Supply and demand function competition in input-output networks*

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Abstract

This paper presents a model of non-infinitesimal firm-to-firm trade via competition in supply and demand functions relating quantities to prices. The main features of the model are two: first, firms have endogenous market power in both input and output markets; second, firms internalize their position in the supply chain. The former is important to rank market power across firms: in models in which firms are restricted to affect only output or only input prices the ranking of market power can be reversed. The latter is important for the assessment of aggregate distortions: final prices and distortions are higher than in a model where firms do not take their position in the supply chain into account. An equilibrium exists for general non-parametric technology, provided the best replies are convex-valued, under suitable regularity and boundedness assumptions. Under a suitable parametric functional form, the equilibrium is in linear schedules.

Keywords: production networks, oligopoly, double auction, supply function equilibrium

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1 Introduction

Production of goods in modern economies typically features long and interconnected supply chains.[1] Moreover, many authors find that market power is a sizable phenomenon, some even argue increasing,[2] and many firms are large relative to their sector or even the whole economy.[3] How are prices formed in an input-output network of non-price-taking firms? How is surplus split? How efficient is the process?

This paper provides a strategic non-cooperative model of large firms interacting in an input-output network consisting of many specific supply-customer relationships. It does so introducing the technique of competition in schedules, or supply and demand functions, to the modeling of general equilibrium oligopoly. The main interest lies in the fact that such a technique allows to have a fully strategic model in which firms understand and take into account their position in the network, and have market power on both inputs and outputs markets simultaneously, in an endogenously determined way. These features are rarely both present in input-output models of the macroeconomy,[4] but I argue that they are important to analyze market power in input-output networks. In particular, the fact that firms are fully strategic and take their position in the network into account can generate large differences in the magnitude of distortions due to imperfect competition. The fact that firms have market power on both input and output markets, as opposed to only outputs, can generate large differences in the ranking of market power across firms or sectors. Both effects are particularly stark especially when supply chains are long. These results suggest that the technique of competition in supply and demand functions can be important for the researcher interested in analyzing and estimating economies with market power, especially in the presence of complex input-output networks.

Formally, firms have each a set of input and output goods, some of which are in turn outputs or inputs of other firms,[5] and these trade relationships, or *input-output links*, are exogenous. Firms play a simultaneous game in which the available actions are supply and demand schedules, relating quantities of the traded goods to prices: as in a double auction, the realized price on every trade relationship is

[1] Recently the focus of a large literature, see Carvalho and Tahbaz-Salehi (2018), Bernard et al. (2018).

[2] See De Loecker et al. (2020), Berry et al. (2019).

[3] Now known as *superstar firms* since Autor et al. (2020).

[4] With some exceptions, see the discussion in the literature section.

[5] For the purpose of the model, two different instances of the same good, that are traded between different producers at possibly different prices are labeled as different "goods"

the one where demand and supply cross. The classic metaphor for the price-taking general equilibrium behavior is that a "walrasian" auctioneer proposes prices and collects supply and demand "bids", until all markets clear. The approach followed in this paper takes this metaphor one step further, applying it to non-infinitesimal firms. The auctioneer acts as a market maker in financial markets, collecting firms' conditional schedules. Firms, being non-infinitesimal, fully internalize the mechanism and submit their schedules to affect prices in their favor. Such schedules are meant not as a literal description of the workings of the market[6], but as an abstraction of a bargaining procedure, parsimonious but powerful enough for the complexity of the problem.

The detailed contributions are the following. First, I show that an equilibrium exists under general regularity and boundedness conditions on the set of feasible schedules (Theorem 1), I provide necessary conditions for equilibrium in the form of a system of partial differential equations (Theorem 2), and a condition under which the equilibrium is ex-post, in the sense that firms would not change their decisions even after the realization of uncertainty (Corollary 3.1). Then, I provide a parameterized functional form for the transformation functions of firms that allows to considerably simplify the analysis, allowing the existence of an equilibrium in *linear strategies*.[7] I use this parameterized functional form to show the qualitative effect of the two aforementioned features of the model, namely: the fact that firms have market power on both input and output markets simultaneously, and that firms explicitly take their position in the supply chain into account. Theorem 4 shows that the former fact can completely reverse the relative ranking of market power in a supply chain compared to a more standard model in which firms have market power only on outputs (Section 5). The latter fact has the effect of predicting larger distortions compared to models in which firms take as given prices and quantities on markets in which they are not directly participating (Theorem 5). Proposition 2 shows that such increased distortions can be arbitrarily high, hence potentially of empirical relevance. In the rest of the Introduction, I expand on each of the contributions.

Theorem 1 shows existence of an equilibrium under general regularity and boundedness assumptions on the technology, consumer demands, and the set of feasible schedules, and under the assumption that the best reply correspondences are convex-valued (or, in particular, single-valued). As in the seminal Klemperer and Meyer (1989) paper on Supply function equilibrium, uncertainty in the realized

[6] Although they are in some cases, e.g. the electricity or financial markets.

[7] By which I mean that the best response over *all* feasible schedules is linear.

prices is key to avoiding a huge multiplicity of best replies. In this paper, the uncertainty comes from stochastic parameters in the transformation function of firms, which can be seen as input (and output)-specific productivity shocks. This is sufficient to generate enough variation in the schedules so that the equilibrium prices span all the feasible set, and the best reply is not indeterminate. The result departs from other existing results in the literature in that it does not impose parametric functional forms,[8] and the presence of firm-to-firm trade.[9] The regularity and boundedness assumptions on the set of feasible schedules allow to use Banach spaces techniques and the Ky Fan fixed point theorem to show existence, rather than looking for an equilibrium as a solution to a system of differential equations.[10] The existence result provided can be of interest also to the modeling of financial markets where traders have price impacts, departing from the standard CARA - gaussian setting.

Theorem 2 indeed expresses the necessary conditions for an equilibrium in the form of a system of partial differential equations, and clarifies that the equilibrium in this model is not ex-post (as in Klemperer and Meyer (1989)), due to network effects. Corollary 3.1 illustrates that the equilibrium is ex-post under a measurability condition, stating that the residual demand and supply depend on a number of uncertain parameters equal to the degree of each firm in the network. This condition says that the degrees of freedom of each firm are as many as the independent sources of uncertainty. The condition is satisfied if the residual schedules are linear (the case to which the parametric model in the following section is dedicated), or if the network is a sequence of sectors linearly connected, a network I label *regular layered supply chain*. This shows that the ex-post or ex-ante nature of the equilibrium depends on an interaction of the schedules' functional form and the network structure.[11]

In Section 4 I introduce a parametric functional form for the technology such that the equilibrium is in linear schedules. This delivers a tractable framework

[8]As Malamud and Rostek (2017)

[9]Contrary to Wilson (2008) or Holmberg and Philpott (2018) that study an oligopoly where a transmission network affects the demand.

[10]As Klemperer and Meyer (1989), Glebkin et al. (2020).

[11] Rostek and Yoon (2021a) show that even in some cases when the equilibrium is not ex-post, if schedules are linear the optimization can still be expressed as a *pointwise* optimization over quantities given price impacts. This is not true anymore in the model of the present paper for nonlinear schedules: the crucial problem is that the way prices respond to a change in schedules (or quantities) is itself uncertain, because it depends on the realization of the stochastic parameters in the other markets. As clarified by Theorem 2, the optimization over quantities alone would miss this effect, even conditioning on the price impact. The consequence is that the optimization cannot be performed on prices anymore, but has to be done directly on schedules.

that can be used to derive the further insights discussed in Sections 5 and Section 6, and is amenable to numerical simulations and estimation. Tractability is also a consequence of the fact that with linear schedules the measurability assumption discussed above applies, and the equilibrium is ex-post, for any network structure.[12] The technology introduced is flexible enough to incorporate different degrees of complementarity and substitutability. Such a parametric functional form also has an independent interest, since it is to the best of my knowledge the only alternative to Cobb-Douglas and Leontief allowing a completely analytic solution of general equilibrium with perfect competition in an IO network; and has the advantage to allow a solution for different degrees of substitutability. In what follows, I use such a parametric functional form to derive further insights on the implications of the model for the study of market power.

In Section 5 I discuss the mechanisms behind differnces in market power across firms, and the relation to network position. In particular, Theorem 4 shows that if in the S&D equilibrium firms are constrained to have price impact only on the output market the result is a completely reversed ranking of market power (as measured by markups or markdowns) with respect to a situation where firms are constrained to have price impact only on the input market. The reason is that, when input prices are taken as given, the markups are determined by the elasticity of the residual demand alone, and depending on network position this can have radically different effects with respect to a situation in which output prices are taken as given, and markdowns are determined by the elasticity of the residual supply alone. As already mentioned, in the full-blown S&D equilibrium, instead, firms have market power on *both* inputs and outputs, in an endogenously determined way. These considerations suggest that models that impose restrictions on which prices a firm can affect might be problematic when we are concerned with the relative ranking of market power across firms. Naturally, if the modeler has strong reasons to assume that firms have direct control over certain prices but not others, building these restrictions into the models is the reasonable thing to do. If these assumptions are just a modeling device, though, it might be important to use models in which firms have a priori ability to affect all prices. This is true in particular when modeling general input-output networks that connect many firms that are very different in terms of the nature of their processes and products, and so very specific assumptions on which prices firms can or cannot control are harder to justify.

[12] A case widely used in financial market models, e.g. Malamud and Rostek (2017) and others discussed in the literature.

Theorem 5 compares the baseline model to a model in which firms fail to internalize the network structure in computing the residual demand they face: this is operationalized by assuming that firms take as given the prices in the markets further down or upstream from their direct customers or suppliers. The theorem shows that, in this case, the welfare loss due to oligopolies is smaller: namely the price impact matrices are smaller (in the positive semidefinite sense).[13] In particular, if a firm does not internalize some reactions in the network, this amounts to that firm perceiving a larger elasticity of demand and supply and, as a consequence, is able to charge smaller markups and markdowns. This is because, in the S&D equilibrium, the elasticity of demand depends on the schedules chosen by directly connected firms, but also *indirectly* connected firms. The reason is that, in equilibrium, a change in a price triggers a change in all other prices of connected firms: failing to account for some of these pass-through effects means firms perceive a different elasticity of demand.[14] The exercise also has an independent interest in that it shows that the model can easily incorporate restrictions to the rationality of the firms in cases in which the complete rationality assumptions maintained so far seem extreme. Proposition 2 further shows that the effect can be arbitrarily large when a supply chain is very long. These considerations suggest that when in a supply chain firms are large and have a sizable price impact,[15] having a model that properly accounts for all strategic effects is important to correctly assess the magnitude of distortions.

The rest of the paper is organized as follows. Section 2 defines the general nonparametric model. Section 3 presents the existence results. Section 4 describes the parametric version of the model. Sections 5 and 6 explore the insights the model yields about relative and aggregate market power. Section 7 concludes. The main proofs are in the Appendix, the others are in the Online Appendix.

1.1 Related literature

This paper contributes to three lines of literature: the literature on competition in supply and demand functions, the literature on production networks or networked

[13] A different interpretation would arise from the perspective of estimation of some structural parameters: for the same final price a model assuming that firms internalize their position in the supply chain would imply that market power is smaller. The important point is still that the difference can be large and this calls for a model that properly accounts for these effects.

[14] The literature on outsourcing and endogenous supply chains provides evidence that firms are aware of their supply chain and take its structure and their position in it into account in their decisions, see e.g. Berlingieri et al. (2020), Alfaro et al. (2019).

[15] Firms that have large dimension with respect to their own sector or the whole economy are often called *superstar firms* since Autor et al. (2020), and are the subject of a large literature.

markets, and the literature on general equilibrium oligopoly.

My contribution to the literature on competition on supply and demand functions is to introduce the technique to the modeling of general equilibrium oligopoly, and providing a general existence result. The literature has studied the situation where the demand firms receive comes from a network structure with a large dimension of uncertainty, in Wilson (2008), Holmberg and Philpott (2018), Ruddell et al. (2017), but their firms only supply to a node in the network, do not trade among themselves. Firm-to-firm trade is studied in a bilateral setting in Weretka (2011) and Hendricks and McAfee (2010), always constraining the schedules to a parametric functional form. In the finance literature the model is used to study simultaneous demand and supply of heterogeneous assets: Malamud and Rostek (2017), as well as Rostek and Yoon (2021a), Rostek and Yoon (2021b) and Rostek and Weretka (2012) analyze a parametric model yielding an equilibrium in linear strategies akin to the one in Section 4; Glebkin et al. (2020) and Du and Zhu (2017) study general functional forms, but in a centralized market (corresponding to a trivial network). Ausubel et al. (2014) and Woodward (2021) study general functional forms in the context of centralized auctions. Vives (2011) studies market power arising from asymmetric information, rather than network position.

My contribution to the production networks literature is to provide a model of competition in an input-output network in which all firms have market power on both input and output markets, and are fully strategic internalizing their position in the supply chain. Many models explicitly assume that firms have power to decide/affect prices only on one side of the market. To this class belong the workhorse sequential oligopoly games in Spengler (1950), Salinger (1988), Ordover et al. (1990), Hart et al. (1990).[16] and the recent Carvalho et al. (2020). These models all feature sequential moves in which downstream firms take input prices as given and, hence, one-sided market power. In another class of models authors assume that output prices are equal to the marginal cost times a markup. The concept of the marginal cost itself implicitly implies price-taking in the input market: indeed, it arises from the price-taking cost minimization problem of the firm. Hence, it is implicitly assuming unilateral market power. To this category belong Grassi (2017), Bernard et al. (2022), Baqaee (2018), Baqaee and Farhi (2019), Baqaee and Farhi (2020), Huremovic and Vega-Redondo (2016), Magerman et al. (2020), Dhyne et al. (2019), Huneeus et al. (2021), Arkolakis et al. (2021), Pasten et al. (2020), Pellegrino (2019). In Galeotti et al. (2021) only primary producers charge a markup, while the intermediate firms behave competitively, thus abstract-

[16] And used in classic textbook treatments, such as Tirole (1988).

ing from the balance of market power among firms that trade with each other. The exception is Acemoglu and Tahbaz-Salehi (2020), that follows a mixed approach: input prices are taken as given when firms decide their input mix, but are then determined in equilibrium through a link-level alternating offers game, relyin on exogenously specified bargaining weights. My results complements theirs, providing a model that does not rely on the choice of exogenously specified bargaining weights.

Except for Acemoglu and Tahbaz-Salehi (2020), all these papers feature also the implicit or explicit assumption that firms do not internalize the effect of their decisions on sectors/firms further downstream beside the direct customers. Sometimes this is a consequence of the assumption of a continuum of firms in each sector (and so sector-level aggregates are taken as given by every individual firm),[17] other times it is explicitly assumed.[18] This is the motivation for the exercise of Section 6, as described in the Introduction.

I contribute to the literature on general equilibrium with market power by providing a fully strategic model of the production side with endogenous market power and firm-to-firm trade; furthermore, the game does not depend on price normalization, and can incorporate general assumptions on owner's preferences as in Azar and Vives (2021). In the recent literature on "general oligopolistic competition" (Azar and Vives (2021), Azar and Vives (2018) and Ederer and Pellegrino (2022)) do not consider firm-to-firm trade, while in the while in the literature on general equilibrium matching Fleiner et al. (2019) study firm-to-firm trade with distortions that are exogenous wedges rather than the outcome of a strategic setting as in the present paper.

2 The Model

In this section I introduce the primitives of the model, that is the firms and their technology, the input-output network, and the utility of the consumer. Firms play a game in which the strategies are supply and demand schedules. Finally, I introduce the technical assumptions needed for the subsequent results.

[17] This is the case in, e.g. Baqaee (2018) and various others listed in the literature.

[18] E.g., in Grassi (2017), Kikkawa et al. (2019).

2.1 General setting

Firms and Production Network There are N firms and M goods: their sets are respectively denoted $\mathcal{N}$ and $\mathcal{M}$. Each good might be produced by more firms, and each firm may produce more than one good. I write $i \to g$ if firm i produces good g, and $g \to i$ if firm i needs good g for production. Each firm produces using labor, and a set of inputs produced by other firms, which I denote as $\mathcal{N}_i^{in}$. The set of goods produced by firm i is $\mathcal{N}_i^{out}$. The consumers' utility depends directly on a subset of goods, denoted $\mathcal{C} \subseteq \mathcal{M}$. Firms, goods and the connections defined above define a directed bipartite graph $\mathcal{G} = (\mathcal{N}, \mathcal{M}, E)$, where $E = \{(i, g) \mid i \to g \text{ or } g \to i\}$ is the set of existing connections. I refer to $\mathcal{G}$ as the *input output network* of this economy. Note that in this setting, a *good* is identified by the fact that is exchanged by a number of firms for a specified price. That is, the framework can accommodate for a firm selling the same physical good to different groups of customers for different prices: simply, this case would show up in the model as two distinct goods. I denote $d_i^{out} = |\mathcal{N}_i^{out}|$ the *out-degree* (number of outputs) of i, and $d_i^{in} = |\mathcal{N}_i^{in}|$ the *in-degree* (number of inputs) of firm i, excluding labor.

Remark 2.1. In the case in which each firm produces only one good, and the goods are all distinct, we can identify the sets of firms and goods and say that two firms are connected if one is a customer of the other. This is the more standard approach in the literature. Figure 1 illustrates the standard (Left) and the bipartite (Right) representation followed here, in the example of a tree network.

The production possibilities available to firm i are described by a *transformation function* Φ_i. This is a function of the input and output quantities, and also on a vector of stochastic parameters $\boldsymbol{\varepsilon}_i = (\varepsilon_{ig})_{g \in \mathcal{N}_i}$, one for each good traded by i. These can be thought of as technological shocks, increasing or decreasing the input quantity needed to achieve a certain level of output. As in Mas-Colell et al. (1995), input quantities are negative, while output quantities are positive. The production possibility set of firm i is thus $\{(q_{gi})_{i \to g}, (q_{ig})_{g \to i}, \ell_i \mid \Phi_i((q_{gi})_g, (-q_{ig})_g, \ell_i, \boldsymbol{\varepsilon}_i) \leq 0\}$. The reason to describe the technology as a transformation function is, besides generality, to treat symmetrically inputs and outputs: goods are allowed to be both, depending on what is more convenient given the market conditions and the implied prices. This a standard approach taken also in Mas-Colell et al. (1995). In our contest it allows a considerable technical simplification, allowing to abstract from corner solutions: negative quantities are allowed, they simply mean trade in the opposite direction.

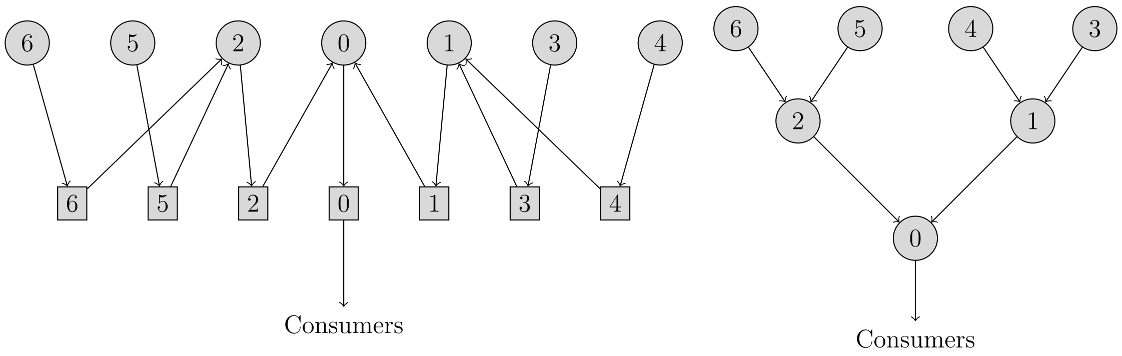

Figure 1: Left: bipartite representation of the production network: the circles are the firms, the squares are goods. An arrow from a good to a firm means the firm buys the good, an arrow from a firm to a good means that the firm sells the good. Right: classic representation of the network, where nodes are firms and links represent the flow of goods. In this example, in which each firm has one distinct output good, the two are equivalent. In general this representation is ambiguous, because it does not allow to see whether, e.g. 5 and 6 output is the same good or two distinct goods.

The price of good g is denoted p_g, so that for a firm buying and producing quantities $(q_{gi})_{i\to g}, (q_{ig})_{g\to i}, \ell_i$, the nominal profit is:

$$\Pi_i = \sum_{g,i\to g} p_g q_{gi} - \sum_{g,g\to i} p_g q_{ig} - w\ell_i$$

Consumers There is a continuum of identical consumers or, equivalently, a representative consumer. She gets utility $U((c_g)_{g\in\mathcal{C}}, L, \varepsilon_{i,c})$ from a subset of goods $\mathcal{C} \subseteq \mathcal{M}$, and disutility from labor L; similarly to the firms, I am going to assume that the utility also depends on a vector of stochastic parameters $\boldsymbol{\varepsilon}_c = (\varepsilon_{g,c})_{g\in\mathcal{C}}$, one for each good consumed. Denote the demand for good i derived by U as $D_{i,c}$, and the labor supply as L. The profits of the firms are rebated to the representative consumer, so that the total income is $wL + Pro$, where $Pro = \sum_i Pro_i$ is the aggregate profit. Welfare in this economy is the utility of the consumers in equilibrium: $U(c^*, L^*)$, where c^* and L^* are the equilibrium values of consumption and labor: since the profits are rebated, such welfare also includes the producers surplus.

Notation Bold symbols are used to denote vectors of prices and stochastic parameters: $\boldsymbol{p}$ is the vector of all prices, except the wage w, $\boldsymbol{p}_i^{in} = ((p_g)_{g\in\mathcal{N}_i^{in}})$ are the prices of all input goods of firm i, and similarly $\boldsymbol{p}_i^{out}$ is the price vector for the outputs, so that $\boldsymbol{p}_i' = (\boldsymbol{p}_i^{out}, \boldsymbol{p}_i^{in})'$. Similarly, $\boldsymbol{p}_c = (p_g)_{g\in\mathcal{C}}$ is the vector of prices of goods consumed by the consumer. The analogous notations hold for stochastic parameters, so that, e.g., $\boldsymbol{\varepsilon}$ is the vector that stacks all the stochastic parameters of all firms.

When A is a function of many variables, $\nabla A = (\partial_1 A, \ldots, \partial_n A)'$ denotes the (column) vector of partial derivatives (the gradient). HA denotes the matrix of second derivatives, that is the Hessian of A. When A is a vector function of x, $\partial_x A$ denotes the square matrix with on each row the gradient of A_i with respect to x (the Jacobian matrix).

If B is a matrix, B_{-i} denotes the same matrix to which row and column i have been removed. If $\boldsymbol{b}$ is a vector, $\boldsymbol{b}_{-i}$ denotes the same vector to which element i has been removed. $B \geq C$ denotes the fact that $B - C$ is positive semidefinite (even when they are not symmetric).

The Game I: players and actions The competition among firms take the form of a game in which firms compete choosing in supply and demand functions. This means that the players of the game are the firms, and the actions available to each firm i are a family of functions defined over a set $\mathcal{F}_i$ of tuples of input-output prices, wage, and a set of firm-specific stochastic parameters $\mathcal{E}_i$: $\mathcal{S}_i : (w, \boldsymbol{p}_i, \boldsymbol{\varepsilon}_i) \in \mathcal{F}_i \times \mathcal{E}_i \to \mathbb{R}^{d_i+1}$, where $\mathcal{F}_i \times \mathcal{E}_i \subset \mathbb{R}^{2(d_i+1)}$. Such functions are called *schedules*, and $\mathcal{S}_i = (S_i, -D_i, -\ell_i)$, composed by profiles of supply functions for outputs $S_i = (S_{gi})_{i \to g}$, demand functions for intermediate inputs $D_i = (D_{ig})_{g \to i}$, and for labor ℓ_i. [19] The set of feasible supply and demand schedules for firm i (defined below) is denoted $\mathcal{A}_i$, and $\mathcal{A} = \prod_{i \in \mathcal{N}} \mathcal{A}_i$.

In the general model of this section we are not restricting traded quantities to be positive. This is a matter of interpretation: since trade has a direction, negative quantities can simply be interpreted as trade flowing in the opposite direction.[20] This approach simplifies the analysis because rules out corner solutions in which firms decide not to buy some inputs (or sell some outputs) at all. Section 4 takes a more applied stance, and studies the existence of an equilibrium not allowing for negative quantities (that is, trade flows in the opposite direction as prescribed).

The Game II: prices and payoffs To complete the definition of the game, we have to define the payoffs. These are, in short, the expected profits calculated in the prices that satisfy the market clearing conditions. The market clearing

[19] The sign convention makes formulas simpler allowing the derivative to be positive semidefinite.

[20] Indeed, this is the interpretation followed by classic treatments of production theory, such as Mas-Colell et al. (1995).

conditions are:

$$\begin{aligned}
\sum_{j,g\to j} D_{jg}(\boldsymbol{p}_j, w, \boldsymbol{\varepsilon}_j) &= \sum_{k,k\to g} S_{gk}(\boldsymbol{p}_k, w, \boldsymbol{\varepsilon}_k) \quad \text{if } g \in \mathcal{M} \\
D_{cg}(\boldsymbol{p}_c, w, \boldsymbol{\varepsilon}_c) &= \sum_{k,k\to g} S_{gk}(\boldsymbol{p}_k, w, \boldsymbol{\varepsilon}_k) \quad \text{if } g \in \mathcal{C} \\
\sum_i \ell_i(\boldsymbol{p}_i, w, \boldsymbol{\varepsilon}_i) &= L(w, \boldsymbol{p}_c, \boldsymbol{\varepsilon}_c)
\end{aligned}$$

Define a function $MC : \mathbb{R}^M \times \mathcal{E} \to \mathbb{R}^M$ such that (normalizing the wage to 1):

$$\begin{aligned}
MC_g &= \sum_{k,k\to g} S_{gk}(\boldsymbol{p}_k, w, \boldsymbol{\varepsilon}_k) - \sum_{j,g\to j} D_{jg}(\boldsymbol{p}_j, w, \boldsymbol{\varepsilon}_j) \quad \text{if } g \in \mathcal{M} \\
MC_g &= \sum_{k,k\to g} S_{gk}(\boldsymbol{p}_k, w, \boldsymbol{\varepsilon}_k) - D_{cg}(w, p_c, \boldsymbol{\varepsilon}_c) \quad \text{if } g \in \mathcal{C} \qquad (1)
\end{aligned}$$

Throughout the paper I am going to assume that Walras' law is specified and the schedules are homogeneous of degree zero. Hence the market clearing conditions can be stated as $MC(\boldsymbol{p}, w, \boldsymbol{\varepsilon}) = \mathbf{0}$. Formally, we have the following definition.

Definition 2.1 (Pricing function and payoffs)**.** Call $\mathcal{E} = \times_{i\in\mathcal{N}}\mathcal{E}$. Define a *feasible pricing function* as a function $(\boldsymbol{p}^*, w^*) : \mathcal{E} \to \mathbb{R}^M$ such that $MC(\boldsymbol{p}^*(\boldsymbol{\varepsilon}), w^*(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}) = \mathbf{0}$ for all $\boldsymbol{\varepsilon} \in \mathcal{E}$.

The payoff of firm (player) i is the mapping from supply and demand schedules in $\mathcal{A}_i$ to real numbers defined by the profits, normalized by the wage:

$$\begin{aligned}
\pi_i(S_i, D_i, \ell_i) = \mathbb{E}_F \Bigg(&\sum_{g,i\to g} p_g^*(\boldsymbol{\varepsilon}) S_{g,i}(\boldsymbol{p}_i^*(\boldsymbol{\varepsilon}), w^*(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}_i) - \sum_{g,g\to i} p_g^*(\varepsilon) D_{ig}(\boldsymbol{p}_i^*(\boldsymbol{\varepsilon}), w^*(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}_i) \\
&- w^*(\boldsymbol{\varepsilon}) \ell_i(\boldsymbol{p}_i^*(\boldsymbol{\varepsilon}), w^*(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}_i) \Big) / w
\end{aligned}$$

In summary, we defined a game: $G = (\mathcal{N}, (\mathcal{A}_i)_{i\in\mathcal{N}}, (\pi_i)_{i\in\mathcal{N}})$. Proposition 1 below shows that the pricing function exists, so the payoffs are well defined, and moreover that the equilibrium does not depend on the normalization of prices. I call a Nash equilibrium of this game a Supply and Demand function equilibrium. Notice that the equilibrium defines a probability distribution over all the endogenous objects; prices, quantities, hence welfare. In Sections 5 and 6, concerned with the implications for market power and welfare, I am going to consider the limit of the equilibrium of the game for uncertainty that vanishes. Namely, I am going to consider profiles of prices and quantities of traded goods $\boldsymbol{p}_g \, \forall g \in \mathcal{M}$, and q_{gi}, q_{jg} for all links $i \to g$ and $g \to j$, that realize in a Nash Equilibrium of the game G_n defined identically as above, but where the sequence of distributions F_n converges to a distribution with mass in 0: $\boldsymbol{\varepsilon} \xrightarrow{\mathcal{D}} 0$ (always mantaining support $\mathcal{E}$.[21]

Generalizations: objectives of the firm While the *price* normalization is inconsequential, the *uniform* normalization of profits is. Hence, in the appendix (Section 3) Theorem 1 and 2 are proven under the more general assumption that there are two distinct types of agents, workers and owners, and *firms optimize the indirect utility of shareholders*, following Azar and Vives (2021). The construction in the main text corresponds to the case in which owners are identical and only value a good produced independently from the network, whose price is in fixed proportion with the wage. This is equivalent to the approach followed in Ederer and Pellegrino (2022), and the polar opposite of the assumption mantained in Azar and Vives (2021), in which owners have the same utility as consumers. Both are evidently abstractions: in the main text I follow the former to for two reasons: first, our focus is on the effect of endogenous market power on firm-to-firm trade, rather than the interactions of market power and owner's incentives, that are instead the focus of Azar and Vives (2021). Second, such an assumption simplifies the parametric solution of the model in Section 4, since it allows to obtain a linear equilibrium.

2.2 Assumptions

In this paragraph I collect all the assumptions needed for Theorems 1 and 2.

Assumption 1 - Demand Consumers have aggregate demand D_c that has negative semidefinite jacobian with corank 1 with respect to both prices $\boldsymbol{p}_c, w$. It is positive definite with respect to stochastic parameters $\boldsymbol{\varepsilon}_c$; moreover all demands are positive and differentiable.

Assumption 2 - Technology The transformation function Φ_i is differentiable, convex and increasing in the quantities $\boldsymbol{q}_i = (\boldsymbol{q}^{out}, -\boldsymbol{q}^{in})$: $\nabla\Phi_i >> 0$. It satisfies the Inada condition that $\lim_{q_j \to 0} \partial_{q_j}\Phi_i = +\infty$. The joint support of the distribution F of all stochastic parameters $\boldsymbol{\varepsilon} = ((\boldsymbol{\varepsilon}_i)_{i\in\mathcal{N}}, \boldsymbol{\varepsilon}_c)$, call it $\mathcal{E}$, is the closure of an open set, bounded in norm by K_e (hence compact), and the distribution admits a differentiable density f.

[21] So the stochastic variation in $\boldsymbol{\varepsilon}$ is used to "identify" the equilibrium schedules, but when computing the equilibrium predictions I am considering the case in which the shock vanishes.

Assumption 3 - Feasible schedules Define $\mathcal{A}_i$ as the set of schedules such that:

a) *Homogeneity* each schedule $\mathcal{S}_i$ is homogeneous of degree 0 in $\boldsymbol{p}_i, w$;

b) *Feasibility* each schedule $\mathcal{S}_i$ satisfies the **technology constraint**, that is, for any possible $(\boldsymbol{p}_i, w, \boldsymbol{\varepsilon}_i)$, it must be:

$$\Phi_i(\mathcal{S}_i(\boldsymbol{p}_i, w, \boldsymbol{\varepsilon}_i), \boldsymbol{\varepsilon}_i) \leq 0 \tag{2}$$

c) *Regularity* the schedules $\mathcal{S}_i$ are infinitely differentiable and have Jacobian derivative with respect to prices $\partial_{\boldsymbol{p}_i,w}\mathcal{S}_i$ that is positive semidefinite with rank $d_i - 1$[22]; the derivative with respect to stochastic parameters $\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}_i$ is positive definite.

d) *Bounds* The feasible schedules are bounded in the following norm: $\|\mathcal{S}_i\|_g = \mathbb{E}\int |\mathcal{S}_i(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i)| g(\boldsymbol{p}) \mathrm{d}\boldsymbol{p}$, where g is a positive integrable function whose integral is 1. Namely, there is a K_S such that $\|\mathcal{S}_i\|_g \leq K_S$. There exist constants k and K such that for all $\boldsymbol{p}, w, \boldsymbol{\varepsilon}$ $\|\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}_i\|_2 \leq K$ and $\partial_{\boldsymbol{p}_i,w}\mathcal{S}_i \geq kI_i$, where $\|\cdot\|_2$ is the spectral matrix norm, and I_i is the identity matrix of appropriate dimension.

e) *Limits* If $\boldsymbol{p}/w \to \infty$, or $p_g/w \to 0$ for some g, there is at least one i, g such that $\mathcal{S}_{i,g} \to \infty$.

Denote $\mathcal{A} = \prod_i \mathcal{A}_i$.

Most of these assumptions are technical in nature: in particular, the boundedness and regularity assumptions are crucial in establishing compactness of the feasible set. Assumption e) says that the schedules are such that for extreme values of prices, at least on demand or supply diverges: this is used to show existence of a positive market clearing price vector. As part of the proof of Theorem 1 I am going to show that there is a bounded set of prices, bounded away from zero, where we can focus without loss of generality: the sup norm in the definition above is to be considered in such a bounded set.

For a given vector of parameters $\boldsymbol{\varepsilon}_i$, the assumptions on the transformation function are quite standard: if the firm has a single output y produced with a strictly concave increasing production function f_i (for example a CES with decreasing returns to scale), then $\Phi_i(y, q_1, \ldots, q_n) = y - f_i(-q_1, \ldots, -q_n, \boldsymbol{\varepsilon}_i)$ (remembering that negative quantities represent inputs) is indeed convex and increasing

[22] The rank cannot be maximum because of homogeneity in prices.

in the q variables. The assumptions on stochastic parameters guarantee that they represent productivity parameters, each of which has an independent effect.

Example 1. In the single output case, consider a strictly concave production function $\hat{f}_i$, and the schedule adopted by a price taking profit maximizing firm $\hat{S}_i$. Now consider the transformation function $\Phi_i(y, q_1, \ldots, q_n, \boldsymbol{\varepsilon}_i) = y - \varepsilon_i - \hat{f}_i(-\varepsilon_{i1} - q_1, \ldots, -\varepsilon_{in} - q_n)$. In this context an example of a schedule satisfying the assumptions above is:

$$D_{ig} = -\varepsilon_{ig} + \hat{D}_{ig}\,\forall g \to i, \quad S_i = \varepsilon_i + \hat{S}_i$$

It is immediate to verify that it satisfies the technology constraint. The derivative $\partial_{\boldsymbol{p}_i}\mathcal{S}_i$ is the same as $\partial_{\boldsymbol{p}_i}\hat{\mathcal{S}}_i$, and is a standard property of production theory that this is positive semidefinite. If the production function satisfies the Inada conditions, the limits in Assumption 3e) are also satisfied. The bounds are satisfied in every set bounded and bounded away from zero. Moreover, $\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}_i$ is the identity, hence positive definite, and satisfies the bounds.[23] In the multiple output case, the reasoning is analogous.

The regularity and boundedness assumptions 3c) $-$ d) guarantee that the demand and schedules are well behaved, enough to solve the market clearing system. The various boundedness assumptions are useful for various technical steps, and ultimately to guarantee compactness of the set of schedules, that is necessary to use the Schauder fixed point theorem in Theorem 1.

Example 2. Standard Supply Function Equilibrium

The model by Klemperer and Meyer (1989) can be seen as a special case of this setting, in which there is only one sector, the network $\mathcal{G}$ is empty, the only uncertainty is on the consumers, and the labor market is competitive. Their setting is a "partial" equilibrium one, in which the consumers do not supply labor to firms but appear only through a demand function $D(\cdot)$, and firms have a cost function for production $C(\cdot)$, that does not explicitly represent payments to anyone. Nonetheless, under the simplifying assumption of a competitive labor market (introduced later in Section 4), the game played by the firms is precisely the same: if the transformation function is $\Phi(q_i, -\ell_i) = q_i - C(\ell_i)$, and the consumer utility gives rise to a demand of the form $D_c + \varepsilon_c$, the game G played by firms is precisely the same as in Klemperer and Meyer (1989).

[23] For an example in which $\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}_i$ is positive definite but not diagonal we can consider the transformation function $\Phi_i(y, q_1, \ldots, q_n, \boldsymbol{\varepsilon}_i) = y - \frac{\varepsilon_i}{\varepsilon_i + \sum_g \varepsilon_{ig} + 1} - f_i(-\frac{\varepsilon_{i1}}{\varepsilon_i + \sum_g \varepsilon_{ig} + 1} - q_1, \ldots, -\frac{\varepsilon_{in}}{\varepsilon_i + \sum_g \varepsilon_{ig} + 1} - q_n)$.

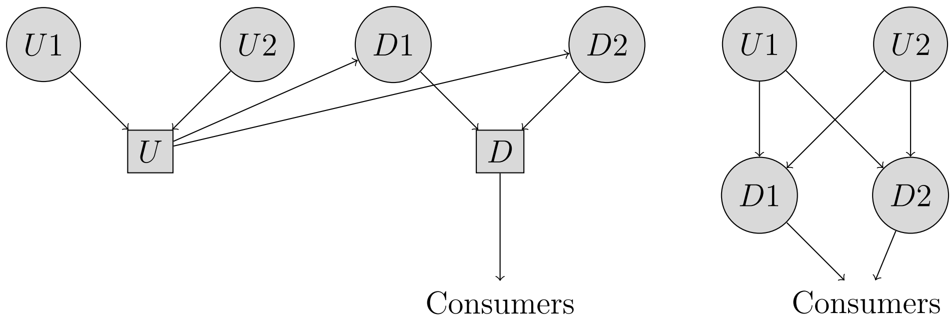

Figure 2: A layered supply chain. Left: bipartite representation, the squares represent goods, the circles firms. Right: firm-only representation.

Example 3 (Regular layered supply chain)**.** A regular layered supply chain is a production structure in which firms are divided in m *layers*, as in Figure 2. There are m goods, each produced by all the firms in a layer; there are n firms per layer. Firms in layer $i+1$ sell to firms in layer i, firm 0 sells its output to the consumer, and firms in layer m are the only ones to use labor.

3 Existence

In this section I present Theorems 1, 2 and Corollary 3.1. First, I prove as a preliminary result that a pricing function exists and is unique, hence the payoffs above are well-defined (Proposition 1 below), and moreover the set of feasible prices is bounded, that is going to be important for the argument of Theorem 1.

Proposition 1 (Feasible pricing and price normalization)**.** *1. There exist a feasible pricing function* $(\boldsymbol{p}, w) : \mathcal{E} \times \mathcal{A} \to \mathbb{R}^M_+$*, and is unique up to normalization. Moreover, the payoffs are independent of price normalization.*

2. Normalizing the wage to 1, the image of the pricing function $\mathcal{P} = \boldsymbol{p}(\mathcal{E} \times \mathcal{A})$ *is bounded, that is there is a* $k_p > 0$ *such that for any* $\boldsymbol{p} \in \mathcal{P}$ $\|\boldsymbol{p}\|_2 < k_p$*.*

The proof relies on the regularity assumptions 3c) and the limits in 3e) to show that the pricing function exists thanks to a global form of the implicit function theorem. The uniqueness up to normalization follows from homogeneity of the schedules, that translates into homogeneity of the excess supply MC. The second part follows from the bounds in Assumption 3d) and an application of the mean value theorem.

Thanks to the normalization by the wage, the profits depend only on price ratios, and so the game does not depend on the specific price normalization. For this reason, from now on, I am going to focus on homogenized schedules obtained normalizing the wage to 1, writing, with a slight abuse of notation, $\mathcal{S}_i(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i)$ for $\mathcal{S}_i(\boldsymbol{p}_i, 1, \boldsymbol{\varepsilon}_i)$. Moreover, since the technology constraint is binding, from now on we focus on $\mathcal{S}_{i,-\ell} = (S_i, -D_i)$, that is the profile of schedules for input and output goods *excluding* labor. Because of the above assumptions $\partial_{\boldsymbol{p}_i}\mathcal{S}_i$ is positive definite.

3.1 Existence

The main argument is a fixed point theorem. The main obstacle is establishing compactness of the set of feasible schedules. In order to do this it is crucial first to limit the domains of the schedules to a compact set. In general, for a compact domain D, define $\mathcal{A}(D)$ as the set of schedules in $\mathcal{A}$ that are restricted to $D \times \mathcal{E}$. To be precise, $\mathcal{S}_i$ is restricted to the projection of D on the space of input and output prices of i, call it D_i. Second, it is necessary to consider the *closure* of $\mathcal{A}(D)$, denoted $\overline{\mathcal{A}}_i$, with respect to the $\|\cdot\|_\infty$-norm on the set of schedules: $\|\mathcal{S}_i\|_\infty = \max_{D\times\mathcal{E}} |\mathcal{S}_i(\boldsymbol{p}_i, w, \boldsymbol{\varepsilon}_i)|$, which is well defined thanks to the compactness of $D\times\mathcal{E}$. Lemma A.2 in the Appendix shows that the pricing function is Lipschitz, and so can be extended without problems to $\overline{\mathcal{A}(D)}$. To obtain compactness, thanks to the Ascoli-Arzelà theorem, the last piece we need is to choose an upper bound K to the norm of the price derivatives $\|\partial_{\boldsymbol{p}_i}\mathcal{S}_i\|_2 < K$:[24] denote $\overline{\mathcal{A}(D)}^K$ the set of schedules that satisfie this bound. The formal statement of the theorem is as follows.

Theorem 1. *If the best reply correspondences are convex-valued, there exists a compact domain $\tilde{\mathcal{P}} \subseteq \overline{\mathcal{P}}$ such that the game G has a pure strategy Nash equilibrium in $\overline{\mathcal{A}(\tilde{\mathcal{P}})}^K$.*

Furthermore, all prices in $\tilde{\mathcal{P}}$ can arise for some value of $\boldsymbol{\varepsilon}$, and $\tilde{\mathcal{P}}$ is the closure of an open set (in particular, it has positive measure).

The second part of the statement guarantees that, thanks to our assumptions on stochastic parameters, the equilibrium spans a set of prices that is "large" enough, in particular in which derivatives are meaningful.

The proof of the first part applies the Ky Fan fixed point theorem to $\overline{\mathcal{A}(\tilde{P})}^K$. Via a standard argument the differentiability and boundedness assumptions on the schedules in $\mathcal{A}(\tilde{P})^K$ are enough to guarantee equicontinuity, and applying the Ascoli-Arzelà theorem we obtain that the closure is compact. Assumption 2 on

[24]This assumption cannot be included directly in the definition of $\mathcal{A}$ because it can be incompatible with 3e).

the technology is also sufficient to show that $\overline{\mathcal{A}(\tilde{\mathcal{P}})}^K$ is convex. Hence, if the best reply is convex-valued, there exist a fixed point by Ky Fan's fixed point theorem. For the second part, thanks to the assumptions of positive definiteness of $\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}_i$ we can show that the pricing function is locally (right-)invertible, and this allows to conclude that the set of feasible prices is the closure of an open set.

3.2 Necessary conditions for equilibrium

In this section I derive necessary conditions for best replies and describe the insights that emerge on the structure of the equilibrium.

The necessary conditions are best expressed in terms of the *residual schedule*, the schedule that collects the residual demands and supplies that the firm faces on all its input-output connections. It can be formally constructed as follows.

Definition 3.1 (Conditional pricing function and residual schedule)**.** Given a profile of schedules $(\mathcal{S}_i)_{i\in\mathcal{N}}$, the *pricing function conditional on* i is the function $\boldsymbol{p}_{-i}(\cdot \mid i)$, defined on $\boldsymbol{p}_i, w, \boldsymbol{\varepsilon}$ that satisfies the market clearing conditions 1, excluding those relative to the input and output prices of i.:

$$\boldsymbol{MC}_g(\boldsymbol{p}_{-i}(\boldsymbol{p}_i, w, \boldsymbol{\varepsilon} \mid i), \boldsymbol{p}_i, \boldsymbol{\varepsilon}) = 0 \quad \forall g \notin \mathcal{N}_i$$

The *residual schedule* of firm i is:

$$\mathcal{S}^r(\boldsymbol{p}_i, w, \boldsymbol{\varepsilon}) = -\sum_{j\neq i} \mathcal{S}_j(\boldsymbol{p}_j(\boldsymbol{p}_i, w, \boldsymbol{\varepsilon} \mid i), \boldsymbol{\varepsilon}_j)$$

The next lemma sums up some properties of the residual schedules that are going to be useful.

Lemma 3.1. Under Assumptions 1,2 and 3, the residual schedule is homogeneous of degree zero in $\boldsymbol{p}_i, w$, differentiable, has positive semidefinite derivative $\partial_{\boldsymbol{p}_i,w}\mathcal{S}_i^r$ of corank 1 (i.e. has maximum rank minus 1).

Theorem 2. *Remember that $\mathcal{S}_{i,-\ell}$ denotes the schedule played by firm i excluding labor demand (and similarly for $\mathcal{S}^r_{i,-\ell}$). Assume a schedule profile $\mathcal{S}_i \in \mathcal{A}$ is twice differentiable, the spectral norm $\|\|_2$ of the schedules is differentiable, and the boundary of $\tilde{\mathcal{P}}$ is differentiable. $\mathcal{S}_i$ is a best reply to the profile $\mathcal{S}_{-i}$ only if satisfies the following partial differential equation for all $(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i) \in \mathcal{P} \times \mathcal{E}$:*

$$\mathbb{E}\left[([\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}] + [\partial_{\boldsymbol{p}_i}\mathcal{S}^r_{i,-\ell}])^{-1}\left(-\mathcal{S}_{i,-\ell} + \partial_{\boldsymbol{p}_i,w}\mathcal{S}^r_i((\boldsymbol{p}_i, 1)' - \lambda\nabla\Phi_i)\right) + \mathcal{K}_i \mid \boldsymbol{p}_i, \boldsymbol{\varepsilon}_i\right] = 0 \tag{3}$$

and the technology constraint: $\Phi_i(\mathcal{S}_i, \boldsymbol{\varepsilon}_i) = 0$.

$\mathcal{K}_i$ *is a term, whose expression can be found in the Appendix, that collects all the terms involving the boundedness and positive definiteness constraints; it is equal to 0 if and only if none is binding.*

The first order condition can be understood as follows. The term $-\mathcal{S}_{i,-\ell} + \partial_{\boldsymbol{p}_i}\mathcal{S}_i^r((\boldsymbol{p}_i, 1)' - \lambda\nabla\Phi_i)$ represents the sensitivity of the profit to a variation in the prices. In this context the "marginal cost" of producing an additional unit of output is an ill-suited concept: indeed, the standard marginal cost is intimately connected with the assumption of taking input prices as given, being the multiplier in the standard cost minimization problem. In our setting, where firms have some market power on all input and output markets, the relevant generalization is the marginal value of relaxing the technology constraint, which is exactly the multiplier λ_i, times $\nabla\Phi_i$, that represents the marginal product of each input/output. Hence the vector $(\boldsymbol{p}_i', 1)' - \lambda_i\nabla\Phi_i$ can be thought as the vector of markups (for outputs) and markdowns (for inputs). The reason why the schedule without labor demand $\mathcal{S}_{i,-\ell}$ appears in the expression is because we normalized the wage to 1: this is inconsequential, as we showed that price normalization does not affect the payoffs nor the schedules. Then, we can see that this term of the FOC has a very similar intuition to the standard Lerner equation: the higher the responsiveness of demand/supply to prices, the smaller the markups/markdowns that can be charged.

The term $([\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}] + [\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}^r])^{-1}$ represents the sensitivity of the prices to a variation in the schedules. Again, the schedules without labor demand appear because of the normalization of the wage. The key difference from Klemperer and Meyer (1989) is the presence of the expectation in the expression. The reason is somewhat different from Holmberg and Philpott (2018) and Wilson (2008), in which the equilibrium is not ex-post because of the possibility of binding transmission capacities in an otherwise linear transmission network. To understand why this is the case, consider Figure 3. A seller faces a residual demand of the form $\varepsilon_D D_U^r + \varepsilon_c$, where ε_D and ε_c are two distinct sources of uncertainty. Computing first the optimal prices for given ε_D, and varying ε_c, we find the red curve (Left panel). This is what happens computing the best reply in a standard supply function competition. But now note that ε_D changes the slope of the residual demand, so is also affecting the optimal price, and in such a way that *the optimal price realizes a different demand quantity*. Hence if we represent on the same graph (Right panel) the optimal price quantity pairs varying ε_D, they do not lie on the red line, they form another curve. Hence, no single supply function can touch

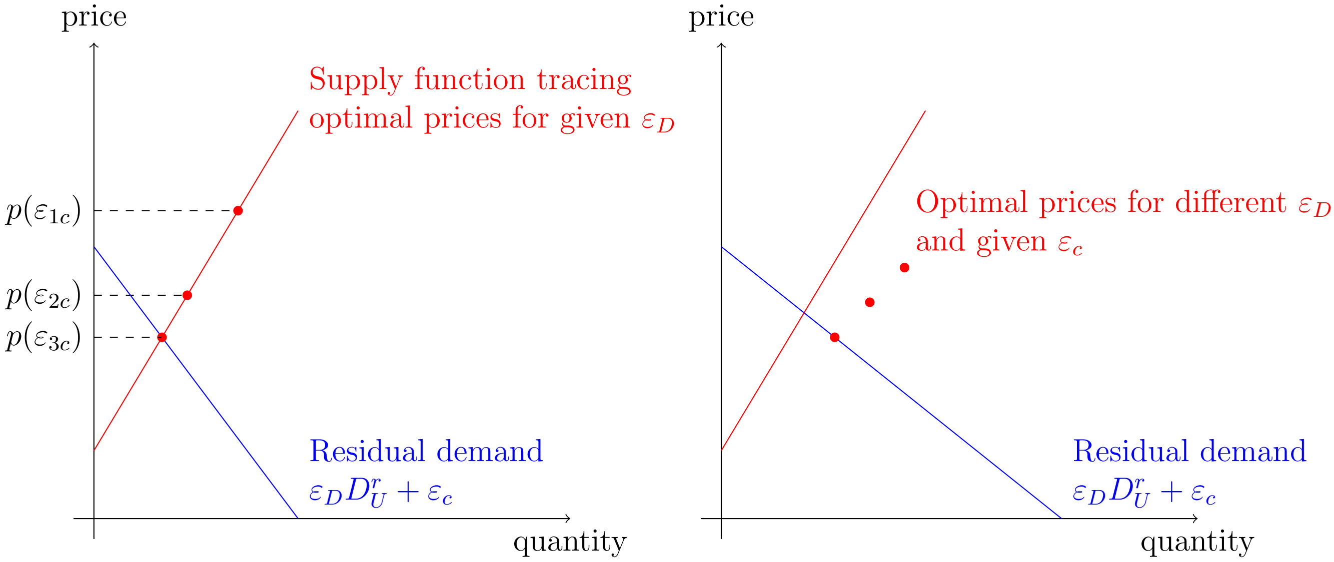

Figure 3: A supply function is not equivalent to ex-post price setting when uncertainty has enough dimensions.

all the ex-post optimal points, but has to trade-off between them, depending on the relative probability. This is the reason why the expectation appears in the necessary conditions. Moreover, in general the optimization is also not *pointwise*, in the sense described by Rostek and Yoon (2021a). Namely, the optimization in schedules is not equivalent to a pointwise optimization in quantities traded, taking the price impacts as given. It would be only in case the price sensitivity term $([\partial_{\boldsymbol{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\boldsymbol{p}_i} \mathcal{S}^r_{i,-\ell}])^{-1}$ drops from equation 3, which happens only when is measurable with respect to $\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i$, as discussed in the following section. In general such a sensitivity might depend on the realization of the residual uncertain parameters in a way that correlates with the slope of the residual demand, modifying the marginal impact of changing the schedule, and hence the optimal choice.

The proof proceeds computing the Gateaux derivative along a direction, then imposing that all Gateaux derivatives are zero: since this is true for any direction η_i, this allows to conclude that the expression in the Theorem is zero. The term $\mathcal{K}_i$ is obtained differentiating the constraints, and applying integration by parts to transform the terms with the derivatives of η_i in terms depending only on η. The additional regularity conditions, such as second order differentiability, and differentiability of the spectral norm, are necessary to deal with these constraints, since the constraints involve derivatives of the schedules. The differentiability of the boundary of the feasible price set $\tilde{\mathcal{P}}$ is necessary to apply the divergence theorem, that is key step in obtaining the expression above.

3.3 Unique best reply

In case the degrees of freedom of the firms are exactly the same as the uncertain parameters they face we can prove that best replies are single-valued. In this case the equilibrium is *ex-post*, and the partial differential equation 3 boils down to an implicit equation. The key assumption needed for this is the following:

Assumption 4-Measurability for each firm i, there exist a function f_i such that the residual demand is measurable with respect to $(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i)$, that is it satisfies $\mathcal{S}_i^r(\boldsymbol{p}_i, \boldsymbol{\varepsilon}) = f_i(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i)$.

The immediate consequence of this assumption is that the residual schedule is completely known once we know $\boldsymbol{p}_i$ and $\boldsymbol{\varepsilon}_i$, hence there is no residual uncertainty and hence the expectation in 3 is trivial. So, for an interior solution for which the positive definite constraints are not binding, the FOC reduces to:

$$\left(-\mathcal{S}_{i,-\ell} + \partial_{\boldsymbol{p}_i}\mathcal{S}_i^r(\boldsymbol{p}_i' - \lambda\nabla\Phi_i(-\mathcal{S}_i^r, \boldsymbol{\varepsilon}_i))\right)\left([\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}] + [\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}^r]\right)^{-1} = 0$$

where now the term $\frac{1}{P_i}([\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}] + [\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}^r])^{-1}$ simplifies away, and we are left with:

$$\mathcal{S}_{i,-\ell} = \partial_{\boldsymbol{p}_i}\mathcal{S}_i^r((\boldsymbol{p}_i, 1) - \lambda_i\nabla\Phi_i(-\mathcal{S}_i^r, \boldsymbol{\varepsilon}_i)) \tag{4}$$

This is an equation that directly defines the best reply schedule $\mathcal{S}_{i,-\ell}$ as a function of prices and schedules played by competitors. Hence it is immediate to conclude that in this context the best reply is unique. Moreover we recover both the point-wise optimization and the ex-post equilibrium as in Klemperer and Meyer (1989). We can summarize the above discussion as follows.

Corollary 3.1. Under Assumptions 1, 2, 3 and 4, if the constraints are not binding ($\mathcal{I}_i = \mathcal{J}_i = 0$), the best reply is single valued in the interior of $\mathcal{A}(\tilde{\mathcal{P}})$.

The measurability assumption is not vacuous. An example that satisfies it for any network is when the profile of schedules played is linear, case to which are devoted the Sections from 4 on. In this case the function f_i is actually a *constant*, independent of $\boldsymbol{\varepsilon}$ and $\boldsymbol{p}_i$.[25] The following is another example, where it is not the functional form, but the structure of the network that determines the measurability.

Example 4 (Regular layered supply chain)**.** In the context of a regular layered supply chain each firm has 1 degree of freedom, because it has to decide a schedule

[25] In the notation of Section 4, it is the inverse of the price impact matrix, Λ_i^{-1}.

for inputs and outputs, constrained by the technology. Hence, it is sufficient one stochastic parameter to generate enough variation in the realized prices to span the whole feasible set. Assume that the only stochastic parameter is the one of consumer demand ε_c, while the trasformation functions of firms, and the schedules, are all deterministic.[26] In this case the measurability assumption is satisfied, because, under the assumptions above, realizations of the stochastic parameter ε_c are one to one with price variation, for any firm. Details are in Online Appendix D.6.

4 Parametric version

In this Section I introduce a specific parametric functional form for the technology that allows to have an equilibrium in linear schedules.[27] In order for the game to have this type of equilibrium, another simplifying assumption is needed, the assumption that *labor markets are competitive*. As clarified below, this means that the quantity of labor bought by firms is not determined via a double auction in which firms internalize the effect on the labor supply via the wage, but it is chosen taking the wage as given.[28]

4.1 Technology

As in other contexts with supply and demand functions, the most tractable case is when the equilibrium schedules are linear: linearity allows to have fixed point equations in coefficient matrices, and to reduce the problem to a finite dimensional one. For the best reply to be linear, we need the best reply optimization problem to be quadratic in prices (or in quantities), but the technology constraints to remain *linear*. Standard functional forms as the translog or the CES do not satisfy this requirement, and lead to complex nonlinear differential equations for the determination of equilibrium schedules.

[26]Or, equivalently, the distribution of $\boldsymbol{\varepsilon}_i$ is a Dirac for all i.

[27]Naturally, no linear function is homogeneous of degree zero. What is meant here is that schedules are linear when choosing a normalization: in the present case, we are going to normalize the wage w to 1. We already proved that the equilibrium is independent of the price normalization chosen, hence this normalization is inconsequential. Nevertheless, it is going to greatly simplify our task analytically.

[28]Competitiveness of the labor market is an abstraction, as we know that monopsony on the labor market is a relevant phenomenon (see e.g. the discussion in Azar and Vives (2021)). However, our main focus is on modeling market power in firm-to-firm trade. Moreover, one of the main takeaways of this paper is that full blown strategic interaction across the supply chain increase the aggregate impact of market power (Theorem 5): in this regard, neglecting monopsony power in the labor market is a conservative assumption.

In the main text, I introduce the technology in the simplest way, introducing what I define "handling costs" below. In Online Appendix E.4, I show that such handling costs can be microfounded with a standard constant returns technology nesting perfect substitutes and perfect complements.

I first introduce a useful concept to deal with technologies with multiple outputs.

Definition 4.1. Given a profile of input quantities $\boldsymbol{q}_i^{in}$), a *production allocation* $\boldsymbol{z}_i = ((z_{i,kj})_{j\to i, i\to k}) \in \mathbb{R}^{d_i^{in} d_i^{out}}$ is an allocation of input quantities to output production lines. That is, it has to satisfy the resource constraint: $q_{ij} = \sum_k z_{i,kj}$ for all $j \in \mathcal{N}_i^{in}$.

Given a production allocation, the vector of outputs produced is $q_{ki} = \sum_j \omega_{ij} z_{i,kj}$.

We sum up these relationship in matrix form as: $\boldsymbol{q}_i = U_i \boldsymbol{z}_i$, where $qv_i = (-(q_{ki})_k, (q_{ij})_j)$, and and U_i vertically stacks $I_{out,in} \otimes \omega_i$, and $-I_{in,out} \otimes u_{in}$. The idea is that with multiple outputs, intermediate input quantities q_{ij} have to be allocated to the production of one among the output goods: $z_{i,kj}$ is the amount of input j allocated to the production of the output k.

Definition 4.2 (Handling costs)**.** In addition to raw input payments, choosing the production allocation $\boldsymbol{z}_i$ has "handling" costs, paid in labor units, and quadratic in the quantities chosen:

$$\boldsymbol{\varepsilon}_i' \boldsymbol{q}_i + \frac{1}{2k_i} \boldsymbol{z}_i' \Sigma_i \boldsymbol{z}_i + \frac{1}{2} \sum_k \ell_{i,k}^2 \tag{5}$$

for some positive definite matrix Σ_i.

The matrix Σ_i codifies the (purely technological) patterns of substitutability and complementarity: $\Sigma_{i,jk} > 0$ means that, ceteris paribus, buying both inputs j and k has a higher cost than only using one of the two: this captures substitutability; on the contrary, $\Sigma_{i,jk} < 0$ captures complementarity. When $\Sigma_i = I$ inputs are nor substitutes nor complements.

Example 5. Consider a case with a single output. If $\Sigma_i = (1 - \sigma_i) I + \sigma J$, where J is a matrix of ones, the profit becomes:

$$\pi_i = p_i \sum \omega_{ij} q_{ij} - \sum_j p_j q_{ij} - \varepsilon_i \sum_j q_{ij} - \frac{1}{2k_i} \sum q_{ij}^2 - \frac{\sigma_i}{2k_i} \sum_{j,k} q_{ij} q_{ik}$$

If σ_i is close to 1, the quadratic term becomes close to $(\sum q_{ij})^2$, that is only the aggregate quantity matters: inputs are good substitutes. If σ_i is negative, inputs instead are complements.

If there is just one sector as in 2, the game is the same as the linear-quadratic example of Klemperer and Meyer (1989).

The "handling costs" can be interpreted as all the costs connected with storage, transportation, inventory, and general management related tasks that have to be performed in order to use that input in production. This is the same assumption followed in Bimpikis et al. (2019), and can be seen as the extension to an input-output setting of the standard quadratic cost function commonly used. Indeed, if a sector uses no intermediate inputs but only labor, the costs reduce to $\frac{1}{2}l_i^2$, and $q_i = l_i$, so that in this case the functional form reduces to a standard technology with quadratic cost function, used for example by Pellegrino (2019), Klemperer and Meyer (1989) and many others.

The parameters $\boldsymbol{\varepsilon}_i = ((\varepsilon_{ig})_{i\to g}, (\varepsilon_{ig})_{g\to i})$ act reducing the productivity of labor, and so increasing the amount of labor necessary to achieve the same level of production. They are stochastic, and follow a joint distribution F with also the corresponding consumer parameters introduced below. Each firm knows only its own vector $\boldsymbol{\varepsilon}_i$, but not the one of others.[29] As common in supply and demand function models, the uncertainty will be crucial in achieving a unique best response.

Consumers Similarly, the utility function of the consumers is quadratic in consumption and (quasi-)linear in disutility of labor L:

$$U(C, L) = (\boldsymbol{A}_c + \varepsilon_c)' B_c^{-1} C - \frac{1}{2} C' B_c^{-1} C - L$$

where remember $C = (C_g)_{g \in \mathcal{C}}$ is the vector of quantities consumed, and we assume B_c positive definite. This means that the consumer has demands of the form (on the support): $D_c = \boldsymbol{A} + \boldsymbol{\varepsilon}_c - B_c \frac{\boldsymbol{p}_c}{w}$. The parameter vector $\boldsymbol{\varepsilon}_c = (\varepsilon_{gc})_{g\to c}$ and $\boldsymbol{\varepsilon}$ follow a joint distribution F. The only assumption on F we make is that its support contains an open set around 0.

4.2 Existence

The parametric version is formally not a special case of the previous setting, because the transformation function that generates it is not differentiable (see

[29] Or, equivalently, the uncertainty is realized after the choice of supply and demand functions.

Online Appendix E.4), and the assumption of competitive labor market. Hence, a separate existence result is required. In exchange, it is not necessary to assume ex-ante a bound on the norms of the derivatives, as in Theorem 1.[30] Moreover, since the parametric functional form is meant to facilitate quantitative analysis, in this section I provide a result in which traded quantities are positive, that is trade happens in the direction specified by the network. For that is useful to introduce a sector level version of the model, as follows.

Sector level version Define a *sector* as a set of identical firms: all the firms with the same technology and with identical input-output connections, call their number n_i. They all produce the same goods and sell them to the same other firms. For an example, consider the layered supply chain in Example 3. This allows to simplify the analysis analyzing the *sector-level* network, solving for only one coefficient matrix per sector. This simplification is useful in a number of ways. First, it allows to show that there is an equilibrium with positive trade, as explained below. It is also useful to generate insights, as in the analysis of the layered supply chain in the next section. Moreover, since we only need to solve for one coefficient matrix B_i per sector, hence the number of equations one needs to solve is (typically) much smaller, and this can be important for numerical work.

Given the above technology, and normalizing the wage to 1, the best reply problem of firm i is:

$$\max_{(z_{i,kj})_{k,j},p_i,\ell_i} \sum_k p_{ki} D^r_{ki}(p_i,\varepsilon) - \sum_j p_{ij} S^r_{ij}(p_i,\varepsilon) - \varepsilon_i \sum z_{i,kj} - \frac{1}{2}\sum z^2_{i,kj} - \ell_i \quad (6)$$

subject to: $D_{ij} = \sum_k z_{i,kj}$ for all $j \in \mathcal{N}_i^{in}$ and $S_{ki} = \sum_j \omega_{ij} z_{i,kj} + \alpha_i \sqrt{\ell_i}$ for all $k \in \mathcal{N}_i^{out}$. The choice is both over the schedules, the optimal combination of inputs and outputs, and the quantity of labor (because in this section labor markets are assumed competitive).

The residual schedule arising from a profile of linear schedules is still linear, and has the form:

$$\mathcal{S}^r_i = -\tilde{\boldsymbol{A}}_i - \Lambda_i^{-1} \boldsymbol{p}^a_i + \Lambda_{\boldsymbol{\varepsilon},i} \boldsymbol{\varepsilon}$$

for a coefficient matrix Λ_i that we label the *price impact* of i. The expression arises from the partial solution of the market clearing equations, that here are a linear system $\boldsymbol{Mp} = \boldsymbol{A} + \boldsymbol{\varepsilon_c}$, where the matrix M is build from the profile of

[30] In Appendix E.5 I show that the parametric model of Section 4 has an equilibrium with nonnegative trade, namely the direction of trade is the one specified rather than being endogenous.

coefficients $(B_i)_i$. A formal proof of this fact is in Online Appendix E. Because of linearity, the Measurability assumption is satisfied. Hence, the best response can be written as a finite dimensional optimization in the following form. [31] Define the *perfect competition matrix* for sector i as $C_i = U_i \Sigma_i^{-1} U_i' + \alpha_i^2$, where α_i is a diagonal matrix with on the diagonal $\alpha_{i,k}$, and U_i vertically stacks $I_{out,in} \otimes \omega_i$, and $-I_{in,out} \otimes u_{in}$. A calculation shows that this is the matrix of demands and supplies chosen by a firm that takes prices as given.[32]

The following theorem states conditions for existence, and a characterization of the equilibrium.

Theorem 3. *Assume that at least one of the α_i, with i connected to the final consumer, is positive.*[33]

1. *If each good is traded by at least 3 firms, there are sets $\mathcal{E}_i$, $\tilde{\mathcal{P}}_i$ and matrices $\tilde{B}_i$, B_i^{ε} such that $\mathcal{S}_i = \tilde{B}_i \boldsymbol{p}_i + B_i^{\varepsilon} \boldsymbol{\varepsilon}_i$ are a Supply and Demand function equilibrium;*

2. *The equilibrium coefficient matrices satisfy: that satisfies the following fixed-point equation:*

$$B_i = \Lambda_i^{-1} - \Lambda_i^{-1} \left(C_i + \Lambda_i^{-1} \right)^{-1} \Lambda_i^{-1} \tag{7}$$

3. *in the sector level model where $n_i \geq 2$ for all i, there is a subset of links E_0 such that a Supply and Demand function equilibrium exists in the subnetwork defined by E_0 and all traded quantities are positive.*[34]

The condition that each good is traded by at least three firms is common to linear equilibria in supply or demand functions (e.g. Malamud and Rostek (2017), Woodward (2021)).

[31] Notice that this does not follow directly from Theorem 3.1 for the usual reason that here the technology is not differentiable. For an extended proof see the Appendix.

[32] See Appendix E.1.

[33] If all the α_i are 0, a trivial equilibrium in which every supply and demand function are constantly 0, and so no unilateral deviation yields any profit because there would not be trade anyway, is always present. To avoid this, it is sufficient to assume that at least one of the α_i, with i connected to the final consumer, is positive. The theorem states that even with all $\alpha_i = 0$ a non-trivial equilibrium will also exist.

[34] In some specific networks it is possible to prove that there are no corner solutions in equilibrium, that is no links need to be cancelled. See Appendix .

The expression for the best reply highlights the role of the price impact. If $\alpha_i > 0$, that implies C_i is invertible, the equation simplifies to:

$$B_i = \left(C_i^{-1} + \Lambda_i\right)^{-1}$$

The proof builds from Theorem 2 in Malamud and Rostek (2017), showing that there exist a profile of matrices satisfying equation 7. The important addition is to show that there exist an equilibrium where trade is positive. In such a case we allow the possibility of *inactive links*, that is links over which the specific schedules are identically 0. This is easier to analyze in the sector level model because in such a case each group of firm has exactly one schedule involved in each link. As a result, if all other firms do not trade on a given link, there is no unilateral deviation that can generate trade. Hence, restricting the analysis to a subset of links does not affect equilibrium reasoning: this allows to cancel links where the unconstrained equilibrium calculation would yield negative trade. The proof shows that the recursive elimination always ends, and so an equilibrium exists. Details are in the Appendix.

5 Relative market power

In this section I use the parametric functional form just introduced to discuss the implications of the supply and demand function equilibrium for the assessment of relative market power among firms. For example, consider a competition authority that wants to understand in which sector of the production network market power is stronger, and where the antitrust efforts should be focused. I show that the fact that in this model firms have market power on both input and output markets (in an endogenous way) can fundamentally change the ranking of market power with respect to a more standard model in which market power is only on outputs.

5.1 Markups and markdowns

The first order conditions for the best reply problem give us the following expression:

$$\mathcal{S}_{i,-\ell} = \Lambda_i^{-1}(\boldsymbol{p}_i - \boldsymbol{\lambda}_i) \tag{8}$$

where $\boldsymbol{\lambda}_i$ is the vector of the multipliers relative to each input and output constraint in 6. We can see that the equation is very similar to equation 3, where the positive definiteness constraints and the expectation are not needed. The vector

$\boldsymbol{\lambda}_i$ represents the marginal values of each input/outputs and, as discussed in the context of the general necessary conditions, $\boldsymbol{\mu}_i = \boldsymbol{p}_i - \boldsymbol{\lambda}_i$ represents a measure of market power in each input and output market.

In equilibrium, they are equal to:

$$\boldsymbol{\mu}_i = \begin{pmatrix} \mu_i^{out} \\ -\mu_i^{in} \end{pmatrix} = \Lambda_i \boldsymbol{q}_i^* = (B_i^{-1} - C_i^{-1}) q v_i^*$$

The expression above suggests that, for given quantities produced, the markups are intuitively "decreasing" in B_i: the steeper the schedules, the smaller the ability of firms to affect prices, the smaller the markups. The expression above pins down the exact relationship, taking into account all the cross-price effects. In the case of the layered supply chain, where B_i is a number, this intuition can be made precise.

5.2 The layered supply chain

The *layered supply chain* introduced in Example 3, allows sharper characterizations.[35] Since all firms in each layer are identical, this is an instance of the sector level model described in the previous section, and B_i is a number. In this case the best reply equation reduces to:

$$B_i = \frac{\overline{\Lambda}_i + (n_i - 1)B_i}{\overline{\Lambda}_i + (n_i - 1)B_i + 1}$$

and where $\overline{\Lambda}_i$ is the "sector level" price impact and is: $\overline{\Lambda}_i = \left(\frac{1}{\Lambda_i^{out}} + \frac{1}{\Lambda_i^{in}}\right)^{-1}$ for $i < N$, where Λ_i^{in} is the slope of the aggregate residual supply for layer i, and Λ_i^{out} is the aggregate residual demand. $\overline{\Lambda}_N = \Lambda_N^{out}$ is the slope of the residual demand for layer N (as the effect on the input price is not internalized). The next proposition characterizes markups and markdowns in this case. Moreover, it shows what is the effect of firms having bilateral market power on both inputs and outputs.

Theorem 4. *In a symmetric Supply and Demand Function Equilibrium for the layered supply chain, if $n_i = n_j$ for any i, j, then markups are larger the more*

[35] The case analyzed here is slightly different from the one analyzed in Example 3, because there no firms use labor but firm N at the beginning of the chain. Here, instead, due to the specific form of the technology, intermediate firms also use labor, in the form of payment of handling costs. Because the labor market is competitive, this has no effect and firms still have to determine one input and one output schedule.

upstream the layer is, while markdowns are larger the more downstream a layer is.

If firms do not internalize their price impact on the input price, markups are increasing going upstream*, while there are no markdowns. If firms do not internalize their price impact on the output price, but only the input, then markdowns are increasing going* downstream*, and there are no markups.*

The intuition for the result above is simple: upstream firms perceive a smaller elasticity of the residual demand on output markets the more they are upstream, and so charge higher markups. The opposite happens with residual supply and markdowns. In a supply chain as the one described, if n_i is constant across layers, the situation is completely symmetric, and so increase in markups and markdowns exactly offset each other: $m_i + M_i$ is constant. Hence, each layer extracts the same surplus, if they have the same level of competition. If some layer is more competitive, the corresponding firms have lower profits. This yields insights on what happens in general networks: markups still tend to be higher upstream and markdowns downstream, and the general pattern of interactions determines which effect prevails.

If firms instead do not internalize their effect on input prices, but only outputs, the symmetry is broken, because firms consider the effect of network position on the elasticitiy of demand only on, e.g., the output side. Below, I discuss how the same insight can be gathered from the classic model of sequential oligopoly that is well known in IO. The important similarity is that firms take input prices as given, but upstream firms perceive a smaller elasticity of demand, internalizing the pass-through of price changes: this yields the prediction that markups are higher upstream.

This result yields important insights on the hidden consequences of using models in which competition is artificially constrained to be unilateral. If such a modeling strategy is not motivated by the specifics of the market studied, but is just an assumption imposed for tractability, as in production network models, the result above suggests that implication for the relative ranking of market power among firms can be severely changed. The supply and demand function equilibrium provides a setting in which the modeler does not have to choose on which side of the market firms can affect prices, rather the price impact is an additional prediction that can be asked to the model.

Benchmark: sequential oligopoly This setting provides a good setting to gain intuition on the differences of the Supply and Demand Function Equilibrium with the more classic sequential competition models.

Consider the following classic IO model:[36] at each stage of the supply chain firms compete à la Cournot, taking as given the input price they face. Hence market power is unilateral by construction. This means that firms in sectors 1 and 2 play first, simultaneously, committing to supply a certain quantity. Then firms in sector 0 do the same, taking the price of good 1 and 2 as given. The model can then be solved by backward induction.

Call p_0 the inverse demand of the consumer, and assume for simplicity it is concave (this can be sometimes relaxed, as shown below). Assume the technology is linear: $f(q) = Aq$. Capital letters mean sector level quantities, lower case letters are used for firm level quantities.

The markups of firms in sector 0 is equal to the elasticity of the inverse demand, in absolute value. Throughout, I denote elasticities by η:

$$\mu_0 = -\eta_{p_0}$$

What is the markup of upstream sectors? The first order conditions of firms in sector 0 imply that the inverse demand faced by firms in sector 1 is:

$$p_1 = \left(p_0'\left(AQ_1\right)AQ_1 + p_0\left(AQ_1\right)\right)A$$

The markup of firms in sector 1 are then:

$$\begin{aligned}
\mu_1 = &- \eta_{p_1} = - \left(\frac{p_0' AQ}{p_0' AQ + p_0} (\eta_{p_0'} + 1) + \frac{p_0}{p_0' AQ + p_0} \eta_{p_0} \right) \\
&= \underbrace{\frac{-p_0' AQ}{p_0' AQ + p_0}}_{>0} (\underbrace{\eta_{p_0'}}_{>0} + 1) + \frac{p_0}{p_0' AQ + p_0} \mu_0 \\
&> \frac{p_0}{p_0' AQ + p_0} \mu_0 > \mu_0
\end{aligned}$$

which puts in evidence that the optimization introduces a force that tends to increase the markup, through the *pass-through*, represented by the term $\frac{p_0}{p_0' f + p_0}$.

The reasoning can be similarly extended to a chain of any lenght.

[36] This is a version of the simplest setting e.g. in Salinger (1988). A similar model, in prices, is Ordover et al. (1990)

5.3 General networks

This section describes how the network of input-output relationships affects the equilibrium of the model. The matrix of coefficients of the market clearing system, M, contains the fundamental network information in this setting.

Indeed, inverting the matrix M and collecting the diagonal D on both sides we get:

$$M^{-1} = D^{-1/2}(I - D^{1/2}LD^{1/2})^{-1}D^{-1/2}$$

which shows that M^{-1} is, modulo a normalization, has the familiar form of a *Leontief inverse* matrix. It is standard that entries of matrices of this form constitute a measure of the (weighted) number of *undirected* paths connecting the goods in the network.

To understand how the price impact Λ_i (whose expression is obtained in Proposition 3) relates to the input-output connections consider as an example a tree, as in Figure 1. To obtain the price impact of, say, node 2 we have first to eliminate the links of the graph connecting input and output goods of 2. Since the network is a tree now we have two separate subnetworks, as illustrated for sector 2 in Figure 4. The entries of the matrix Λ_2 count the number of weighted paths between input and outputs of 2. But since in the reduced network input and output links are disconnected, the matrix is diagonal, and can be partitioned into:

$$\Lambda_i = \begin{pmatrix} \Lambda_i^{out} & \mathbf{0} \\ \mathbf{0} & \Lambda_i^{in} \end{pmatrix}$$

where Λ_i^{out} is the (weighted) number of self loops of the output link in the reduced graph, while Λ_i^{in} is the matrix with on the diagonal the number of self loops of the input links. Figure 4 illustrates the situation for $i = 2$. The mechanism is similar to the line network: the more upstream the sector is, the larger the portion of the network in which the "self-loops" have to be calculated, hence the more elastic the demand the node is facing. This is because a larger portion of the network is involved in the determination of the demand, and each price variation will distribute on a larger fraction of firms.

Consider instead the network in Figure 5. What is the price impact of firm 2? In Figure 6 is represented the network induced between goods: 12 denotes the good sold by firm 1 to 2 and 10 the good sold by firm 2 to 0, that are supposed different. In the right panel, is represented the subnetwork useful to calculate the price impact of firm 2, that is the links that remain once eliminating firm 2.

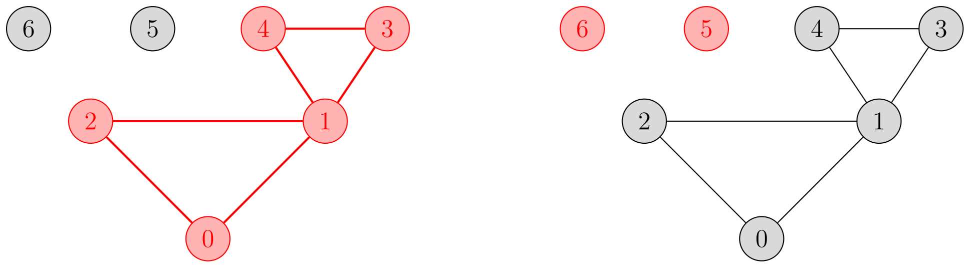

Figure 4: The relevant subnetworks for the calculation of the price impact for node 2. Left: output, right: inputs.

Since now even in the reduced graph input and output goods of firm 2 are still connected, this means that Λ_2 is not diagonal anymore.

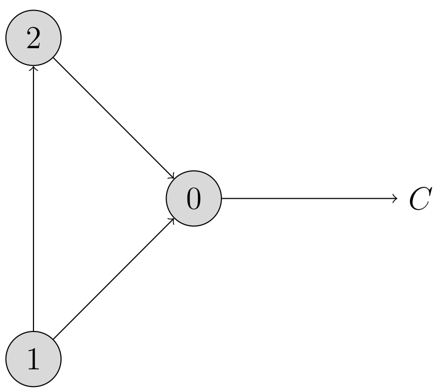

Figure 5: A simple production network: c represents the consumer demand, while the other numbers index the firms.

6 Aggregate impact of market power

In this section I explore the question of how the strategic interactions along the supply chains affect welfare. In the supply and demand function competition model we can easily explore the case in which firms, instead of internalizing the full effects of their commitment to a schedule on other firms and markets, are boundedly rational and are just able to take into account the behavior (schedule) of their direct neighbors, but not of other nodes of the network at larger distance.[37] More generally, in this section I explore the consequence of firms that have a limited ability to correctly internalize the behavior of the other indirectly connected firms at T_i steps in the network. If $T_i = \infty$ we are back in the model

[37] Ideally, we want to explore the consequence of including in models such as Baqaee (2018) the fact that firms internalize the impact of their decision on further downstream customers and upstream suppliers. In such models it is not clear how to do so.

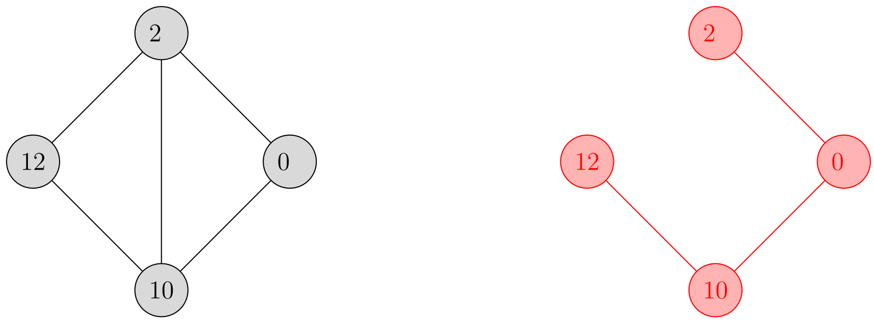

Figure 6: (Left) the goods network of the production network depicted in Figure 5. (Right) The subnetwork of the goods network in Figure 5 for the calculation of the price impact of firm 2.

of the previous sections. To do that, I change the definition of residual demand perceived by firms, as follows.

Definition 6.1 (Local residual schedule)**.** Suppose all firms other than i play linear symmetric schedules $\mathcal{S}_{-i}$. We can decompose the market clearing matrix as: $M = D - L$, where L is a matrix such that $\lambda_1(D^{-1/2}LD^{-1/2}) < 1$. Define $\tilde{L} = D^{-1/2}LD^{-1/2}$. The *local realized prices conditional on i at level T_i* are the function defined as:

$$\boldsymbol{p}_{-i}(\boldsymbol{p}_i, \boldsymbol{\varepsilon} \mid i, T_i) = D_i^{-1/2} \sum_{k=0}^{T_i} \tilde{L}_{-i}^k D_i^{-1/2} (\boldsymbol{A}_{-i} - M_{C_i} \boldsymbol{p}_i) \tag{9}$$

The local residual schedule at level T_i is the profile of demand and supply functions faced by i, $\mathcal{S}^r(\cdot \mid i, T_i) = (-D_i^r(\cdot \mid i, T_i), S_i^r(\cdot \mid i, T_i))$, when evaluating prices in other markets following $\boldsymbol{p}_{-i}(\cdot \mid i, T_i)$:

$$D_{gi}^r(\boldsymbol{p}_i, \boldsymbol{\varepsilon} \mid T_i) = \underbrace{\sum_{g \to k} D_{kg}(\boldsymbol{p}_k(\boldsymbol{p}_i, \boldsymbol{\varepsilon} \mid i, T_i), \boldsymbol{\varepsilon}_k)}_{\text{demand for good } g} - \underbrace{\sum_{k \to g, k \neq i} S_{gk} \boldsymbol{p}_k(\boldsymbol{p}_i, \boldsymbol{\varepsilon} \mid i, T_i), \varepsilon_k)}_{\text{supply by competitors}}$$

$$S_{ig}^r(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i \mid T_i) = \underbrace{\sum_{j \to g} S_{gj}(\boldsymbol{p}_j(\boldsymbol{p}_i, \boldsymbol{\varepsilon} \mid i, T_i), \boldsymbol{\varepsilon}_j)}_{\text{supply of good } j} - \underbrace{\sum_{g \to j, j \neq i} D_{jg}(\boldsymbol{p}_j(\boldsymbol{p}_i, \boldsymbol{\varepsilon} \mid i, T_i), \boldsymbol{\varepsilon}_j)}_{\text{demand by competitors}} \quad \forall g \in \mathcal{N}_i$$

Because of linearity, its expression is:

$$\mathcal{S}_i^r(\cdot \mid i, T_i) = \begin{pmatrix} -D_i^r(\cdot \mid i, T_i) \\ S_i^r(\cdot \mid i, T_i) \end{pmatrix} = \left(\hat{B}_i - M'_{C_i} D_{-i}^{-1/2} \sum_{k=0}^{T_i} \tilde{L}_{-i} D_{-i}^{-1/2} M_{C_i} \right) \boldsymbol{p}_i$$

The expression above simply states that firm i, when reasoning about the impact of a variation in its output price (that it can directly affect via the choice of a supply function) on other prices through the network, it taking into account only up to T_i steps. To see this, note that $M_{-i}^{-1} = D_i^{-1/2} \sum_{k=0}^{\infty} \tilde{L}_{-i}^k D_i^{-1/2}$, that is the coefficient matrix of the full residual schedule, cfr 3. The coefficient matrix of the conditional prices at level T_i is a truncation of this sum at level T_i, and can be understood in this way. Rewrite the system of market clearing equations:

$$M_{-i}\boldsymbol{p}_{-i} = \boldsymbol{A}_{-i} - M_{C_i}\boldsymbol{p}_i \Longrightarrow D_{-i}\boldsymbol{p}_{-i} = L_{-i}\boldsymbol{p}_{-i} + \boldsymbol{A}_{-i} - M_{C_i}\boldsymbol{p}_i$$

and solve it to get: $\boldsymbol{p}_{-i} = D_{-i}^{-1}(L_{-i}\boldsymbol{p}_{-i} + \boldsymbol{A}_{-i} - M_{C_i}\boldsymbol{p}_i)$.

This expression means that firm i understands the prices of goods sold, e.g., downstream by its customers depend in turn on the prices those customers face in input and output markets. If $T_i = 0$, these prices at distance 1 are considered constants, and as a result $\boldsymbol{p}_{-i} = \boldsymbol{A}_{-i} - M_{C_i}\boldsymbol{p}_i$. If T_i is higher, instead we can substitute iteratively the expression for prices, getting exactly:

$$\boldsymbol{p}_{-i} = D_i^{-1/2}\tilde{L}_{-i}^{T_i+1}D_i^{-1/2}\boldsymbol{p}_{-i} + D_i^{-1/2}\sum_{k=0}^{T_i}\tilde{L}_{-i}^k D_i^{-1/2}(\boldsymbol{A}_{-i} - M_{C_i}\boldsymbol{p}_i)$$

Now, to get the equation 9 in the definition, we neglect the higher order term $D_i^{-1/2}\tilde{L}_{-i}^{T_i+1}D_i^{-1/2}\boldsymbol{p}_{-i}$. Since the equation is linear, and the only thing affecting the best reply equations are the derivatives, this is the simplest way to capture exactly the fact that firms internalize the impact of its decisions up to distance T_i in the network. For example $T_i = 0$ means that the firm is considering all $\boldsymbol{p}_{-i}$ as constants.

Definition 6.2. A Local Supply and Demand Function equilibrium at levels $(T_i)_{i\in\mathcal{N}}$ is a profile of supply and demand schedules $(\mathcal{S}_i)_{i\in I}$ such that:

1. for any firm i, $\mathcal{S}_i$ is a best reply to $\mathcal{S}_{-i}$, given the residual schedule at level T_i, $\mathcal{S}_i^r(\cdot \mid T_i)$:
2. the prices and quantities $(\boldsymbol{p}(\boldsymbol{\varepsilon}), q(\boldsymbol{\varepsilon}))$ solve the market clearing conditions for any feasible realization of $\boldsymbol{\varepsilon}$.

The basic difference with the game analyzed so far is that in the firms optimization the *prices of sectors not directly connected with i are taken as given.* Indeed, in the constraints of the optimization there are only the market clearing conditions relative to the links directly connected to i. This is the analogous in this setting of models such as Baqaee (2018), Grassi (2017), Levchenko et al. (2016).

The next theorem explores the welfare implications of this behavioral assumption.

Theorem 5. *In a Local S&D equilibrium at levels $(2T_i)_{i\in\mathcal{N}}$, all price impacts are smaller than in the maximal S&D, and increasing in all the parameters T_i.*

If there is just one consumer good, its price is increasing in all the parameters T_i.

The result says that limitations to internalize nodes further away in the network is detrimental to firms, because it lowers their ability to charge higher (lower) prices for outputs (inputs). The reason why we focus on even levels ($2T_i$) is because if one compares the equilibrium with T_i to the one with T_i+1, in the one with T_i+1 firm internalize the effect on firms that are directly in competition with them for surplus: hence they might want to reduce the price impact. If the increment is sufficiently large instead this does not happen. This is a direct consequence of the fact that the slope (in matrix sense) of the residual schedule perceived at level $2T_i$ is decreasing in T_i in the positive semidefinite sense, that is, if $T_i > T_i'$:

$$\hat{B}_i - M_{C_i}' D_{-i}^{-1/2} \sum_{k=0}^{2T_i} \tilde{L}_{-i} D_{-i}^{-1/2} M_{C_i} < \left(\hat{B}_i - M_{C_i}' D_{-i}^{-1/2} \sum_{k=0}^{2T_i'} \tilde{L}_{-i} D_{-i}^{-1/2} M_{C_i} \right)$$

as proven in the appendix.

Strategic complementarities once again allow to transform this in an equilibrium statement.

Theorem 5 is a qualitative result. The following proposition shows that in some examples the inefficiencies due to market power can be very different for different short-sightedness parameters T_i. Unsurprisingly, this bites in particular in networks in which the supply chain has many steps further downstream and upstream, as in long production chains.

Proposition 2. *Consider a layered production chain of N layers, with 2 firms per layer. Denote W^l the welfare if $T_i = 0$ for all i, and W^g the welfare loss if $T_i = \infty$ for all i.*

If N goes to infinity, the relative welfare loss $\frac{W^l - W^g}{W^l}$ goes to 1.

7 Conclusion

This paper provides a way to model oligopoly in general equilibrium as a game in which firms fully internalize their position in the supply chain and have market power both over inputs and outputs, in an endogenously determined way. I show that such features are desirable in a input-output model with market power: if absent, both the aggregate and the relative ranking of distortions due to imperfect competitions is crucially affected. This suggests that, when modeling complex networks of large firms with market power, simplifying assumptions might affect in a sizable way the results. The parametric functional form introduced is suitable for quantitative work, and the strategic complementarity structure of the equilibrium makes it computationally tractable: the exploration of the quantitative implications of the supply and demand function equilibrium for the analysis of market power is an interesting avenue for future research.

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Appendix

A Proofs of Section 3

As anticipated in the text, the proofs of this section are done under a more general assumption for the payoffs, consistent with the literature on general equilibrium oligopoly (in particular Azar and Vives (2021)), namely, that firms optimize the indirect utility of their owners. The details are as follows.

Workers and Owners As in Azar and Vives (2021), there are two types of agent: workers, and owners. There is a continuum of identical workers or, equivalently, there is a representative worker, whose utility is $U((c_g)_{g\in\mathcal{C}}, L, \varepsilon_{i,c})$. The workers have aggregate demand D^w_c that has negative semidefinite jacobian with maximum rank (which is $|\mathcal{C}|-1$) with respect to both prices $\boldsymbol{p}_c$ and stochastic parameters $\boldsymbol{\varepsilon}_c$

The owners, instead, do not work, but own the firms. They are a continuum, partitioned in N groups, and owners in group i collectively own all the shares of firm i. They have utility functions homogeneous of degree 1, generating aggregate indirect utilities $V_i = \frac{\Pi}{P_i}$, where Π_i is the profit of firm i, P_i is a function of prices, homogeneous of degree 1 (the *price index* relative to owners of group i) and differentiable. These assumptions are enough to generate an aggregate demand that is differentiable and has negative definite jacobian as in the main text.

As anticipated, firms optimize the indirect utility of shareholders. Hence the payoff of firm i is:

$$\pi_i(\mathcal{S}) = \mathbb{E}\frac{\Pi_i}{P_i} = \mathbb{E}\left(\sum_{i\to g}\frac{\boldsymbol{p}_g}{P_i}S_{g,i}(\boldsymbol{p}_i,\boldsymbol{\varepsilon}_i) - \sum_{g\to i}\frac{\boldsymbol{p}_g}{P_i}D_{ig}(\boldsymbol{p}_i,\boldsymbol{\varepsilon}_i) - \frac{w}{P_i}\ell_i(\boldsymbol{p}_i,\boldsymbol{\varepsilon}_i)\right)$$

Note that this depends only on ratios p_g/P_i, hence not on price normalization. The assumption followed in the main body, of firms maximizing profits Π_i, can be recovered as a special case of this setting assuming that the owners' utility only depends on only one good, o, and, moreover, such a good is produced from a continuum of firms that use only labor as input (hence are isolated from the network). Hence in this case the price indices are all $P_i = w$, and we recover the main text formulation.

A.1 Proof of Proposition 1

We are going to need the following Lemmas, proved in the online Appendix.

Lemma A.1. Under Assumptions 3c, d) the map MC has positive definite jacobian derivative $\partial_{\boldsymbol{p}}MC$. Moreover, there are $\underline{k}$ and $\overline{K}$ such that $\|\partial_{\boldsymbol{p}}MC\|_2 \le \overline{K}$ and $\|\partial_{\boldsymbol{p}}MC^{-1}\|_2 \le \underline{k}^{-1}$.

Lemma A.2. There is a constant K_p such that the derivatives of the pricing function with respect to the stochastic parameters, and the (Fréchet) derivatives with respect to the schedules are bounded above: $\|\partial_{\boldsymbol{\varepsilon}} \boldsymbol{p}\| \leq K_p$ and $\|\partial_{\mathcal{S}_i} \boldsymbol{p}\|_g^{op} \leq K_p$. Here $\|\cdot\|_g^{op}$ denotes the operator norm: relative to the $\|\cdot\|_g$ norm in the domain: if A is a linear operator $\mathcal{A}_i \to \mathbb{R}^M$, $\|A\|_g^{op} = \max\{\|A\mathcal{S}_i\| \mid \|\mathcal{S}_i\|_g = 1\}$.

Part 1 First, focus on schedules in $\mathcal{A}$. Fix w. As defined in the text, the market clearing conditions are: $MC(\boldsymbol{p}, \boldsymbol{\varepsilon}) = \mathbf{0}$.

Assumption 3d) guarantees that the map $MC(\cdot, w, \boldsymbol{\varepsilon}) : \mathbb{R}^M_+ \to \mathbb{R}^M$ is proper. Indeed, it is continuous, and $\lim\|MC\| = \infty$ if $\boldsymbol{p}/w$ tends to the boundary of $\mathbb{R}^M_+$. Hence the counterimage of a bounded set is bounded away from the boundary: hence the counterimage of a compact is compact and the map is proper. Hence, we can use a global inversion theorem (Theorem 1.8 in Ambrosetti and Prodi (1995)) to conclude that MC is invertible and onto $\mathbb{R}^M$, hence in particular for each $\boldsymbol{\varepsilon} \in \mathring{\mathcal{E}}$ there is a unique $\boldsymbol{p}(\boldsymbol{\varepsilon})$ st $MC(\boldsymbol{p}(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}) = 0$. Now, using the implicit function theorem applied to the map $(\boldsymbol{p}, \boldsymbol{\varepsilon}) \mapsto (MC(\boldsymbol{p}, \boldsymbol{\varepsilon}), \boldsymbol{\varepsilon})$, we can conclude that $\boldsymbol{p}(\boldsymbol{\varepsilon})$ is differentiable on $\mathring{\mathcal{E}}$, the interior of $\mathcal{E}$. The Lemma A.2 guarantees that it is Lipschitz, so it can be extended uniquely to the whole of $\mathcal{E}$.

Now let us consider schedules in the closure of $\mathcal{A}$. Lemma A.2 guarantees that the map $\boldsymbol{p} : \mathcal{E} \times \mathcal{A} \to \mathbb{R}^M$ is Lipschitz, hence it can be extended in a unique way to the closure of the domain.

So far, we produced a unique function $\boldsymbol{p}(w, \boldsymbol{\varepsilon})$ for each fixed w. Now consider two functions such that $MC(\boldsymbol{p}(w, \boldsymbol{\varepsilon}), w, \boldsymbol{\varepsilon}) = 0$ and $MC(\boldsymbol{p}'(w', \boldsymbol{\varepsilon}), w', \boldsymbol{\varepsilon}) = 0$. Since MC is homogeneous of degree zero $MC(\boldsymbol{p}'(w', \boldsymbol{\varepsilon}), w', \boldsymbol{\varepsilon}) = MC(\boldsymbol{p}'(w', \boldsymbol{\varepsilon})w/w', w, \boldsymbol{\varepsilon}) = 0$, and so $\boldsymbol{p}'(w', \boldsymbol{\varepsilon})w/w' = \boldsymbol{p}(w, \boldsymbol{\varepsilon})$, that is, the functions are the same up to a positive normalization. Since the payoffs only depend on price ratios, they are independent of the normalization chosen.

Part 2 Fix a schedule $\overline{\mathcal{S}}_i \in \mathcal{A}_i$ and a value $\boldsymbol{\varepsilon}$, and call $\mathcal{S}_t = t\mathcal{S} + (1-t)\overline{\mathcal{S}}$ and $\boldsymbol{\varepsilon}_t = t\boldsymbol{\varepsilon}' + (1-t)\boldsymbol{\varepsilon}$. By the mean value theorem in Banach spaces (e.g., Proposition 7.2 in Luenberger (1997)):

$$\|\boldsymbol{p}(\mathcal{S}, \boldsymbol{\varepsilon}') - \boldsymbol{p}(\overline{\mathcal{S}}, \boldsymbol{\varepsilon})\|_2 \leq \sup_{t\in[0,1]} \|\partial_{\mathcal{S}} \boldsymbol{p}(\boldsymbol{\varepsilon}_t, \mathcal{S}_t)\|_g^{op} \|(\mathcal{S} - \overline{\mathcal{S}})\|_g + \sup_{t\in[0,1]} \|\partial_{\boldsymbol{\varepsilon}} \boldsymbol{p}(\boldsymbol{\varepsilon}_t, \mathcal{S}_t)\|_2 \|(\boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon})\|_2$$

Now by the Lemma A.2 and Assumption 3d, such a norm is bounded above by $k_p = 2K_pK_S + 2K_pK_e$, and in particular the image of $\mathcal{E} \times \mathcal{A}$ via $\boldsymbol{p}$ is bounded by k_p. $\square$

A.2 Proof of Theorem 1

Thanks to Proposition 1 the set $\overline{\mathcal{P}}$ is bounded; since it is closed by definition, it is compact, hence all schedules and their derivatives have upper bounds on it. The set $\mathcal{A}(\mathcal{P})^K$ is nonempty, because the schedules introduced in Example 1 satisfy all the assumptions, if the values K, K_S are large enough and k is small enough.

The set of all differentiable schedules that are bounded, with bounded derivatives, and compact domain $\mathcal{P} \times \mathcal{E}$ is equicontinuous (Theorem 14.2 in Treves (2016)), hence, by the Ascoli-Arzelà theorem, its closure is compact in the sup-norm. The set $\overline{A(\mathcal{P})}^K$ is a subset of such a compact set. Moreover, it is closed by definition, being the closure of $\mathcal{A}(\mathcal{P})^K$. Hence it is a closed subspace of a compact set, and so is compact. Since the pricing function is Lipschitz, it can be extended uniquely to such closure: hence the game is well defined also on $\overline{\mathcal{A}(\overline{\mathcal{P}})}^K$.

Since the profit function is continuous, the best reply problem has a solution. Moreover, by the maximum theorem the solution correspondence is upper-hemicontinuous (in particular, if single valued, is a continuous function).

It remains to prove that $\mathcal{A}(\mathcal{P})^K$ is convex. Consider $\mathcal{S}_i$ and $\mathcal{S}_i'$ in $\mathcal{A}_i(\mathcal{P})$. All the regularity assumptions are inherited by any convex combination, and it has the same domain by definition. The bounds are also inherited:

$$kI_i \leq \alpha \partial_{\boldsymbol{p}_i} \mathcal{S}_i + (1-\alpha)\partial_{\boldsymbol{p}_i} \mathcal{S}_i' \text{ and } \|\alpha \partial_{\boldsymbol{p}_i, w} \mathcal{S}_i + (1-\alpha)\partial_{\boldsymbol{p}_i, w} \mathcal{S}_i'\|_2 \leq K$$

and similarly for $\partial_{\boldsymbol{\varepsilon}_i} \mathcal{S}_i$. By convexity of Φ, the technology constraint is also satisfied:

$$\Phi_i(\alpha \mathcal{S}_i + (1-\alpha)\mathcal{S}_i', \boldsymbol{\varepsilon}_i) \leq \alpha \Phi_i(\mathcal{S}_i, \boldsymbol{\varepsilon}_i) + (1-\alpha)\Phi_i(\mathcal{S}_i', \boldsymbol{\varepsilon}_i) \leq 0$$

which is what we wanted to show.

So, if best replies are convex-valued (or in particular single valued), the best reply map is continuous on a set $\overline{A(\mathcal{P})}^K$ that is compact and convex, hence by the Ky Fan fixed point theorem the game has an equilibrium.

Denote the equilibrium profile as $\mathcal{S}^*$. Now, it is possible to further restrict the domain of each schedule to $\tilde{\mathcal{P}}_i = \boldsymbol{p}_i(\mathcal{E}, \mathcal{S}^*)$, that is the image of $\mathcal{E}$ via the equilibrium profile. This in general might be smaller than $\mathcal{P}$. Nevertheless, the profile $\mathcal{S}^*$ remains an equilibrium. Indeed, all the price values in $\mathcal{P} \setminus \boldsymbol{p}_i(\mathcal{E}, \mathcal{S}^*)$ have probability zero, so they do not affect the payoffs. Hence we can restrict each schedule to $\tilde{\mathcal{P}}_i = \boldsymbol{p}_i(\mathcal{E}, \mathcal{S}^*)$, to have an equilibrium in which the whole domain is spanned. Finally, the following Lemma (proven in the Online Appendix) uses the positive definiteness of $\partial_{\boldsymbol{\varepsilon}_i} \mathcal{S}_i$ to guarantee that each $\tilde{\mathcal{P}}_i$ is the closure of an open set. $\square$

Lemma A.3. Under Assumptions 1-3, $\mathcal{P}_i$ is the closure of an open set.

A.3 Proof of Theorem 2

We derive necessary conditions for an solution in $\mathcal{A}$. The positive definiteness constraints on the Jacobian can be expressed as: $\det[\partial_{\boldsymbol{p}_i,w}\mathcal{S}^H - kI]_j \geq 0$, where $[A]_j$, $j = 1, \dots d_i$ are the principal minors of a matrix A, and $\partial_{\boldsymbol{p}_i,w}\mathcal{S}^H$ is the symmetric part of the Jacobian. The others are $\det[\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}^H]_j \geq 0$, always for all $j = 1, \dots d_i$. The necessary conditions for optimization are the usual Lagrange multiplier equations (Luenberger (1997)). The Lagrangian is:

$$\mathcal{L}_i(\mathcal{S}_i) = \mathbb{E}\left[\boldsymbol{p}'_{i,-\ell}\mathcal{S}_{i,-\ell} + \mathcal{S}_1 - \lambda_i\Phi_i(\mathcal{S}_i, \boldsymbol{\varepsilon}_i) - \sum_j \lambda^p_j \det[\partial_{\boldsymbol{p}_i,w}\mathcal{S}^H - kI]_j - \sum_j \lambda^e_j \det[\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}^H]_j \right.$$

$$\left. -\lambda^p_+(K^2 - \|\partial_{\boldsymbol{p}_i,w}\mathcal{S}\|^2_2) - \lambda^e_+(K^2 - \|\partial_{\boldsymbol{p}_i,w}\mathcal{S}_i\|^2_2)\right]$$

where $\boldsymbol{p}_i(\mathcal{S}, \boldsymbol{\varepsilon})$ is the unique pricing function (from Proposition 1) such that the wage is 1.

We have to show that this is Fréchet differentiable, and the necessary condition is setting the Fréchet differential to 0. To do so, in the following Lemma (proven in the Online Appendix) we compute the Gateaux differential in the direction eta_i. Under the assumption we made on η_i, it is always possible to choose h small enough such that $\mathcal{S}_i + h\eta_i \in \mathcal{A}$.

Lemma A.4. Assume that $\partial\tilde{\mathcal{P}}$ has differentiable boundary. The Gateaux differential of the Lagrangian in a direction η_i, satisfying the above assumptions, is:[38] $\mathbb{E}\left[\eta'_{i,-\ell}G_i\right]$, where:

$$G_i = ([\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}] + [\partial_{\boldsymbol{p}_i}\mathcal{S}^r_{i,-\ell}])^{-1}\left(\mathcal{S}^r_i + \partial_{\boldsymbol{p}_{i,-\ell}}\mathcal{S}^r_{i,-\ell}((\boldsymbol{p}_i, 1) - \lambda_i\nabla\Phi_i) - \frac{\boldsymbol{p}'_i\mathcal{S}_i}{P_i}\nabla_{\boldsymbol{p}_i,-1}P_i\right) + \mathcal{K}_i$$

where $\mathcal{K}_i = \sum_j \mathcal{I}^p_{i,j} + \sum_j \mathcal{I}^e_{i,j} + \mathcal{Q}^p_i + \mathcal{Q}^e_i$ is the term with the derivatives of the

[38]Note that the component of η relative to labor does not directly enter the equation, but this is not strange because it is implicitly determined by the technology constraint.

four constraints.

$$
\begin{aligned}
\mathcal{I}^p_{i,j} =& D^p_j(0)\lambda^p_j T^p_{1j} + \tilde{T}^p_{2j} \quad \text{ where } \tilde{T}^p_{2j,m} = div_{\boldsymbol{p}_i}\left(\lambda^p_j D^p_j(0)\tilde{f}_i T^p_{2,m}\right)/\tilde{f}_i \\
& \text{and } T^p_{1j} \text{ and } T^p_{2j,m} \text{ are the vectors such that:} \\
& tr\left([\partial_{\boldsymbol{p}_i}(\mathcal{S}^H_i)]^{-1}_j\left(\partial_h[\partial_{\boldsymbol{p}_i}(\mathcal{S}^H_i + h\eta^H_i)]_j\right)\right) = T^p_{1j}\eta_{i,-\ell} + \sum_{m=1}^{j} T^p_{2j,m}\partial_{\boldsymbol{p}_i}(\eta_i)_m \\
\mathcal{I}^e_{i,j} =& D^e_j(0)\lambda^e_j T^e_{1j} + \tilde{T}^e_{2j} \quad \text{ where } \tilde{T}^e_{2j,m} = \partial_{\boldsymbol{\varepsilon}_i}\left(\lambda^e_j D^e_j(0) f_i(\boldsymbol{\varepsilon}_i \mid \boldsymbol{\varepsilon}_{-i}) T^e_{2j}\right)/f(\boldsymbol{\varepsilon}_i \mid \boldsymbol{\varepsilon}_{-i}) \\
& \text{and } T^e_{1j} \text{ and } T^e_{2j,m} \text{ are the vectors such that:} \\
& tr\left([\partial_{\boldsymbol{\varepsilon}_i}(\mathcal{S}^H_i)]^{-1}_j\left(\partial_h[\partial_{\boldsymbol{\varepsilon}_i}(\mathcal{S}^H_i + h\eta^H_i)]_j\right)\right) = T^e_{1j}\eta_{i,-\ell} + \sum_{m=1}^{j} T^e_{2j,m}\partial_{\boldsymbol{\varepsilon}_i}(\eta_i)_m \\
\mathcal{Q}^p_m =& \lambda^p_+ Q^p_1 + \tilde{Q}^p_2 \quad \text{ where } \tilde{Q}^p_{2,m} = div_{\boldsymbol{p}_i}(\lambda^p_+ Q^p_{2,m}\tilde{f}_i)/\tilde{f}_i \\
& \text{and the vectors } Q^p_1,\ Q^p_{2,m} \text{ are such that:} \\
& \partial_h \|\partial_{\boldsymbol{p}_i,w}\mathcal{S}_i\|^2_2 \mid_{h=0} = Q^e_1\eta_{i,-\ell} + \sum_m Q^p_{2,m}\partial_{\boldsymbol{p}_i}\eta_{i,m} \\
\mathcal{Q}^e_m =& \lambda^e_+ Q^e_1 + \tilde{Q}^e_2 \quad \text{ where } \tilde{Q}^e_{2,m} = div_{\boldsymbol{\varepsilon}_i}(\lambda^e_+ Q^e_{2,m} f_i)/f_i \\
& \text{and the vectors } Q^e_1,\ Q^e_{2,m} \text{ are such that:} \\
& \partial_h \|\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}_i\|^2_2 \mid_{h=0} = Q^e_1\eta_{i,-\ell} + \sum_m Q^e_{2,m}\partial_{\boldsymbol{\varepsilon}_i}\eta_{i,m}
\end{aligned}
$$

The assumption of differentiable boundary is necessary to apply the divergence theorem, and integrate by parts the derivative of the constraints, eliminating the derivatives of η_i from the expression.

Now by the law of iterated expectations we can rewrite the expectation as $\mathbb{E}[\eta'_{i,-\ell}(\boldsymbol{p}_i, \varepsilon_i)\mathbb{E}[G_i \mid \boldsymbol{p}_i, \boldsymbol{\varepsilon}_i]]$, and by the arbitrariness of η_i the FOC is equivalent to $\mathbb{E}[G_i \mid \boldsymbol{p}_i, \boldsymbol{\varepsilon}_i] = 0$. Using Lemma A.3 to conclude that $\mathcal{S}^r_i = -\mathcal{S}_i$ for all the possible prices, we obtain the expression in the main text, noting that in that case $\nabla_{\boldsymbol{p}_{i,-\ell}} P_i = 0$. □

B Proofs of Section 5

B.1 Proof of Theorem 4

The proof follows from the following lemmas, proven in the Online Appendix F.

Lemma B.1.

$$\overline{\Lambda}_i = ((\Lambda_i^{out})^{-1} + (\Lambda_i^{in})^{-1})^{-1} = \frac{\prod_{k\neq i} n_k B_k B_c}{\prod_{k\neq i} n_k B_k + B_c \sum_{j\neq i} \prod_{k\neq i, k\neq j} n_k B_k} \tag{10}$$

$$\overline{\Lambda}_N = ((\Lambda_N^{out})^{-1} + (\Lambda_N^{in})^{-1})^{-1} = \frac{\prod_{k\neq 1} n_k B_k B_c}{\prod_{k\neq 1} n_k B_k + B_c \sum_{j\neq 1} \prod_{k\neq 1, k\neq j} n_k B_k} \tag{11}$$

Lemma B.2. In equilibrium $n_i \geq n_j$ implies $B_i^* \geq B_j^*$.

Calculations reveal that:

$$\mu_i^{out} = p_i - \lambda_i^{out} = \frac{\frac{\Lambda_i^{in}}{\Lambda_i^{out}+\Lambda_i^{in}}}{(1+B_i) + \frac{\Lambda_i^{out}\Lambda_i^{in}}{\Lambda_i^{out}+\Lambda_i^{in}}}(p_i - p_{i-1})$$

$$\mu_i^{in} = \lambda_i^{in} - p_{i-1} = \frac{\frac{\Lambda_i^{out}}{\Lambda_i^{out}+\Lambda_i^{in}}}{(1+B_i) + \frac{\Lambda_i^{out}\Lambda_i^{in}}{\Lambda_i^{out}+\Lambda_i^{in}}}(p_i - p_{i-1})$$

Now by the previous lemma $B_i = B_j$ for all sectors and so market clearing conditions imply that $p_i - p_{i-1}$ is constant across sectors. Moreover by Lemma B.2 also $\frac{\Lambda_i^{out}\Lambda_i^{in}}{\Lambda_i^{out}+\Lambda_i^{in}}$ is. Now inspecting the right hand side of the expressions we see that the markup is decreasing with Λ_i^{out}, which is itself decreasing as one goes upstream. Then it follows that the markup is increasing going upstream, and symmetrically for the markdown.

If the firms do not take the price impact into account on input markets, the best reply equations become:

$$B_i = \frac{\overline{\Lambda}_i + (n-1)B_i}{\overline{\Lambda}_i + (n-1)B_i + 1} \quad \text{where } \overline{\Lambda}_i = \frac{\Lambda_i^{out}}{1+\Lambda_i^{out}}$$

and Λ_i^{out} is increasing upstream. Hence, in equilibrium, B_i is decreasing upstream, which means that markups are increasing. $\square$

C Proofs of Section 6

C.1 Proof of Theorem 5

The best reply matrix for firm i at level T_i is: $BR_i(B_{-i}, T_i) = \left(C_i^{-1} + \Lambda_i^{T_i}\right)^{-1}$ where $\Lambda_i^{T_i} = \overline{B}_i - M'_{C_i} D_{-i}^{-1/2} \sum_{k=0}^{T_i'} \tilde{L}_{-i} D_{-i}^{-1/2} M_{C_i}$, where $\overline{B}_i$ is the diagonal matrix that on the diagonal has the coefficient $B_{k,ii}$ for all the neighbors k of i.

The equilibrium profile of matrix coefficients satisfies $B_i = BR_i(B_{-i}, T_i)$ for each i. The result follows by applying the theory of monotone comparative statics, and the following Lemma, proven in the Online Appendix.

Lemma C.1. If $T_i > T_i'$ then $BR_i(B_{-i}, 2T_i) < BR_i(B_{-i}, 2T_i')$.

Hence, by standard arguments we can conclude that in the maximal equilibrium $B^*(T_i') \geq B^*(T_i)$, which is our first thesis. From this it follows that $M^*(T_i') \geq M^*(T_i)$, and so the price of the unique consumption good satisfies:

$$p_c(T_i) = \boldsymbol{A}'(M^*(T_i))^{-1}\boldsymbol{A} \geq \boldsymbol{A}'(M^*(T_i'))^{-1}\boldsymbol{A} = p_c(T_i')$$

which is what we wanted to show.

Online Appendix

D Additional proofs of section 3

D.1 Proof of Lemma A.1

Part I: Positive definite

By the *lifting* procedure as in Malamud and Rostek (2017), we can consider every supply function as defined on the set of all prices instead then the prices of the neighboring goods, and similarly having values in tuples of all the goods: $\hat{\mathcal{S}}_i : \mathbb{R}^M_+ \to \mathbb{R}^M$. The consistency required is, of course,

$$\hat{\mathcal{S}}_i(\boldsymbol{p}_i, \boldsymbol{p}_{-i}, \varepsilon) = \mathcal{S}_i(\boldsymbol{p}_i, \varepsilon)\, \forall g \in \mathcal{N}_i \quad \hat{\mathcal{S}}_{gi}(\boldsymbol{p}_i, \boldsymbol{p}_{-i}, \varepsilon) = 0\, \forall g \notin \mathcal{N}_i$$

With this notation, we can write the excess supply function as:

$$MC_g = \sum_i \hat{\mathcal{S}}_{gi} - \hat{D}_c + \hat{\ell}_c$$

Denote $\mathcal{S}_{i,-\ell}$ the schedule of i excluding (if present) labor demand. Moreover, MC is homogeneous of degree zero, hence naturally we cannot invert it as a full function of prices. For convenience we consider it a function of $\boldsymbol{p}_{-w}$, the vector of prices excluding the wage.

The Jacobian derivative is:

$$\partial_{\boldsymbol{p}} MC = \sum_i \partial_{\boldsymbol{p}} \hat{\mathcal{S}}_{i,-\ell} - \partial_{\boldsymbol{p}} \hat{D}_c$$

This is symmetric if all the derivatives are symmetric. We are going to prove that, once we normalize by a price, this is also positive definite. By Theorem 6 in Gale and Nikaido (1965), this implies that the realized prices are well defined on any convex domain.

Considering any vector $\boldsymbol{x} \in \mathbb{R}^M \setminus \{0\}$, we have

$$\boldsymbol{x}' \partial_{\boldsymbol{p}} MC \boldsymbol{x} = \sum_i \boldsymbol{x}' (\partial \boldsymbol{p}_i \hat{\mathcal{S}}_{i,-\ell} - \partial_{\boldsymbol{p}} \hat{D}) \boldsymbol{x} = \sum_i \boldsymbol{x}_i' \partial \boldsymbol{p}_i \mathcal{S}_{i,-\ell} \boldsymbol{x}_i + \boldsymbol{x}_c' (-\partial_{\boldsymbol{p}} D_c) \boldsymbol{x}_c$$

where, as for the prices, we denote $\boldsymbol{x}_i = (x_g)_{g \in \mathcal{N}_i}$. Now if there is a λ_i such that $\boldsymbol{p}_i = \lambda_i \nabla \Phi_i$ for each i, then $\mathcal{S}_{i,-\ell}$ is positive definite, because the original schedules have co-rank 1. In this case, it follows that $\partial_{\boldsymbol{p}} MC$ is positive definite. If not, $\mathcal{S}_{i,-\ell}$

has co-rank 1, and satisfies:

$$\sum_g u_{ig}[\mathcal{S}_{i,-\ell}]_{hg} = 0\,\forall h \quad u_{ig} = \left(\frac{1}{p_h}p_g - \frac{1}{\partial_{p_h}\Phi_i}\nabla_{\boldsymbol{p}}\Phi_i\right) \text{ for some } h \in \mathcal{N}_i$$

So, if there is a nonzero vector $\boldsymbol{x}$ such that $\boldsymbol{x}'\partial_{\boldsymbol{p}_{-w}}MC\boldsymbol{x} = 0$, it must be $\boldsymbol{x}_i = \boldsymbol{u}_i$ for some i, and $\boldsymbol{x}_i = 0$ otherwise. Where, since $\boldsymbol{u}_i$ is nonzero, and the sum is null, at least two entries of the vector u_i are nonzero, corresponding to, say, good g and h. Then $x_g = u_{ig} \neq 0$, and x_g is also an element of $\boldsymbol{x}_j$, so also $\boldsymbol{x}_j = \boldsymbol{u}_j$. Repeating the reasoning, we can go on until we reach a firm k such that the good g such that $u_{kg} \neq 0$, and g is a good consumed by the consumer: in that case x_g cannot be zero, and we reach a contradiction. Hence the quantity $\boldsymbol{x}'\partial_{\boldsymbol{p}}MC\boldsymbol{x}$ is positive, and the jacobian $\partial_{\boldsymbol{p}}MC$ is positive definite.

Part II: bounds

For the lower bound, by Assumption $3d$) we have:

$$k\sum_i \hat{I}_i \leq \partial_{\boldsymbol{p}}MC = \sum_i \partial_{\boldsymbol{p}_i}\hat{\mathcal{S}}_{i,-\ell} - \partial_{\boldsymbol{p}}\hat{D}_c$$

where $\hat{I}_i$ is the lifting of the identity matrix relative to i, having a 1 on the diagonal whenever g, h are both traded by firm i, and zero otherwise. The sum of such matrices is still diagonal. In particular, the entry in position g, h is $n_g k$, where n_g is the sum of firms that trade good g, plus (eventually) the consumer. Anyhow this is larger than $2k$, so the matrix is bounded below, and so it can be found a $\underline{k}$ such that $\underline{k}I \leq K\sum_i I_i$. Now by definition this is the same as $\underline{k}I \leq H(\partial_{\boldsymbol{p}}MC)$, where $H(A) = (A + A')/2$ denotes the symmetric part of a matrix. For a property of the positive semidefinite ordering, it follows that $\underline{k}^{-1}I \geq H(\partial_{\boldsymbol{p}}MC)^{-1}$, that implies $\underline{k}^{-1} \geq \|H(\partial_{\boldsymbol{p}_i}MC)^{-1}\|_2$. By Lemma 2.1 in Mathias (1992) it follows that $\|\partial_{\boldsymbol{p}_i}MC^{-1}\|_2 \leq \|H(\partial_{\boldsymbol{p}}MC)^{-1}\|_2 \leq \underline{k}^{-1}$.

Concerning the upper bound, it is sufficient to apply subadditivity of the norm and again Assumption $3d$)

$$\|\sum_i \partial_{\boldsymbol{p}_i}\hat{\mathcal{S}}_{i,-\ell} - \partial_{\boldsymbol{p}}\|_2 \leq (N+1)K = \overline{K}$$

$\square$

D.2 Proof of Lemma A.2

We have to prove that the Fréchet derivative of $\boldsymbol{p}$ with respect to the schedules is bounded. By the implicit function theorem is:

$$\partial_{\mathcal{S}}\boldsymbol{p} = -(\partial_{\boldsymbol{p}}MC)^{-1}\partial_{\mathcal{S}}MC$$

we have to compute the Gateaux derivatives in all the directions η that satisfy the constraints:

$$\partial_{\mathcal{S}}MC(\eta) = \partial_h MC(\mathcal{S} + h\eta, \boldsymbol{p}, \boldsymbol{\varepsilon}) = MC(\eta, \boldsymbol{p}, \boldsymbol{\varepsilon})$$

Now by Assumption 3d) $\|\eta\|_g < K_S$, and so $\|\partial_{\mathcal{S}}MC\|_g^{op} \leq \sum_i \|\eta_i\|_2 = NK_S$. Moreover, from Lemma A.1 follows $\|(\partial_{\boldsymbol{p}}MC)^{-1}\|_2 \leq \underline{k}^{-1}$. Hence, for any η $\|\partial_{\mathcal{S}}MC(\eta)\|_2 \leq \underline{k}^{-1}NK_S$. By definition of operator norm, the operator norm of $\|\partial_{\mathcal{S}}\boldsymbol{p}\|$ is bounded above by the same constant.

Similarly,

$$\|\partial_{\boldsymbol{\varepsilon}}\boldsymbol{p}\|_2 = \|-(\partial_{\boldsymbol{p}}MC)^{-1}\partial_{\boldsymbol{\varepsilon}}MC\|_2 \leq \underline{k}^{-1}NK$$

and now define $K_p = \max\{\underline{k}^{-1}NK, \underline{k}^{-1}NK_S\}$. □

D.3 Proof of Lemma 3.1

Consider the excess supply function, neglecting all g that are produced or used by firm i. We obtain a function:

$$MC_g^i : (\boldsymbol{p}, \boldsymbol{\varepsilon}) \mapsto MC_g(\boldsymbol{p}, \boldsymbol{\varepsilon})\, \forall g \notin \mathcal{N}_i$$

With a reasoning totally analogous, this is a function that can be inverted, expressing p_{-i} as a function of $\boldsymbol{p}_i$ (including labor). Moreover, this function is homogeneous of degree 1 in prices.

Now, for $g \in \mathcal{N}$, the residual schedule is simply:

$$\mathcal{S}_g^r(\boldsymbol{p}_i, \boldsymbol{\varepsilon}) := MC_g(\boldsymbol{p}_{-i}(\boldsymbol{p}_i), \boldsymbol{p}_i, \boldsymbol{\varepsilon}) - \mathcal{S}_g(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i)$$

Homogeneity follows immediately. Hence, we normalize the wage to 1.

Define $\hat{MC}^i$ the function such that $\hat{MC}^i = MC_g(\boldsymbol{p}_{-i}(\boldsymbol{p}_i), \boldsymbol{p}_i, \boldsymbol{\varepsilon}) - \mathcal{S}_g(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i)$. Notice that by definition of the excess supply function this is actually independent of $\mathcal{S}$. Now, we can compute the derivative of the partially solved prices:

$$\partial_{\boldsymbol{p}_i}\boldsymbol{p}_{-i} = -(\partial M\hat{C}_{-i_{\boldsymbol{p}_{-i}}})^{-1}\partial M\hat{C}_{\boldsymbol{p}_i}$$

and so define:

$$\partial_{\boldsymbol{p}_i}\mathcal{S}^r = \partial_{\boldsymbol{p}_i}\mathcal{S}^r = \partial\hat{MC}_{i\boldsymbol{p}_i} - \partial\hat{MC}_{i\boldsymbol{p}_{-i}}(\partial\hat{MC}_{\boldsymbol{p}_{-i}})^{-1}\partial\hat{MC}_{\boldsymbol{p}_i}$$

that is the Schur complement of $\partial\hat{MC}_{-i\boldsymbol{p}_{-i}}$ in the jacobian $\partial\hat{MC}$, appropriately reordered to have all $g \in \mathcal{N}$ in the upper left corner, and all others in the rest:

$$\partial\hat{MC} = \begin{pmatrix} \partial\hat{MC}_{i\boldsymbol{p}_i} & \partial\hat{MC}_{i\boldsymbol{p}_{-i}} \\ \partial\hat{MC}_{-i\boldsymbol{p}_i} & \partial\hat{MC}_{-i\boldsymbol{p}_{-i}} \end{pmatrix}^{-1} = \begin{pmatrix} (\partial\hat{MC}_{i\boldsymbol{p}_i} - \partial\hat{MC}_{i\boldsymbol{p}_{-i}}(\partial\hat{MC}_{-i\boldsymbol{p}_{-i}})^{-1}\partial\hat{MC}_{\boldsymbol{p}_i})^{-1} & B \\ C & D \end{pmatrix}$$

Hence we conclude that if all schedules have positive definite derivatives then $\partial_{\boldsymbol{p}_i}\mathcal{S}^r$ is positive definite beacause principal submatrices of positive definite matrices are still positive definite. $\square$

D.4 Proof of Lemma A.3

Fix $\mathcal{S}$. Since the stochastic parameters are $\sum_i d_i \geq M$, the map $\boldsymbol{p}$ is not invertible. We can consider a restriction such that it is. Namely, impose that the uncertain parameters relative to the same good are the same across firms: $\varepsilon_{gi} = \varepsilon_{gj}$ for all i, j and g. Let us denote the stochastic parameters remained independent as $\tilde{\boldsymbol{\varepsilon}}$, and their domain as $\tilde{\mathcal{E}} \subset \mathbb{R}^M$. This is a compact set, because it is a closed subset of a compact set. This way, the uncertain parameters behave formally exactly like prices, and with analogous reasoning as in Proposition 1 we obtain that $\partial_{\tilde{\boldsymbol{\varepsilon}}}MC$ is positive definite. Moreover, repeating the reasoning in the proof of Proposition 3.1, we obtain that if we consider constants the parameters relative to one firm i, this is equivalent to calculate the matrix $\partial_{\boldsymbol{\varepsilon}/i}MC = \partial_{\tilde{\boldsymbol{\varepsilon}}}MC - \partial_{\tilde{\boldsymbol{\varepsilon}}}\hat{\mathcal{S}}_i$, and this is still positive definite, exactly as $\partial_{\boldsymbol{p}}MC - \partial_{\boldsymbol{p}}\hat{\mathcal{S}}_i$ is still positive definite. In particular, it is invertible.

Hence, in the interior of $\tilde{\mathcal{E}}$:

$$\partial_{\boldsymbol{\varepsilon}/i}\boldsymbol{p} = -(\partial_{\boldsymbol{p}}MC)^{-1}\partial_{\boldsymbol{\varepsilon}/i}MC$$

is invertible, and so the map $\boldsymbol{p} : \tilde{\mathcal{E}} \to \mathbb{R}^M$ is locally invertible: for any $\tilde{\boldsymbol{\varepsilon}} \in \mathring{\tilde{\mathcal{E}}}$ there is an open $U_{\tilde{\boldsymbol{\varepsilon}}}$ such that $\boldsymbol{p}\mid_{U_{\tilde{\boldsymbol{\varepsilon}}}}$ is invertible. In particular, $\boldsymbol{p}(U_{\tilde{\boldsymbol{\varepsilon}}})$ is open, and so $\boldsymbol{p}(\mathring{\tilde{\mathcal{E}}}) = \cup_{\tilde{\boldsymbol{\varepsilon}}}U_{\tilde{\boldsymbol{\varepsilon}}}$ is open too; hence $\boldsymbol{p}(\tilde{\mathcal{E}})$ is the closure of an open set. $\square$

D.5 Proof of Lemma A.4

Consider the perturbation in the direction of η: $\mathcal{S}_i + h\eta_i$. Write $\boldsymbol{p}_i(h)$ for $\boldsymbol{p}_i(\mathcal{S}_i + h\eta_i, \varepsilon_i)$. Define the functions:

$$N(h) = -\mathbb{E}\left[\frac{\boldsymbol{p}_i'(h)}{P_i(\boldsymbol{p}_i(h))}\mathcal{S}_i^r(\boldsymbol{p}_i(h), \varepsilon_i)\right] \tag{12}$$

$$M(h) = \Phi_i(\mathcal{S}_i^r(\boldsymbol{p}_i(h), \varepsilon_i)) \tag{13}$$

The Gateaux derivatives in direction η_i are $N'(0)$, $M'(0)$. Note that we can exchange derivatives and integrals since all the functions involved have bounded derivative (and the price space is supposed compact), hence dominated (because a probability space has finite measure) (see Billingsley (2008), Theorem 16.8). We have first to compute the derivative of $\boldsymbol{p}_i(\mathcal{S}_i + h\eta_i, \varepsilon)$ with respect to h, that by the chain rule is:

$$\partial_h \boldsymbol{p}_i(\mathcal{S}_{i,-\ell} + h\eta_{i,-\ell}, \varepsilon) = \partial_{\mathcal{S}_{i,-\ell}}\boldsymbol{p}_i \eta_{i,-\ell}$$

where $\partial_{\mathcal{S}_{i,-\ell}}\boldsymbol{p}_i$ is the Gateaux derivative of the prices as functions of the schedules chosen, that can be computed via the implicit function theorem:[39]

$$[\partial_{\mathcal{S}_{i,-\ell}}\boldsymbol{p}_i(\eta_i)] = -([\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}] + [\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}^r])^{-1}\eta_{i,-\ell}$$

where I use the fact that the both the submatrices are positive semidefinite, and the residual demand is positive definite. Remember that $\mathcal{S}_{i,-\ell}$ denotes the components of the schedule $\mathcal{S}_i$ excluding the labor demand, and similarly for $\mathcal{S}_{i,-\ell}^r$ and $\eta_{i,-\ell}$. Hence, now:

$$\begin{aligned}
N'(h) &= -\frac{\partial}{\partial h}\mathbb{E}\left[\frac{\boldsymbol{p}_i'(h)}{P_i(\boldsymbol{p}_i(h))}\mathcal{S}_i^r(\boldsymbol{p}_i(h), \varepsilon_i)\right] \\
&= -\mathbb{E}\left[\frac{\partial_h \boldsymbol{p}_i(h)'\mathcal{S}_i^r + \boldsymbol{p}_i'\partial_p \mathcal{S}^r \partial_h \boldsymbol{p}_i(h)}{P_i} - \frac{\boldsymbol{p}_i'(h)(\mathcal{S}_i^r(\boldsymbol{p}_i(h), \varepsilon_i))}{P_i^2}\nabla_{\boldsymbol{p}_i}P_i'\partial_h \boldsymbol{p}_i(h)\right] \\
&= \mathbb{E}\left[\left((\mathcal{S}_{i,-\ell}^r)' + \boldsymbol{p}_i'\partial_p \mathcal{S}_{i,-\ell}^r + \frac{(\mathcal{S}_i(\boldsymbol{p}_i(h), \varepsilon_i))}{P_i}\nabla_{\boldsymbol{p}_i,-1}P_{i,-\ell}\right)\frac{1}{P_i}([\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}] + [\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}^r])^{-1}\eta_{i,-\ell}\right]
\end{aligned}$$

[39] Alternatively, we can compute directly $\partial_h \boldsymbol{p}_i$ using the implicit function theorem, the procedures are identical.

for any direction η_i. Moreover:

$$\begin{aligned} M'(h) &= \Phi_i(\mathcal{S}_i^r(\boldsymbol{p}_i(h), \boldsymbol{\varepsilon}_i)) \\ &= \nabla\Phi_i \partial_p \mathcal{S}_i^r \partial_h \boldsymbol{p}_i(h) \\ &= -\nabla\Phi_i \partial_p \mathcal{S}_{i,-\ell}^r ([\partial_{\boldsymbol{p}_i} \mathcal{S}_{i,-\ell}] + [\partial_{\boldsymbol{p}_i} \mathcal{S}_{i,-\ell}^r])^{-1} \eta_{i,-\ell} \end{aligned}$$

for any direction η_i.

Now we turn to the positive definiteness constraints. Define $D_j^p(h) = \det[\partial_{\boldsymbol{p}_i}(\mathcal{S}_i^H + h\eta_i^H)]_j$. A standard result in linear algebra gives $\partial det A = det A tr(A^{-1}\partial A)$. Hence:

$$\frac{\mathrm{d}}{\mathrm{d}h} D_j^p(h) = \det[\partial_{\boldsymbol{p}_i}(\mathcal{S}_i + h\eta_i) - kI]_j tr\left([\partial_{\boldsymbol{p}_i}(\mathcal{S}_i^H + h\eta_i) - kI]_j^{-1} \partial_h [\partial_{\boldsymbol{p}_i}(\mathcal{S}_i^H + h\eta_i^H)]_j\right)$$

Now the term in the trace is a matrix, and linear in the components of η (via $\partial_h \boldsymbol{p}_i$) and its derivatives. Hence there are row vectors T_{1j}^p and $T_{2j,m}^p$ such that:

$$tr\left([\partial_{\boldsymbol{p}_i}(\mathcal{S}_i) - kI]_j^{-1}\left(\partial_h[\partial_{\boldsymbol{p}_i}(\mathcal{S}_i^H + h\eta_i^H)]_j\right)\right) = T_{1j}^p \eta_{i,-\ell} + \sum_{m=1,\neq\ell}^{j} T_{2j,m}^p \partial_{\boldsymbol{p}_i}(\eta_i)_m$$

and so, for $h = 0$:

$$\partial_h D_j^p(h) \mid_{h=0} = D_j^p(0)\left(T_{1j}^p \eta_{i,-\ell} + \sum_{m=1,\neq\ell}^{j} T_{2j,m}^p \partial_{\boldsymbol{p}_i}(\eta_i)_m\right)$$

Now let us analyze the term with $\partial_{\boldsymbol{p}_i}(\eta_i)_m$. By the law of iterated expectations:

$$\mathbb{E}\lambda_j^p D_j^p(0)(T_{2,m}\partial_{\boldsymbol{p}_i}(\eta_i)_m) = \mathbb{E}\left(\mathbb{E}\left(\lambda_j^p D_j(0) T_{2j,m} \partial_{\boldsymbol{p}_i}(\eta_i)_m \mid \boldsymbol{\varepsilon}_i\right)\right)$$

Now in the integral $\mathbb{E}\left(\lambda_j^p D_j(0) T_{2j,m} \partial_{\boldsymbol{p}_i}(\eta_i)_m \mid \boldsymbol{\varepsilon}_i\right)$ $\boldsymbol{\varepsilon}_i$ is constant. We can do a change of variables expressing some variables $\tilde{\boldsymbol{\varepsilon}}_i \subset \boldsymbol{\varepsilon}_{-i}$ as functions of prices:

$$\mathbb{E}\left(\lambda_j^p D_j(0) T_{2j,m} \partial_{\boldsymbol{p}_i}(\eta_i)_m \mid \boldsymbol{\varepsilon}_i\right) = \int \int_{\tilde{\mathcal{P}}} \lambda_j^p D_j^p(0) T_{2j,m}^p \partial_{\boldsymbol{p}_i}(\eta_i)_m(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i) \tilde{f}_i(\boldsymbol{\varepsilon}\backslash(\boldsymbol{\varepsilon}_i, \tilde{\boldsymbol{\varepsilon}}_i), \boldsymbol{p}_i \mid \boldsymbol{\varepsilon}_i) \mathrm{d}\boldsymbol{p}_i \mathrm{d}\boldsymbol{\varepsilon}\backslash(\boldsymbol{\varepsilon}_i, \tilde{\boldsymbol{\varepsilon}}_i)$$

Now in the innermost integral we can integrate by parts the term involving the derivative of η_i. By the divergence theorem:

$$\int_{\tilde{\mathcal{P}}} \lambda_j^p D_j^p(0) T_{2j,m}^p \partial_{\boldsymbol{p}_i}(\eta_i)_m(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i) \tilde{f}_i \mathrm{d}\boldsymbol{p}_i =$$

$$\int_{\partial\tilde{\mathcal{P}}} \lambda_j^p D_j^p(0) T_{2j,m}^p (\eta_i)_m(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i) \tilde{f}_i \mathrm{d}\sigma(\boldsymbol{p}_i) - \int_{\tilde{\mathcal{P}}} div_{\boldsymbol{p}_i}\left[\lambda_j^p D_j^p(0) \tilde{f}_i T_{2j,m}^p\right](\eta_i)_m(\boldsymbol{p}_i, \boldsymbol{\varepsilon}_i) \mathrm{d}\boldsymbol{p}_i$$

where $\partial\tilde{\mathcal{P}}$ is the boundary of $\tilde{\mathcal{P}}$, and the operator $div(v_1,\dots,v_n)=\sum_i \partial_i v_i$ is the divergence. Now by assumption $\eta_i=0$ on $\partial\tilde{\mathcal{P}}$, so the middle integral disappears. Hence, integrating also over the $\tilde{\boldsymbol{\varepsilon}}_i$ and $\boldsymbol{\varepsilon}_i$ we obtain:

$$\mathbb{E}\lambda_j^p D_j^p(0)(T_{2j,m}^p\partial_{\boldsymbol{p}_i}(\eta_i)_m)=-\mathbb{E}\partial_{\boldsymbol{p}_i}\left(\lambda_j^p D_j^p(0)T_{2j,m}^p\tilde{f}_i\right)/\tilde{f}_i(\eta_i)_m$$

Finally, the additional term due to the constraint that $\partial_{\boldsymbol{p}_i}\mathcal{S}_{i,-\ell}$ is positive definite can be written:

$$\mathbb{E}\left(\mathcal{I}_{i,j}^p\eta_{i,-\ell}\right)\ \text{where}\ \mathcal{I}_{i,j}^p=D_j^p(0)\lambda_j^p T_{1j}^p+\tilde{T}_{2j}^p$$

where $\tilde{T}_{2j}^p$ is a vector whose components are: $\tilde{T}_{2j,m}^p=div_{\boldsymbol{p}_i}\left(\lambda_j^p D_j^p(0)\tilde{f}_i T_{2,m}^p\right)/\tilde{f}_i$.

With analogous calculations (even simpler, because we need not perform the change of variables $\boldsymbol{\varepsilon}_i\to\boldsymbol{p}_i$ done above), the term relative to $\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}_{i,-\ell}$ is:

$$\mathbb{E}\left(\mathcal{I}_{i,j}^e\eta_i\right)\ \text{where}\ \mathcal{I}_{i,j}^e=D_j^e(0)\lambda_j^e T_{1j}^e+\tilde{T}_{2j}^e$$

where $\tilde{T}_{2j}^e$ is a vector whose components are $\tilde{T}_{2j,m}^e=\partial_{\boldsymbol{\varepsilon}_i}\left(\lambda_j^e D_j^e(0)f_i(\boldsymbol{\varepsilon}_i\mid\boldsymbol{\varepsilon}_{-i})T_{2j}^e\right)/f(\boldsymbol{\varepsilon}_i\mid\boldsymbol{\varepsilon}_{-i})$ and T_{1j}^e and $T_{2j,m}^e$ are the vectors such that:

$$tr\left([\partial_{\boldsymbol{\varepsilon}_i}(\mathcal{S}_i^H)]_j^{-1}\left(\partial_h[\partial_{\boldsymbol{\varepsilon}_i}(\mathcal{S}_i^H+h\eta_i^H)]_j\right)\right)=T_{1j}^e\eta_{i,-\ell}+\sum_{m=1}^{j}T_{2j,m}^e\partial_{\boldsymbol{\varepsilon}_i}(\eta_i)_m$$

The bounds in the norm yield, remembering that $\|A\|_2^2=\rho(A'A)$, where ρ is the maximum eigenvalue, that is simple by assumption, and so by Theorem 4.4 in Demmel (1997):

$$\partial_h\|\partial_{\boldsymbol{p}_i}\mathcal{S}_i\|_2^2=\frac{1}{v_p'u_p}v_p'\partial_h(\partial_{\boldsymbol{p}_i}\mathcal{S}_i'\partial_{\boldsymbol{p}_i}\mathcal{S}_i)u_p$$

where v_p and u_p are respectively the right and left eigenvectors relative to ρ. Now:

$$\partial_h(\partial_{\boldsymbol{p}_i}\mathcal{S}_i'\partial_{\boldsymbol{p}_i}\mathcal{S}_i)=\partial_h\partial_{\boldsymbol{p}_i}\mathcal{S}_i'\partial_{\boldsymbol{p}_i}\mathcal{S}_i+\partial_{\boldsymbol{p}_i}\mathcal{S}_i'\partial_h\partial_{\boldsymbol{p}_i}\mathcal{S}_i=2H(\partial_h\partial_{\boldsymbol{p}_i}\mathcal{S}_i'\partial_{\boldsymbol{p}_i}\mathcal{S}_i)$$

Again, this is linear in the components of $\eta_{i,-\ell}$ and its derivative. Hence there are vectors Q_1^p, $Q_{2,m}^p$ such that:

$$\partial_h\|\partial_{\boldsymbol{p}_i}\mathcal{S}_i\|_2^2\mid_{h=0}=Q_1^p\eta_{i,-\ell}+\sum_m Q_{2,m}^p\partial_{\boldsymbol{p}_i}\eta_{i,m})$$

Now integrating by parts $\partial_{\boldsymbol{p}_i}\eta_{i,m}$ as above we obtain:

$$\mathbb{E}(\mathcal{Q}^p\eta_{i,-\ell}) = 0 \text{ where } \mathcal{Q}^p_m = \lambda^p_+ Q^p_1 + \tilde{Q}^p_2$$

where $\tilde{Q}^p_{2,m} = div_{\boldsymbol{p}_i}(\lambda^p_+ Q^p_{2,m}\tilde{f}_i)/\tilde{f}_i$.

Analogously, for the constraint on $\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}_i$ we obtain: Now integrating by parts $\partial_{\boldsymbol{p}_i}\eta_{i,m}$ as above we obtain:

$$\mathbb{E}(\mathcal{Q}^e\eta_{i,-\ell}) = 0 \text{ where } \mathcal{Q}^p_m = \lambda^p_+ Q^e_1 + \tilde{Q}^e_2$$

where $\tilde{Q}^e_{2,m} = div_{\boldsymbol{\varepsilon}_i}(\lambda^e_+ Q^e_{2,m} f_i)/f_i$, and the vectors Q^e_1, $Q^e_{2,m}$ are such that:

$$\partial_h \|\partial_{\boldsymbol{\varepsilon}_i}\mathcal{S}_i\|^2_2 \mid_{h=0} = Q^e_1\eta_{i,-\ell} + \sum_m Q^e_{2,m}\partial_{\boldsymbol{\varepsilon}_i}\eta_{i,m})$$

□

D.6 Details of Example 3

Write $D_i = \sum_{j\in i} D_{ij}$ for the aggregate demand function from firms in layer i, and similarly $S_i = \sum_{j\in i} S_{ij}$ for the supply. Hence $MC = (S_1 - D_2, \ldots, S_n - D_c)$. Now consider the matrix:

$$diag(\boldsymbol{p})^{-1}\partial_{\boldsymbol{p}} MC diag(\boldsymbol{p}) =$$

$$\begin{pmatrix} \partial_{p_1}S_1 - \partial_{p_1}D_2, & -\partial_{p_2}D_2\frac{p_2}{p_1}, & 0 & \ldots & 0 \\ -\partial_{p_1}S_2\frac{p_1}{p_2} & \partial_{p_2}S_2 - \partial_{p_2}D_3, & -\partial_{p_3}D_3\frac{p_3}{p_2}, & \ldots & 0 \\ 0 & \ldots & 0 & \partial_{p_{n-1}}S_n\frac{p_{n-1}}{p_n} & \partial_{p_n}S_n - \partial_{p_n}D_c \end{pmatrix}$$

By homogeneity, $\frac{\partial_{p_{i-1}}D_i}{\partial_{p_i}D_i} = -\frac{p_i}{p_{i-1}}$, and $\frac{\partial_{p_{i-1}}S_i}{\partial_{p_i}S_i} = -\frac{p_i}{p_{i-1}}$, so on each row of this matrix the sum of the off-diagonal terms is equal to $\partial_{p_i}S_i + \partial_{p_i}D_{i+1}$, which is exactly equal to the diagonal element, but for row 1 and 2, in which one of the addenda is missing and so the diagonal element is larger. Hence the matrix is weakly chained diagonally dominant, so positive definite. So, by similarity, also $\partial_p MC$ is positive definite, and since it has negative off-diagonal elements, it is an M-matrix and $\partial_p MC^{-1}$ has all positive entries. Now $\partial_\varepsilon MC = (0, \ldots, -\partial_\varepsilon D_c)$, and so $\partial_\varepsilon p_i > 0$ for all i. Moreover if $\varepsilon \to \infty$ $D_c \to \infty$ and $\varepsilon \to \infty$ $D_c \to 0$, so the whole price space is reached.

Finally, $\partial_\varepsilon p_i \neq 0$ implies $\partial_{p_i}\varepsilon \neq 0$, that is there exist g_i such that $\varepsilon = g_i(p_i)$, hence the measurability assumption is satisfied.

E Additional proofs of section 4

Consider first the market clearing equations. If all the firms are using symmetric linear schedules with coefficients $(B_i)_i$, then the market clearing equations 1 define a linear system for the prices of goods traded (on active links), because all equations are linear in prices. That is, (1) translates into:

$$\boldsymbol{M}\boldsymbol{p}^a = \boldsymbol{A} + M_\varepsilon \boldsymbol{\varepsilon} \tag{14}$$

$$\tag{15}$$

where the vector of constants $\boldsymbol{A}$ is zero but for the entries corresponding to links to the consumer (that have value $\boldsymbol{A}_{ci} = A_{ci}$). $\boldsymbol{p}^a$ is a vector that stacks all the prices of the goods traded on active links.

This matrix M is the fundamental source of network information in this setting: it is a matrix indexed on the set of *goods* traded on the network, that codifies the dependence of good g on the price of h.

Under the maintained assumptions, the linear system 1 can be partially solved to yield the residual conditional prices $\boldsymbol{p}_{-i}(\cdot \mid i)$:

$$\boldsymbol{p}_{-i}(\boldsymbol{p}, \varepsilon \mid i) = (M_{-i})^{-1}(-M_{C_i}\boldsymbol{p}_i + \boldsymbol{A}_{-i} + M_\varepsilon \boldsymbol{\varepsilon})$$

where M_{-i} and $\boldsymbol{A}_{-i}$ are what remains from the matrices M and $\boldsymbol{A}$ after canceling all the rows corresponding to the inputs and outputs of i, while M_{C_i} is the set of columns of M relative to inputs and outputs of i. This can be substituted in the supply and demand functions of suppliers and customers of i to yield the expression in the next proposition.

Proposition 3. *If all firms $j \neq i$ are using symmetric linear supply and demand schedules with symmetric positive semidefinite coefficients $(B_j)_j$ and positive definite $(B_{i,\varepsilon})_i$, generically in the values of $(B_j)_j$ there exist a domain $\mathcal{F}_i$ containing a neighborhood of $\varepsilon_i = 0$ such that the residual supply and demand schedule for active links of sector i is linear and can be written as:*

$$\mathcal{S}_i^r = \begin{pmatrix} -D_i^r \\ S_i^r \end{pmatrix} = -\tilde{\boldsymbol{A}}_i - \Lambda_i^{-1} p_i^a + \Lambda_{\varepsilon,i}\varepsilon$$

Moreover, $\Lambda_{i,\varepsilon}$ has full rank, and Λ_i is symmetric positive definite and equal to the matrix $[M_i^{-1}]_i$, where:

- *M_i is the matrix obtained by M by setting B_i to 0;*

- *if A is a matrix indexed by edges, $[A]_i$ is the submatrix of A relative to all the links that are either entering or exiting i.*

The coefficient Λ_i can be thought as a (sector level) *price impact*[40]: the slope coefficients of the (inverse) supply and demand schedules, describing what effect on prices firms in sector i can have. It is a measure of market power: the larger the price impact, the larger the rents firms in that sector can extract from the market.

E.1 Perfect competition benchmark

If a firm takes prices as given will optimize:

$$\max_{q_{ki}, q_{ij}, z_{i,kj}} \sum_k p_k q_{ki} - \sum_j p_j q_{ij} - \frac{1}{2} \sum_{k,h} z_{i,jk} \Sigma_{i,jk,hm} z_{i,hm}$$

subject to:

$$q_{ki} = \sum_j \omega_{ij} z_{i,kj}, \quad q_{ij} = \sum_k z_{i,kj}$$

In vector notation:

$$\max_{q_{ki}, q_{ij}, z_{i,kj}} \boldsymbol{p}_i' \boldsymbol{q}_i - \frac{1}{2} \boldsymbol{z}_i' \Sigma_i \boldsymbol{z}_i$$

subject to:

$$U_i \boldsymbol{z}_i + \alpha_i \ell_i = \boldsymbol{q}_i$$

where α_i is a diagonal matrix with on the diagonal $\alpha_{i,k}$, and U_i vertically stacks $I_{out,in} \otimes \omega_i$, and $-I_{in,out} \otimes u_{in}$

The FOC yield, defining λ_i as he vector of multipliers:

$$\begin{aligned} 0 &= -\boldsymbol{z}_i' \Sigma + \lambda_i' U_i \\ 0 &= -l_i' + \lambda_i' \alpha_i \\ \boldsymbol{p}_i &= \lambda_i \end{aligned}$$

so, solving for the quantities: $\boldsymbol{z}_i = \Sigma_i^{-1} U_i' \boldsymbol{p}_i$, $l_i = \alpha_i \lambda_i$, and plugging them into the constraint yields:

$$\boldsymbol{q}_i = U_i \boldsymbol{z}_i + \alpha_i \ell_i = \left(U_i \Sigma_i^{-1} U_i' + \alpha_i^2 \right) \boldsymbol{p}_i$$

That is the demand of firms under perfect competition. Define the matrix $C_i = U_i \Sigma_i^{-1} U_i + \alpha_i^2$ as the *perfect competition matrix* for i. If at least one $\alpha_i > 0$, it

[40] Using a financial terminology. It is also the reason for the notation: in finance it is common to denote Λ the price impact of traders.

follows that it is positive definite and symmetric. If $\Sigma_i = I$ it reduces to:

$$C_i = \begin{pmatrix} \omega_i' \omega_i I_i^{out} & u_{out} \omega_i' \\ \omega_i u_{out}' & d_i^{out} I_i^{in} \end{pmatrix}$$

Moreover, the profit is:

$$\pi_i = \boldsymbol{p}_i' \left(U_i \Sigma_i^{-1} U_i + \alpha_i^2 \right) \boldsymbol{p}_i - \frac{1}{2} \boldsymbol{p}_i' \left(U_i \Sigma_i^{-1} U_i + \alpha_i^2 \right) \boldsymbol{p}_i = \frac{1}{2} \boldsymbol{p}_i' C_i \boldsymbol{p}_i$$

and since C_i is positive definite, is always nonnegative. We can see that if firms are all producing the same quantities, as in Section **??**, the profits are the same for all.

It is possible to explicitly compute the walrasian equilibrium allocation with this technology. Indeed, in this context the market clearing system is $M\boldsymbol{p} = \boldsymbol{A}$, and the matrix M is the sum of the lifted C_is. Now, having the prices, we can just compute the quantities from the supply/demand functions computed above. This gives analytic expressions for prices and quantities as functions of the fundamental parameters.

E.2 Proof of Proposition 3

M is the Jacobian J of Proposition 1 specialized in this linear setting. By the same Proposition, it is invertible and positive definite.

Lemma E.1. Consider the matrix $\tilde{M}_i$ obtained from the matrix M by eliminating the coefficient B_i. It is positive definite (hence invertible) if:

- Assume $\alpha_i > 0$, and that all goods are traded by at least three firms;
- If $\alpha_i = 0$ for some i, and for each of its inputs and outputs there exists other 2 firms that produce and buy it.

(A sufficient condition for the second assumption is that $n_i \geq 2$ in the sector level model).

Moreover, the matrix $\boldsymbol{A}_{-i}$ is also positive definite and invertible.

Proof. With an analogous reasoning as in Proposition 1, we obtain that $x' \tilde{M}_i x = \sum_{j \neq i} x_j' B_j x_j$.

If $\alpha_i > 0$ for all i, all the B_m are positive definite. Hence, the only possibility for $x' \tilde{M}_i x$ to be 0 is if some good g is not traded by any of the j firms, in which case the vector x identically zero for all $h \neq g$, and an arbitrary number different from zero at g nullifies the quadratic form. If all the goods are traded by at least one other firm, this can never happen.

If $\alpha_i = 0$, then some of the B_i are positive semidefinite, and nullified by the positive vectors $\tilde{u}_i$. Since B_c is nullified by the zero vector, the only possibility for the sum above to be zero is that subtracting firm i creates a disconnected network, whose matrix now is nullified by the vector that has all zeros for the goods produced in the component where the consumer is, and all the relevant elements from $\tilde{u}_i$ in the other component. (This might be impossible depending on the coefficients, but we want to keep the proof independent of specific parameters). Under the above assumptions canceling firm i never creates a disconnected network. Since all $B_{i,\varepsilon}$ matrices are positive semidefinite, the same reasoning proves that $\boldsymbol{A}_{-i}$ is positive definite, hence invertible.

□

Now we prove the main Proposition.

I prove explicitly that there are sets $\mathcal{F}_i$ such that $p_i^*(0)$ is feasible, such that the partial solution $p_{-i}^*(\varepsilon, p_i)$ is linear. Hence, the residual demand is linear on some set $\mathcal{F}_i$.

Let us calculate it explicitly. The market clearing system of equations conditional on i is:

$$MC_g(p, \varepsilon) = 0 \, \forall g \notin \mathcal{N}_i$$

that, under the assumption that all schedules of firms are linear, is:

$$\begin{pmatrix} M_{C_i} & M_{-i} \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_i \\ \boldsymbol{p}_{-i} \end{pmatrix} = \boldsymbol{A}_{-i}(\varepsilon)$$

where we reorder the entries of the matrix M to have in the leading upper left position all the rows that represent equations involving input and output goods of firm i, and all the columns relative to prices of input and output of i. M_{-i} is M from which we cancelled all the rows and columns relative to i, M_{C_i} is the matrix of all columns relative to inputs and outputs of i. Write M_i for the matrix M subject to this reordering. And solving:

$$M_{-i} p_{-i} = -M_{C_i} \boldsymbol{p}_i + \boldsymbol{A}_{-i} \Longrightarrow p_{-i} = M_{-i}^{-1}(-M_{C_i} \boldsymbol{p}_i + \boldsymbol{A}_{-i}(\varepsilon))$$

The remaining market clearing conditions are:

$$S_{gi}(\boldsymbol{p}_i, \varepsilon_i) = D^r_{gi}(\boldsymbol{p}_i, \varepsilon) = \underbrace{\sum_{g \to k} D_{kg}(\boldsymbol{p}_k(\boldsymbol{p}_i, \varepsilon \mid i), \varepsilon_k)}_{\text{demand for good } g} - \underbrace{\sum_{k \to g, k \neq i} S_{gk} \boldsymbol{p}_k(\boldsymbol{p}_i, \varepsilon \mid i), \varepsilon_k)}_{\text{supply by competitors}}$$

$$D_{ig}(\boldsymbol{p}_i, \varepsilon_i) = S^r_{ig}(\boldsymbol{p}_i, \varepsilon) = \underbrace{\sum_{j \to g} S_{gj}(\boldsymbol{p}_j(\boldsymbol{p}_i, \varepsilon \mid i), \varepsilon_j)}_{\text{supply of good } j} - \underbrace{\sum_{g \to j, j \neq i} D_{jg}(\boldsymbol{p}_j(\boldsymbol{p}_i, \varepsilon \mid i), \varepsilon_j)}_{\text{demand by competitors}} \quad \forall g \in \mathcal{N}_i$$

Using the matrices just defined, and changing the sign of the first equation, we get:

$$\begin{aligned}
\mathcal{S}^r_i = \begin{pmatrix} -D^r_i \\ S^r_i \end{pmatrix} &= ([M]_i - B_i)\boldsymbol{p}_i + M'_{C_i} \boldsymbol{p}_{-i} \\
&= ([M]_i - B_i - M'_{C_i} M^{-1}_{-i} M_{C_i})\boldsymbol{p} + M'_{C_i} M^{-1}_{-i} \boldsymbol{A}_{-i} \boldsymbol{\varepsilon}_{-i}
\end{aligned}$$

and denoting $\tilde{M}_i$ the matrix $M - E_i B_i E'_i$, we have that the coefficient matrix can be rewritten as $[M]_i - B_i - M'_{C_i} M^{-1}_{-i} M_{C_i} = [(\tilde{M}_i)^{-1}]^{-1}_i$, via block matrix inversion, and moreover is positive definite. The fact that $\tilde{M}_i$ is positive definite and hence invertible follows from the Proposition 1.

Now defining: $\Lambda^{-1}_i = [(\tilde{M}_i)^{-1}]^{-1}_i$, we obtain the expression in the main text:

$$\mathcal{S}^r_i = \Lambda^{-1}_i \boldsymbol{p}_i + \tilde{\boldsymbol{A}}_i(\varepsilon)$$

where $\tilde{\boldsymbol{A}}_i(\varepsilon) = M'_{C_i} M^{-1}_{-i} \boldsymbol{A}_{-i}(\varepsilon)$

E.3 Proof of Theorem 3

Rewrite best reply as a finite dimensional optimization This paragraph essentially proves that in the linear case the measurability condition is satisfied.

Assume all other firms in all other sectors are playing a profile of symmetric linear schedules that for the prices relative to active links have coefficients $(B_j)_j$ which are positive semidefinite. Consider the best reply problem of firm α in sector i. This is:

$$\max_{(S_{ki})_k, (D_{ij})_j, (z_{i,kj})_{k,j}} \mathbb{E}\left(\sum_k p^*_{ki} S_{ki} - \sum_j p^*_{ij} D_{ij} - \varepsilon_i \sum z_{i\alpha,kj} - \frac{1}{2}\sum_{k,j} z^2_{\alpha,kj}\right)$$

subject to the market clearing conditions 1. All the sums run over active links: prices relative to inactive links do not affect the objective function nor the constraints. I already used the fact that at the optimum it must be $l_{i\alpha,kj} = \varepsilon_i z_{i\alpha,kj} + \frac{1}{2} z_{i\alpha,kj}^2$.

Using the residual demand, we can rewrite the optimization as:

$$\max_{(S_{ki})_k,(D_{ij})_j,(z_{i,kj})_{k,j}} \mathbb{E}\left(\sum_k p_{ki}^* S_{ki} - \sum_j p_{ij}^* D_{ij} - \varepsilon_i \sum z_{i,kj} - \frac{1}{2}\sum z_{i,kj}^2\right)$$

subject to:

$$D_{ki}^r((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_k z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall i \to k \tag{16}$$

$$S_{ij}^r((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_j \omega_{ij} z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall j \to i \tag{17}$$

$$D_{ki}^r(p_i^{out}, p_i^{in}, \varepsilon) = S_{ki}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall i \to k \tag{18}$$

$$S_{ij}^r(p_i^{out}, p_i^{in}, \varepsilon) = D_{ij}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall j \to i \tag{19}$$

Now assume ε is in the set $\mathcal{E}_i$ where Proposition 3 applies. Then since Λ_i^{-1} is invertible the last two conditions in 16 define uniquely a function for the prices of active links $p_i^*(\varepsilon) : \mathcal{E}_i \to \mathbb{R}^{d_i}$. Then we can rewrite the optimization as:

$$\max_{(S_{ki})_k,(D_{ij})_j,(z_{i,kj})_{k,j},p_i^*} \mathbb{E}\left(\sum_k p_{ki}^* D_{ki}^r - \sum_j p_{ij}^* S_{ij}^r - \varepsilon_i \sum z_{i,kj} - \frac{1}{2}\sum z_{i,kj}^2\right)$$

subject to:

$$D_{ki}^r((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_k z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall i \to k \tag{20}$$

$$S_{ij}^r((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_j \omega_{ij} z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall j \to i \tag{21}$$

$$\tag{22}$$

Now (S, D) do not appear explicitly in the problem any more. For the active links, we can recover them using the information in the pricing function. Indeed, for any x in the range of p_i^*, define:

$$S_{ki}(x, \varepsilon_i) = D_{ki}^r(x, \varepsilon),\ \forall i \to k$$
$$D_{ij}(x, \varepsilon_i) = S_{ij}^r(x, \varepsilon),\ \forall j \to i$$

for some $\varepsilon \in (p_i^*)^{-1}(x)$. By definition of p_i^*, the relation above must be satisfied for all the elements in the counterimage. For all the non-active links, they are both identically zero.

Finally, optimizing with respect to a function of the stochastic variable is equivalent to optimizing ex-post, for any fixed value of ε, as in Klemperer and Meyer (1989). Hence we can write the best reply problem in its final form:

Optimization The best reply problem here is:

$$\max_{p_{ki}, p_{ij}, z_{i,kj}} \sum_k p_k D_{ki}^r - \sum_j p_j S_{ij}^r - \sum_{j,k} \varepsilon_{i,jk} z_{i,jk} - \frac{1}{2} \sum_{k,h} z_{i,jk} \Sigma_{i,jk,hm} z_{i,hm}$$

subject to:

$$D_{ki}^r = \sum_j \omega_{ij} z_{i,kj} + \sum_k \alpha_{ik} \ell_{ik}, \quad S_{ij}^r = \sum_k z_{i,kj}$$

In vector notation:

$$\max_{p_{ki}, p_{ij}, z_{i,kj}} -\boldsymbol{p}_i' \mathcal{S}^r - \boldsymbol{\varepsilon}_i' \boldsymbol{z}_i - \frac{1}{2} \boldsymbol{z}_i' \Sigma_i \boldsymbol{z}_i$$

subject to:

$$U_i \boldsymbol{z}_i + \alpha_i \ell_i = -\mathcal{S}_i^r$$

where α_i is a diagonal matrix with on the diagonal $\alpha_{i,k}$, and U_i vertically stacks $I_{out,in} \otimes \omega_i$, and $-I_{in,out} \otimes u_{in}$.

Call λ_i the vector of multipliers for input and output constraints respectively. The Hessian of the problem is a block diagonal matrix with blocks $-(J_i + J_i')$ and minus Σ_i (with respect to the zs), so the problem is concave.

The FOCs yield:

$$\begin{aligned}
\boldsymbol{z}_i &: 0 = -\boldsymbol{\varepsilon}_i' U_i - \boldsymbol{z}_i' \Sigma + \lambda_i' U_i \\
l_i &: 0 = -\ell_i' + \lambda_i' \alpha_i \\
\boldsymbol{p}_i &: 0 = (\lambda_i' - \boldsymbol{p}_i') \Lambda_i^{-1} - \mathcal{S}^r
\end{aligned}$$

The first conditions are identical to the perfect competition, except for the multipliers replaced by prices: $\boldsymbol{z}_i = \Sigma_i^{-1} U_i' (\lambda_i - \boldsymbol{\varepsilon}_i)$, $l_i = \alpha_i \lambda_i$. Plugging them into the constraint yields:

$$-\mathcal{S}_i^r = U_i \boldsymbol{z}_i + \alpha_i \ell_i = C_i \lambda_i - U_i \Sigma_i^{-1} U_i' \boldsymbol{\varepsilon}_i$$

which is the exact analogue. From the expressions we have we can solve for λ_i:

$$\lambda_i = (C_i + \Lambda_i^{-1})^{-1}(\Lambda_i^{-1}\boldsymbol{p}_i + U_i\Sigma_i^{-1}U_i'\boldsymbol{\varepsilon}_i)$$

We have first to show that the FOC define a function in an open set around $\varepsilon = 0$. Using the expression above and the FOC we can solve for the prices:

$$-\Lambda_i^{-1}\boldsymbol{p}_i - M_{C_i}'M_{-i}^{-1}\boldsymbol{A}_{-i}\boldsymbol{\varepsilon}_{-i} = -\mathcal{S}_i^r = \Lambda_i^{-1}(\boldsymbol{p}_i - \lambda_i)$$

$$-\Lambda_i^{-1}\boldsymbol{p}_i - M_{C_i}'M_{-i}^{-1}\boldsymbol{A}_{-i}\boldsymbol{\varepsilon}_{-i} = \Lambda_i^{-1}(\boldsymbol{p}_i - (C_i + \Lambda_i^{-1})^{-1}(\Lambda_i^{-1}\boldsymbol{p}_i + U_i\Sigma_i^{-1}U_i'\boldsymbol{\varepsilon}_i))$$

$$\boldsymbol{p}_i = (2\Lambda_i^{-1} - \Lambda_i^{-1}(C_i + \Lambda_i^{-1})^{-1}\Lambda_i^{-1})^{-1}(\Lambda_i^{-1}(C_i + \Lambda_i^{-1})^{-1}U_i\Sigma_i^{-1}U_i'\boldsymbol{\varepsilon}_i - M_{C_i}'M_{-i}^{-1}\boldsymbol{A}_{-i}\boldsymbol{\varepsilon}_{-i})$$

The inverse exists because $\Lambda_i^{-1} - \Lambda_i^{-1}(C_i + \Lambda_i^{-1})^{-1}\Lambda_i^{-1}$ is positive definite: $\|\Lambda_i^{-1}(C_i + \Lambda_i^{-1})^{-1}\| = \|(C_i\Lambda_i + I)^{-1}\| < 1$ and the result follows from Theorem 7.7.3 of Horn and Johnson (2012). Hence is by construction full rank, and also M_{C_i} and $\boldsymbol{A}_{-i}$. It follows that the map from the $\boldsymbol{\varepsilon}_{-i}$ to prices is surjective, and spans an open set in the region without corner solutions. Hence the first order conditions pin down the values of the best reply schedule in an open region of the price space, and we can write:

$$\mathcal{S}_i = -\mathcal{S}^r = \Lambda_i^{-1}(\boldsymbol{p}_i - \lambda_i)$$

and using the expression we got for λ_i:

$$\mathcal{S}_i = (\Lambda_i^{-1} - \Lambda_i^{-1}(C_i + \Lambda_i^{-1})^{-1}\Lambda_i^{-1})\boldsymbol{p}_i + B_{\varepsilon,i}\boldsymbol{\varepsilon}_i$$

where we defined $B_{\varepsilon,i} = \Lambda_i^{-1}(C_i + \Lambda_i^{-1})^{-1}U_i\Sigma_i^{-1}U_i'$, that has full rank minus 1, and moreover, by results in Anderson Jr and Duffin (1969), is positive definite. Moreover, $C_i = B_{i,\varepsilon}U_i' + \alpha_i'\alpha_i$. This is the expression of our best response, and the coefficient matrix is positive definite with a reasoning similar to the one above. Moreover, if C_i is invertible, by the Woodbury formula the coefficient matrix can be written as $(C_i^{-1} + \Lambda_i)^{-1}$.

The equilibrium equations have the same form of the equilibrium equations in Malamud and Rostek (2017). From their Theorem 2 it follows that there exists a fixed point, and can be reached iterating the best reply equations from above or below, for appropriately chosen initial conditions. $\square$

There exist a profile of coefficients implying positive trade Now I prove that there exist one that yields positive trade if we limit ourselves to a subset of links - that will be the active links in equilibrium.

Start from the original network $G = (N, E)$. Set $n = 0$ and $L_1 = E$.

1. Find the unconstrained equilibrium profile B_n^* in the network $G_i = (N, L_n)$. Identify the set of links that have negative trade or negative price $E_{n,0}$.

2. set $L_{n+1} = L_n / E_{n,0}$;

The set of links shrink at each step, and when the network is empty there are no negative trades. Hence there must exist an index $\hat{\imath}$ such that for all $i > \hat{\imath}$ $L_i = L_{\hat{\imath}}$. The equilibrium $B_{\hat{\imath}}^*$, augmented with identically zero functions for all excluded links, is an equilibrium of the original game.

Generic Equilibrium existence It remains to prove that the profile of matrices $(B_i^*)_i$ identified above constitute the coefficient matrices of a profile of linear schedules for an open set $\mathcal{F}$ that contains $(p^*(0), 0)$. To prove this, consider the linear functions defined by $(B_i^*)_i$ and extend them to the whole price space.

Consider:

$$(S_{-1}, D)_i = \tilde{B}_i(-p_1 \tilde{u}_{-1} + p_{i,-\ell} + t_i) + \varepsilon_i B_{\varepsilon,i,-1}$$

where t_i solves the Linear Complementarity problem:

$$\tilde{B}_i(-p_1\tilde{u}_{-1}+p_{i,-\ell}+t_i)+\varepsilon_i B_{\varepsilon,i,-\ell} \geq 0 \quad t'_{i,-\ell}(\tilde{B}_i(-p_1\tilde{u}_{-1}+p_{i,-\ell}+t_i)+\varepsilon_i B_{\varepsilon,i}) = 0 \quad t_i \geq 0$$

This corresponds to the form of the solution of the Optimization 6, where t_i is a function of the Lagrange multipliers on the nonnegativity constraints.

Using this form we see that the market clearing conditions can be written as a Linear Complementarity Problem:

$$B_{ij}(p_i + t_i) + \varepsilon_i B_{\varepsilon,i} = B_{ij}(p_j + t_j) + \varepsilon_i B_{\varepsilon,j} \tag{23}$$

$$B_i(p_i + t_i) + \varepsilon_i B_\varepsilon \geq 0 \tag{24}$$

$$t'_i(B_i(p_i + t_i) + \varepsilon_i B_\varepsilon) = 0 \tag{25}$$

$$t_i \geq 0 \tag{26}$$

The first set of equations can be rewritten as $M(p+t) = \boldsymbol{A} + M_\varepsilon \varepsilon$ and solved for $p + t$ since M is invertible. So to compute which t variables are not zero it is sufficient to use the complementary slackness condition. Moreover, it is a standard result (Cottle et al. (2009), Proposition 1.4.6) that the solution as a function of ε is piecewise linear. Positive definiteness proves that the solution is unique and so non-ambiguous (Cottle et al. (2009) Theorem 3.1.6).

Now the fact that we can express the residual demand as a linear function for all i relies on the fact that $(0, p^*(0))$ lies in one of the regions where the function is linear and not on one of the boundary regions. Now the boundary regions are identified by a set of equations $F_j((B_i)_i, \varepsilon) = 0$ for some indices j, where the F are affine functions (see Cottle et al. (2009), Prop. 1.4.6.). Each boundary region, hence, is an hyperplane, and has measure zero. This means that the set of profiles of coefficients such that 0 is in one of the boundary regions:

$$\mathcal{B}_F = \{(B_i)_i \mid F_j((B_i)_i, 0) = 0 \forall j\}$$

has itself measure zero (it is a finite union).

Now consider the map $\mathcal{O} : (\omega_i)_i \to (B_i^*)$ that maps the values of the parameters to the B_i^* that solve 7. I prove that this is one-to-one. To see this, suppose $\mathcal{O}((\omega_i)_i) = \mathcal{O}((\omega_i')_i)$. Then by the construction of 3 we get that $\Lambda_i((\omega_i)_i) = \Lambda_i((\omega_i')_i)$, and by the equation 7 we get that the perfect competition matrices must agree too: $(C_i)_i = (C_i')_i$. From this, inspecting the matrix, it follows that $(\omega_i)_i = (\omega_i')_i$. Moreover it is continuous. By the implicit function theorem, is also differentiable. It follows that the inverse image of a measure zero set has measure zero.

Since $\mathcal{O}$ is a homeomorphism the preimage of a rare set is rare, and so we conclude that the property of existence of a linear eqilibrium is generic in $(\omega_i)_i$.

E.4 Neoclassical microfoundation of handling costs

The structure for the labor costs can be rationalized via a neoclassical production function, as the next proposition clarifies.

Proposition 4. *If the production possibility set of firm i is the set of $(\boldsymbol{q}^{out}, \boldsymbol{q}^{in}, \ell) \in \Phi_i$ such that $\exists \boldsymbol{z}_i \in \mathbb{R}^{d_i^{out} d_i^{in}}$ such that:*

$$q_{ki} = \sum_j \omega_{ij} \min\{f_{i,kj}(\ell_{i,k1}, \ldots, \ell_{i,kd_i^{in}}), z_{i,kj}\} + \alpha_{ki}\sqrt{\ell_{ki}} \quad q_{ij} = \sum_k z_{i,kj}$$

where, for every k, $(f_{i,kj})_j$ is implicitly defined by the equations:

$$\ell_{i,kj} = (\varepsilon_{i,k}\omega_{ij} - \varepsilon_{i,j}) f_{i,kj} + \frac{1}{2k_i} \sum_h \sigma_{i,k,jh} f_{i,kj} f_{i,kh} \quad \forall j$$

where the matrices $\Sigma_{i,k} = (\sigma_{i,k,jh})_{j,h \in \mathcal{N}_i}$, and if the ε parameters are small enough then, at the optimum, the payments to labor and the quantities demanded have *exactly the relation in equation 4.2, defining Σ_i as the block matrix that has as k-th diagonal blocks the matrix $\Sigma_{i,k}$.*

Proof. First, we need to prove that the function $f_{i,kj} : \mathbb{R}^{d_i^{out} d_i^{in}} \to \mathbb{R}^{d_i^{out} d_i^{in}}$ describing the combination of labor tasks to be allocated to deal with $z_{i,jk}$, is well defined. It is implicitly defined by:

$$\ell_{i,kj} = (\varepsilon_{i,k}\omega_{ij} - \varepsilon_{i,j})f_{i,kj} + \frac{1}{2k_i}\sum_h \sigma_{i,k,jh} f_{i,kj} f_{i,kh} \quad \forall k, j$$

For every fixed output k, the vector function $(f_{i,jk})_j$ is defined implicitly by:

$$F_{ik}((\ell_{i,kj})_j, (f_{i,kj})_j) = -\ell_{i,kj} + (\varepsilon_{i,k}\omega_{ij} - \varepsilon_{i,j})f_{i,kj} + \frac{1}{2k_i}\sum_h \sigma_{i,k,jh} f_{i,kj} f_{i,kh} = 0 \quad \forall j$$

The Jacobian of F_{ik} with respect to $(f_{i,kj})_j$ is the matrix with entries $(\varepsilon_{i,k}\omega_{ij} + \varepsilon_{i,j})\delta_{hj} + \frac{1}{2k_i}\sigma_{i,k,jh} f_{i,kj} + \frac{1}{2k_i}\delta_{jh}\sigma_{i,k,jj} f_{i,kj}$. This is:

$$J_{i,k} = \varepsilon_{i,k} diag(\omega_i) - diag(\varepsilon_i^{in}) + \frac{1}{2k_i}\Sigma_{i,k} diag(f_{i,k\cdot}) + \frac{1}{2k_i} diag(\Sigma_{i,k}) diag(f_{i,k\cdot})$$

The matrix $\Sigma_{i,k}$ is positive definite by assumption, and $f_{i,k\cdot} > 0$, then it follows that $\Sigma_{i,k} diag(f_{i,k\cdot})$ is positive definite, in the sense that $\boldsymbol{x}'\Sigma_{i,k} diag(f_{i,k\cdot})\boldsymbol{x} > 0$ if $\boldsymbol{x} \neq 0$, even if not symmetric.[41] If the ε are small enough the other terms of the sum are negligible, so it follows that J_{ik} is positive definite. Hence, on any convex set we can apply the Gale Nikaido theorem (Gale and Nikaido (1965)) and conclude that the equation above can be inverted. Hence, this proves that the equations above define f as a function of the ℓ. This is true on any convex set in which the f are strictly positive. But $f_{i,kj} = 0$ if and only if $\ell_{i,kj} = 0$, so we can extend the uniqueness also to the case in which the variables reach 0. So the above is a well-defined function.

The expression for the handling costs at the optimum is immediate, because at the optimum it must be $f_{i,kj} = z_{i,kj}$, and so:

$$\sum_{k,j} \ell_{i,kj} = \sum_k \varepsilon_{i,k} \sum_j \omega_{ij} z_{i,kj} - \sum_j \varepsilon_{ij} \sum_k z_{i,kj} + \frac{1}{2k_i} \sum_k \Sigma_{i,k} \boldsymbol{z}_i' \boldsymbol{z}_i$$

in matrix notation:

$$\sum_{k,j} \ell_{i,kj} = \boldsymbol{\varepsilon}_i' U_i \boldsymbol{z}_i + \frac{1}{2k_i} \boldsymbol{z}_i' \Sigma_i \boldsymbol{z}_i$$

[41] This follows from Theorem 7.6.3 in Horn and Johnson (2012), because is the product of two symmetric positive definite matrices, hence has all positive eigenvalues.

which is what we wanted to show. □

□

The technology above codifies the idea that each input, to be productive, needs to be complemented with a quantity of labor ℓ_{ij}. Hence, the firm also needs to hire a quantity of labor $\sum_j \ell_{ij}$ in addition to ℓ_i, that we can think as labor allocated to generic tasks (e.g. management, organization). Because of the Leontief term, at the optimum it must be $-k_i\varepsilon_{ij} + \sqrt{k_i^2\varepsilon_{ij}^2 + 2k_i\ell_{ij}} = q_{ij}$. This is independent of the specific market structure or optimization the firm is performing, and simply follows from avoiding waste of inputs. Inverting this equation we find that the quantity of labor associated to input j must be: $\ell_{ij} = \varepsilon_{ij}q_{ij} + \frac{1}{2}q_{ij}^2$. Summing across all j we recover exactly the handling cost formulation in 4.2.

The idea that allows to incorporate substitutability across inputs is to assume that labor has to be divided into "tasks" $l_{i1}, l_{i2}, \ldots$, and each input, to be productive, needs to be complemented with a specific combination of labor allocated to different tasks f_{ij}. The overlap among tasks used for different inputs will produce the substitutability or complementarity, as the following proposition illustrates.

The parameter k_i is a constant that can be thought as the fixed endowment of capital that the firm has: under this interpretation, the above technology has the usual property of constant returns to scale.

The profit of the firm, hence, can be written as:

$$\pi_i = p_i \sum \omega_{ij}q_{ij} + \alpha_i\sqrt{\ell_i} - \sum_j p_j q_{ij} - \sum_j \varepsilon_{ij}q_{ij} - \frac{1}{2k_i}(\sum q_{ij}^2 + \ell_i^2)$$

This makes it clear that, *for given prices*, inputs are neither substitutes nor complements: a competitive firm taking prices as given would not change its demand for q_{ij} when p_k changes. Under imperfect competition this is not true anymore.

E.5 Networks with no corner solutions

If the network is a tree such that each sector has just one customer sector, as in Figure 1, then it is easy to prove that in equilibrium there is trade on all links. Indeed, in this case is possible to prove that equilibrium prices are all strictly positive. Then, if i has 0 suppliers, then in equilibrium produces $q_i = B_i p_i > 0$. If sector j has only roots as suppliers, since they all produce strictly positive quantities it follows that $q_j = \sum \omega_{jk}q_k > 0$. Iterating the reasoning we obtain that on all links there is positive trade.

F Additional proofs of section 5

F.1 Proof of Lemma B.1

By induction, I prove that:

$$\overline{\Lambda}_i = \frac{\prod_{k\neq i} n_k B_k B_c}{\prod_{k\neq i} n_k B_k + B_c \sum_{j\neq i} \prod_{k\neq i, k\neq j} n_k B_k} \tag{27}$$

$$\overline{\Lambda}_N = \frac{\prod_{k\neq 1} n_k B_k B_c}{\prod_{k\neq 1} n_k B_k + B_c \sum_{j\neq 1} \prod_{k\neq 1, k\neq j} n_k B_k} \tag{28}$$

By induction on the size of the line N. If $N = 2$ it can be checked by calculation. Assume it holds for a line of size $N-1$. To get the corresponding expressions for a line of size N we must substitute B_c with the objective demand of the last but one layer, which is $\Lambda_i^{out} = \frac{n_N B_N B_c}{n_N B_N + B_c}$. If we do it we get that for $i \leq N-1$:

$$\overline{\Lambda}_i = \frac{\prod_{k\neq i} n_k B_k \frac{n_N B_N B_c}{n_N B_N + B_c}}{\prod_{k\neq i} n_k B_k + \frac{n_N B_N B_c}{n_N B_N + B_c} \sum_{j\neq i} \prod_{k\neq i, k\neq j} n_k B_k}$$

and reordering and simplifying the denominator we get the expression above. Analogously can be done for Λ_1^{out}. Moreover, always by induction we can find:

$$\Lambda_1^{in} = \frac{\prod_{k\neq 1} n_k B_k}{\sum_{k\neq 1} n_k B_k}$$

and $\Lambda_1^{out} = B_c$, so substituting in the corresponding expression:

$$\overline{\Lambda}_1 = \frac{\Lambda_1^{out}\Lambda_1^{in}}{\Lambda_1^{out} + \Lambda_1^{in}} = \frac{\frac{\prod_{k\neq 1} n_k B_k}{\sum_{k\neq 1} n_k B_k} B_c}{B_c + \frac{\prod_{k\neq 1} n_k B_k}{\sum_{k\neq 1} n_k B_k}}$$

and simplifying we get the desired result. $\square$

F.2 Proof of Lemma B.2

To apply the theory of monotone comparative statics, I will prove that if $n_i \geq n_j$ then $BR_i(x, B_{-i,j}) \geq BR_j(x, B_{-i,j})$, that is the best reply of i dominates the best reply of j conditional on the coefficients of all other sectors.

We have that $BR_i \geq BR_j$ if and only if:

$$\overline{\Lambda}_i^{-1} + (n_i - 1)x \geq \overline{\Lambda}_j^{-1} + (n_j - 1)x$$

In particular, using the characterization of $\overline{\Lambda}_i^{-1}$ above, we have that this is true if and only if:

$$\frac{\mathcal{B}B_c n_j x}{\mathcal{B}(n_j x + b_C) + n_j x \mathcal{F}} + (n_i - 1)x \geq \frac{\mathcal{B}B_c n_i x}{\mathcal{B}(n_i x + b_C) + n_i x \mathcal{F}} + (n_j - 1)x$$

where $\mathcal{B}$ and $\mathcal{F}$ are only functions of the coefficients $B_{-i,j}$ and their respective number of firms. This is true if and only if

$$\frac{\mathcal{B}B_c n_j}{\mathcal{B}(n_j x + b_C) + n_j x \mathcal{F}} - (n_j - 1) \geq \frac{\mathcal{B}B_c n_i}{\mathcal{B}(n_i x + b_C) + n_i x \mathcal{F}} - (n_i - 1)$$

$$\frac{\mathcal{B}B_c n_j - (n_j - 1)(\mathcal{B}(n_j x + B_C) + n_j x \mathcal{F})}{\mathcal{B}(n_j x + B_C) + n_j x \mathcal{F}} \geq \frac{\mathcal{B}B_c n_i - (n_i - 1)(\mathcal{B}(n_j x + B_C) + n_j x \mathcal{F})}{\mathcal{B}(n_i x + B_C) + n_i x \mathcal{F}}$$

$$\frac{\mathcal{B}B_c - (n_j - 1)(\mathcal{B}(n_j x) + n_j x \mathcal{F})}{\mathcal{B}(n_j x + b_C) + n_j x \mathcal{F}} \geq \frac{\mathcal{B}B_c - (n_i - 1)(\mathcal{B}(n_j x) + n_j x \mathcal{F})}{\mathcal{B}(n_i x + B_C) + n_i x \mathcal{F}}$$

which is true if and only if $n_i \geq n_j$ because the function is decreasing.

Then we can conclude that if $n_i \geq n_j$ then $BR_i(x, B_{-i,j}) \geq BR_j(x, B_{-i,j})$, and so, using a result from Lazzati (2013) we can conclude that in equilibrium $B_i^* \geq B_j^*$. □

G Additional proofs of Section 6

G.1 Proof of Lemma C.1

$BR_i(B_{-i}, 2T_i) < BR_i(B_{-i}, 2T_i')$ is equivalent to $\Lambda_i^{2T_i} < \Lambda_i^{2T_i'}$. Then, with algebraic manipulations we find:

$$\begin{aligned}&\Lambda_i^{2T_i} - \Lambda_i^{2T_i'} = \\ &- M_{C_i}' D_{-i}^{-1/2} \tilde{L}_{-i}^{T_i'} \left(M_{-i}^{-1} - \tilde{L}_{-i}^{T_i - T_i'} M_{-i}^{-1} \tilde{L}_{-i}^{T_i - T_i'} \right) \tilde{L}_{-i}^{T_i'} D_{-i}^{-1/2} M_{C_i}\end{aligned}$$

Now, by Schur theorem, the matrix $M_{-i}^{-1} - \tilde{L}_{-i}^{T_i - T_i'} M_{-i}^{-1} \tilde{L}_{-i}^{T_i - T_i'}$ is positive definite if and only if the matrix

$$\begin{pmatrix} M_{-i}^{-1} & \tilde{L}_{-i}^{T_i - T_i'} \\ \tilde{L}_{-i}^{T_i - T_i'} & M_{-i} \end{pmatrix}$$

is, and this, in turn, applying the same theorem to the other diagonal matrix, is positive definite if and only if $M_{-i} - \tilde{L}_{-i}^{T_i - T_i'} M_{-i} \tilde{L}_{-i}^{T_i - T_i'}$ is. By a standard property, this is positive definite if and only if $\lambda_1(M_{-i}^{-1} \tilde{L}_{-i}^{T_i - T_i'} M_{-i} \tilde{L}_{-i}^{T_i - T_i'}) < 1$, where by λ_1

we denote the largest eigenvalue. We have:

$$\lambda_1(M_{-i}^{-1}\tilde{L}_{-i}^{T_i-T_i'}M_{-i}\tilde{L}_{-i}^{T_i-T_i'}) = \lambda_1(M_{-i}^{-1/2}\tilde{L}_{-i}^{T_i-T_i'}M_{-i}\tilde{L}_{-i}^{T_i-T_i'}M_{-i}^{-1/2}) = \lambda_1(M_{-i}^{-1/2}\tilde{L}_{-i}^{T_i-T_i'}M_{-i}^{1/2})^2$$

the first equality by similarity, the second because the matrix $M_{-i}^{-1/2}\tilde{L}_{-i}^{T_i-T_i'}M_{-i}^{1/2}$ is normal. Then, by similarity again we can conclude that $\lambda_1(M_{-i}^{-1/2}\tilde{L}_{-i}^{T_i-T_i'}M_{-i}^{1/2})^2 = \lambda_1(\tilde{L}) < 1$. Hence we proved that $\Lambda_i^{2T_i} < \Lambda_i^{2T_i'}$.

G.2 Proof of Proposition 2

If $T_i = \infty$, we are in the case covered by Proposition 8, hence all coefficients $B_i = B$ are the same. The best reply equations are:

$$B = \frac{\overline{\Lambda} + (n-1)B}{\overline{\Lambda} + (n-1)B + 1} \quad \text{where } \overline{\Lambda} = \frac{nBB_c}{(N-1)B_c + nB}$$

so that:

$$\left(\frac{nBB_c}{(N-1)B_c + nB} + (n-1)B + 1\right) B = \frac{nBB_c}{(N-1)B_c + nB} + (n-1)B$$

From this we see that if N goes to infinity (the chain becomes longer and longer), then $B \to \frac{n-2}{n-1}$. If $n = 2$, $B \to 0$.

If $T_i = 0$ the equations are the same, but the price impacts are:

$$\overline{\Lambda}_0 = \frac{n_1 B_1 B_c}{n_1 B_1 + B_c} \quad \overline{\Lambda}_i = \frac{n^2 B_{i-1} B_{i+1}}{nB_{i-1} + nB_{i+1}} \quad \overline{\Lambda}_N = \frac{nB_{N-1}}{nB_{N-1} + 1}$$

In equilibrium, B is bounded below by a positive quantity B^*. This is because if we calculate the best reply to the profile in which all coefficients are set to be ε, then, for all sectors but the first and the last:

$$BR \geq \frac{(2n-1)\varepsilon}{(2n-1)\varepsilon + 1}$$

and the RHS is larger than ε if $\varepsilon < \frac{2n-2}{2n-1}$. For the first and last sector a similar condition obtains $\varepsilon < B_0^*$, $\varepsilon < B_N^*$. So, an equilibrium cannot involve all coefficients smaller than $\min\{B_0^*, B_1^*, \frac{2n-2}{2n-1}\}$.

The ratio of welfares is:

$$\frac{W^g}{W^l} = \frac{Q^g}{Q^l} \frac{\frac{A_c}{B_c} - \frac{1}{2B_c}Q^g - \frac{1}{2}\frac{N}{n}Q^g}{\frac{A_c}{B_c} - \frac{1}{2B_c}Q^l - \frac{1}{2}\frac{N}{n}Q^l}$$

Now:

$$Q^g = \frac{\Lambda_c^g}{B_c + \Lambda_c^g} \quad Q^l = \frac{\Lambda_c^l}{B_c + \Lambda_c^l}$$

where $\Lambda_c^g = \frac{nB^g B_c}{NB_c + nB^g}$, and $\Lambda_c^l = \frac{B_c \prod_i nB_j}{B_c \sum_i \prod_{j \neq i} nB_j + \prod_i nB_j}$. It follows that both quantities go to 0, but:

$$\lim_{N \to \infty} \frac{Q^g}{Q^l} = \lim_{N \to \infty} \frac{\frac{nB^g B_c}{NB_c + nB^g}}{\frac{B_c \prod_i nB_j}{B_c \sum_i \prod_{j \neq i} nB_j + \prod_i nB_j}} = \lim_{N \to \infty} nB^g \frac{B_c \frac{1}{N} \sum_i \frac{1}{nB_i} + 1}{B_c} = 0$$

and so also the ratio of welfares goes to 0.