The Welfare Effects of the Generic Competition Paradox: Patent Expiry May Harm Consumers

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January 31, 2024

Abstract

In pharmaceutical markets, entries of generic firms are often followed by an increase in brand-name prices. This paper provides a theoretical explanation for this so-called "generic competition paradox" and measures its welfare effects. We present a model of a horizontally differentiated market with a high and a low segment. We consider two different settings: a monopoly game where the product is sold by a single incumbent, and a competitive game where an entrant also enters the market. For low-segment consumers, the two firms differ only in their prices and locations, but high-segment consumers are brand-loyal (i.e. they are willing to purchase the incumbent's product only). We find that, under certain parametrizations, a monopolist incumbent is active in both segments, but the entry makes it focus only on its brand-loyal consumers, leaving the entire low segment for the entrant. We refer to this latter equilibrium as "strategic separation", as it means that the two firms become local monopolists in two different segments. It is the shift in the incumbent's focus that explains the generic competition paradox: with the low segment lost, it no longer needs to keep its price at a lower level. Our welfare analysis suggests that although the entry increases the aggregate welfare, it also decreases the surplus of consumers. This finding has important policy implications: if a social planner focuses on the consumers' side, pushing for patent expiry may be counterproductive.

JEL codes: D43, L11, L43, L65

Keywords: generic competition paradox, patents, pharmaceuticals, brand loyalty, segmentation, product differentiation, competition policy

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1 Introduction

It is considered common knowledge that competition decreases prices, hence patents are often seen as a necessary evil. In order to create an incentive for firms to spend on research and development, we are willing to suffer a higher price for the product for a limited amount of time. Thus, we do not want the patent to last “too long”, since we hope that the appearing competitors will drive the price down after the expiry of the patent, thereby increasing social welfare.

However, facts in the pharmaceutical industry do not always agree with this assumption. Grabowski and Vernon (1992) showed that brand-name prices increase as a result of patent expiry. This result is known as the “generic competition paradox”. Although debated by some (Caves, Whinston, Hurwitz, Pakes, & Temin, 1991; Wiggins & Maness, 2004; Saha & Xu, 2021), it remains a well-established finding that has been confirmed by several empirical papers (Frank & Salkever, 1997; Regan, 2008). Vandoros and Kanavos (2013) showed it even in the case of regulated markets, where the problem might be exacerbated by barriers of entry. Magazzini, Pammolli, and Riccaboni (2004) conclude that the preexisting price regulations can in effect pose an obstacle for generic entry.

The generic competition paradox can in itself is problematic, since the most loyal consumers of brand-name drugs tend to be members of more vulnerable groups (elderly or less educated), as shown by Fraeyman et al. (2015). Moreover, generic drugs are perceived to be of inferior quality both by patients and doctors (Colgan et al., 2015). Nevertheless, most papers implicitly suggest that these negative effects would be outweighed by the cheapness of generic drugs, and hence the net welfare effect of generic entries is seen as positive. However, Tessema, Kesselheim, and Sinha (2019) point out that even generic prices may be high in certain cases. This is especially a problem if the number of generic firms is low (Reiffen & Ward, 2005).

But even with the prices of generics being sufficiently low, there may be cases where this cheapness does not compensate the increase in brand-name prices. Therefore, the net effect of generic entries on welfare is ambiguous. In this paper, we aim to measure this welfare effect using a theoretical model.

In the theoretical literature, several models have been developed that provided diverse explanations for the generic competition paradox (Ferrara & Kong, 2008; Kong, 2009; Nabin, Mohan, Nicholas, & Sgro, 2012; Papanastasiou, 2016). However, the explanation we consider the most intuitive is the one by Frank and Salkever (1992). In their model, they divide the market into a loyal and a cross-price sensitive segment. Loyal consumers are only willing to purchase the brand-name product, while cross-price sensitive consumers may buy from generic firms as well. The authors show that under such settings, the generic competition paradox can be explained by the brand-name firm shifting its focus to the loyal segment.

The model we present follows exactly the same intuition as Frank and Salkever (1992), but there are many technical differences. While they assume a sequential game where the brand-name firm sets its price first, followed by n generic firms, in our model there is only one generic firm that decides simultaneously with the brand-name firm. Moreover, we also introduce horizontal product differentiation into the model using Desai's (2001) framework.

The results from our model provide a sufficient explanation for the generic competition paradox. Under certain parametrizations, the incumbent firm does indeed increase its price after the entry. Similarly to the model of Frank and Salkever (1992), the mechanism behind this is a shift in the incumbent's focus (from price sensitive to brand-loyal consumers). But unlike their model, ours also enables us to conduct a welfare analysis. We find that while the aggregate welfare increases after the entry, the consumer surplus falls. This is our most important finding which may have important policy implications. It shows us that patents are not always just a necessary evil, they can actually be beneficial for the consumer side.

The paper is organized as follows. In the next section we present our model's assumptions. Then in section 3 we show its equilibrium results and the corresponding surplus values. Finally we conclude our paper with policy considerations in section 3. The calculations for the model's equilibria are included in appendix A for the monopoly game, and in appendix B for the competitive game.

2 Model setting

We assume that there is a continuum of consumers ($\mathcal{C}$) with uniformly distributed locations along a line segment. We normalize the mass of consumers and the length of the segment to one unit and denote the location of consumer $i \in \mathcal{C}$ by $x_i \in [0,1]$. Consumers can be of two different types, denoted by $T_i \in L, H$. The probability that consumer i is of type L is $\lambda \in [\frac{1}{2}, 1[$, and this probability is independent of i's location[1]. We refer to the set of type-L consumers as the *low segment* and the set of type-H consumers as the *high segment*[2].

Each consumer demands either 1 or 0 units of a product. We consider two different settings: one where this product is sold by a single monopolist (referred to as the incumbent, I), and one where an entrant (E) also enters the market. Firms are located at $x_I, x_E \in [0,1]$ and offer their products at prices $p_I, p_E \in \mathbb{R}_+$ without being able to price discriminate. For a consumer j who is located exactly at the same point as one of the firms ($x_j = x_F$ for some $F \in \{I, E\}$), the reservation price for F's product is $v_{F,T_j} \in \mathbb{R}_+$. However, other consumers face some linear transportation costs. In general, consumer $i \in \mathcal{C}$'s utility gain from consuming the product of firm $F \in \{I, E\}$ is:

$$u_i(F) = v_{F,T_i} - |x_i - x_F| - p_F \tag{1}$$

[1]The independence implies that the uniform distribution of locations is preserved within the two groups as well.

[2]The assumption that $\lambda \geq \frac{1}{2}$ implies that the low segment is at least as large as the high segment.

We assume that $v_{I,L} = v_{E,L} \triangleq v > 0$, that is, low-segment consumers do not care about which firm they buy the product from, only about its price and location. However, high-segment consumers are brand-loyal, and they are only willing to buy the incumbent's product. Formally, $v_{E,H} = 0$ and $v_{I,H} \triangleq V > 0$. We further assume that $v < V$, i.e. high-segment consumers derive more utility from the incumbent's product than low-segment consumers, and that $1 \leq v$, i.e. even when transportation costs are maximal (one unit), a low-segment consumer would still accept the product for free.

The games we consider are as follows. In the monopoly game, the incumbent chooses the price-location pair (p_I^*, x_I^*) that maximizes its profit, and then each consumer decides whether she wants to consume the product. In the competitive game, the incumbent and the entrant simultaneously choose the price-location pairs (p_I^*, x_I^*) and (p_E^*, x_E^*) that maximize their profits. Then consumers choose whether they want to consume from I, from E, or not consume at all. For the sake of simplicity, we assume that the costs of production are zero for both firms.

Solving the game for the entire range of possible parameterizations is unnecessary for our purpose of demonstrating that the strategic separation is an equilibrium in the competitive game. Instead, we focus only on a certain subset of this domain where the separation occurs, and show that it is indeed a Nash equilibrium. In particular, we make the following assumption:

$$V \geq \frac{2 \cdot v - \lambda}{2 \cdot (1 - \lambda)} \tag{2}$$

3 Results

Since deriving the equilibria of the two games is rather complicated, here we only show the final results. Our calculations are included in appendix A for the monopoly game, and in appendix B for the competitive game. In appendix A we find that in the monopoly game, the incumbent's equilibrium location and price is:

$$x_I^* = \frac{1}{2}, \quad p_I^* = \begin{cases} \frac{v}{2} + \frac{1-\lambda}{4 \cdot \lambda} & \text{if } v \leq \frac{\lambda+1}{2 \cdot \lambda} \text{ and } V < \frac{(2 \cdot \lambda \cdot v + 1 - \lambda)^2}{8 \cdot \lambda \cdot (1-\lambda)} + \frac{1}{2} \\ \left\{ \frac{v}{2} + \frac{1-\lambda}{4 \cdot \lambda}, V - \frac{1}{2} \right\} & \text{if } v \leq \frac{\lambda+1}{2 \cdot \lambda} \text{ and } V = \frac{(2 \cdot \lambda \cdot v + 1 - \lambda)^2}{8 \cdot \lambda \cdot (1-\lambda)} + \frac{1}{2} \\ V - \frac{1}{2} & \text{otherwise} \end{cases} \tag{3}$$

As we can see, the formula for the incumbent's optimal price depends on the parametrization. In the first case it covers the entire high segment and also serves part of the low segment (as shown in the left side of figure 1). However, in the third[3] case it focuses only on the high segment, leaving its extreme consumers with exactly zero utility (see the right side of figure 1).

[3]The second case from this piecewise function is just the turning point between the other two cases, where the incumbent is indifferent between the two values. We can ignore this.

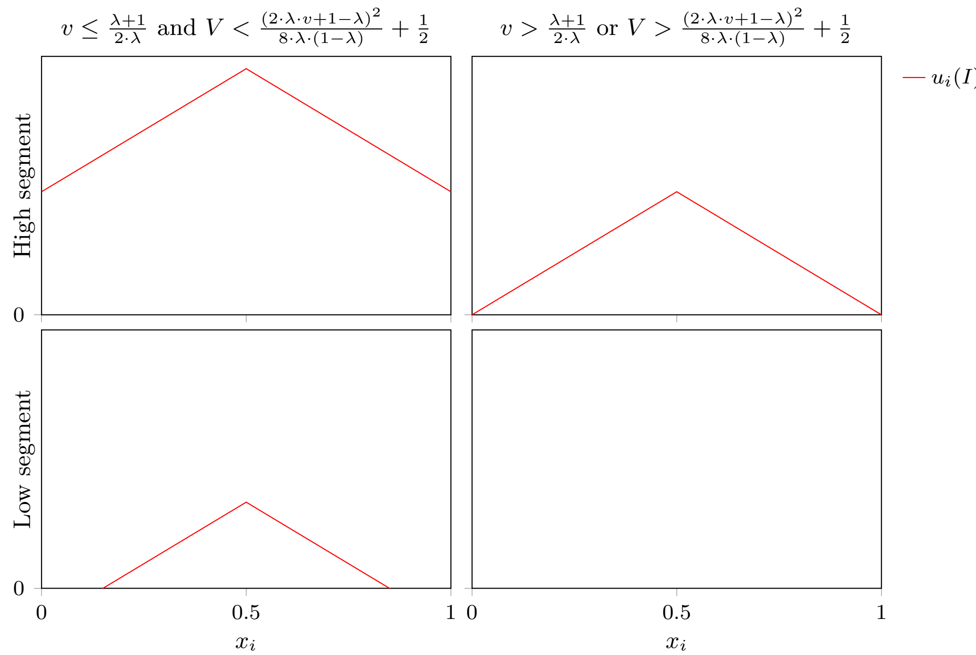

Figure 1: The equilibrium of the monopoly game (see equation (3)). The red lines show the consumers' utility gain from consuming the incumbent's product, by location and segment.

In appendix B we find that in the competitive game's equilibrium, the two firms will have the following locations and prices:

$$x_I^* = x_E^* = \frac{1}{2}, \quad p_I^* = V - \frac{1}{2}, \quad p_E^* = v - \frac{1}{2} \tag{4}$$

This is the equilibrium which we refer to as "strategic separation", as it means that the two firms are active in different segments of the market. The incumbent operates only in the high segment, while the entrant operates only in the low segment. Moreover, they both cover their "own" segment, leaving exactly zero utility gain for the consumers at the two extreme points (see figure 2).

Note that when $v > \frac{\lambda+1}{2\cdot\lambda}$ or $V \geq \frac{(2\cdot\lambda\cdot v+1-\lambda)^2}{8\cdot\lambda\cdot(1-\lambda)} + \frac{1}{2}$, the entrant's emergence does not change the incumbent's behavior at all: it chooses the same price-location pair in (3) and (4). These cases are not that interesting from our point of view since they show no strategic interaction between the two firms. However, the entry does change the incumbent's decision if the following criteria both hold:

$$v \leq \frac{\lambda+1}{2\cdot\lambda}, \quad V < \frac{(2\cdot\lambda\cdot v+1-\lambda)^2}{8\cdot\lambda\cdot(1-\lambda)} + \frac{1}{2} \tag{5}$$

For the specific case when $\lambda = \frac{1}{2}$, the (v, V) pairs that satisfy these constraints (and (2)) are

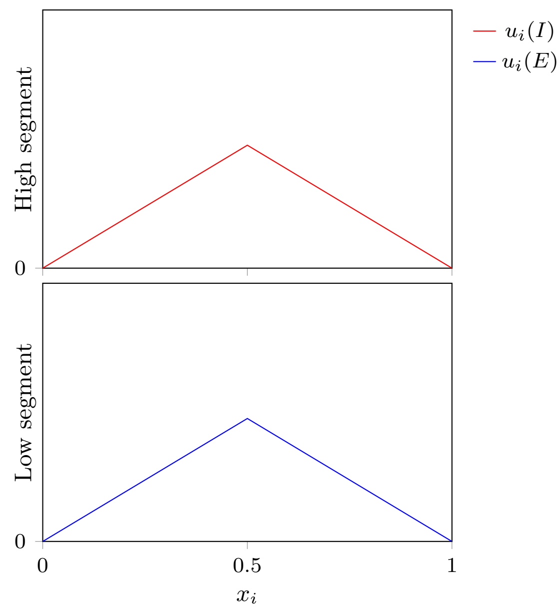

Figure 2: The equilibrium of the competitive game (see equation (4)). The red (blue) lines show the consumers' utility gain from consuming the incumbent's (entrant's) product, by location and segment.

shown in figure 3[4]. Under these circumstances, the incumbent covers part of the low segment when it is a monopolist, but the new entry makes it leave the entire low segment to the entrant and focus only on its brand-loyal high-segment consumers. Moreover, we can easily see that $\frac{v}{2} + \frac{1-\lambda}{4 \cdot \lambda} < V - \frac{1}{2}$, so this shift in the incumbent's focus also means a price increase. Therefore, our model provides a sufficient explanation for the generic competition paradox.

The economic intuition behind this phenomenon is quite straightforward. When the incumbent is a monopolist, it can attract some low-segment consumers and make a higher profit by maintaining a slightly lower price level. However, when the entrant emerges, it undercuts the incumbent, making itself more attractive on the low segment. Therefore, it is no longer worthwhile for the incumbent to have low-segment consumers, as this would require a further price decrease. Instead, it focuses only on the high segment, where it faces no competitors. Moreover, now that the low segment is "lost", keeping the price at the monopoly level also becomes unnecessary. The incumbent can raise its price as long as the high segment remains covered.

[4] To see how these sets change for different values of λ, visit `https://www.geogebra.org/calculator/h9g7sexs`

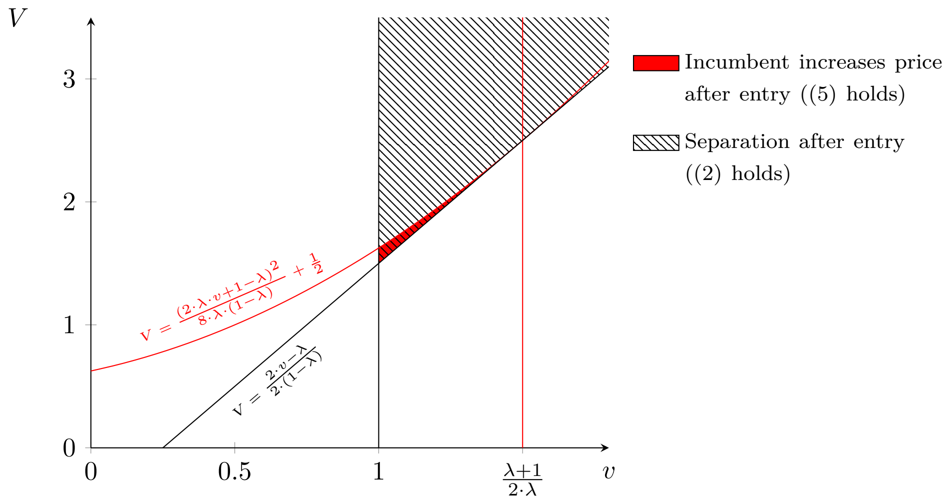

Figure 3: The possible (v, V) pairs that satisfy (2) and (5) when $\lambda = \frac{1}{2}$.

3.1 Welfare analysis

Having obtained the equilibria of the monopoly and competitive games, we now conduct a welfare analysis. For each group of agents, we compare the equilibrium surpluses from the two games. We have shown that a monopolist incumbent is active in the low segment iff equation (5) holds, and this also influences the surplus values we get in the monopoly setting. Therefore, we need to treat the parameterizations that satisfy this criterion separately from those that do not. Table 1 shows the surpluses (by group and game) when equation (5) does not hold, while table 2 shows the surpluses when it does.

The values from these tables can be derived quite simply. In fact, we have already obtained the profits in the appendices. For the monopoly game the incumbent's profit is shown in equation (11) or (12) (depending on the parametrization), and for the competitive game the profits of the two firms are shown in equations (20) and (21). In both games, we can easily get the aggregate profit by adding up the profits of the two firms[5]. Next, the consumer surpluses in the two segments can also be calculated. We simply need to substitute the equilibrium values into the consumer i's utility from equation (1) and integrate with respect to x_i (paying attention to the limits). Finally, the social surplus is given as the sum of the aggregate profit and the consumer surplus.

The results from table 1 are fairly standard. We have seen that under parametrizations that satisfy equation (5), the incumbent focuses only on the high segment regardless of the entry. Therefore, its profit remains unaffected, and so does the consumer surplus on the high segment. However, the entry increases both the entrant's profit and the surplus of low-segment consumers. Conse-

[5] In the monopoly game, the entrant has zero profits.

Group	Surplus under monopoly		Surplus under competition
Incumbent	$(1-\lambda)\cdot\left(V-\frac{1}{2}\right)$	$=$	$(1-\lambda)\cdot\left(V-\frac{1}{2}\right)$
Entrant	0	$<$	$\lambda\cdot\left(v-\frac{1}{2}\right)$
Firms	$(1-\lambda)\cdot\left(V-\frac{1}{2}\right)$	$<$	$(1-\lambda)\cdot\left(V-\frac{1}{2}\right)+\lambda\cdot\left(v-\frac{1}{2}\right)$
L segment	0	$<$	$\lambda\cdot\frac{1}{4}$
H segment	$(1-\lambda)\cdot\frac{1}{4}$	$=$	$(1-\lambda)\cdot\frac{1}{4}$
Consumers	$(1-\lambda)\cdot\frac{1}{4}$	$<$	$\frac{1}{4}$
Society	$(1-\lambda)\cdot\left(V-\frac{1}{4}\right)$	$<$	$(1-\lambda)\cdot\left(V-\frac{1}{4}\right)+\lambda\cdot\left(v-\frac{1}{4}\right)$

Table 1: The surpluses of each group when the entry does not change the incumbent's behavior (i.e. (5) does not hold). The middle column shows the relation between the two values.

quently, the aggregate surpluses (of producers, consumers and the whole society) are all higher in the competitive setting. In such parametrizations, competition does not harm anyone: it is a Pareto-improvement compared to a monopoly.

However, the results we see in table 2 are more surprising. Here the incumbent is worse off after the entry because it cannot collect profit from the low segment anymore. As we have discussed earlier, this makes it raise its price and collect more from the surplus of high-segment consumers. Hence these high-segment consumers are also made worse off by the entry. Moreover, the consumer surplus lost on the high segment exceeds the gain on the low segment. In other words, competition is harmful to consumers on the aggregate level. This finding is not in line with the classic economic intuition that competition on the supply side is beneficial to the demand side. The reason for this unusual result is the strategic separation: the entry forces the incumbent to focus on the high segment only and raise its prices.

Nevertheless, on the level of the whole society, the losses of consumers and the incumbent are compensated by the gains of the entrant. Aggregate welfare is higher in the competitive game than in the monopoly one, which is again a standard result. But since it is debatable whether a social planner should focus on aggregate welfare or only on consumer surplus, our results raise an important question from a policy perspective. If we argue that a policymaker should focus only on the

Group	Surplus under monopoly		Surplus under competition
Incumbent	$\frac{(2\cdot\lambda\cdot v+1-\lambda)^2}{8\cdot\lambda}$	$>$	$(1-\lambda)\cdot\left(V-\frac{1}{2}\right)$
Entrant	0	$<$	$\lambda\cdot\left(v-\frac{1}{2}\right)$
Firms	$\frac{(2\cdot\lambda\cdot v+1-\lambda)^2}{8\cdot\lambda}$	$<$	$(1-\lambda)\cdot\left(V-\frac{1}{2}\right)+\lambda\cdot\left(v-\frac{1}{2}\right)$
L segment	$\lambda\cdot\left(\frac{v}{2}-\frac{1-\lambda}{4\cdot\lambda}\right)^2$	$<$	$\lambda\cdot\frac{1}{4}$
H segment	$(1-\lambda)\cdot\left(V-\frac{1}{4}-\frac{v}{2}-\frac{1-\lambda}{4\cdot\lambda}\right)$	$>$	$(1-\lambda)\cdot\frac{1}{4}$
Consumers	$(1-\lambda)\cdot\left(V-\frac{1}{4}-\frac{v}{2}-\frac{1-\lambda}{4\cdot\lambda}\right)+\lambda\cdot\left(\frac{v}{2}-\frac{1-\lambda}{4\cdot\lambda}\right)^2$	$>$	$\frac{1}{4}$
Society	$(1-\lambda)\cdot\left(V-\frac{1}{4}\right)+\lambda\cdot\left(\frac{v}{2}-\frac{1-\lambda}{4\cdot\lambda}\right)\cdot\left(\frac{3\cdot v}{2}+\frac{1-\lambda}{4\cdot\lambda}\right)$	$<$	$(1-\lambda)\cdot\left(V-\frac{1}{4}\right)+\lambda\cdot\left(v-\frac{1}{4}\right)$

Table 2: The surpluses of each group when the incumbent increases its price after the entry (i.e. (5) holds). The middle column shows the relation between the two values.

consumer side, then stimulating competition and promoting new entries may be counterproductive.

4 Conclusion

Our paper's aim was to demonstrate that the so-called "generic competition paradox" can pose a greater danger than formerly believed, and the fact the generics are offered at a lower price might not counterbalance the empirically oft-observed phenomenon that the originator reacts with a price increase to the expiry of the patent. We have shown that, in the case of one generic competitor and complete demarketing from the more elastic segment, consumer surplus declines after the appearance of a competitor. This is a surprising result and might have relevance to patent policy.

However, two other factors not explicitly addressed in our model might exacerbate the situation. The ones most hurt by the expiry of the patent usually belong to more vulnerable groups, such as the elderly and the less educated. Secondly, the consumers of the generics might perceive the non-brand versions as inferior, which further decreases consumer welfare.

The above arguments might warn policymakers to be less eager to urge patent expiry in those cases when we do not expect significant competition in the generics market. One solution would be to offer an opportunity for the originators to extend their patents, and this can be decided on a case-by-case basis. This system could have other beneficial consequences as well: since originators would assign a positive probability to being able to extend their patents, this could lead to a higher level of research and development.

It would be interesting to verify our results for the case of two or more generics firm. This, however would involve lengthy analysis of mixed strategy equilibria, and hence remains (for now) a topic for further research.

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A Equilibrium in the monopoly game

We first consider the case when the incumbent is the only seller of the product. Due to the sequential nature of the game, we need to use backward induction to find its subgame perfect Nash equilibrium.

A.1 The decision of consumers in the second stage subgame

In the second stage subgame, the decision of consumer $i \in \mathcal{C}$ is quite simple. She will purchase the incumbent's product iff her utility gain $u_i(I)$ is non-negative. Using equation (1) this yields:

$$v_{I,T_i} - p_I \geq |x_i - x_I|$$

When $p_I > v_{I,T_i}$, this can never be true and so in such cases no consumer from type T_i will purchase the product. However, when $p_I \leq v_{I,T_i}$, we have:

$$x_I + p_I - v_{I,T_i} \leq x_i \leq x_I + v_{I,T_i} - p_I$$

This means that the consumers of the incumbent are within a $v_{I,T_i} - p_I$ radius of the x_I. However, the two ends of the segment may also fall within this radius. Hence the incumbent's leftmost and rightmost consumers from segment $T_i \in \{L, H\}$ are located at $\max\{x_I + p_I - v_{I,T_i}, 0\}$ and $\min\{x_I + v_{I,T_i} - p_I, 1\}$ (respectively, and if they exist). Therefore, the ratio that the incumbent covers from segment $T_i \in \{L, H\}$ is:

$$\begin{aligned} D_{T_i}(p_I, x_I) &= \max\Big\{\min\{x_I + v_{I,T_i} - p_I, 1\} - \max\{x_I + p_I - v_{I,T_i}, 0\}, 0\Big\} = \\ &= \max\Big\{\min\Big\{(2 \cdot v_{I,T_i} - 2 \cdot p_I), (1 - x_I + v_{I,T_i} - p_I), (x_I + v_{I,T_i} - p_I), 1\Big\}, 0\Big\} \end{aligned} \tag{6}$$

Note that the variable x_I occurs only twice in this equation: in $1 - x_I + v_{I,T_i} - p_I$ and in $x_I + v_{I,T_i} - p_I$. These two expressions are equal when $x_I = \frac{1}{2}$, but when $x_I \neq \frac{1}{2}$, one of the two values becomes smaller than $\frac{1}{2} + v_{I,T_i} - p_I$, and so does their minimum. Hence we know that $D_{T_i}(p_I, \frac{1}{2}) \geq D_{T_i}(p_I, x_I)$ for any $p_I \in \mathbb{R}_+$ and $x_I \in [0, 1]$.

A.2 The decision of the incumbent in the first stage subgame

Turning to the incumbent's decision in the first stage subgame, we know that it wants to choose a price-location pair that maximizes the following function:

$$\Pi(p_I, x_I) = \lambda \cdot D_L(p_I, x_I) \cdot p_I + (1 - \lambda) \cdot D_H(p_I, x_I) \cdot p_I \tag{7}$$

Using our result from above, we can easily see that $\Pi(p_I, \frac{1}{2}) \geq \Pi(p_I, x_I)$ for any $p_I \in \mathbb{R}_+$ and

$x_I \in [0, 1]$. This means that the incumbent can never be worse off when it sets $x_I = \frac{1}{2}$, so we can say without loss of generality[6] that $x_I^* = \frac{1}{2}$. By substituting this value into the D_{T_i} function we get:

$$D_{T_i}\left(p_I, \frac{1}{2}\right) = \max\left\{\min\left\{(2 \cdot v_{I,T_i} - 2 \cdot p_I), \left(\frac{1}{2} + v_{I,T_i} - p_I\right), 1\right\}, 0\right\}$$

However, due to the fact that $\frac{1}{2} + v_{I,T_i} - p_I < 2 \cdot v_{I,T_i} - 2 \cdot p_I \iff 1 < \frac{1}{2} + v_{I,T_i} - p_I$, this can be further reduced:

$$D_{T_i}\left(p_I, \frac{1}{2}\right) = \max\left\{\min\left\{(2 \cdot v_{I,T_i} - 2 \cdot p_I), 1\right\}, 0\right\} \tag{8}$$

Our assumption that $V > v$ implies that $D_L\left(p_I, \frac{1}{2}\right) \leq D_H\left(p_I, \frac{1}{2}\right)$ for any $p_I \in \mathbb{R}_+$. We can also use our additional restriction from (2) together with the assumptions that $\lambda \geq \frac{1}{2}$ and $v \geq 1$ to show that:

$$V \geq \frac{2 \cdot v - \lambda}{2 \cdot (1 - \lambda)} = v + \frac{2 \cdot v - 1}{2} \cdot \frac{\lambda}{1 - \lambda} \geq v + \frac{2 \cdot v - 1}{2} \geq v + \frac{1}{2} \tag{9}$$

Now assume that $D_L\left(p_I, \frac{1}{2}\right) > 0$, and hence $0 < 2 \cdot v - 2 \cdot p_I$. Using our previous result we can show that $0 < 2 \cdot v - 2 \cdot p_I \leq 2 \cdot (V - \frac{1}{2}) - 2 \cdot p_I$. This implies that $1 < 2 \cdot V - 2 \cdot p_I$, so $D_H\left(p_I, \frac{1}{2}\right) = 1$. In other words, the high segment must necessarily be fully covered whenever the low segment is not fully uncovered. Therefore, we only need to consider the following five cases (also shown in figure 4):

1. Both segments are fully uncovered, i.e. $D_L\left(p_I, \frac{1}{2}\right) = D_H\left(p_I, \frac{1}{2}\right) = 0$
2. The low segment is fully uncovered and the high segment is partially covered, i.e. $D_L\left(p_I, \frac{1}{2}\right) = 0$ and $D_H\left(p_I, \frac{1}{2}\right) = 2 \cdot V - 2 \cdot p_I$
3. The low segment is fully uncovered and the high segment is fully covered, i.e. $D_L\left(p_I, \frac{1}{2}\right) = 0$ and $D_H\left(p_I, \frac{1}{2}\right) = 1$
4. The low segment is partially covered and the high segment is fully covered, i.e. $D_L\left(p_I, \frac{1}{2}\right) = 2 \cdot v - 2 \cdot p_I$ and $D_H\left(p_I, \frac{1}{2}\right) = 1$
5. Both segments are fully covered, i.e. $D_L\left(p_I, \frac{1}{2}\right) = D_H\left(p_I, \frac{1}{2}\right) = 1$

Moreover, the first one of these cases can never be an optimum since $D_L\left(p_I, \frac{1}{2}\right) = D_H\left(p_I, \frac{1}{2}\right) = 0$ also means that $\Pi\left(p_I, \frac{1}{2}\right) = 0$, and the incumbent can clearly do better than reaching zero profits.

[6] Although there may be other optimal values of x_I, these cases must by definition yield the same maximized profit. Moreover, even if there are several optimal values of p_I, we can find all of them by maximizing the profit under the assumption that $x_I^* = \frac{1}{2}$. This can be proven simply. If there is an optimal price-location pair (p'_I, x'_I) such that $x'_I \neq \frac{1}{2}$, our findings yield that $\Pi(p'_I, \frac{1}{2}) \geq \Pi(p'_I, x'_I)$. Therefore, the pair $(p'_I, \frac{1}{2})$ is also optimal, so we must find this p'_I value also when we set $x_I^* = \frac{1}{2}$.

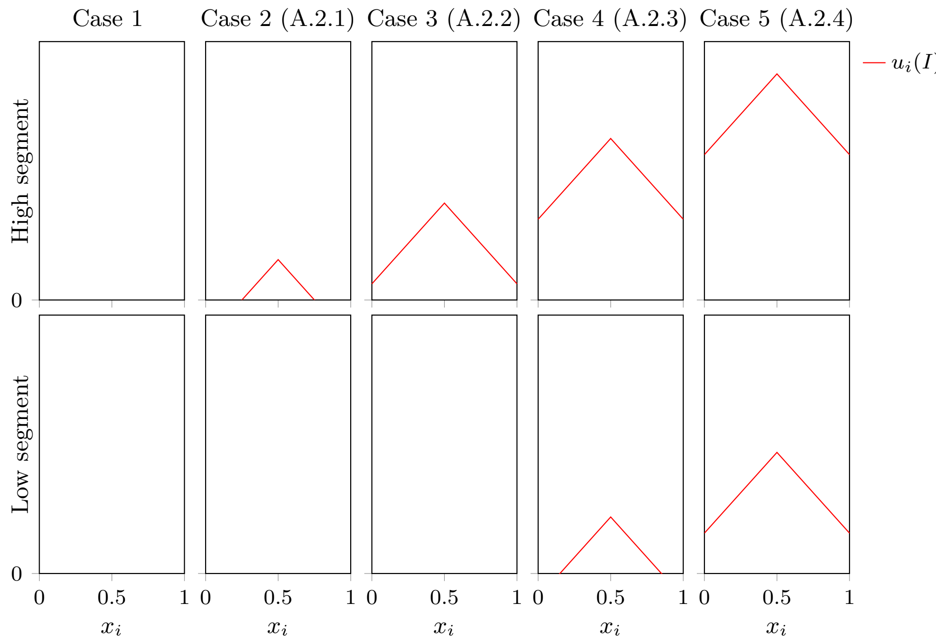

Figure 4: The five cases to consider when the incumbent is a monopolist. The red lines show the consumers' utility gain from consuming the incumbent's product, by location and segment.

Therefore we can ignore this case. We deal with the remaining four cases one by one, find their local optima and then see which one of these is the global optimum.

A.2.1 The low segment is fully uncovered and the high segment is partially covered

In this case we have $D_L\left(p_I, \frac{1}{2}\right) = 0$ and $D_H\left(p_I, \frac{1}{2}\right) = 2 \cdot V - 2 \cdot p_I$. Putting this together with equation (8) we get that $2 \cdot v - 2 \cdot p_I \leq 0$ and $0 \leq 2 \cdot V - 2 \cdot p_I \leq 1$. Since we know from equation (9) that $V - \frac{1}{2} > v$, for p_I this gives us the following constraint:

$$V - \frac{1}{2} \leq p_I \leq V$$

We can use this constraint to rewrite the incumbent's profit maximization problem from (7) for this specific case:

$$\max_{p_I \in [V - \frac{1}{2}, V]} \quad \lambda \cdot 0 \cdot p_I + (1 - \lambda) \cdot (2 \cdot V - 2 \cdot p_I) \cdot p_I \tag{10}$$

By solving the first order condition we get $p_I = \frac{V}{2}$. However, since we assumed that $V > 1$, we have $\frac{V}{2} < V - \frac{1}{2}$. Therefore this case yields a corner solution: $p_I = V - \frac{1}{2}$. In this case the incumbent's profit is:

$$\Pi\left(V-\frac{1}{2},\frac{1}{2}\right)=(1-\lambda)\cdot\left(V-\frac{1}{2}\right) \tag{11}$$

A.2.2 The low segment is fully uncovered and the high segment is fully covered

This is the case where $D_L\left(p_I,\frac{1}{2}\right)=0$ and $D_H\left(p_I,\frac{1}{2}\right)=1$. Together with equation (8) this means that $2\cdot v-2\cdot p_I\leq 0$ and $2\cdot V-2\cdot p_I\geq 1$. For p_I this means that:

$$v\leq p_I\leq V-\frac{1}{2}$$

Therefore, in this case the incumbent's profit maximization problem from (7) becomes:

$$\max_{p_I\in[v,V-\frac{1}{2}]}\quad \lambda\cdot 0\cdot p_I+(1-\lambda)\cdot 1\cdot p_I$$

The objective function is strictly increasing in p_I, so we have a corner solution: $p_I=V-\frac{1}{2}$. This is exactly the same as what we found in the previous case.

A.2.3 The low segment is partially covered and the high segment is fully covered

Here we have $D_L\left(p_I,\frac{1}{2}\right)=2\cdot v-2\cdot p_I$ and $D_H\left(p_I,\frac{1}{2}\right)=1$, so together with equation (8) we obtain that $0\leq 2\cdot v-2\cdot p_I\leq 1$ and $2\cdot V-2\cdot p_I\geq 1$. Since we know from (9) that $V-\frac{1}{2}>v$, for p_I this implies:

$$v-\frac{1}{2}\leq p_I\leq v$$

By rewriting the incumbent's profit maximization problem from (7) we get:

$$\max_{p_I\in[v-\frac{1}{2},v]}\quad \lambda\cdot(2\cdot v-2\cdot p_I)\cdot p_I+(1-\lambda)\cdot 1\cdot p_I$$

The first order condition yields that $p_I=\frac{v}{2}+\frac{1-\lambda}{4\cdot\lambda}$. Since $\lambda\geq\frac{1}{2}$ and $v\geq 1$, we clearly have that $\frac{v}{2}+\frac{1-\lambda}{4\cdot\lambda}\leq\frac{v}{2}+\frac{1}{4}\leq\frac{v}{2}+\frac{v}{4}<v$, so the upper limit on p_I holds. However, we cannot be sure that the lower limit holds as well. Hence the optimal price in this case depends on the parametrization. When $v\leq\frac{\lambda+1}{2\cdot\lambda}$, we have the interior solution $p_I=\frac{v}{2}+\frac{1-\lambda}{4\cdot\lambda}$. But when $v>\frac{\lambda+1}{2\cdot\lambda}$, we have a corner solution $p_I=v-\frac{1}{2}$. The corresponding two profits are (respectively):

$$\Pi\left(\frac{v}{2}+\frac{1-\lambda}{4\cdot\lambda},\frac{1}{2}\right)=\frac{(2\cdot\lambda\cdot v+1-\lambda)^2}{8\cdot\lambda} \tag{12}$$

$$\Pi\left(v-\frac{1}{2},\frac{1}{2}\right)=v-\frac{1}{2} \tag{13}$$

A.2.4 Both segments are fully covered

In this last case we have that $D_L\left(p_I, \frac{1}{2}\right) = D_H\left(p_I, \frac{1}{2}\right) = 1$, so using equation (8) we get $2\cdot v - 2\cdot p_I \geq 1$ and $2 \cdot V - 2 \cdot p_I \geq 1$. Since we know that $v < V$, this leaves us with the following constraint on p_I:

$$p_I \leq v - \frac{1}{2}$$

Hence the incumbent's profit maximization problem from (7) becomes:

$$\max_{p_I \in [0, v-\frac{1}{2}]} \quad \lambda \cdot 1 \cdot p_I + (1-\lambda) \cdot 1 \cdot p_I$$

This is strictly increasing in p_I, so in this case we have a corner solution $p_I = v - \frac{1}{2}$. The corresponding profit can be seen in equation (13).

A.2.5 Global optimum

What is left for us now is to compare the local optima from the four cases and decide which one is the global optimum. In particular, we need to compare the three values from equations (11), (12) and (13). However, note that our additional restriction from equation (2) can be rearranged as $v - \frac{1}{2} \leq (1-\lambda) \cdot \left(V - \frac{1}{2}\right)$. This means that in the domain we are looking for solutions in, the profit in (13) will never be greater than the profit in (11). Hence we can ignore the possibility of $p_I = v - \frac{1}{2}$ being the global optimum.

This also implies that when $v > \frac{\lambda+1}{2\cdot\lambda}$, the optimum must necessarily[7] be $p_I^* = V - \frac{1}{2}$. However, when $v \leq \frac{\lambda+1}{2\cdot\lambda}$, we also need to compare the profits in (11) and (12). The latter can only be greater if the following inequality holds[8]:

$$(1-\lambda) \cdot \left(V - \frac{1}{2}\right) \leq \frac{(2 \cdot \lambda \cdot v + 1 - \lambda)^2}{8 \cdot \lambda}$$

This inequality simplifies to:

$$V \leq \frac{(2 \cdot \lambda \cdot v + 1 - \lambda)^2}{8 \cdot \lambda \cdot (1-\lambda)} + \frac{1}{2} \tag{14}$$

Therefore, putting all of our previous results together, we have found that a monopolist incum-

[7] Under such parametrizations, the profit from (12) is not even a local optimum, so we only need to compare (11) and (13), which we have just done.

[8] When it holds with equality, the incumbent is indifferent between choosing $p_I = \frac{v}{2} + \frac{1-\lambda}{4\cdot\lambda}$ and $p_I = V - \frac{1}{2}$.

bent's optimal price-location pair is[9]:

$$x_I^* = \frac{1}{2}, \quad p_I^* = \begin{cases} \frac{v}{2} + \frac{1-\lambda}{4\cdot\lambda} & \text{if } v \leq \frac{\lambda+1}{2\cdot\lambda} \text{ and } V < \frac{(2\cdot\lambda\cdot v+1-\lambda)^2}{8\cdot\lambda\cdot(1-\lambda)} + \frac{1}{2} \\ \left\{\frac{v}{2} + \frac{1-\lambda}{4\cdot\lambda}, V - \frac{1}{2}\right\} & \text{if } v \leq \frac{\lambda+1}{2\cdot\lambda} \text{ and } V = \frac{(2\cdot\lambda\cdot v+1-\lambda)^2}{8\cdot\lambda\cdot(1-\lambda)} + \frac{1}{2} \\ V - \frac{1}{2} & \text{otherwise} \end{cases} \tag{15}$$

B Equilibrium in the competitive game

We now turn to the case when the entrant is also present at the market. For symmetry reasons, we can assume without loss of generality that $x_E \leq x_I$. Like in the previous game, we use backward induction to find the subgame perfect Nash equilibrium.

B.1 The decision of consumers in the second stage subgame

In the second stage subgame, consumers of course choose the option that yields the highest utility gain. Since $v_{E,H} = 0$, we have that $u_i(E) \leq 0 \quad \forall i \in \mathcal{C}, T_i = H$. Therefore, high-segment consumers will never buy the entrant's product, i.e. $D_{E,H}(p_E, p_I, x_E, x_I) = 0$. This also means that the incumbent remains a (local) monopolist on the high segment, so the ratio it covers from this segment is the same as in equation (6):

$$D_{I,H}(p_I, p_E, x_I, x_E) = \max\Big\{\min\Big\{(2 \cdot V - 2 \cdot p_I), (1 - x_I + V - p_I), (x_I + V - p_I), 1\Big\}, 0\Big\}$$

However, in the low segment consumers may choose any of the two firms. A low-segment consumer i will consume the product from $F \in \{I, E\}$ iff $u_i(F) \geq 0$ and $u_i(F) \geq u_i(-F)$ both hold. Like in the monopoly game, the first one of these inequalities will not hold for any x_i when $p_F > v$, and when $p_F \leq v$ it implies that:

$$x_F + p_F - v \leq x_i \leq x_F + v - p_F \tag{16}$$

Now consider the second inequality $u_i(F) \geq u_i(-F)$ and assume there exists a unique consumer $j \in \mathcal{C}, T_j = L$ for whom it holds with equality (meaning that she is indifferent between choosing I and choosing E). This consumer must necessarily be located strictly between the two firms[10]. Together with our assumption that $x_E \leq x_I$, this means that $x_E < x_j < x_I$. Hence the $u_j(E) =$

[9] As we have discussed earlier, the incumbent is only weakly better off by setting $x_I^* = \frac{1}{2}$. This means that there may be other optimal values of x_I as well. However, in footnote 6 we have shown that this does not cause any problems. Hence we stick to this "lazy" description of the equilibrium, which is sufficient for our purposes.

[10] This is because for values of x_i where $x_i - x_F$ and $x_i - x_{-F}$ have the same sign, the difference between $u_i(F)$ and $u_i(-F)$ is constant in x_i. If this difference is zero, it must be zero for a continuum of consumers, which would contradict our assumption that j is unique.

$u_j(I)$ equality becomes:

$$v - (x_j - x_E) - p_E = v - (x_I - x_j) + p_I$$

We can solve this for x_j:

$$x_j = \frac{x_I + x_E + p_I - p_E}{2}$$

If this value is inside the $]x_E, x_I[$ interval, a unique j exists. In such cases, consumers to the left of j will prefer the entrant's product to the incumbent's, and consumers to the right of her will have the opposite preferences. Putting this result together with our finding from equation (16), we find that the incumbent's leftmost type-L consumer is located at $\max\{x_I + p_I - v, \frac{x_I+x_E+p_I-p_E}{2}, 0\}$, while its rightmost type-L consumer is at $\min\{x_I + v - p_I, 1\}$ (if they exist). Similarly, if the entrant's leftmost and rightmost type-L consumers exist, they are located at $\max\{x_E + p_E - v, 0\}$ and $\min\{x_E + v - p_E, \frac{x_I+x_E+p_I-p_E}{2}, 1\}$ (respectively). This means that the ratios that the two firms cover from the low segment are:

$$D_{I,L}(p_I, p_E, x_I, x_E) = \max\left\{\min\left\{x_I + v - p_I, 1\right\} - \max\left\{x_I + p_I - v, \frac{x_I + x_E + p_I - p_E}{2}, 0\right\}, 0\right\}$$

$$D_{E,L}(p_E, p_I, x_E, x_I) = \max\left\{\min\left\{x_E + v - p_E, \frac{x_I + x_E + p_I - p_E}{2}, 1\right\} - \max\left\{x_E + p_E - v, 0\right\}, 0\right\}$$

However, this is only true in cases where j exists, i.e. $\frac{x_I+x_E+p_I-p_E}{2} \in]x_E, x_I[$. When $\frac{x_I+x_E+p_I-p_E}{2} < x_E$, the incumbent's product is better for all consumers, so we have $D_{E,L}(p_E, p_I, x_E, x_I) = 0$. In this case the incumbent is a monopolist on this segment as well, implying that:

$$D_{I,L}(p_I, p_E, x_I, x_E) = \max\left\{\min\left\{(2 \cdot v - 2 \cdot p_I), (1 - x_I + v - p_I), (x_I + v - p_I), 1\right\}, 0\right\}$$

Similarly, when $x_I < \frac{x_I+x_E+p_I-p_E}{2}$, we have $D_{I,L}(p_I, p_E, x_I, x_E) = 0$ and:

$$D_{E,L}(p_E, p_I, x_E, x_I) = \max\left\{\min\left\{(2 \cdot v - 2 \cdot p_E), (1 - x_E + v - p_E), (x_E + v - p_E), 1\right\}, 0\right\}$$

Finally, it is possible that $x_F = \frac{x_I+x_E+p_I-p_E}{2}$ for some $F \in \{I, E\}$. In such cases there is a continuum of consumers who are indifferent between the two firms, so it is reasonable to assume that half of them chooses the entrant, and the other half chooses the incumbent. However, in a case like that, a firm would be incentivized to unilaterally decrease its price by an infinitesimally small amount, hence causing a discrete jump in its demand and profit. In other words, these cases can never be in the Nash equilibrium of this game, and are therefore irrelevant from our perspective.

From now on, we ignore them.

Summarizing our results from this subsection, we have found that the demands are:

$$D_{I,H}(p_I, p_E, x_I, x_E) = \max\Big\{\min\Big\{(2 \cdot V - 2 \cdot p_I), (1 - x_I + V - p_I), (x_I + V - p_I), 1\Big\}, 0\Big\}$$

$$D_{I,L}(p_I, p_E, x_I, x_E) =$$

$$= \begin{cases} 0 & \text{if } p_I - p_E < x_E - x_I \\ \max\left\{\min\left\{(2 \cdot v - 2 \cdot p_I), (1 - x_I + v - p_I), (x_I + v - p_I), 1\right\}, 0\right\} & \text{if } x_I - x_E < p_I - p_E \\ \max\left\{\min\left\{x_I + v - p_I, 1\right\} - \max\left\{x_I + p_I - v, \frac{x_I + x_E + p_I - p_E}{2}, 0\right\}, 0\right\} & \text{otherwise} \end{cases} \tag{17}$$

$$D_{E,H}(p_E, p_I, x_E, x_I) = 0$$

$$D_{E,L}(p_E, p_I, x_E, x_I) =$$

$$= \begin{cases} 0 & \text{if } x_I - x_E < p_I - p_E \\ \max\left\{\min\left\{(2 \cdot v - 2 \cdot p_E), (1 - x_E + v - p_E), (x_E + v - p_E), 1\right\}, 0\right\} & \text{if } p_I - p_E < x_E - x_I \\ \max\left\{\min\left\{x_E + v - p_E, \frac{x_I + x_E + p_I - p_E}{2}, 1\right\} - \max\left\{x_E + p_E - v, 0\right\}, 0\right\} & \text{otherwise} \end{cases} \tag{18}$$

B.2 The decision of firms in the first stage subgame

From these equations it can be seen that the first stage game in this competitive setting is much more complicated than in the monopoly one, so we take a different approach. Instead of trying to find the equilibrium in a general form, we show that the following strategy profile is a Nash equilibrium:

$$x_I^* = x_E^* = \frac{1}{2}, \quad p_I^* = V - \frac{1}{2}, \quad p_E^* = v - \frac{1}{2} \tag{19}$$

The profits of the two firms in this case are:

$$\Pi_I(p_I^*, p_E^*, x_I^*, x_E^*) = (1 - \lambda) \cdot \left(V - \frac{1}{2}\right) \tag{20}$$

$$\Pi_E(p_E^*, p_I^*, x_E^*, x_I^*) = \lambda \cdot \left(v - \frac{1}{2}\right) \tag{21}$$

To show that this is indeed a Nash equilibrium, we need to prove that neither firm is incentivized to unilaterally deviate from it. For the incumbent, the possible forms of unilateral deviation (see figure 5) are the following:

1. Cover the high segment partially (serving neither extreme points) and leave the low segment to

the entrant. In this case we have $D_{I,H}(p_I, p_E, x_I, x_E) = 2 \cdot V - 2 \cdot p_I$, $D_{I,L}(p_I, p_E, x_I, x_E) = 0$, $D_{E,H}(p_E, p_I, x_E, x_I) = 0$, and $D_{E,L}(p_E, p_I, x_E, x_I) = 1$.

2. Cover the high segment partially (serving exactly one extreme point) and leave the low segment to the entrant. In this case we have $D_{I,H}(p_I, p_E, x_I, x_E) = 1 - x_I + V - p_I$, $D_{I,L}(p_I, p_E, x_I, x_E) = 0$, $D_{E,H}(p_E, p_I, x_E, x_I) = 0$, and $D_{E,L}(p_E, p_I, x_E, x_I) = 1$.

3. Cover the high segment partially (serving exactly one extreme point) and take part of the low segment from the entrant. In this case we have $D_{I,H}(p_I, p_E, x_I, x_E) = 1 - x_I + V - p_I$, $D_{I,L}(p_I, p_E, x_I, x_E) = 1 - \frac{x_I + x_E + p_I - p_E}{2}$, $D_{E,H}(p_E, p_I, x_E, x_I) = 0$, and $D_{E,L}(p_E, p_I, x_E, x_I) = \frac{x_I + x_E + p_I - p_E}{2}$.

4. Cover the high segment fully and leave the low segment to the entrant. In this case we have $D_{I,H}(p_I, p_E, x_I, x_E) = 1$, $D_{I,L}(p_I, p_E, x_I, x_E) = 0$, $D_{E,H}(p_E, p_I, x_E, x_I) = 0$, and $D_{E,L}(p_E, p_I, x_E, x_I) = 1$.

5. Cover the high segment fully and take part of the low segment from the entrant. In this case we have $D_{I,H}(p_I, p_E, x_I, x_E) = 1$, $D_{I,L}(p_I, p_E, x_I, x_E) = 1 - \frac{x_I + x_E + p_I - p_E}{2}$, $D_{E,H}(p_E, p_I, x_E, x_I) = 0$, and $D_{E,L}(p_E, p_I, x_E, x_I) = \frac{x_I + x_E + p_I - p_E}{2}$.

6. Cover both segments fully. In this case we have $D_{I,H}(p_I, p_E, x_I, x_E) = 1$, $D_{I,L}(p_I, p_E, x_I, x_E) = 1$, $D_{E,H}(p_E, p_I, x_E, x_I) = 0$, and $D_{E,L}(p_E, p_I, x_E, x_I) = 0$.

Luckily, we do not need to consider all six of these deviations. First of all, when the incumbent deviates to case 2, its demand can certainly be increased by keeping p_I the same and setting $x_I = \frac{1}{2}$, and hence arriving either in case 1 or case 4. But by increasing the demand without changing the price, the incumbent's profit must also increase, meaning that this is beneficial for it. Therefore, the incumbent's deviation to case 2 can only be beneficial if a deviation to case 1 or 4 is also beneficial. This means that we can ignore this case. Furthermore, the fourth case means that the incumbent would have the same overall demand as in (19), but with a lower price. This would clearly nor be beneficial for it, so this case can be ignored as well. We only deal with the remaining four deviations of the incumbent.

We also need to check whether the entrant is incentivized to deviate unilaterally from (19). Since this firm can never reach a positive demand in the high segment, here we need to consider less cases. In particular, the incumbent can deviate in the following ways (see figure 6):

1. Cover the low segment partially (serving neither extreme points). In this case we have $D_{I,H}(p_I, p_E, x_I, x_E) = 1$, $D_{I,L}(p_I, p_E, x_I, x_E) = 0$, $D_{E,H}(p_E, p_I, x_E, x_I) = 0$, and $D_{E,L}(p_E, p_I, x_E, x_I) = 2 \cdot v - 2 \cdot p_I$.

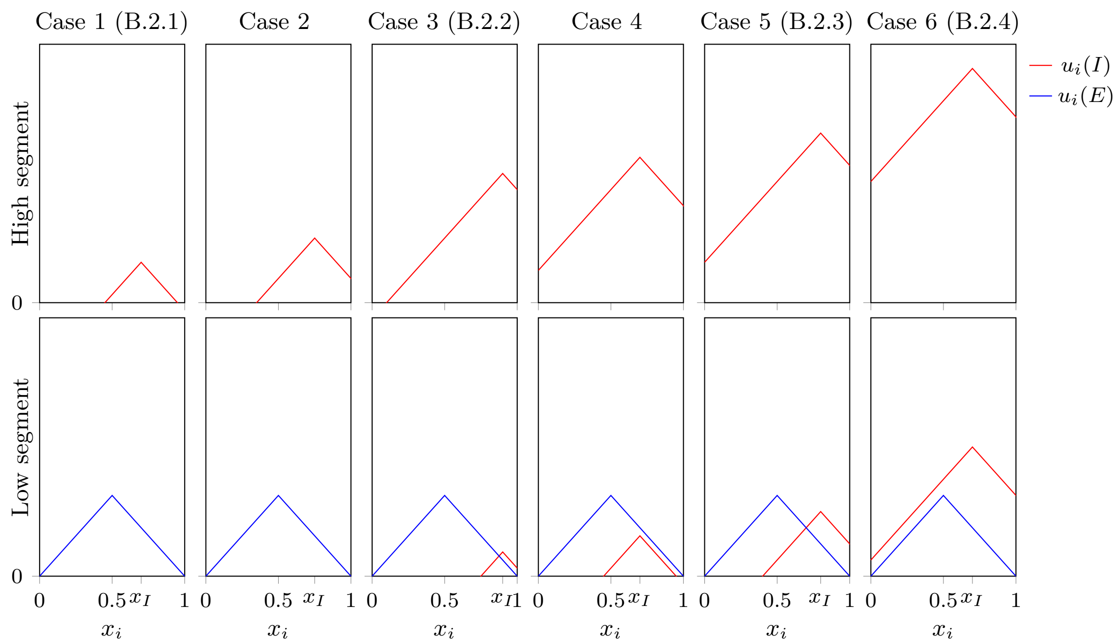

Figure 5: The incumbent's possible forms of unilateral deviation from the equilibrium in (19). The red (blue) lines show the consumers' utility gain from consuming the incumbent's (entrant's) product, by location and segment.

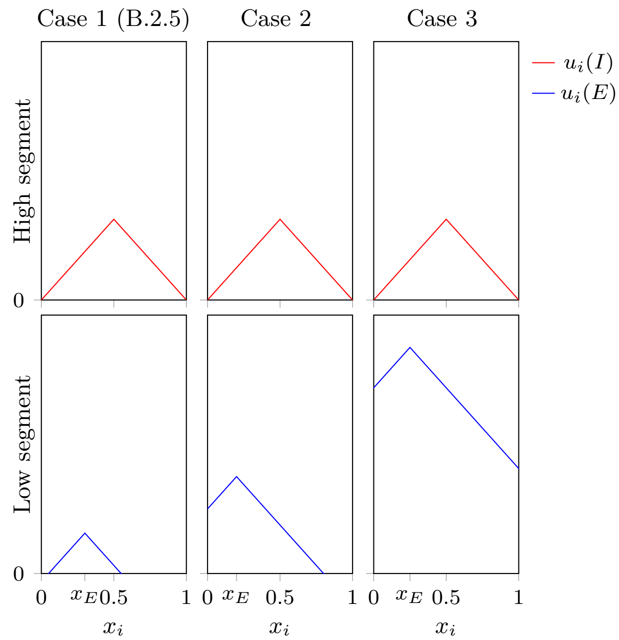

Figure 6: The entrant's possible forms of unilateral deviation from the equilibrium in (19). The red (blue) lines show the consumers' utility gain from consuming the incumbent's (entrant's) product, by location and segment.

2. Cover the low segment partially (serving exactly one extreme point). In this case we have $D_{I,H}(p_I, p_E, x_I, x_E) = 1$, $D_{I,L}(p_I, p_E, x_I, x_E) = 0$, $D_{E,H}(p_E, p_I, x_E, x_I) = 0$, and $D_{E,L}(p_E, p_I, x_E, x_I) = x_I + V - p_I$.

3. Cover the low segment fully. In this case we have $D_{I,H}(p_I, p_E, x_I, x_E) = 1$, $D_{I,L}(p_I, p_E, x_I, x_E) = 0$, $D_{E,H}(p_E, p_I, x_E, x_I) = 0$, and $D_{E,L}(p_E, p_I, x_E, x_I) = 1$.

However, for the same reasons we ignored cases 2 and 4 from the incumbent's deviations, from the entrant's deviations we can ignore cases 2 and 3 (respectively). Therefore, we are left with four relevant forms of deviation for the incumbent, and one for the entrant. In the following, we are going to consider these deviations one by one, and show that neither one makes the deviating agent better off, and hence (19) is indeed a Nash equilibrium. To make the titles shorter, we suppress the terms "*partially (serving neither extreme points)*" and "*partially (serving exactly one extreme point)*" as "*partially*$_0$" and "*partially*$_1$" (respectively).

B.2.1 I covers the high segment partially$_0$ and leaves the low segment to E

In this case the incumbent's demands are $D_{I,H}(p_I, p_E, x_I, x_E) = 2 \cdot V - 2 \cdot p_I$ and $D_{I,L}(p_I, p_E, x_I, x_E) = 0$. Note that neither functions have x_I in them, so as long as the incumbent remains in this case, its choice of location does not have an effect on its demand (and profit). Therefore, we can assume without loss of generality that $x_I = \frac{1}{2}$. This is practical since this is the location that gives the incumbent the most flexibility in choosing the price. However, the inequality that $0 < 2 \cdot V - 2 \cdot p_I < 1$ must hold even under these flexible conditions. Otherwise the value of the $D_{I,H}$ function would be zero or one. But this constraint also makes sure that the low segment remains empty, since we have already shown in equation (9) that the incumbent can never have consumers in the low segment unless the high segment is fully covered. Therefore, the only constraint we need to enforce on p_I in this case is:

$$V - \frac{1}{2} \leq p_I \leq V$$

To show that the incumbent is worse off by this subgroup of possible deviations, it is sufficient to show that the maximal profit it can reach in this case is smaller than the one from (20). We can make this comparison by maximizing this case's profit subject to the above constraint:

$$\max_{p_I \in]V - \frac{1}{2}, V[} \quad \lambda \cdot 0 \cdot p_I + (1 - \lambda) \cdot (2 \cdot V - 2 \cdot p_I) \cdot p_I$$

If the interval in the constraint was closed, the problem would be the same as in (10). The solution to that problem is $p_I = V - \frac{1}{2}$, which is exactly the price we have in (19). Therefore, the incumbent is not incentivized to unilaterally deviate in this manner.

B.2.2 I covers the high segment partially$_1$ and takes part of the low segment from E

After this deviation, the incumbent's demands become $D_{I,H}(p_I, p_E, x_I, x_E) = 1 - x_I + V - p_I$ and $D_{I,L}(p_I, p_E, x_I, x_E) = 1 - \frac{x_I + x_E + p_I - p_E}{2}$. Note that the x_I variable appears with a negative sign in both functions. This implies that as long as it remains in this case, the incumbent is always better off when it moves leftwards (*ceteris paribus*). To reach the highest possible profit here, the incumbent should choose the smallest x_i that makes sure that it remains in this case. "Not remaining in this case" may present itself in two forms, as figure 7 shows. The first possibility is that by moving leftwards, the incumbent reaches the consumer at the leftmost point of the highest segment. This way it would get into case 5 from figure 5. The second possibility is that the incumbent gets so close to the entrant that the entrant "steals" all of its consumers, leading to the situation in case 4 from figure 5.

However, note that in the first one of these scenarios, moving even further left and getting into case 5 is actually beneficial for the incumbent[11]: it can increase its demand in both segments without having to decrease its price. This means that if we found that the incumbent was better off by deviating this way, we would also find that it is (even) better off by deviating to case 5. Since

[11] This is not true in the second scenario where moving further left would mean losing entire segment of consumers.

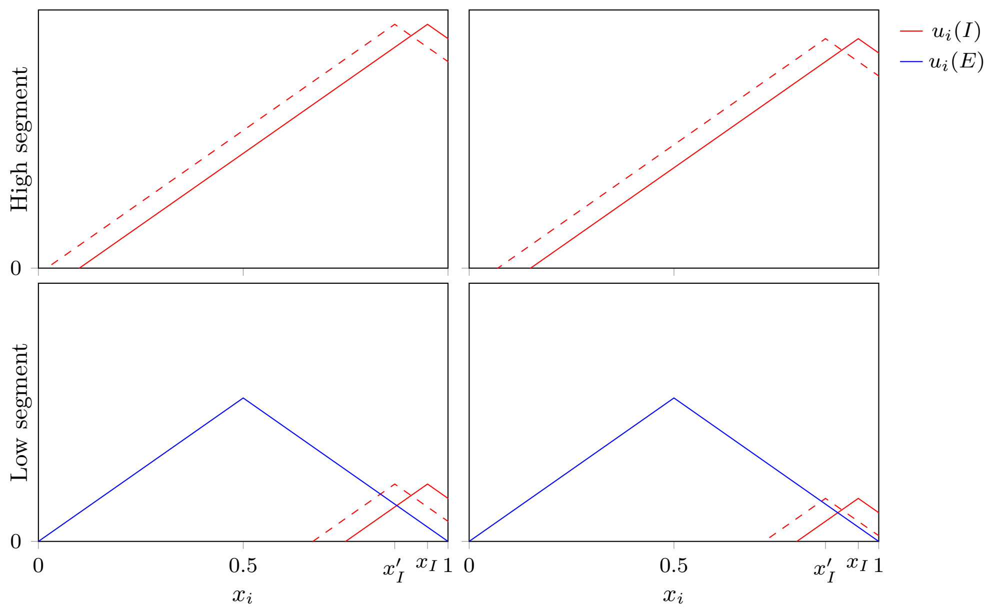

Figure 7: It is always beneficial for the incumbent to move from x_I (solid) to x_I' (dashed), but to remain within this case it must not cover the whole high segment (left) and must not let the entrant cover the whole low segment (right). The red (blue) lines show the consumers' utility gain from consuming the incumbent's (entrant's) product, by location and segment.

we will cover case 5 in subsubsection B.2.3 anyway, we do not need to deal with this first kind of deviation here. What we need to consider is only the second possibility (on the right side of figure 7) where the stricter constraint is the non-zero demand in the low segment.

If this is true, the incumbent can get the highest profit within this case by choosing an infinitesimally higher x_I than what would lead to the case 4 from figure 5. To do this, it should leave the low-segment consumer at 1 with an infinitesimally small positive utility. Let us denote this utility by $\delta > 0$. By definition, we have that $v - (1 - x_I) - p_I = \delta$. For x_I, this means that:

$$x_I = p_I - v + 1 + \delta \tag{22}$$

This x_I must of course fall between x_E^* and 1, so we have $\frac{1}{2} \leq p_I - v + 1 + \delta \leq 1$, which gives us a constraint on p_I:

$$v - \frac{1}{2} - \delta \leq p_I \leq v - \delta \tag{23}$$

Substituting (22) (and the entrant's choices from equation (19)) into the two demand functions, we obtain:

$$D_{I,H}\left(p_I, v-\frac{1}{2}, p_I-v+1+\delta, \frac{1}{2}\right) = V+v-2\cdot p_I-\delta$$
$$D_{I,L}\left(p_I, v-\frac{1}{2}, p_I-v+1+\delta, \frac{1}{2}\right) = v-p_I-\frac{\delta}{2}$$

If the incumbent wants to remain in this case, it also needs to make sure that it does not cover the high segment entirely, i.e. $V+v-2\cdot p_I-\delta<1$. This yields another constraint: $\frac{V+v-1-\delta}{2}<p_I$. Since we know that $v<V$ and $\delta>0$, this is a stricter lower bound than what we had in (23). Therefore, the overall constraint becomes:

$$\frac{V+v-1-\delta}{2}<p_I\leq v-\delta$$

To see which deviation is best for the incumbent from this subset, we maximize its profit subject to the above constraint:

$$\max_{p_I\in]\frac{V+v-1-\delta}{2},v-\delta]} \quad \lambda\cdot\left(v-p_I-\frac{\delta}{2}\right)\cdot p_I+(1-\lambda)\cdot(V+v-2\cdot p_I-\delta)\cdot p_I$$

The first order condition yields that $p_I=\frac{v+(1-\lambda)\cdot V}{2\cdot(2-\lambda)}-\frac{\delta}{4}$. If this is indeed an interior solution, it must hold that:

$$\frac{V+v-1-\delta}{2}<\frac{v+(1-\lambda)\cdot V}{2\cdot(2-\lambda)}-\frac{\delta}{4} \tag{24}$$

We can rearrange this inequality as:

$$V+(1-\lambda)\cdot v-2+\lambda-\frac{(2-\lambda)\cdot\delta}{2}<0$$

However, since we know that $v\geq 1$ and $\lambda\geq\frac{1}{2}$ and hence (using equation (2)) $V\geq\frac{2\cdot v-\lambda}{2\cdot(1-\lambda)}\geq\frac{2\cdot 1-\frac{1}{2}}{2\cdot(1-\frac{1}{2})}=\frac{3}{2}$, we can show that the following chain of inequalities holds:

$$\begin{aligned}V+(1-\lambda)\cdot v-2+\lambda-\frac{(2-\lambda)\cdot\delta}{2} &\geq V+(1-\lambda)\cdot 1-2+\lambda-\frac{(2-\lambda)\cdot\delta}{2}\geq\\ &\geq V+(1-\lambda)\cdot 1-2+\lambda-\frac{(2-\frac{1}{2})\cdot\delta}{2}\geq\\ &\geq\frac{3}{2}+(1-\lambda)\cdot 1-2+\lambda-\frac{(2-\frac{1}{2})\cdot\delta}{2}=\frac{1}{2}-\frac{3\cdot\delta}{4}\end{aligned}$$

Since we defined δ as an infinitesimally small number, $\frac{1}{2}-\frac{3\cdot\delta}{4}$ is definitely positive, which contradicts the inequality from (24). This means that under these settings the incumbent is always incentivized to decrease its price. But since we have an open interval in the profit maximization problem, this price must remain above $\frac{V+v-1-\delta}{2}$. Hence the incumbent's best option within this scenario is to choose an infinitesimally higher price:

$$p_i = \frac{V+v-1-\delta}{2} + \varepsilon \quad |\varepsilon > 0$$

If we substitute this value into the incumbent's profit, we get:

$$\left(\lambda \cdot \frac{v-V+1-2\cdot\varepsilon}{2} + (1-\lambda)\cdot(1-2\cdot\varepsilon)\right) \cdot \frac{V+v-1-\delta+2\cdot\varepsilon}{2}$$

Our next task is to prove that this profit is smaller than what the incumbent obtains in (20). We use proof by contradiction, assuming that:

$$(1-\lambda)\cdot\left(V-\frac{1}{2}\right) \leq \left(\lambda \cdot \frac{v-V+1-2\cdot\varepsilon}{2} + (1-\lambda)\cdot(1-2\cdot\varepsilon)\right) \cdot \frac{V+v-1-\delta+2\cdot\varepsilon}{2} \tag{25}$$

The profit on the right hand side is decreasing[12] in both δ and ε, but these values must by definition be greater than zero. This gives us an upper limit on the incumbent's profit:

$$\left(\lambda \cdot \frac{v-V+1-2\cdot\varepsilon}{2} + (1-\lambda)\cdot(1-2\cdot\varepsilon)\right) \cdot \frac{V+v-1-\delta+2\cdot\varepsilon}{2} <$$
$$< \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left(\left(\lambda \cdot \frac{v-V+1-2\cdot\varepsilon}{2} + (1-\lambda)\cdot(1-2\cdot\varepsilon)\right) \cdot \frac{V+v-1-\delta+2\cdot\varepsilon}{2}\right) =$$
$$= \left(\lambda \cdot \frac{v-V+1}{2} + 1 - \lambda\right) \cdot \frac{V+v-1}{2}$$

Therefore, if we want (25) to hold, we must also have that:

$$(1-\lambda)\cdot\left(V-\frac{1}{2}\right) \leq \left(\lambda \cdot \frac{v-V+1}{2} + 1 - \lambda\right) \cdot \frac{V+v-1}{2}$$

This inequality can be rearranged as:

$$0 \leq v^2 + 2\cdot\frac{1-\lambda}{\lambda}\cdot v - (V-1)^2 - 2\cdot\frac{1-\lambda}{\lambda}\cdot V$$

The right hand side is quadratic in v, with v^2 having a positive coefficient. Therefore, we can use the quadratic formula to obtain a solution for v. This yields:

$$v \leq -\sqrt{\left(V-1+\frac{1-\lambda}{\lambda}\right)+2\cdot\frac{1-\lambda}{\lambda}} - \frac{1-\lambda}{\lambda} \text{ or } \sqrt{\left(V-1+\frac{1-\lambda}{\lambda}\right)+2\cdot\frac{1-\lambda}{\lambda}} - \frac{1-\lambda}{\lambda} \leq v$$

[12] We defined the two variables as infinitesimally small precisely because it is best for the incumbent to set these values as low as possible, so this is not at all surprising. However, this can be shown analytically as well.

However, since we know that v is positive, the first inequality can never hold. Hence we are left with the second one, giving us a lower limit on v. Since our assumption from (2) can be rearranged as $v \leq V - \lambda \cdot V + \frac{\lambda}{2}$, this also gives us an upper limit. If we want (25) to hold, this upper limit must necessarily exceed the lower one, otherwise such values of v would not exist. This yields:

$$\sqrt{\left(V - 1 + \frac{1-\lambda}{\lambda}\right) + 2 \cdot \frac{1-\lambda}{\lambda}} - \frac{1-\lambda}{\lambda} \leq V - \lambda \cdot V + \frac{\lambda}{2}$$

This inequality simplifies to:

$$\lambda \cdot (2-\lambda) \cdot \left(V - \frac{1}{2}\right) \cdot \left(V - \frac{1}{2} - \frac{1}{2-\lambda}\right) \leq 0$$

Since we know that $\lambda > 0$, $\lambda < 2$, and $V > \frac{1}{2}$, this can only hold if $V \leq \frac{1}{2} + \frac{1}{2-\lambda}$. However, since $\frac{1}{2-\lambda}$ is increasing in λ and we know that $\lambda < 1$, it must also hold that:

$$V \leq \frac{1}{2} + \frac{1}{2-\lambda} < \frac{1}{2} + \frac{1}{2-1} = \frac{3}{2}$$

This can never be true because equation (2) implies that the lowest possible value of V (when $V = 1$ and $\lambda = \frac{1}{2}$) is exactly $\frac{2 \cdot 1 - \frac{1}{2}}{2 \cdot (1 - \frac{1}{2})} = \frac{3}{2}$. Therefore, we have finally reached a contradiction and showed that our assumption from (25) was incorrect, hence proving that the incumbent is not incentivized to deviate in this form.

B.2.3 I covers the high segment fully and takes part of the low segment from E

In this case the incumbent's demands are $D_{I,H}(p_I, p_E, x_I, x_E) = 1$ and $D_{I,L}(p_I, p_E, x_I, x_E) = 1 - \frac{x_I + x_E + p_I - p_E}{2}$. Like in the previous case, the demand on the low segment is decreasing in x_I (and on the high segment it is constant). Therefore, we can make a similar argument that the incumbent is always incentivized to move leftwards as long as it remains within this case. Also similarly to the previous case, we have two scenarios to consider (see figure 8). The first one is that by moving leftwards, the incumbent finally reaches a point where it takes all consumers from the entrant, hence covering the entire[13] low segment and getting into case 6 from figure 5. This can only happen if the incumbent's price is not higher[14] than the entrant's, i.e. $p_I \leq p_E^* = v - \frac{1}{2}$.

[13]Actually, before the incumbent would cover the whole low segment, there is a point where a continuum of consumers is indifferent between the two firms.

[14]If we wanted to be completely precise, we should have said that there is a third possibility that $p_I = p_E^*$. If the incumbent moved leftwards with this price, it would eventually arrive to the case where all type-L consumers are indifferent between the two firms, and so the low segment demand is split evenly. This would actually be beneficial for the incumbent because by staying on the right side of the entrant it covers strictly less than half of the low segment. However, the incumbent could then go on and increase its profit even further by choosing an infinitesimally lower price than p_E^* and covering the whole market. Therefore, in the end the same argument applies here as when $p_I < p_E^*$.

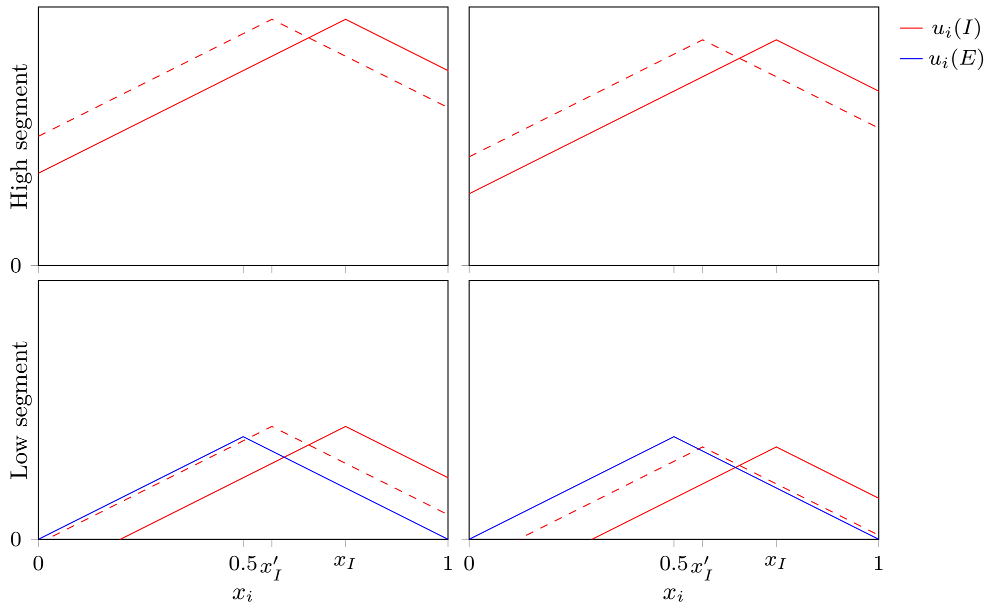

Figure 8: It is always beneficial for the incumbent to move from x_I (solid) to x_I' (dashed), but to remain within this case it must not take the whole low segment from the entrant (left) and must not let the entrant take it (right). The red (blue) lines show the consumers' utility gain from consuming the incumbent's (entrant's) product, by location and segment.

The second option is that it is the entrant who takes the incumbent's consumers, meaning that the incumbent arrives to case 4 from figure 5. This can happen if $p_I > p_E^* = v - \frac{1}{2}$.

However, also analogously to the previous case, we do not need to consider the first one of these options (on the left side of figure 8). The reason for this is that this scenario can be beneficial for the incumbent only if a deviation to case 6 is also beneficial, but we will deal with that case later in subsubsection B.2.4. Therefore, what we are left with is the second option (on the right side of figure 8). Here the incumbent's best option is to leave the low-segment consumer at 1 with an infinitesimally small positive utility, $\zeta > 0$. This means by definition that $v - (1 - x_I) - p_I = \zeta$, so can obtain x_I as:

$$x_I = p_I - v + 1 + \zeta \tag{26}$$

This x_I must again fall between x_E^* and 1, meaning that:

$$v - \frac{1}{2} - \zeta \leq p_I \leq v - \zeta \tag{27}$$

Together with ζ's definition, this constraint already makes sure that the incumbent has at least some consumers on the low segment. But since we are only dealing with the scenario where the incumbent's price is over the entrant's, we have an even stricter lower bound: $p_I > p_E^* = v - \frac{1}{2}$.

Moreover, we also have another upper bound because the incumbent must cover the entire high segment in this case, which means that even the type-H consumer at 0 gets a non-negative utility: $V-(x_I-0)-p_I \geq 0$. Substituting x_I from equation (26), we can rearrange this as $p_I \leq \frac{V+v-1-\zeta}{2}$. Therefore, putting all the constraints together, we have found that the incumbent stays within this case iff:

$$v-\frac{1}{2} < p_I \leq \min\left\{v-\zeta, \frac{V+v-1-\zeta}{2}\right\} \tag{28}$$

By substituting (26) (as well as the entrant's choices from equilibrium (19)) into the incumbent's low-segment demand function, we get:

$$D_{I,L}\left(p_I, v-\frac{1}{2}, p_I-v+1+\zeta, \frac{1}{2}\right) = v-p_I-\frac{\zeta}{2}$$

To see which deviation from this subgroup yields the highest profit, we can solve the incumbent's profit maximization problem subject to (28):

$$\max_{p_I \in]v-\frac{1}{2}, \min\{v-\zeta, \frac{V+v-1-\zeta}{2}\}]} \quad \lambda \cdot \left(v-p_I-\frac{\zeta}{2}\right) \cdot p_I + (1-\lambda) \cdot 1 \cdot p_I$$

The first order condition solves to $p_I = \frac{v}{2} + \frac{1-\lambda}{2 \cdot \lambda} - \frac{\zeta}{4}$. This will be an interior solution iff it falls between the lower and upper bounds, i.e. the parametrization satisfies the following conditions:

$$\begin{aligned} v &< \frac{1}{\lambda} - \frac{\zeta}{2} \\ \frac{1-\lambda}{\lambda} + \frac{3 \cdot \zeta}{2} &\leq v \\ \frac{1}{\lambda} + \frac{\zeta}{2} &\leq V \end{aligned} \tag{29}$$

First, let us assume that these conditions hold and so $p_I = \frac{v}{2} + \frac{1-\lambda}{2 \cdot \lambda} - \frac{\zeta}{4}$ is indeed an interior solution. If this is the case, the incumbent's "maximized" profit becomes:

$$\lambda \cdot \left(\frac{v}{2} + \frac{1-\lambda}{2 \cdot \lambda} - \frac{\zeta}{4}\right)^2 \tag{30}$$

What we need to show is that this value is smaller than the incumbent's equilibrium profit from (20). We use proof by contradiction, assuming that:

$$(1-\lambda) \cdot \left(V-\frac{1}{2}\right) \leq \lambda \cdot \left(\frac{v}{2} + \frac{1-\lambda}{2 \cdot \lambda} - \frac{\zeta}{4}\right)^2$$

Since the function on the right side of the inequality is decreasing in ζ and we know that $\zeta > 0$, this must necessarily mean that:

$$(1-\lambda)\cdot\left(V-\frac{1}{2}\right)\leq\lambda\cdot\left(\frac{v}{2}+\frac{1-\lambda}{2\cdot\lambda}-\frac{\zeta}{4}\right)^2<\lim_{\zeta\to 0}\lambda\cdot\left(\frac{v}{2}+\frac{1-\lambda}{2\cdot\lambda}-\frac{\zeta}{4}\right)^2=\lambda\cdot\left(\frac{v}{2}+\frac{1-\lambda}{2\cdot\lambda}\right)^2$$

This gives us an upper bound on V:

$$V<\frac{\lambda}{1-\lambda}\cdot\left(\frac{v}{2}+\frac{1-\lambda}{2\cdot\lambda}\right)^2+\frac{1}{2}$$

Together with (2), this implies:

$$\frac{2\cdot v-\lambda}{2\cdot(1-\lambda)}\leq V<\frac{\lambda}{1-\lambda}\cdot\left(\frac{v}{2}+\frac{1-\lambda}{2\cdot\lambda}\right)^2+\frac{1}{2}$$

By rearranging, we get that such values of V exist only if the following condition is satisfied:

$$\left(v-\frac{\lambda+1}{\lambda}-\sqrt{\frac{2}{\lambda}}\right)\cdot\left(v-\frac{\lambda+1}{\lambda}+\sqrt{\frac{2}{\lambda}}\right)>0$$

This can happen either if $v<\frac{\lambda+1}{\lambda}-\sqrt{\frac{2}{\lambda}}$ or if $v>\frac{\lambda+1}{\lambda}+\sqrt{\frac{2}{\lambda}}$. However, $\lambda\in[\frac{1}{2},1[$ implies that $\frac{\lambda+1}{\lambda}-\sqrt{\frac{2}{\lambda}}<1$, so the first inequality would contradict our assumption that $v\geq 1$. Similarly, $\lambda\in[\frac{1}{2},1[$ also means that $\frac{\lambda+1}{\lambda}+\sqrt{\frac{2}{\lambda}}>\frac{1}{\lambda}-\frac{\zeta}{2}$. This contradicts the first constraint from (29), so the second inequality means that we cannot have an interior solution. Therefore, we have reached a contradiction and shown that the incumbent is not incentivized to unilaterally deviate from equilibrium (19), though only for parametrizations that yield interior solutions.

Next, let us deal with the parametrizations which yield corner solutions, i.e. where (29) is violated. Note that for parametrizations where $\frac{\lambda+1}{\lambda}-\sqrt{\frac{2}{\lambda}}\leq v\leq\frac{\lambda+1}{\lambda}+\sqrt{\frac{2}{\lambda}}$, we have already shown that the the profit from (30) is below the profit from (20). But we also know that a constrained maximization problem can never yield a higher profit than an unconstrained one with the same objective function, so if any of these parametrizations leads to a corner solution, the corresponding profit must also fall below the profit from (20). Therefore (since we can also ignore cases where $v<1$), we are only left with the parametrizations where $v>\frac{\lambda+1}{\lambda}+\sqrt{\frac{2}{\lambda}}$. We have already shown that under such settings it is p_I's lower bound that is violated. But since this lower bound is strict, the incumbent's best option is to choose an infinitesimally higher price: $p_I=v-\frac{1}{2}+\eta$ for some $\eta<0$. Hence its profit becomes:

$$\left(1-\frac{\lambda}{2}-\frac{\lambda\cdot\zeta}{2}-\lambda\cdot\eta\right)\cdot\left(v-\frac{1}{2}+\eta\right)$$

To prove that this is also not a beneficial deviation, we need to show that this value falls below the profit from (20). Like before, we use proof by contradiction. We assume that:

$$(1-\lambda)\cdot\left(V-\frac{1}{2}\right)\leq\left(1-\frac{\lambda}{2}-\frac{\lambda\cdot\zeta}{2}-\lambda\cdot\eta\right)\cdot\left(v-\frac{1}{2}+\eta\right)$$

The function on the right hand side is decreasing in both ζ and η, so by a similar argument we must have:

$$(1-\lambda)\cdot\left(V-\frac{1}{2}\right)\leq\left(1-\frac{\lambda}{2}-\frac{\lambda\cdot\zeta}{2}-\lambda\cdot\eta\right)\cdot\left(v-\frac{1}{2}+\eta\right)<$$
$$<\lim_{\zeta\to 0}\lim_{\eta\to 0}\left(\left(1-\frac{\lambda}{2}-\frac{\lambda\cdot\zeta}{2}-\lambda\cdot\eta\right)\cdot\left(v-\frac{1}{2}+\eta\right)\right)=\left(1-\frac{\lambda}{2}\right)\cdot\left(v-\frac{1}{2}\right)$$

This can be rearranged as:

$$V<\frac{2\cdot v-\lambda}{2\cdot(1-\lambda)}-\frac{\lambda}{1-\lambda}\cdot(2\cdot v-1)$$

However, since $\lambda\in[\frac{1}{2},1[$ and $v\geq 1$, this inequality contradicts our assumption from equation (2). Therefore, we have reached a contradiction for corner solutions as well, meaning that the incumbent is not incentivized to deviate from equilibrium (19) in this form.

B.2.4 *I* covers both segments fully

If the incumbent deviates this way, its demands on the two segments become $D_{I,H}(p_I,p_E,x_I,x_E)=D_{I,L}(p_I,p_E,x_I,x_E)=1$. Similarly as in subsubsection B.2.1, we can assume without loss of generality that $x_I=\frac{1}{2}$. This gives the incumbent more flexibility in choosing its price than any other location. But even under these flexible conditions, the incumbent must serve most extreme points of the low segment[15]. This yields[16] $v-(x_I-0)-p_I>0$ and $v-(1-x_I)-p_I>0$. Substituting $x_I=\frac{1}{2}$ into any of these equations, we get the following constraint on p_I:

$$p_I<v-\frac{1}{2}$$

To obtain this case's highest achievable profit, we solve the profit maximisation problem subject to this constraint:

$$\max_{p_I\in[0,v-\frac{1}{2}[}\quad\lambda\cdot 1\cdot p_I+(1-\lambda)\cdot 1\cdot p_I$$

[15]If it covers the low segment entirely, the high segment's coverage is also ensured (see equation (9)), so that constraint will not be binding.

[16]The reason we have strict inequalities is that if we allowed for equality, all low-segment consumers would be indifferent between the two firms. We have shown that these cases can be ignored.

The objective function is increasing in p_I on the whole domain, but we have a strict upper bound. This means that the incumbent's best option is to choose an infinitesimally lower price, $v - \frac{1}{2} - \theta$ for some $\theta > 0$. This way its profit becomes:

$$v - \frac{1}{2} - \theta$$

If this kind of deviation was beneficial for the incumbent, the above profit would be at least as high as the one from (20), i.e.:

$$(1 - \lambda) \cdot \left(V - \frac{1}{2}\right) \leq v - \frac{1}{2} - \theta$$

This can be rearranged as:

$$V \leq \frac{2 \cdot v - \lambda}{2 \cdot (1 - \lambda)} - \frac{\theta}{1 - \lambda}$$

However, this would contradict our assumption from (2). Therefore the incumbent is not incentivized to deviate in this manner either.

B.2.5 E covers the low segment partially$_0$

When the entrant deviates this way, its demands on the two segments become $D_{E,H}(p_E, p_I, x_E, x_I) = 0$ and $D_{E,L}(p_E, p_I, x_E, x_I) = 2 \cdot v - 2 \cdot p_I$. Note that this case is completely analogous with what we had in subsubsection B.2.1, we only need to replace E, I, v, H, and L with I, E, V, L, and H (respectively). Therefore, this kind of unilateral deviation will not be beneficial for the entrant either. Since this was the last case we needed to consider, this ends our proof that the strategy profile in equation (19) is indeed a Nash equilibrium of the competitive game.

B.3 Payoff dominance

Although we have shown that the strategy profile in equation (19) is a Nash equilibrium, this does not necessarily mean that it is the *only* Nash equilibrium. In theory, there could be another equilibrium that Pareto dominates this one. If this was the case, the two firms would most probably coordinate to reach that dominant equilibrium instead, so we should not expect (19) to be the actual outcome of the game. Therefore, if we want our results to have practical relevance, it is not sufficient to show that (19) is a Nash equilibrium. We also need to prove that such dominating equilibria cannot possibly exist. This is what we do in this subsection.

When proving that (19) is a Nash equilibrium, in each case we have shown that a unilateral deviation is *strictly* worse for the players[17]. This implies that there cannot be other equilibria which make only one firm better off and leave the other one with the same profit. If an equilibrium Pareto dominates the one from (19), that equilibrium must yield a *strictly* higher profit for *both* firms. However, it can easily be shown[18] that the entrant can never obtain a higher profit than what it has in (21). We have reached a contradiction, meaning that no other Nash equilibrium can Pareto dominate the one from (19). Our findings are therefore relevant.

[17]In other words, we have found (19) to be a *strict* Nash equilibrium.

[18]In (19), the entrant already covers the entire lower segment, so its demand (and profit) cannot be increased by lowering the price. The other option would be to choose a higher price and cover a smaller portion of the low segment, but we have already shown in subsubsection B.2.5 that this is not beneficial either.