{R} \in \mathcal{R}(I)$ such that, for every $i \in E$ and $y \in C$, if $x\ R_i(J_i)\ y$ then $x\ \hat{R}_i(\hat{J}_i)\ y$. Because Γ implements F in Nash equilibrium, there exists $m \in N(\Gamma, J, R)$ such that $g(m) = x$. Then, for every $i \in E$ and every $\hat{m}_i \in M_i$, $x = g(m_i, m_{-i})\ R_i(J_i)\ g(\hat{m}_i, m_{-i})$. Hence, for every $i \in E$ and every $\hat{m}_i \in M_i$, $x = g(m_i, m_{-i})\ \hat{R}_i(\hat{J}_i)\ g(\hat{m}_i, m_{-i})$. Therefore, $m \in N(\Gamma, \hat{J}, \hat{R})$, which contradicts that Γ implements F in Nash equilibrium because $g(m) = x \notin F(\hat{J})$.

Claim 2. Let $J, \hat{J} \in C^n$ and $x \in F(J)$ be such that $x \notin F(\hat{J})$. Then, there exists $i \in E$ such that, for every $R_i, \hat{R}_i \in \mathcal{R}(I_i)$ there is some $y \in C$ such that $x\ R_i(J_i)\ y$ and $y\ \hat{P}_i(\hat{J}_i)\ x$.

It follows from Claim 1 and the fact that $\mathcal{R}(I)$ has a cartesian product structure, *i.e.*, $\mathcal{R}(I) \equiv \times_{i \in E} \mathcal{R}(I_i)$.

Claim 3. Let $i \in E$ and $x, J_i, \hat{J}_i \in C$ be such that, for every $R_i, \hat{R}_i \in \mathcal{R}(I_i)$ there is some $y \in C$ such that $x\ R_i(J_i)\ y$ and $y\ \hat{P}(\hat{J}_i)\ x$. Then, $J_i = x \neq \hat{J}_i$ and $J_i\hat{J}_i \in I_i$.

From the definition of $\mathcal{R}(I_i)$, the only possibility that for every $R_i \in \mathcal{R}(I_i)$ there is some $y \in C$ such that $x\ R_i(J_i)\ y$ is that $J_i = x$ and $xy \in I_i$. In this case, from the definition of $\mathcal{R}(I_i)$, the only possibility that $y\ \hat{P}_i(\hat{J}_i)\ x$ for every $\hat{R}_i \in \mathcal{R}(I_i)$ is that $\hat{J}_i = y$.

Claim 4. Let $J, \hat{J} \in C^n$, and $x \in F(J)$ be such that $x \notin F(\hat{J})$. Then there exists $i \in E$ with $J_i = x \neq \hat{J}_i$ and $J_i\hat{J}_i \in I_i$.

By Claim 2, there exists $i \in E$ such that, for every $R_i, \hat{R}_i \in \mathcal{R}(I_i)$ there is some $y \in C$ such that $x\ R_i(J_i)\ y$ and $y\ \hat{P}_i(\hat{J}_i)\ x$. Then, $i \in E$ and $x, J_i, \hat{J}_i \in C$ are such that for every $R_i, \hat{R}_i \in \mathcal{R}(I_i)$ there is some $y \in C$ such that $x\ R_i(J_i)\ y$ and $y\ \hat{P}_i(\hat{J}_i)\ x$. Hence, by Claim 3, $J_i = x \neq \hat{J}_i$ and $J_i\hat{J}_i \in I_i$. ■

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