The Equilibrium-Value Convergence for the Multiple-Partners Assignment Game*

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Abstract

We study the *multiple-partners assignment game* (Sotomayor, 1992), the simplest many-to-many extension of the assignment game. Our main result is that the Shapley value of a replicated multiple-partners assignment game converges to a competitive equilibrium payoff when the number of replicas tends to infinity. Furthermore, the result also holds for a large subclass of semivalues since we prove that they converge to the same value as the replica becomes large. For

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the analysis, we introduce and study multiple-partners assignment games with types, which are games where several agents are identical. We show, in particular, that every competitive equilibrium outcome of a “large” game with types satisfies equal treatment of equals and equal treatment of partnerships.

Keywords: Assignment game; Shapley value; Replica; Semivalues

JEL Classification: C78; C71; D78

1 Introduction

We study the *multiple-partners assignment game* (the *multiple-partners game*, for short), introduced by Sotomayor (1992), which is the simplest many-to-many extension of the assignment game (Shapley and Shubik, 1972). In this two-sided matching market, a set of possibly heterogeneous players from one side meet with another set of possibly heterogeneous players from the other side. Each player in a multiple-partners game has a quota and can have as many partnerships with different players from the other side as her quota allows. If two players form a partnership, they produce a gain, which can be divided between them in any manner they decide. The total surplus of a player is the sum of the surpluses she obtains in all her partnerships. This model can represent markets with sellers and buyers, firms and workers, or venture capital firms and startups.

An outcome of the multiple-partners game is a matching, which specifies a set of partners for each player meeting the quotas, and a payoff vector that stipulates the sharing of the surplus in each partnership. Stability and competitive equilibrium are the main solution concepts in matching models. In a *competitive equilibrium outcome* (Sotomayor, 2007), the objects a seller offers have associated a non-negative price, and, given the price vector, each buyer chooses a bundle that maximizes her total surplus. Sotomayor (2007) proved that a competitive equilibrium is a *stable outcome* where each seller obtains the same payoff in each transaction.

By considering the multiple-partners game as a coalitional game with transferable utility (a *TU game*), one can also apply to multiple-partners games single-valued solution concepts for TU games. The *Shapley value* (Shapley, 1953) is the most popular value in TU games. It has been studied in assignment games by Hoffmann and Sudhölter (2007) and van den Brink and Pinter (2015).[1] A major drawback of the Shapley value of an assignment game (hence, also of a multiple-partners game) is that it may not be a stable payoff. The main purpose of our paper is to show that this drawback may be fixed through replication of the game.

We prove that the Shapley value of a replicated multiple-partners game converges to a competitive equilibrium payoff (hence, to a stable payoff) when the number of replicas tends to infinity. Thus, our result generalizes Shapley and Shubik's (1969) and Liggett, Lippman, and Rumelt's (2009) theorems on the asymptotic behavior of the Shapley value of expanding 1-to-1 *glove markets* and expanding 1-to-k *glove markets*.

Furthermore, our result applies to a large subclass of *semivalues* (Dubey, Neyman, and Weber, 1981), which are single-valued solutions obtained by relaxing the axioms that characterize the Shapley value. Our theorem states that all the semivalues of this subclass of a replicated game converge to the same value as the replica becomes large, and the players' payoffs converge to a competitive equilibrium payoff.

For the proof of our theorem, we introduce and study *multiple-partners games with types*, which are games where several agents are identical, and we provide properties of *large multiple-partners games with types*, where the number of players of any type is larger than any players' quota. We show, for instance, that every competitive equilibrium outcome of a large game with types satisfies *equal treatment of equals* (that is, two

[1] Núñez and Rafels (2019) reviewed the contributions that study the Shapley value of the assignment game. The nucleolus (Schmeidler, 1969) is another popular solution concept for TU games that has been considered for the assignment game, e.g., Llerena, Núñez, and Rafels (2015).

identical players obtain the same vector of payoffs) and *equal treatment of partnerships* (that is, the payoff obtained by a player is the same in all her partnerships).[2]

The line of research on the value-equilibrium convergence was initiated by Shapley (1964), who showed the convergence of the Shapley value of replicated exchange economies with transferable utility to a competitive allocation.[3] Champsaur (1975) proved that the NTU Shapley value allocations (Shapley, 1967) are asymptotically included in the set of competitive allocations for exchange economies with production (see also Shapley and Shubik, 1969, and Mas-Colell, 1977).[4] Relatedly, Wooders and Zame (1987) formulated a fairly general class of games with transferable utility and proved that the Shapley value allocation can get arbitrarily close to an element in the core for sufficiently large games.

In multiple-partners games, Sotomayor (2019) introduced multi-stage cooperative games, defined the concepts of sequential stability and perfect competitive equilibrium, and studied the effect of the replications of the market on the cooperative and the competitive structures of the extended markets. She proved, in particular, that the sets of stable and competitive equilibrium allocations shrink to the set of stable allocations that satisfy equal treatment of equals and equal treatment of partnerships when the replica is large.[5] In contrast, our focus is the convergence of the Shapley value and

[2] These results generalize previous results by Sotomayor (2010), who showed that the replicated market satisfies equal treatment of equals and equal treatment of partnerships.

[3] See Hart (2002) for a survey on this topic.

[4] In parallel with the asymptotic approach, the equivalence between the Shapley value allocation and the set of competitive allocations is proved for economic environments with a continuum of agents. In particular, Aumann and Shapley (1974) established the equivalence for market games with a continuum of traders. Aumann (1975) obtained a similar result for pure exchange economies with a continuum of traders.

[5] See Massó and Neme (2014) for a class of theorems regarding the finite convergence of the set of competitive allocations and other stability concepts for a different type of many-to-many assignment game.

other semivalues to a competitive payoff.

The remainder of the paper is organized as follows. Section 2 describes the environment and the solution concepts that we will study. Section 3 introduces the concept of multiple-partners games with types, and Section 4 states and proves properties of the competitive equilibria for those games. Section 5 states and proves our main convergence result. Section 6 concludes the paper. All the proofs except the one for the theorem are in the Appendix.

2 The multiple-partners game

2.1 The model

We study the multiple-partners assignment game (that we call the *multiple-partners game*) introduced by Sotomayor (1992), a generalization of the assignment game (Shapley and Shubik, 1972). In this model, there are two finite and disjoint sets of players: a set of *buyers* $B = \{b_1, \ldots, b_{n_b}\}$ and a set of *sellers* $S = \{s_1, \ldots, s_{n_s}\}$. We use b and s to represent, generically, any element of B and S, respectively.

Each player has a *quota* representing the maximum number of partnerships he/she can enter. Each buyer can only acquire one object from each seller. Thus, the quota $r(b) > 0$ of buyer $b \in B$ is an integer representing the maximum number of objects buyer b can acquire (or the maximum number of sellers he can buy from). Similarly, each seller $s \in S$ owns $r(s) > 0$ identical objects. Hence, the quota $r(s) > 0$ represents the maximum number of buyers the seller s can sell to.[6] Since the objects owned by each seller are identical, occasionally, we use s also to refer to an object sold by seller s.

Without loss of generality, we assume that every player has a reservation utility

[6] A multiple-partners game is an *assignment game* if $r(b) = 1$ for all $b \in B$ and $r(s) = 1$ for all $s \in S$.

of 0; that is, a seller assigns a worth of 0 to any object that she does not sell, and a buyer obtains a worth of 0 from any unfilled spot in his quota. Players can obtain non-negative payoffs when they form partnerships. For each pair $(b, s) \in B \times S$, there is a non-negative number $a_{bs} \geq 0$, representing the *worth* generated from a partnership between the buyer b and the seller s. The surplus a_{bs} can be shared between b and s any way they decide. If buyer b acquires the object s at the price p_{bs}, then his individual payoff in this transaction is $u_{bs} = a_{bs} - p_{bs}$ whereas seller s receives p_{bs}. Finally, we assume that players' preferences are separable across pairs in that the payoff from a partnership does not depend on the other partnerships formed.

The market described above is $M := \langle B, S, \mathbf{a}, \mathbf{r} \rangle$. It will often be denoted by M when this simplification does not lead to confusion.

A feasible matching for M assigns at most $r(b)$ distinct sellers to each buyer b and at most $r(s)$ distinct buyers to each seller s. We represent it through a matrix.

Definition 1. *A* ***feasible matching*** *for $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r} \rangle$ is a $n_b \times n_s$ matrix $\boldsymbol{x} = (x_{bs})_{(b,s) \in B \times S}$ of non-negative integer entries such that $x_{bs} \in \{0, 1\}$ for all $b \in B$ and $s \in S$. Furthermore, $\sum_{b \in B} x_{bs} \leq r(s)$ for all $s \in S$ and $\sum_{s \in S} x_{bs} \leq r(b)$ for all $b \in B$.*

We denote by $\mathcal{A}(B, S, \mathbf{r})$ the set of feasible matchings between B and S.

For each feasible matching $\mathbf{x}$ for M, we denote by $C_b(\mathbf{x})$ the set of partners assigned to buyer b according to $\mathbf{x}$. Formally, for a given $\mathbf{x} \in \mathcal{A}(B, S, \mathbf{r})$, we define $C_b(\mathbf{x}) = \{s \in S \mid x_{bs} = 1\}$. Similarly, we define by $C_s(\mathbf{x})$ the set of partners assigned to seller s at $\mathbf{x}$. Therefore, $C_b(\mathbf{x})$ has a most $r(b)$ elements, for each $b \in B$, and $C_s(\mathbf{x})$ has at most $r(s)$ elements, for each $s \in S$. The set of pairs $(b, s) \in B \times S$ that are assigned to each other at $\mathbf{x}$ is denoted by $C(\mathbf{x})$. That is, $(b, s) \in C(\mathbf{x})$ if $x_{bs} = 1$. We say that buyer b and seller s are (respectively, are not) matched at $\mathbf{x}$ if $(b, s) \in C(\mathbf{x})$ (respectively, $(b, s) \notin C(\mathbf{x})$).

An outcome of the market M involves not only a matching but also a vector of payoffs:

Definition 2. *A feasible outcome for* $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r} \rangle$*, denoted by* $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$*, is a feasible matching* $\boldsymbol{x}$ *and a pair of payoff vectors* $(\mathbf{u}, \mathbf{v})$*, where* $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^{C(\boldsymbol{x})}$ *satisfy* $u_{bs} \geq 0$*,* $v_{bs} \geq 0$*, and* $u_{bs} + v_{bs} = a_{bs}$ *for all* $(b, s) \in C(\boldsymbol{x})$*.*

Definition 3. *A feasible payoff vector* $(\mathbf{u}, \mathbf{v})$ *for* $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r} \rangle$ *is the projection of some feasible outcome* $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ *for* M *on* $\mathbb{R}_+^{C(\boldsymbol{x})} \times \mathbb{R}_+^{C(\boldsymbol{x})}$*.*

In a feasible outcome $(\mathbf{u}, \mathbf{v}; \mathbf{x})$, the individual payoffs of each $b \in B$ and $s \in S$ are given by the arrays of numbers $u_{bs} \geq 0$ and $v_{bs} \geq 0$, respectively, only defined if $x_{bs} = 1$, and such that $u_{bs} + v_{bs} = a_{bs}$.

Given a feasible outcome $(\mathbf{u}, \mathbf{v}; \mathbf{x})$, we denote by $u_{b,\min}$ and $v_{s,\min}$ the *minimum payoff* of buyer b and seller s, respectively, among his/her payoffs. That is, $u_{b,\min} := \min_{s \in C_b(\mathbf{x})} u_{bs}$ and $v_{s,\min} := \min_{b \in C_s(\mathbf{x})} v_{bs}$. The *total payoff* of buyer b and seller s are given by $\sum_{s \in C_b(\mathbf{x})} u_{bs}$ and $\sum_{b \in C_s(\mathbf{x})} v_{bs}$, respectively.

2.2 Stability and competitive equilibrium

Stability is a natural solution concept for the multiple-partners game. Sotomayor (1992) proved that the notion of setwise-stability[7] is equivalent to the following notion of pairwise stability, which we will refer to simply as *stability*:

Definition 4. *The feasible outcome* $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ *for* $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r} \rangle$ *is stable if* $u_{b,\min} + v_{s,\min} \geq a_{bs}$ *for all* $b \in B$ *and all* $s \notin C_b(\boldsymbol{x})$*.*

The interpretation of Definition 4 is standard: For a feasible outcome to be stable, there cannot exist some pair of players (b, s) who are not matched but who could both get a higher payoff by forming a partnership while at the same time dissolving one of their current partnerships, if it is necessary to stay within their quotas.

[7] A feasible outcome is setwise-stable if there is no coalition of players who, by forming new partnerships only among themselves, − possibly dissolving some partnerships to remain within their quotas and possibly keeping other partnerships− can all obtain a higher payoff.

Another natural solution concept for the multiple-partners game is the *competitive equilibrium* (Sotomayor, 2007). Under the competitive approach, each object s is associated with a non-negative price p_s.[8] We denote by $\mathbf{p} = (p_s)_{s \in S} \in \mathbb{R}^S_+$ a *price vector*.

In a competitive equilibrium, given a price vector $\mathbf{p}$, each buyer b maximizes his total payoff over the sets of objects that are feasible to him. We say that a *set* Q is *feasible to buyer* b if it has at most $r(b)$ elements. Therefore, we define buyer b's *demand set* $D_b(\mathbf{p})$ as:

$$D_b(\mathbf{p}) := \underset{Q \text{ feasible to } b}{\operatorname{argmax}} \sum_{s \in Q} (a_{bs} - p_s).$$

Thus, in a competitive equilibrium, every agent is assigned a set of partners in their demand set, and the competitive pressure leads the price of every unsold object to be zero:

Definition 5. *A competitive equilibrium (CE) of* $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r} \rangle$ *is a pair* $(\mathbf{p}, \boldsymbol{x})$*, where* $\mathbf{p} \in \mathbb{R}^S_+$ *and* $\boldsymbol{x}$ *is a feasible matching for* M*, such that:*

1. $C_b(\boldsymbol{x}) \in D_b(\mathbf{p})$ *for all* $b \in B$ *and*

2. $p_s = 0$ *if* $|C_s(\boldsymbol{x})| < r(S)$.

Each competitive equilibrium $(\mathbf{p}, \mathbf{x})$ of M is a projection of some feasible outcome $(\mathbf{u}, \mathbf{p}; \mathbf{x})$, where we only keep one copy of each seller's price p_s and $u_{bs} = a_{bs} - p_s$ if $s \in C_b(\mathbf{x})$. We refer to such an outcome as a *CE outcome* of M. Similarly, a *CE payoff vector* $(\mathbf{u}, \mathbf{p})$ is a projection of some CE outcome $(\mathbf{u}, \mathbf{p}; \mathbf{x})$. We say that the matching $\mathbf{x}$ is *compatible* with the payoff vector $(\mathbf{u}, \mathbf{p})$. The set of CE payoff vectors for M is denoted by $\mathcal{CE}(M)$.

[8] The prices of two objects owned by the same seller in a competitive equilibrium must be the same. If two objects owned by the same seller had different prices, no buyer would demand the most expensive object.

Sotomayor (2007) proved that the set of CE outcomes is a subset of the set of stable outcomes. She characterized a CE outcome as a stable outcome where all the prices of the objects a seller owns are equal.[9]

2.3 Representation as a TU game and semivalues

A multiple-partners game may be viewed as a coalitional game with transferable utilities (a *TU game*). Therefore, solution concepts proposed for TU games can also be used in the multiple-partners game.

A TU game is a vector (N, v) where N is the player set and the function $v : 2^N \to \mathbb{R}$ satisfies that $v(\varnothing) = 0$. Given (N, v), a subset T of N is called a coalition, and $v(T)$ represents the worth of T. For any player $i \in N$ and coalition $T \subseteq N \setminus \{i\}$, player i's marginal contribution to T is $\mathrm{D}^i\, v(T) := v(T \cup \{i\}) - v(T)$.

Consider the multiple-partners game $M = \langle B, S, \mathbf{a}, \mathbf{r} \rangle$. We may associate M with a TU game $(B \cup S, v^M)$ by letting

$$v^M(T) = \max_{\mathbf{x} \in \mathcal{A}(T \cap B, T \cap S, \mathbf{r})} \sum_{b \in T \cap B} \sum_{s \in C_b(\mathbf{x})} a_{bs} \tag{1}$$

for all $T \in 2^{B \cup S} \setminus \{\varnothing\}$ and setting $v^M(\varnothing) = 0$. That is, $v^M(T)$ is the maximum surplus that the coalition T can obtain by forming feasible partnerships between the set of buyers in T (i.e., $T \cap B$) and the set of sellers in T (i.e., $T \cap S$).

The most important single-valued solution for TU games is the *Shapley value*. The Shapley value was originally defined as the unique single-valued solution satisfying efficiency, additivity, equal treatment, and null player.[10]

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[9] Sotomayor (2007) also proved that the sets of stable outcomes and CE outcomes are endowed with a complete lattice structure.

[10] Let ψ be a single-valued solution. Efficiency of ψ requires $\sum_{i \in N} \psi_i(N, v) = v(N)$ for any (N, v). The solution ψ is additive if $\psi(N, v + v') = \psi(N, v) + \psi(N, v')$ for any two games (N, v) and (N, v'). It satisfies the null player axiom if $\psi_i(N, v) = 0$ for any null player i in (N, v) (that is, for a player

Dubey, Neyman, and Weber (1981) relaxed the axiom system of the Shapley value and define the class of semivalues. A *semivalue* is a single-valued solution satisfying positivity, additivity, equal treatment, null player, and null player out.[11] Interestingly, each semivalue can be uniquely identified by a probability distribution λ over $[0,1]$. The prescription of the semivalue ψ^λ to a player i in a TU game (N,v) can be expressed as i's expected marginal contribution, which depends on the distribution λ over $[0,1]$:

$$\psi_i^\lambda(N,v) = \mathbb{E}[\mathrm{D}^i\, v(\tilde{T})], \tag{2}$$

where the random coalition $\tilde{T}$ follows the probability distribution

$$P(\tilde{T} = T) = \int_{[0,1]} z^{|T|}(1-z)^{|N\setminus(T\cup\{i\})|}\lambda(dz) \tag{3}$$

for all $T \subseteq N\setminus\{i\}$. In particular, the Lebesgue measure over $[0,1]$ identifies the Shapley value.

Going back to the multiple-partners game, the prescription of a semivalue ψ^λ to the buyer $b \in B$ in M (denoted as $\psi_b^\lambda(M)$) is defined as the semivalue of the player b in the induced TU game $(B\cup S, v^M)$:

$$\psi_b^\lambda(M) := \psi_b^\lambda(B\cup S, v^M). \tag{4}$$

The prescription $\psi_s^\lambda(M)$ to a seller s is defined similarly, for all $s \in S$.

We close this section by introducing and discussing equal treatment properties in the multiple-partners game. We adapt the definition of the property for TU games. In a TU game (N,v), two distinct players $i,j \in N$ are said to be equal players if $v(T\cup\{i\}) = v(T\cup\{j\})$ for all $T \subseteq N\setminus\{i,j\}$. That is, two players are equal if they

$i \in N$ such that $\mathrm{D}^i\, v(T) = 0$ for any $T \subseteq N\setminus\{i\}$), for any (N,v). We introduce the equal treatment property at the end of this section.

[11] The single-valued solution ψ satisfies the null player out property if $\psi_j(N\setminus\{i\},v) = \psi_j(N,v)$ for any $j \in N\setminus\{i\}$ if i is a null player in (N,v), for any game (N,v). The Shapley value satisfies the null player out property.

have the same vector of marginal contributions. This binary relation can be easily adapted for multiple-partners games.

Two distinct buyers $b, b' \in B$ are said to be *equal buyers* in M if $r(b) = r(b')$ and $a_{bs} = a_{b's}$ for all $s \in S$. Similarly, two distinct sellers $s, s' \in S$ are said to be *equal sellers* in M if $r(s) = r(s')$ and $a_{bs} = a_{bs'}$ for all $b \in B$. Notice that the notions of equal players for TU games and multiple-partners games are compatible: If two distinct buyers (or sellers) are equal in the game M, they are also equal in the induced TU game $(B \cup S, v^M)$.

We introduce some notation to state the definition of equal treatment of equals. Consider a feasible outcome $(\mathbf{u}, \mathbf{v}; \mathbf{x})$. We denote $\mathbf{u}_b$ the buyer b's vector of payments $(u_{bs})_{s \in C_b(\mathbf{x})}$, where for convenience we list the individual payments (u_{bs}) in a non-increasing order. Similarly, we denote $\mathbf{v}_s$ the seller s's vector of payments $(v_{bs})_{b \in C_s(\mathbf{x})}$, where the individual payments (v_{bs}) are listed in a non-increasing order.

Definition 6. *A feasible outcome $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ for $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r} \rangle$ satisfies equal treatment of equals if $\boldsymbol{u}_b = \boldsymbol{u}_{b'}$ for all equal buyers b and b' and $\boldsymbol{v}_s = \boldsymbol{v}_{s'}$ for all equal sellers s and s'.*

We also define the equal treatment of partnerships. For the buyers, for instance, we say that a feasible outcome satisfies the property if any buyer obtains the same payoff in all their partnerships.

Definition 7. *Consider a feasible outcome $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ for $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r} \rangle$:*

(a) $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ satisfies equal treatment of partnerships among buyers if $u_{bs} = u_{bs'}$ for all $b \in B$ and all $s, s' \in C_b(\boldsymbol{x})$.

(b) $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ satisfies equal treatment of partnerships among sellers if $v_{bs} = v_{b's}$ for all $s \in S$ and all $b, b' \in C_s(\boldsymbol{x})$.

(c) $(\mathbf{u}, \mathbf{v}; \boldsymbol{x})$ satisfies equal treatment of partnerships if it satisfies equal treatment of partnerships among buyers and sellers.

We note that CE outcomes satisfy equal treatment of partnerships among sellers since the price is the same for all the objects owned by a seller.

2.4 The related simple assignment game

In this last subsection, we follow Sotomayor (1992) and briefly explain how to connect a multiple-partners game M and a simple (one-to-one) assignment game $\hat{M}$. We will use this connection in the proofs of some of our results.

Given $M = \langle B, S, \mathbf{a}, \mathbf{r}\rangle$, we split each buyer into several agents with unitary demand and each seller into several indivisible objects. We denote $\hat{B} := \{(b, l) \mid b \in B \text{ and } l = 1, \ldots, r(b)\}$ and $\hat{S} := \{(s, f)) \mid s \in S \text{ and } f = 1, \ldots, r(s)\}$ the sets of agents and objects, respectively. Each agent (b, l) is identified by the buyer b and her index l in b's quota. Similarly, each object (s, f) is identified by its owner s and its index f.

Moreover, given a matching $\mathbf{x} \in \mathcal{A}(B, S, \mathbf{r})$, it is possible to construct a one-to-one feasible matching $\hat{\mathbf{x}} \in \mathcal{A}(\hat{B}, \hat{S}, \mathbf{r})$.[12] Then, given $M = \langle B, S, \mathbf{a}, \mathbf{r}\rangle$ and $\mathbf{x} \in \mathcal{A}(B, S, \mathbf{r})$, we define the simple assignment game $\hat{M} = \langle \hat{B}, \hat{S}, \hat{\mathbf{a}}\rangle$, where:

$$\hat{a}_{(b,l)(s,f)} = \begin{cases} 0 & \text{if } x_{bs} = 1 \text{ and } \hat{x}_{(b,l)(s,f)} = 0 \\ a_{bs} & \text{otherwise.} \end{cases}$$

3 The multiple-partners game with types

The main objective of this paper is to study replicated multiple-partners games. In a such a replicated game, for each player in the original game, we add a new player who is an equal to him/her. We say these two equal players are of the same "type." In this section, we introduce a notation that can conveniently represent multiple-partners games where several players are of the same type. In the replica of a game, there is the same number of players of each type. However, in the proof of our main result

[12] See Sotomayor (1992) for details.

(Theorem 1), we require properties of games where the numbers of players of each type are different. Moreover, we use properties of a specific class of games with types that we call "uneven multiple-partners games with types." Hence, we also define these games.

In a *multiple-partners game with types*, there are two finite and disjoint sets of types: a set of *buyer types* $\underline{B} = \{\underline{b}_1, \ldots, \underline{b}_{t_b}\}$ and a set of *seller types* $\underline{S} = \{\underline{s}_1, \ldots, \underline{s}_{t_s}\}$. Hence, the numbers of types in $\underline{B}$ and $\underline{S}$ are t_b and t_s, respectively. We use $\underline{b}$ and $\underline{s}$ to represent a generic member of $\underline{B}$ and $\underline{S}$, respectively. There can be several buyers or sellers of the same type. If two distinct buyers are of type $\underline{b}$, they are equal: their quotas are equal, and their surplus with any seller is identical. Similarly, if two distinct sellers are of type $\underline{s}$, then the objects they own are identical, and the number of their objects are the same. We indicate the number of buyers and sellers that are of a certain type through the function y, that is, $y : \underline{B} \cup \underline{S} \to \mathbb{Z}_+$, and $y(\underline{b})$ and $y(\underline{s})$ denote the number of type-$\underline{b}$ buyers and the number of type-$\underline{s}$ sellers, respectively. Therefore, the total numbers of buyers and sellers in the game are $\sum_{\underline{b} \in \underline{B}} y(\underline{b})$ and $\sum_{\underline{s} \in \underline{S}} y(\underline{s})$, respectively.

The *multiple-partners game with types* is denoted as $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{r} \rangle$, where $\mathbf{a} = (a_{\underline{b}\underline{s}})_{\underline{b} \in \underline{B}, \underline{s} \in \underline{S}}$ is the matrix that represents the surplus from a partnership between any buyer of type $\underline{b}$ and any seller of type $\underline{s}$, $r(\underline{b}) > 0$ is the maximum number of objects each buyer of type $\underline{b}$ can acquire, and $r(\underline{s}) > 0$ is the number of identical objects owned by each seller of type $\underline{s}$.

In the multiple-partners game with types $\underline{M}$, each type-$\underline{b}$ buyer is denoted by $\underline{b}(h)$, where $h = 1, \ldots, y(\underline{b})$, and each type-$\underline{s}$ seller is denoted by $\underline{s}(g)$, where $g = 1, \ldots, y(\underline{s})$. We also denote $B_{\underline{b}} = \{\underline{b}(h) \mid h = 1, \ldots, y(\underline{b})\}$ the set of type-$\underline{b}$ buyers and $S_{\underline{s}} = \{\underline{s}(g) \mid g = 1, \ldots, y(\underline{s})\}$ the set of type-$\underline{s}$ sellers.

The simplest example of a multiple-partners game with types is a glove market (in fact, the glove market is an assignment game with types):

Example 1. *A glove market satisfies that* $\underline{B} = \{\underline{b}_1\}$, $\underline{S} = \{\underline{s}_1\}$, $r(\underline{b}_1) = r(\underline{s}_1) = 1$, $a_{\underline{b}_1\underline{s}_1} = 1$. *The number of buyers of the unique buyer type is* $y(\underline{b}_1)$, *and the number of* *sellers of the unique seller type is $y(\underline{s}_1)$. The interpretation of a glove market is that each buyer owns a left glove, while each seller owns a right glove. A single glove is worthless. Pairing a left glove with a right glove generates one unit of worth.*

We notice that we have not introduced a new class of games but just a notation. We have seen that a multiple-partners game with types can be easily written as a multiple-partners game where the set of players is larger: Given $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{r} \rangle$, we can define the game $M = \langle B, S, \mathbf{a}, \mathbf{r} \rangle$, with $B = \bigcup_{\underline{b} \in \underline{B}} B_{\underline{b}}$, $S = \bigcup_{\underline{s} \in \underline{S}} S_{\underline{s}}$, and $a_{bs} = a_{\underline{bs}}$, $r(b) = r(\underline{b})$, and $r(s) = r(\underline{s})$ if $b \in B_{\underline{b}}$ and $s \in S_{\underline{s}}$. On the other hand, a multiple-partners game is a multiple-partners game with types where the number of each type of player is one.

We now introduce the concept of an "uneven game," which will play a key role in proving our convergence result. This concept will facilitate the decomposition of the asymptotic semivalues because, as we will show, every asymptotic semivalue is representable as a convex combination of marginal contributions to different uneven games.

Definition 8. *A multiple-partners game with types $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{r} \rangle$ is uneven if*

$$\sum_{\underline{b} \in H} y(\underline{b}) r(\underline{b}) \neq \sum_{\underline{s} \in G} y(\underline{s}) r(\underline{s}) \tag{5}$$

for all non-empty $H \subseteq \underline{B}$ and all non-empty $G \subseteq \underline{S}$.

In an uneven game, for any sets of buyer types and seller types, the total number of partnerships that the buyers (of those buyer types) can make is different from the total number of partnerships that the sellers (of those seller types) can make.

Remark 1. *For fixed sets of buyer and seller types $\underline{B}$ and $\underline{S}$, the unevenness of a multiple-partners game is determined by a finite number of inequalities, which is lower than $(2^{t_b} - 1)(2^{t_s} - 1)$.*

4 Properties of the competitive equilibria of large multiple-partners games with types

In this section, we provide properties of the CE of the multiple-partners games with types when the number of buyers and sellers of each type is large. We will refer to such a game as a "large multiple-partners game with types."

Formally, for a game $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{r} \rangle$, we denote $r^{\max} := \max\{r_b^{\max}, r_s^{\max}\}$, where $r_b^{\max} := \max\{r(\underline{b}) \mid \underline{b} \in \underline{B}\}$ and $r_s^{\max} := \max\{r(\underline{s}) \mid \underline{s} \in \underline{S}\}$. Hence, $r^{\max}$ is the greatest quota among the types in $\underline{M}$. Also, we denote $y^{\min} := \min\{y_b^{\min}, y_s^{\min}\}$, where $y_b^{\min} := \min\{y(\underline{b}) \mid \underline{b} \in \underline{B}\}$ and $y_s^{\min} := \min\{y(\underline{s}) \mid \underline{s} \in \underline{S}\}$. Hence, $y^{\min}$ is the smallest number of players of a given type among the types in $\underline{M}$. Then, we say that $\underline{M}$ is large if the number of players of any type is higher than the greatest quota of all the players:

Definition 9. *A multiple-partners game with types $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{r} \rangle$ is large if $y^{\min} > r^{\max}$.*

Two players of the same type can obtain different payoffs in a CE. However, Proposition 1 ensures that two buyers of the same type, or two sellers of the same type, have the same payoff vectors if the multiple-partners game is large. Moreover, not only each seller obtains the same payoff in her partnerships in a CE, but also each buyer obtains the same payoff in all his partnerships. This result generalizes lemmas 3.1 and 3.2 in Sotomayor (2019), who showed that the property holds if the number of players of each type is the same.

Proposition 1. *Every CE outcome of a large multiple-partners game with types $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{r} \rangle$ satisfies equal treatment of equals and equal treatment of partnerships.*

Proposition 1 allows a characterization of the CE payoff vectors of large multiple-partners games with types, which we state in Remark 2:

Remark 2. *Combining Proposition 1 and Sotomayor's (2007) characterization of a CE payoff vector as a stable payoff vector satisfying equal treatment of partnerships among sellers, we obtain a symmetric characterization of CE payoff vectors for large games: every CE payoff vector of a large multiple-partners game with types is a stable payoff vector satisfying equal treatment of equals and equal treatment of partnerships.*

As a consequence of Proposition 1, we can conveniently simplify the notation for a CE outcome of a large multiple-partners game with types since a seller's price and a buyer's utility in a transaction only depend on their types; they do not depend on the identity of the player (as long as their type is the same) or the identity of their partner. Therefore, we denote a CE outcome of a large game $\underline{M}$ by $(\underline{\mathbf{u}}, \underline{\mathbf{p}}; \mathbf{x})$, where $\underline{\mathbf{u}} = (u_{\underline{b}})_{\underline{b} \in \underline{B}} \in \mathbb{R}^{\underline{B}}$ and $\underline{\mathbf{p}} = (p_{\underline{s}})_{\underline{s} \in \underline{S}} \in \mathbb{R}^{\underline{S}}$. Similarly, we consider the set $\mathcal{CE}(\underline{M})$ of CE payoff vectors for $\underline{M}$ a set of vectors of $\mathbb{R}^{\underline{B}} \times \mathbb{R}^{\underline{S}}$, when $\underline{M}$ is large.

To introduce our next result, we go back to the example of the glove market (Example 1). Remark 3 provides helpful information about the CE outcomes of some glove markets.

Remark 3. *Let us call a glove market asymmetric if it satisfies $y(\underline{b}_1) \neq y(\underline{s}_1)$; that is, the number of buyers (of the unique type) is different from the number of sellers. It is easy to check that the CE payoff vector of an asymmetric glove market is unique. Moreover, the dependence of the stable payoff vector on $y(\underline{b}_1)$ and $y(\underline{s}_1)$ is ordinal rather than cardinal. Indeed, $(u_{\underline{b}_1}, v_{\underline{s}_1}) = (1, 0)$ if $y(\underline{b}_1) < y(\underline{s}_1)$ whereas $(u_{\underline{b}_1}, v_{\underline{s}_1}) = (0, 1)$ if $y(\underline{b}_1) > y(\underline{s}_1)$.*

In the example of the glove market, we could say, for instance, that the glove market with a number of buyers and sellers given by $\mathbf{y}$ is "equivalent" to the glove market characterized by $\mathbf{y}'$ if $y(\underline{b}_1) < y(\underline{s}_1)$ and $y'(\underline{b}_1) < y'(\underline{s}_1)$ because their CE payoff vector is the same. Our next result also relates CE payoffs of multiple-partners games depending on the number of players of each type. For this purpose, we first define an equivalence relation in the set of multiple-partners games.

We partition the set of multiple-partners games with a fixed set of buyer and seller types in equivalence classes using inequalities of the form of equation (5). Formally, fix $\underline{B}$, $\underline{S}$, and $\mathbf{r}$. Two distinct games $\underline{M}$ and $\underline{M}'$ may differ on the number of each type of buyer and seller, that is, $\mathbf{y}$ and $\mathbf{y}'$ may be different (differences in the matrixes of worth $\mathbf{a}$ and $\mathbf{a}'$ are not relevant for our next definition). We define the equivalence relation $\sim$ on multiple-partners games as follows:

Definition 10. *Let $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{r} \rangle$ and $\underline{M}' = \langle \underline{B}, \underline{S}, \boldsymbol{y}', \boldsymbol{a}', \boldsymbol{r} \rangle$ be two multiple-partners games with types. We say that $\underline{M}$ and $\underline{M}'$ are "equivalent," and we write $\underline{M} \sim \underline{M}'$, if:*

$$\sum_{\underline{b} \in H} y(\underline{b}) r(\underline{b}) \leq \sum_{\underline{s} \in G} y(\underline{s}) r(\underline{s}) \iff \sum_{\underline{b} \in H} y'(\underline{b}) r(\underline{b}) \leq \sum_{\underline{s} \in G} y'(\underline{s}) r(\underline{s})$$

for all non-empty $H \subseteq \underline{B}$ and all non-empty $G \subseteq \underline{S}$.

The equivalence relation $\sim$ induces a partition on the set of multiple-partners games with types, given $(\underline{B}, \underline{S}, \mathbf{r})$. We denote this partition by $T(\underline{B}, \underline{S}, \mathbf{r})$. Each cell $\mathcal{M} \in T(\underline{B}, \underline{S}, \mathbf{r})$ is referred to as a *class of games.*

Proposition 2 will show that large multiple-partners games with types of the same class have the same CE payoff vectors if their matrices of worths are the same. The proof of the proposition appeals to Hall's theorem, and we provide a version of this theorem as Lemma 1.[13]

Lemma 1 (Hall, 1935)**.** *Given B and S such that $|B| \leq |S|$, let $\varphi : B \rightsquigarrow S$ be a correspondence. φ satisfies the Hall condition: $|P| \leq |\bigcup_{b \in P} \varphi(b)|$ for all $P \in 2^B \setminus \{\varnothing\}$ if and only if there exists a function $\eta : B \to S$ satisfying (i) $\eta(b) \in \varphi(b)$ for all $b \in B$; (ii) $\eta(b) \neq \eta(b')$ for all $b, b' \in B$ such that $b \neq b'$.*

[13] Hall's theorem was first used by Demange, Gale, and Sotomayor (1986) to characterize stable payoff vectors for an assignment game.

We now state and prove that the CE payoff vector of two large games of the same class is the same if the worth of any buyer-seller partnership is the same. That is, the CE payoff vector does not depend on the number of players of each type in the two games as long as they are of the same equivalence class.

Proposition 2. *Consider two large multiple-partners games with types* $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{r} \rangle$ *and* $\underline{M}' = \langle \underline{B}, \underline{S}, \mathbf{y}', \mathbf{a}, \mathbf{r} \rangle$ *of the same class. Then* $\mathcal{CE}(\underline{M}) = \mathcal{CE}(\underline{M}')$.

Next, we discuss the question of the unicity of the CE payoffs of the multiple-partners games. Not every multiple-partners game has a unique CE payoff. However, we have seen that asymmetric glove markets only have one CE outcome. Clearly, an asymmetric glove market is an uneven multiple-partners game with types. Hence, enquiring whether uneven games have a unique CE payoff is natural. Proposition 3 shows that if the uneven multiple-partners game with types is large, it has a unique CE as an asymmetric glove market.

Proposition 3. *The CE payoff vector* $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ *of a large uneven multiple-partners game with types* $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{r} \rangle$ *is unique.*

Using Proposition 3, we know that the correspondence $\mathcal{CE}$ restricted to large uneven multiple-partners games with types is a function. Hence, we can write the CE payoff vector of such a game $\underline{M}$ as $(\underline{\mathbf{u}}(\underline{M}), \underline{\mathbf{p}}(\underline{M}))$. Moreover, Proposition 2 states that this function does not fully use the information in $\mathbf{y}$. It suffices to know, for all $H \in 2^{\underline{B}} \setminus \{\varnothing\}$ and all $G \in 2^{\underline{S}} \setminus \{\varnothing\}$, whether $\sum_{\underline{b} \in H} y(\underline{b}) r(\underline{b}) < \sum_{\underline{s} \in G} y(\underline{s}) r(\underline{s})$ or $\sum_{\underline{b} \in H} y(\underline{b}) r(\underline{b}) > \sum_{\underline{s} \in G} y(\underline{s}) r(\underline{s})$ (remember that the equality is not possible if the game is uneven). Therefore, we can view a class $\mathcal{M} \in T(\underline{B}, \underline{S}, \mathbf{r})$, if restricted to uneven games, as a specification of $(2^{|\underline{B}|} - 1)(2^{|\underline{S}|} - 1)$ inequalities. We state this fact in Remark 4.

Remark 4. *Fix the sets of buyer types and seller types* $\underline{B}$ *and* $\underline{S}$ *and the vector* r*. A class* $\mathcal{M} \in T(\underline{B}, \underline{S}, \mathbf{r})$*, restricted to uneven games, specifies, for all* $H \in 2^{\underline{B}} \setminus \{\varnothing\}$

and all $G \in 2^{\underline{S}} \setminus \{\varnothing\}$*, either* $\sum_{\underline{b}\in H} y(\underline{b})r(\underline{b}) < \sum_{\underline{s}\in G} y(\underline{s})r(\underline{s})$ *or* $\sum_{\underline{b}\in H} y(\underline{b})r(\underline{b}) > \sum_{\underline{s}\in G} y(\underline{s})r(\underline{s})$.

We close this section by studying the effect of the entrance of a new player of an existing type in a game. Take a large uneven game $\underline{M}$. We know that the $\mathcal{CE}(\underline{M})$ is a singleton. Consider the entrance of one player with an existing type (who can be matched with at most $r^{\max}$ players from the other side of the market); call $\underline{M}'$ the new game. We are sure that the unique equilibrium outcome of $\underline{M}$ is the unique equilibrium outcome of $\underline{M}'$ if the games are equivalent. This is certainly the case (that is, all the equivalences in Definition 10 hold) if $\underline{M}$ is "sufficiently uneven":

Definition 11. *A multiple-partners game with types* $\underline{M} = \langle \underline{B}, \underline{S}, \boldsymbol{y}, \boldsymbol{a}, \boldsymbol{r}\rangle$ *is sufficiently uneven if*

$$\min_{\substack{H\in 2^{\underline{B}}\setminus\{\varnothing\},\\ G\in 2^{\underline{S}}\setminus\{\varnothing\}}} \left| \sum_{\underline{b}\in H} y(\underline{b})r(\underline{b}) - \sum_{\underline{s}\in G} y(\underline{s})r(\underline{s}) \right| > r^{\max}.$$

The game is sufficiently uneven if, after including at most r^{max} players of an existing type, the new game belongs to the same class as the original game. Therefore, if $\underline{M}$ is sufficiently uneven, the entrance of a player of an existing type does not change the original CE payoff vector. We state this comparative statics phenomenon in Corollary 1 when the additional player is a buyer; a similar corollary can be stated if the additional agent is a seller.

Corollary 1. *Let* $\underline{M} = \langle \underline{B}, \underline{S}, \mathbf{y}, \mathbf{a}, \mathbf{r}\rangle$ *be a large sufficiently uneven game and take* $\underline{b} \in \underline{B}$*. Consider* $\underline{M}' = \langle \underline{B}, \underline{S}, \mathbf{y}', \mathbf{a}, \mathbf{r}\rangle$ *that satisfies* $y'(\underline{b}) = y(\underline{b}) + 1$, $y'(\underline{b}') = y(\underline{b}')$ *for all* $\underline{b}' \in \underline{B} \setminus \{\underline{b}\}$*, and* $y'(\underline{s}) = y(\underline{s})$ *for all* $\underline{s} \in \underline{S}$*. Then* $\mathcal{CE}(\underline{M}') = \mathcal{CE}(\underline{M})$ *and it is a singleton.*

Corollary 2 expresses an interesting implication of Corollary 1 in the framework of TU games. It states that the marginal contribution of an entrant (a buyer, in this case) to a large sufficiently uneven multiple-partners game is the CE payoff of a player of the same type as the entrant in the uneven game. In the corollary, we abuse notation and denote $v^{\underline{M}'}$ the characteristic function of the game where the players are all the agents of all the types in M'.

Corollary 2. *Let $\underline{b}$, $\underline{M}$, and $\underline{M}'$ be the same as in Corollary 1. Denote by $\hat{B}$ and $\hat{S}$ the sets of buyers and sellers in $\underline{M}$, respectively. Let $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ be the CE payoff vector for $\underline{M}$. Then $\mathrm{D}^{\underline{b}(y(\underline{b})+1)}\, v^{\underline{M}'}(\hat{B} \cup \hat{S}) = r(\underline{b})\underline{u}_{\underline{b}}$.*

Corollary 2 holds because the players in $\underline{M}$ keep their CE payoff in $\underline{M}'$ and the new buyer $\underline{b}$ obtains his CE payoff in each of his $r(\underline{b})$ partnerships. Hence, the additional surplus in the game is $r(\underline{b})\underline{u}_{\underline{b}}$.

5 Equilibrium-value convergence

In this section, we study the replicas of the multiple-partners game. We use the results of Section 4 to analyze the convergence of the Shapley value and other semivalues as the number of replicas goes to infinity. We show our convergence theorem, which states that the semivalues converge to the same CE payoff.

First, we formally define a replica of a multiple-partners game and apply the results obtained in the previous section to the replicas.

Definition 12. *Consider the multiple-partners game $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r}\rangle$, where $B = \{b_1, \dots, b_{n_b}\}$ and $S = \{s_1, \dots, s_{n_s}\}$. The k-fold replica M^k of M is a multiple-partners game with types $\langle \underline{B}, \underline{S}, \mathbf{y}^k, \mathbf{a}, \mathbf{r}\rangle$, where $\underline{B} = \{\underline{b}_1, \dots, \underline{b}_{n_b}\}$, $\underline{S} = \{\underline{s}_1, \dots, \underline{s}_{n_s}\}$, the characteristics of each buyer of type $\underline{b}$ (respectively, each seller of type $\underline{s}$) in M^k are the same as those of the buyer b (respectively, seller s) in M, and $y^k(\underline{b}) = y^k(\underline{s}) = k$ for all $\underline{b} \in \underline{B}$ and all $\underline{s} \in \underline{S}$.*

The replica of a multiple-partners game is a multiple-partners game with types where the number of buyers and sellers of each type is the same. In the replica M^k of the game $M = \langle B, S, \mathbf{a}, \mathbf{r} \rangle$, $y^{\min} = k$. Therefore, M^k is a large multiple-partners game with types if $k > r^{\max}$. In this case, we say that M^k is a *large replica* of M.

In a CE payoff of a replica M^k, two buyers of the same type, or two sellers of the same type, can have different payoff vectors. Also, a buyer can have different payoffs in different partnerships. However, after Proposition 1, these facts can only happen if the replica is not large. This is a result that was also proven by Sotomayor (2019). We state it here for completeness.

Corollary 3. *Every CE outcome of a large replica M^k of the multiple-partners game $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r} \rangle$ satisfies equal treatment of equals and equal treatment of partnerships.*

Unlike market games and pure exchange economies, the set of CE outcomes of the multiple-partners game is not replication invariant.[14] However, as a corollary of Proposition 2, it follows that the set of CE outcomes of multiple-partners games will eventually become constant through replication since two replicas of the same game always belong to the same equivalence class. As the previous corollary, this result is also derived in Sotomayor (2019).

Corollary 4. *Consider the multiple-partners game $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r} \rangle$ and denote $K \equiv r^{\max} + 1$. Then, $\mathcal{CE}(M^k) = \mathcal{CE}(M^K)$ for all $k \geq K$.*

To introduce our main result, consider the replica M^k of the game $M = \langle B, S, \mathbf{a}, \mathbf{r} \rangle$ and any distribution λ over $[0, 1]$. Since the semivalue ψ^λ satisfies equal treatment of equals, the semivalues of all the k replicas of a player are the same, for any player. Hence, as we do for the CE outcomes of large multiple-partners game with types, we

[14] To see that replication invariance does not hold, consider a multiple-partners game $M = (\{b\}, \{s\}, r(b) = r(s) = 2, a_{bs} = 1)$. It is easy to check that, in the associated TU game, $v^M(\{b, s\}) = 1$, but $v^{M^2}(\{\underline{b}(1), \underline{b}(2), \underline{s}(1), \underline{s}(2)\}) = 4$. Therefore, the sum of the players' payoffs in a CE of M^2 is four times the sum of the payoffs in a CE of M, whereas it should be two times if the set of CE outcomes was replication invariant.

simplify the notation and write $\psi^{\lambda}_{\underline{b}}(M^k)$ to indicate the semivalue of any of the agents of the type $\underline{b} \in \underline{B}$ (that is, any of the replicas of the buyer $b \in B$) and similarly for $\psi^{\lambda}_{\underline{s}}(\underline{M}^k)$.

Theorem 1 states that all the semivalues $\psi^{\lambda}(M^k)$ with $\lambda(\{0,1\}) = 0$ of the replica M^k converge to the same value as the replica becomes large. Moreover, the players' payoffs in these semivalues (in particular, the Shapley value payoffs) converge to a CE payoff.

Theorem 1. *Consider the multiple-partners game $M = \langle B, S, \boldsymbol{a}, \boldsymbol{r}\rangle$ and denote $K \equiv r^{\max} + 1$. There exists $(\underline{\mathbf{u}}^K, \underline{\mathbf{p}}^K) \in \mathcal{CE}(M^K)$ such that:*

$$\lim_{k\to+\infty} \psi^{\lambda}_{\underline{b}}(M^k) = r(\underline{b})\underline{u}^K_{\underline{b}} \text{ and } \lim_{k\to+\infty} \psi^{\lambda}_{\underline{s}}(M^k) = r(\underline{s})\underline{p}^K_{\underline{s}}$$

for all $\underline{b} \in \underline{B}$, $\underline{s} \in \underline{S}$, and $\lambda \in \Delta([0,1])$ such that $\lambda(\{0,1\}) = 0$.

The sketch of the proof of Theorem 1 is the following. We use that the prescription of a semivalue ψ^{λ} to a player can be represented as this player's expected marginal contribution to a random coalition (see equation (2)). With the aid of equation (3), we can connect the distribution of this random coalition to a binomial distribution with unknown parameter z. It will follow from the law of large numbers and the central limit theorem that this random coalition converges to a random large sufficiently uneven game as the original game expands through replication. This will allow us to use the properties of large uneven games that we have stated in section 4. Moreover, we will show that the limit distribution of this random uneven game is independent of the parameter λ, and a player's expected contribution to this random game corresponds to the player's expected CE payoff vector.

We now present the formal proof of the theorem.

Proof. Given the game $M = \langle B, S, \mathbf{a}, \mathbf{r}\rangle$, consider the k-fold replica $M^k = \langle \underline{B}, \underline{S}, \mathbf{y}^k, \mathbf{a}, \mathbf{r}\rangle$, with $k \geq K$. Let $B^k_{\underline{b}} = \{\underline{b}(h) \mid h = 1, \ldots, k\}$ be the set of buyers of type $\underline{b} \in \underline{B}$ and

$S^k_{\underline{s}} = \{\underline{s}(g) \mid g = 1, \ldots, k\}$ the set of sellers of type $\underline{s} \in \underline{S}$. Moreover, denote B^k and S^k the sets of buyers and sellers in M^k, respectively, that is, $B^k = \bigcup_{\underline{b} \in \underline{B}} B^k_{\underline{b}}$ and $S^k = \bigcup_{\underline{s} \in \underline{S}} S^k_{\underline{s}}$.

Without loss of generality, choose an arbitrary buyer type $\underline{b}^* \in \underline{B}$ and take the type-$\underline{b}^*$ buyer $\underline{b}^*(1)$. Given any $\lambda \in \Delta([0,1])$, with $\lambda(\{0,1\}) = 0$, we want to compute $\lim_{k \to +\infty} \psi^\lambda_{\underline{b}^*}(M^k)$, which, because of the equal treatment property of the semivalues, is equal to $\lim_{k \to +\infty} \psi^\lambda_{\underline{b}^*(1)}(M^k)$. First,

$$
\begin{aligned}
\psi^\lambda_{\underline{b}^*(1)}(M^k) =& \psi^\lambda_{\underline{b}^*(1)}(B^k \cup S^k, v^{M^k}) \\
=& \sum_{T \subseteq (B^k \setminus \{\underline{b}^*(1)\}) \cup S^k} \left[\int_{[0,1]} z^{|T|}(1-z)^{k(n_b+n_s)-|T|-1} \lambda(dz) \right] \mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}(T) \\
=& \int_{[0,1]} \sum_{T \subseteq (B^k \setminus \{\underline{b}^*(1)\}) \cup S^k} z^{|T|}(1-z)^{k(n_b+n_s)-|T|-1}\, \mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}(T) \lambda(dz), \qquad (6)
\end{aligned}
$$

where the first equality follows from equation (4), the second from equations (2) and (3), and the third from the linearity of the integration.

To continue the analysis of the previous expression, we construct, for any parameter $z \in (0,1)$ and for each type of buyer and each type of seller, a coalition-valued random variable using binomial distributions on the sets of players of that type, excluding the player $\underline{b}^*(1)$. The probability of a player's presence in the random coalition corresponding to the player's type is z. That is, we define the random variable $\tilde{B}^k_{\underline{b}^*}$ by $P(\tilde{B}^k_{\underline{b}^*} = T) = z^{|T|}(1-z)^{k-|T|-1}$ for all $T \subseteq B^k_{\underline{b}} \setminus \{\underline{b}^*(1)\}$; the random variable $\tilde{B}^k_{\underline{b}}$ by $P(\tilde{B}^k_{\underline{b}} = T) = z^{|T|}(1-z)^{k-|T|}$ for all $T \subseteq B^k_{\underline{b}}$ and buyer type $\underline{b} \in \underline{B} \setminus \{\underline{b}^*\}$; and the random variable $\tilde{S}^k_{\underline{s}}$ by $P(\tilde{S}^k_{\underline{s}} = T) = z^{|T|}(1-z)^{k-|T|}$ for all $T \subseteq S^k_{\underline{s}}$ and seller type $\underline{s} \in \underline{S}$.[15] Moreover, we use the previous random variables on the sets of players of the same type to construct a new random variable, which we denote $\tilde{N}^{k,\underline{b}^*(1)}$, on the subsets of the set of all the players except $\underline{b}^*(1)$, that is, on the set $(B^k \setminus \{\underline{b}^*(1)\}) \cup S^k$. We take the players' presences in any of the previous random coalitions as mutually

[15] For notational convenience, we do not indicate that the random variables depend on z.

independent; hence, the random variable is the following:

$$\tilde{N}^{k,\underline{b}^*(1)} = \Bigg((\bigcup_{\underline{b} \in \underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s} \in \underline{S}} \tilde{S}^k_{\underline{s}}) \Bigg). \tag{7}$$

The probability distribution of $\tilde{N}^{k,\underline{b}^*(1)}$ is as follows:

$$\begin{aligned} P(\tilde{N}^{k,\underline{b}^*(1)} = T) =& \Bigg(\prod_{\underline{b} \in \underline{B}} P(\tilde{B}^k_{\underline{b}} = T \cap B^k_{\underline{b}}) \Bigg) \Bigg(\prod_{\underline{s} \in \underline{S}} P(\tilde{S}^k_{\underline{s}} = T \cap S^k_{\underline{s}}) \Bigg) \\ =& z^{|T|}(1-z)^{k(n_b+n_s)-|T|-1}, \end{aligned} \tag{8}$$

for all $T \subseteq (B^k \setminus \{\underline{b}^*(1)\}) \cup S^k$. Given this probability distribution, we proceed to the analysis of $\lim_{k \to +\infty} \psi^{\lambda}_{\underline{b}^*}(M^k)$, using (6):

$$\begin{aligned} & \lim_{k\to+\infty} \int_{[0,1]} \sum_{T \subseteq (B^k \setminus \{\underline{b}^*(1)\}) \cup S^k} z^{|T|}(1-z)^{k(n_b+n_s)-|T|-1} \, \mathrm{D}^{\underline{b}^*(1)} \, v^{M^k}(T) \lambda(dz) \\ =& \lim_{k\to+\infty} \int_{[0,1]} \sum_{T \subseteq (B^k \setminus \{\underline{b}^*(1)\}) \cup S^k} P(\tilde{N}^{k,\underline{b}^*(1)} = T) \, \mathrm{D}^{\underline{b}^*(1)} \, v^{M^k}(T) \lambda(dz) \\ =& \lim_{k\to+\infty} \int_{[0,1]} \mathbb{E}[\mathrm{D}^{\underline{b}^*(1)} \, v^{M^k}(\tilde{N}^{k,\underline{b}^*(1)})] \lambda(dz) \\ =& \lim_{k\to+\infty} \int_{[0,1]} \mathbb{E}\Bigg[\mathrm{D}^{\underline{b}^*(1)} \, v^{M^k} \Bigg((\bigcup_{\underline{b} \in \underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s} \in \underline{S}} \tilde{S}^k_{\underline{s}}) \Bigg) \Bigg] \lambda(dz) \\ =& \int_{[0,1]} \lim_{k\to+\infty} \mathbb{E}\Bigg[\mathrm{D}^{\underline{b}^*(1)} \, v^{M^k} \Bigg((\bigcup_{\underline{b} \in \underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s} \in \underline{S}} \tilde{S}^k_{\underline{s}}) \Bigg) \Bigg] \lambda(dz) \\ =& \int_{(0,1)} \lim_{k\to+\infty} \mathbb{E}\Bigg[\mathrm{D}^{\underline{b}^*(1)} \, v^{M^k} \Bigg((\bigcup_{\underline{b} \in \underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s} \in \underline{S}} \tilde{S}^k_{\underline{s}}) \Bigg) \Bigg] \lambda(dz) \end{aligned}$$

where the first equality follows from equation (8), the second from the definition of the expectation operator over the random variable $\tilde{N}^{k,\underline{b}^*(1)}$, the third from equation (7), the fourth from the uniform convergence of the functions $z \mapsto \mathbb{E}[\mathrm{D}^{\underline{b}^*(1)} \, v^{M^k}((\bigcup_{\underline{b} \in \underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s} \in \underline{S}} \tilde{S}^k_{\underline{s}}))]$ as $k \to +\infty$,[16] and the last from $\lambda(\{0,1\}) = 0$. Thus, we have

$$\lim_{k\to+\infty} \psi^{\lambda}_{\underline{b}^*}(M^k) = \int_{(0,1)} \lim_{k\to+\infty} \mathbb{E}\Bigg[\mathrm{D}^{\underline{b}^*(1)} \, v^{M^k} \Bigg((\bigcup_{\underline{b} \in \underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s} \in \underline{S}} \tilde{S}^k_{\underline{s}}) \Bigg) \Bigg] \lambda(dz). \tag{9}$$

[16] Notice that the k-th function $z \mapsto \mathbb{E}[\mathrm{D}^{\underline{b}^*(1)} \, v^{M^k}((\bigcup_{\underline{b} \in \underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s} \in \underline{S}} \tilde{S}^k_{\underline{s}}))]$ is continuous and defined on a compact set $[0,1]$ for every $k \in \mathbb{Z}_+$.

Next, we use the coalition-valued random variable $(\bigcup_{\underline{b}\in\underline{B}}\tilde{B}^k_{\underline{b}})\cup(\bigcup_{\underline{s}\in\underline{S}}\tilde{S}^k_{\underline{s}}))$ to define the game-valued random variable $\tilde{M}^k$, which is a multiple-partners game with a random population:

$$\tilde{M}^k := \langle \underline{B}, \underline{S}, \tilde{\mathbf{y}}^k, \mathbf{a}, \mathbf{r}\rangle,$$

where the random variable $\tilde{\mathbf{y}}^k = (\tilde{y}^k(\underline{b}_1),\ldots,\tilde{y}^k(\underline{b}_{n_b});\tilde{y}^k(\underline{s}_1),\ldots,\tilde{y}^k(\underline{s}_{n_s}))$ corresponds to $\tilde{y}^k(\underline{b}) = |\tilde{B}^k_{\underline{b}}|$ for all $\underline{b}\in\underline{B}$ and $\tilde{y}^k(\underline{s}) = |\tilde{S}^k_{\underline{s}}|$ for all $\underline{s}\in\underline{S}$. The components in $\tilde{\mathbf{y}}^k$ are mutually independent, and their probability distributions are

$$P(\tilde{y}^k(\underline{b}^*) = y) = \binom{k-1}{y} z^y(1-z)^{k-y-1}, \tag{10}$$

for all $y = 0,\ldots,k-1$, and

$$P(\tilde{y}^k(\underline{b}) = y) = P(\tilde{y}^k(\underline{s}) = y) = \binom{k}{y} z^y(1-z)^{k-y}, \tag{11}$$

for all $\underline{b}\in\underline{B}\setminus\{\underline{b}^*\}$, $\underline{s}\in\underline{S}$, and $y = 0,\ldots,k$.

Consider a realization $(\bigcup_{\underline{b}\in\underline{B}}\bar{B}^k_{\underline{b}})\cup(\bigcup_{\underline{s}\in\underline{S}}\bar{S}^k_{\underline{s}})$ of the random variable $(\bigcup_{\underline{b}\in\underline{B}}\tilde{B}^k_{\underline{b}})\cup(\bigcup_{\underline{s}\in\underline{S}}\tilde{S}^k_{\underline{s}})$ and let $\bar{\mathbf{y}}^k$ be the corresponding realization of the variable $\tilde{\mathbf{y}}^k$. Denote $\bar{M}^k := \langle \underline{B}, \underline{S}, (\bar{\mathbf{y}}^k)', \mathbf{a}, \mathbf{r}\rangle$ the game corresponding to $(\bar{\mathbf{y}}^k)'$, where $(\bar{\mathbf{y}}^k)'(\underline{b}^*) = \bar{\mathbf{y}}^k(\underline{b}^*)+1$, $(\bar{\mathbf{y}}^k)'(\underline{b}) = \bar{\mathbf{y}}^k(\underline{b})$ for all $\underline{b}\in\underline{B}\setminus\{\underline{b}^*\}$, and $(\bar{\mathbf{y}}^k)'(\underline{s}) = \bar{\mathbf{y}}^k(\underline{s})$ for all $\underline{s}\in\underline{S}$.[17] The cooperative game associated with $\bar{M}^k$ is a subgame of the game associated with M^k. Hence, $\mathrm{D}^{\underline{b}^*(1)}\, v^{\bar{M}^k}\big((\bigcup_{\underline{b}\in\underline{B}}\bar{B}^k_{\underline{b}})\cup(\bigcup_{\underline{s}\in\underline{S}}\bar{S}^k_{\underline{s}})\big) = \mathrm{D}^{\underline{b}^*(1)}\, v^{M^k}\big((\bigcup_{\underline{b}\in\underline{B}}\bar{B}^k_{\underline{b}})\cup(\bigcup_{\underline{s}\in\underline{S}}\bar{S}^k_{\underline{s}})\big)$. Therefore, we can rewrite equation (9) as:

$$\lim_{k\to+\infty}\psi^{\lambda}_{\underline{b}^*}(M^k) = \int_{(0,1)}\lim_{k\to+\infty}\mathbb{E}\Big[\mathrm{D}^{\underline{b}^*(1)}\, v^{\tilde{M}^k}\Big((\bigcup_{\underline{b}\in\underline{B}}\tilde{B}^k_{\underline{b}})\cup(\bigcup_{\underline{s}\in\underline{S}}\tilde{S}^k_{\underline{s}})\Big)\Big]\lambda(dz), \tag{12}$$

where we write $v^{\tilde{M}^k}$ to indicate that $\tilde{M}^k$ is a random game whose players are derived from the random vector that determines the sets of buyers and sellers.

Define the indicator function $I_{\bar{M}^k}$ by

$$I_{\bar{M}^k} = \begin{cases} 1 & \text{if } \bar{M}^k \text{ is large and sufficiently uneven,}\\ 0 & \text{otherwise.}\end{cases}$$

[17] We denote the game $\bar{M}^k$ instead of $(\bar{M}^k)'$, as in Corollary 1, for notational simplicity.

Moreover, denote $I_{\tilde{M}^k}$ the random indicator function, depending on the realization of the random variable $\tilde{M}^k$. Then, inspecting the integrand in (12), we have

$$\begin{aligned}
&\lim_{k\to+\infty} \mathbb{E}\Big[\mathrm{D}^{\underline{b}^*(1)}\, v^{\tilde{M}^k}\Big((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \Big) \Big] \\
&= \lim_{k\to+\infty} \mathbb{E}\Big[\mathrm{D}^{\underline{b}^*(1)}\, v^{\tilde{M}^k}\Big((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \Big) I_{\tilde{M}^k} \\
&\qquad + \mathrm{D}^{\underline{b}^*(1)}\, v^{\tilde{M}^k}\Big((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \Big)(1 - I_{\tilde{M}^k}) \Big] \\
&= \lim_{k\to+\infty} \mathbb{E}[r(\underline{b}^*)\underline{u}_{\underline{b}^*}(\tilde{M}^k) I_{\tilde{M}^k}] \\
&\qquad + \lim_{k\to+\infty} \mathbb{E}\Big[\mathrm{D}^{\underline{b}^*(1)}\, v^{\tilde{M}^k}\Big((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \Big)(1 - I_{\tilde{M}^k}) \Big],
\end{aligned} \tag{13}$$

where the first equality follows from additivity of the expectation operator and the second from Corollary 2, which allows replacing the buyer $\underline{b}^*(1)$'s marginal contribution to a sufficiently large uneven assignment game with $r(\underline{b}^*)\underline{u}_{\underline{b}^*}(\tilde{M}^k)$.

We separately analyze the two addends of (13).

Concerning the second addend, we claim that

$$\lim_{k\to+\infty} \mathbb{E}\Big[\mathrm{D}^{\underline{b}^*(1)}\, v^{\tilde{M}^k}\Big((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \Big)(1 - I_{\tilde{M}^k}) \Big] = 0. \tag{14}$$

To prove (14), we first note that a player's marginal contribution to any coalition is bounded. Indeed, $0 \le \mathrm{D}^{\underline{b}^*(1)}\, v^{\bar{M}^k}\Big((\bigcup_{\underline{b}\in\underline{B}} \bar{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \bar{S}^k_{\underline{s}}) \Big) \le r(\underline{b}^*) \max_{\underline{s}\in\underline{S}} a_{\underline{b}^*\underline{s}}$, for any realization of the random variable.

Therefore, (14) holds if $\mathbb{E}(1-I_{\tilde{M}^k})$ converges to 0 as $k \to +\infty$, that is, the probability that $\tilde{M}^k$ is not sufficiently uneven converges to 0 as $k \to +\infty$. To show this property, it suffices to verify that

$$\lim_{k\to+\infty} P\Big(|\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b}) r(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s}) r(\underline{s})| > r^{\mathrm{Max}} \Big) = 1, \tag{15}$$

for all $H \in 2^{\underline{B}} \setminus \{\varnothing\}$ and $G \in 2^{\underline{S}} \setminus \{\varnothing\}$, where we write $r^{\mathrm{Max}} = r^{\max} + r(\underline{b}^*)$.[18] We

[18] We use r^{Max} instead of $r^{\max}$ in equation (15) because the buyer $\underline{b}^*(1)$ is also in the game.

distinguish two cases to prove (15): Case (a), when $\sum_{\underline{b}\in H} r(\underline{b}) \neq \sum_{\underline{s}\in G} r(\underline{s})$ and Case (b), when $\sum_{\underline{b}\in H} r(\underline{b}) = \sum_{\underline{s}\in G} r(\underline{s})$.

For Case (a), assume, without loss of generality, that $\sum_{\underline{b}\in H} r(\underline{b}) > \sum_{\underline{s}\in G} r(\underline{s})$.[19] By the Chebyshev's inequality,[20] we have

$$P\left(\left|\frac{\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})r(\underline{b})}{k\sum_{\underline{b}\in H} r(\underline{b})} - z\right| \geq \epsilon\right) \leq \frac{z(1-z)}{k\epsilon^2 \sum_{\underline{b}\in H} r(\underline{b})}, \tag{16}$$

for all $\epsilon \in \mathbb{R}_{++}$ and $k \in \mathbb{Z}_+$. Similarly, $P\left(\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})}{k\sum_{\underline{s}\in G} r(\underline{s})} - z\right| \geq \epsilon\right) \leq \frac{z(1-z)}{k\epsilon^2 \sum_{\underline{s}\in G} r(\underline{s})}$ for all $\epsilon \in \mathbb{R}_{++}$ and $k \in \mathbb{Z}_+$. Denote $c := \frac{\sum_{\underline{s}\in G} r(\underline{s})}{\sum_{\underline{b}\in H} r(\underline{b})}$; hence, $c \in (0,1)$. Notice that $\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})}{k\sum_{\underline{s}\in G} r(\underline{s})} - z\right| \geq \epsilon$ if and only if $\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})}{k\sum_{\underline{b}\in H} r(\underline{b})} - cz\right| \geq c\epsilon$. Then,

$$P\left(\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})}{k\sum_{\underline{b}\in H} r(\underline{b})} - cz\right| \geq \epsilon\right) \leq \frac{z(1-z)}{k\epsilon^2 \sum_{\underline{s}\in G} r(\underline{s})}, \tag{17}$$

for all $\epsilon \in \mathbb{R}_{++}$ and all $k \in \mathbb{Z}_+$.

Choose $\epsilon < z - cz$ and pick an arbitrary $\delta \in \mathbb{R}_{++}$. Using (16) and (17), there is $Q \in \mathbb{Z}_+$[21] such that $P\left(\left|\frac{\sum_{\underline{b}\in H} \tilde{y}^q(\underline{b})r(\underline{b})}{q\sum_{\underline{b}\in H} r(\underline{b})} - z\right| \geq \frac{\epsilon}{2}\right) \leq \frac{\delta}{2}$, $P\left(\left|\frac{\sum_{\underline{s}\in G} \tilde{y}^q(\underline{s})r(\underline{s})}{q\sum_{\underline{b}\in H} r(\underline{b})} - cz\right| \geq \frac{\epsilon}{2}\right) \leq \frac{\delta}{2}$, and $\frac{r^{\text{Max}}}{q\sum_{\underline{b}\in H} r(\underline{b})} < z - cz - \epsilon$, for all $q \geq Q$. This implies that $P(\sum_{\underline{b}\in H} \tilde{y}^q(\underline{b})r(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^m q(\underline{s})r(\underline{s}) > r^{\text{Max}}) = P\left(\frac{\sum_{\underline{b}\in H} \tilde{y}^q(\underline{b})r(\underline{b})}{q\sum_{\underline{b}\in H} r(\underline{b})} - \frac{\sum_{\underline{s}\in G} \tilde{y}^q(\underline{s})r(\underline{s})}{q\sum_{\underline{b}\in H} r(\underline{b})} > \frac{r^{\text{Max}}}{q\sum_{\underline{b}\in H} r(\underline{b})}\right) \geq P\left(\frac{\sum_{\underline{b}\in H} \tilde{y}^q(\underline{b})r(\underline{b})}{q\sum_{\underline{b}\in H} r(\underline{b})} - \frac{\sum_{\underline{s}\in G} \tilde{y}^q(\underline{s})r(\underline{s})}{q\sum_{\underline{b}\in H} r(\underline{b})} > z - cz - \epsilon\right) \geq 1 - \delta$. Since this inequality holds for all sufficiently small $\epsilon > 0$, then in the limit,

$$\lim_{k\to+\infty} P(|\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})r(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})| > r^{\text{Max}}) = 1$$

when $\sum_{\underline{b}\in H} r(\underline{b}) \neq \sum_{\underline{s}\in G} r(\underline{s})$.

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[19] We do not distinguish further between $\underline{b}^* \in H$ and $\underline{b}^* \notin H$ because k and $k-1$ are of the same order as $k \to +\infty$ when applying the Chebyshev's inequality and the central limit theorem.

[20] See Shiryaev (2016).

[21] Such a Q exists because, given ϵ, we can select k large enough so that the right-hand side of equations (16) and (17) are arbitrarily small.

For Case (b), where $\sum_{\underline{b}\in H} r(\underline{b}) = \sum_{\underline{s}\in G} r(\underline{s})$, let d-lim be the limit operator with respect to convergence in distribution. By the de Moivre-Laplace central limit theorem,

$$\text{d-lim}_{k\to+\infty} \frac{\tilde{y}^k(\underline{b}) - kz}{\sqrt{kz(1-z)}} = \tilde{\xi}_{\underline{b}} \text{ and } \text{d-lim}_{k\to+\infty} \frac{\tilde{y}^k(\underline{s}) - kz}{\sqrt{kz(1-z)}} = \tilde{\xi}_{\underline{s}},$$

where $\tilde{\xi}_{\underline{b}}$ and $\tilde{\xi}_{\underline{s}}$ follow the standard normal distribution for all $\underline{b} \in \underline{B}$ and $\underline{s} \in \underline{S}$. The components in $\tilde{\xi} = (\tilde{\xi}_{\underline{b}_1}, \ldots, \tilde{\xi}_{\underline{b}_{n_b}}; \tilde{\xi}_{\underline{s}_1}, \ldots, \tilde{\xi}_{\underline{s}_{n_s}})$ are mutually independent. Then

$$\begin{aligned}
&\text{d-lim}_{k\to+\infty} \frac{\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})r(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})}{\sqrt{kz(1-z)}} \\
=&\,\text{d-lim}_{k\to+\infty} \frac{\sum_{\underline{b}\in H} (\tilde{y}^k(\underline{b}) - kz)r(\underline{b}) - \sum_{\underline{s}\in G} (\tilde{y}^k(\underline{s}) - kz)r(\underline{s})}{\sqrt{kz(1-z)}} \\
=&\sum_{\underline{b}\in H} r(\underline{b})\,\text{d-lim}_{k\to+\infty} \frac{(\tilde{y}^k(\underline{b}) - kz)}{\sqrt{kz(1-z)}} - \sum_{\underline{s}\in G} r(\underline{s})\,\text{d-lim}_{k\to+\infty} \frac{(\tilde{y}^k(\underline{s}) - kz)}{\sqrt{kz(1-z)}} \\
=&\sum_{\underline{b}\in H} r(\underline{b})\tilde{\xi}_{\underline{b}} - \sum_{\underline{s}\in G} r(\underline{s})\tilde{\xi}_{\underline{s}}. \qquad (18)
\end{aligned}$$

The random variable $\sum_{\underline{b}\in H} r(\underline{b})\tilde{\xi}_{\underline{b}} - \sum_{\underline{s}\in G} r(\underline{s})\tilde{\xi}_{\underline{s}}$ follows the normal distribution with mean equal to 0 and variance equal to $\sum_{\underline{b}\in H} r(\underline{b})^2 + \sum_{\underline{s}\in G} r(\underline{s})^2$. Therefore,

$$\begin{aligned}
&\lim_{k\to+\infty} P\Bigg(|\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})r(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})| \le r^{\text{Max}}\Bigg) \\
=&\lim_{k\to+\infty} P\Bigg(-\frac{r^{\text{Max}}}{\sqrt{kz(1-z)}} \le \frac{\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})r(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})}{\sqrt{kz(1-z)}} \le \frac{r^{\text{Max}}}{\sqrt{kz(1-z)}}\Bigg) \\
=&P\Bigg(\sum_{\underline{b}\in C} r(\underline{b})\tilde{\xi}_{\underline{b}} - \sum_{\underline{s}\in D} s_{\underline{s}}\tilde{\xi}_{\underline{s}} = 0\Bigg) = 0.
\end{aligned}$$

Thus, the probability that $\tilde{M}^k$ is not sufficiently uneven converges to 0 also in Case (b). Therefore, (14) holds, and the second addend of (13) is equal to zero.

We now analyze the first addend of (13):

$$\begin{aligned}
&\lim_{k\to+\infty} \mathbb{E}[r(\underline{b}^*)\underline{u}_{\underline{b}^*}(\tilde{M}^k)I_{\tilde{M}^k}] = \lim_{k\to+\infty} \mathbb{E}[r(\underline{b}^*)\underline{u}_{\underline{b}^*}(\tilde{M}^k)] \\
=&\lim_{k\to+\infty} \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{r})} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})r(\underline{b}^*)\underline{u}_{\underline{b}^*}(\mathcal{M})
\end{aligned}$$

$$=r(\underline{b}^*) \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{r})} \lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})\underline{u}_{\underline{b}^*}(\mathcal{M}),$$

where the first equality holds because, as we have seen above, the probability that $\tilde{M}^k$ is sufficiently uneven converges to 1 and $r(\underline{b}^*)\underline{u}_{\underline{b}^*}(\tilde{M}^k)$ is bounded, the second (where we denote $P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})$ the probability that the game $\tilde{M}^k$ is in the class $\mathcal{M} \in T(\underline{B},\underline{S},\mathbf{r})$) follows from the finiteness of $T(\underline{B},\underline{S},\mathbf{r})$, by which we can take the summation ranging over each class of games, and the last equality follows from the additivity of the limit operator.

Therefore, going back to equation (13), we write

$$\lim_{k\to+\infty} \mathbb{E}\Big[\mathrm{D}^{\underline{b}^*(1)}\, v^{\tilde{M}^k} \Big((\bigcup_{\underline{b}\in\underline{B}} \tilde{B}^k_{\underline{b}}) \cup (\bigcup_{\underline{s}\in\underline{S}} \tilde{S}^k_{\underline{s}}) \Big) \Big]$$
$$=r(\underline{b}^*) \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{r})} \lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})\underline{u}_{\underline{b}^*}(\mathcal{M}). \tag{19}$$

We now discuss about $(\lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M}))_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{r})}$, which is a probability distribution over the elements of the partition $T(\underline{B},\underline{S},\mathbf{r})$. Remember that an element of $T(\underline{B},\underline{S},\mathbf{r})$ is an equivalence class characterized by inequalities over the pairs (H,G), where $H \in 2^{\underline{B}} \setminus \{\varnothing\}$ and $G \in 2^{\underline{S}} \setminus \{\varnothing\}$ (see Remark 4). However, some equivalence classes have a zero probability in $(\lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M}))_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{r})}$. To see this, consider a pair (H,G) such that $\sum_{\underline{b}\in H} r(\underline{b}) > \sum_{\underline{s}\in G} r(\underline{s})$. According to the weak law of large numbers, the probability of $\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})r(\underline{b}) > \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})$ converges to 1 as k tends to $+\infty$. Therefore, in the distribution $(\lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M}))_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{r})}$, the probability of being in an equivalence class where $\sum_{\underline{b}\in H} y(\underline{b})r(\underline{b}) \le \sum_{\underline{s}\in G} y(\underline{s})r(\underline{s})$ if $\sum_{\underline{b}\in H} r(\underline{b}) > \sum_{\underline{s}\in G} r(\underline{s})$ is zero. A similar argument applies for those pairs (H,G) for which $\sum_{\underline{b}\in H} r(\underline{b}) < \sum_{\underline{s}\in G} r(\underline{s})$.

Consider now a pair (H,G) for which $\sum_{\underline{b}\in H} r(\underline{b}) = \sum_{\underline{s}\in G} r(\underline{s})$. Following the equations (18), $\sum_{\underline{b}\in H} \tilde{y}^k(\underline{b})r(\underline{b}) - \sum_{\underline{s}\in G} \tilde{y}^k(\underline{s})r(\underline{s})$ converges in distribution to $\sum_{\underline{b}\in H} r(\underline{b})\tilde{\xi}_{\underline{b}} - \sum_{\underline{s}\in G} r(\underline{s})\tilde{\xi}_{\underline{s}}$, which follows the normal distribution with mean equal to 0 and variance equal to $\sum_{\underline{b}\in H} r(\underline{b})^2 + \sum_{\underline{s}\in G} r(\underline{s})^2$ as $k \to +\infty$.

Therefore, the equivalence class that $\tilde{M}^k$ belongs to converges in distribution to $\tilde{\mathcal{M}}^{\tilde{\xi}}$, which is determined by the random vector $\tilde{\xi}$ and it is independent of $z \in (0,1)$ and of the choice of player $\underline{b}^*(1)$. Moreover, the $T(\underline{B}, \underline{S}, \mathbf{r})$-valued random variable $\tilde{\mathcal{M}}^{\tilde{\xi}}$ is defined as follows. If the realization of $\tilde{\xi}$ is $\bar{\xi}$, then the realization of $\tilde{\mathcal{M}}^{\tilde{\xi}}$ is the class $\mathcal{M}$ given by:

1. $\mathcal{M}$ specifies $\sum_{\underline{b}\in H} y(\underline{b})r(\underline{b}) < \sum_{\underline{s}\in G} y(\underline{s})r(\underline{s})$ for all $H \in 2^{\underline{B}} \setminus \{\varnothing\}$ and all $G \in 2^{\underline{S}} \setminus \{\varnothing\}$ such that $\sum_{\underline{b}\in H} r(\underline{b}) < \sum_{\underline{s}\in G} r(\underline{s})$;

2. $\mathcal{M}$ specifies $\sum_{\underline{b}\in H} y(\underline{b})r(\underline{b}) > \sum_{\underline{s}\in G} y(\underline{s})r(\underline{s})$ for all $H \in 2^{\underline{B}}\setminus\{\varnothing\}$ and $G \in 2^{\underline{S}}\setminus\{\varnothing\}$ such that $\sum_{\underline{b}\in H} r(\underline{b}) > \sum_{\underline{s}\in G} r(\underline{s})$;

3. $\mathcal{M}$ specifies $\sum_{\underline{b}\in H} y(\underline{b})r(\underline{b}) < \sum_{\underline{s}\in G} y(\underline{s})r(\underline{s})$ for all $H \in 2^{\underline{B}} \setminus \{\varnothing\}$ and all $G \in 2^{\underline{S}} \setminus \{\varnothing\}$ such that $\sum_{\underline{b}\in H} r(\underline{b}) = \sum_{\underline{s}\in G} r(\underline{s})$ if and only if $\sum_{\underline{b}\in H} \bar{\xi}_{\underline{b}} r(\underline{b}) < \sum_{\underline{s}\in G} \bar{\xi}_{\underline{s}} r(\underline{s})$.

Then, going back to equation (12):

$$\begin{aligned}\lim_{k\to+\infty} \psi^{\lambda}_{\underline{b}^*}(M^k) =& \int_{(0,1)} r(\underline{b}^*) \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{r})} \lim_{k\to+\infty} P^{z,\underline{b}^*(1)}(\tilde{M}^k \in \mathcal{M})\underline{u}_{\underline{b}^*}(\mathcal{M})\lambda(dz) \\ =& r(\underline{b}^*) \sum_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{r})} P(\tilde{\mathcal{M}}^{\tilde{\xi}} = \mathcal{M})\underline{u}_{\underline{b}^*}(\mathcal{M}) = r(\underline{b}^*)\mathbb{E}[\underline{u}_{\underline{b}^*}(\tilde{\mathcal{M}}^{\tilde{\xi}})], \end{aligned} \tag{20}$$

where the first equality uses (13) and (19) in equation (12) and the second holds because the equivalence class that $\tilde{M}^k$ belongs to converges in distribution to $\tilde{\mathcal{M}}^{\tilde{\xi}}$.

The proof for the convergence of the payoff of an arbitrary seller is the same. Therefore, we have proven the following:

$$\lim_{k\to+\infty} \psi^{\lambda}_{\underline{b}}(M^k) = r(\underline{b})\mathbb{E}[\underline{u}_{\underline{b}}(\tilde{\mathcal{M}}^{\tilde{\xi}})] \quad \text{and} \quad \lim_{k\to+\infty} \psi^{\lambda}_{\underline{s}}(M^k) = r(\underline{s})\mathbb{E}[\underline{p}_{\underline{s}}(\tilde{\mathcal{M}}^{\tilde{\xi}})], \tag{21}$$

for all $\underline{b} \in \underline{B}$, $\underline{s} \in \underline{S}$, and $\lambda \in \Delta([0,1])$ such that $\lambda(\{0,1\}) = 0$.

Finally, we show that $\lim_{k\to+\infty} \psi^{\lambda}(M^k) \in \mathcal{CE}(M^K)$, where $K = r^{\max} + 1$. We have shown that the limit does not depend on the particular λ; hence, we will do the proof for the Shapley value, corresponding to the case where λ is the Lebesgue measure on

$[0,1]$. Moreover, we also know that an outcome is a CE outcome if and only if it is stable (equivalently, pairwise stable) and all the prices of the objects a seller owns are equal (Sotomayor, 2007). Since the Shapley value satisfies the equal treatment property, we prove that the $\lim_{k\to+\infty}\psi^{\lambda}(M^k)$ is a CE outcome showing that it is feasible and pairwise stable.

We use the linear programming approach introduced by Sotomayor (1992). Let $k \geq K$. The total payoff of any CE outcome is equal to $v^{M^k}(B^k \cup S^k)$, which can be computed through the following primal problem:

$$
\begin{aligned}
v^{M^k}(B^k \cup S^k) = \max_{\mathbf{x}\in\mathbb{R}_+^{B^k\times S^k}} & \sum_{(\underline{b}(h),\underline{s}(g))\in B^k\times S^k} a_{\underline{b}\underline{s}}x_{\underline{b}(h)\underline{s}(g)} \\
\text{s.t.} & \sum_{\underline{s}(g)\in S^k} x_{\underline{b}(h)\underline{s}(g)} \leq r(\underline{b}) \text{ for all } \underline{b}\in\underline{B} \text{ and } h=1,\dots,k, \\
& \sum_{\underline{b}(h)\in B^k} x_{\underline{b}(h)\underline{s}(g)} \leq r(\underline{s}), \text{ for all } \underline{s}\in\underline{S} \text{ and } g=1,\dots,k, \\
& x_{\underline{b}(h)\underline{s}(g)} \leq 1 \text{ for all } \underline{b}\in\underline{B}, \underline{s}\in\underline{S}, \text{ and } h,g=1,\dots,k.
\end{aligned}
$$

The dual problem is:

$$
\begin{aligned}
\min_{\mathbf{y}\in\mathbb{R}_+^{B^k},\mathbf{z}\in\mathbb{R}_+^{S^k},\mathbf{w}\in\mathbb{R}_+^{B^k\times S^k}} & \sum_{\underline{b}(h)\in B^k} r(\underline{b})y_{\underline{b}(h)} + \sum_{\underline{s}(g)\in S^k} r(\underline{s})z_{\underline{s}(g)} + \sum_{\underline{b}(h)\in B^k}\sum_{\underline{s}(g)\in S^k} w_{\underline{b}(h)\underline{s}(g)} \\
\text{s.t. } & y_{\underline{b}(h)} + z_{\underline{s}(g)} + w_{\underline{b}(h)\underline{s}(g)} \geq a_{\underline{b}\underline{s}} \text{ for all } \underline{b}(h)\in B^k \text{ and } \underline{s}(g)\in S^k. \quad (22)
\end{aligned}
$$

In this dual problem, $y_{\underline{b}(h)}$ (resp., $z_{\underline{s}(g)}$) is the utility that each buyer (resp., seller) obtains in each transaction. Then, by equation (21), we show that $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{w}^*)$, defined by $y^*_{\underline{b}(h)} = (\lim_{k\to+\infty}\psi^{\lambda}_{\underline{b}}(M^k))/r(\underline{b})$ for every $\underline{b}(h)\in B^k$, $z^*_{\underline{s}(g)} = (\lim_{k\to+\infty}\psi^{\lambda}_{\underline{s}}(M^k))/r(\underline{s})$ for every $\underline{s}(g)\in S^k$, and $\mathbf{w}^* = \mathbf{0}$, is a solution to the dual problem for sufficiently large k, where λ is the Lebesgue measure on $[0,1]$ (i.e., $\psi^{\lambda}(M^k)$ is the Shapley value of $(B^k\cup S^k, v^{M^k})$).

Define $(\mathbf{y}^k, \mathbf{z}^k, \mathbf{w}^*)$ as $y^k_{\underline{b}(h)} = (\psi^{\lambda}_{\underline{b}}(M^k))/r(\underline{b})$ for every $\underline{b}(h)\in B^k$, $z^k_{\underline{s}(g)} = (\psi^{\lambda}_{\underline{s}}(M^k))/r(\underline{s})$

for every $\underline{s}(g) \in S^k$, and $\mathbf{w}^* = \mathbf{0}$. Then,

$$\begin{aligned}
&\sum_{\underline{b}(h)\in B^k} r(\underline{b}) y^k_{\underline{b}(h)} + \sum_{\underline{s}(g)\in S^k} r(\underline{s}) z^k_{\underline{s}(g)} + \sum_{\underline{b}(h)\in B^k} \sum_{\underline{s}(g)\in S^k} w^*_{\underline{b}(h)\underline{s}(g)} \\
=&\sum_{\underline{b}\in\underline{B}} kr(\underline{b}) \frac{\psi^\lambda_{\underline{b}}(M^k)}{r(\underline{b})} + \sum_{\underline{s}\in\underline{S}} kr(\underline{s}) \frac{\psi^\lambda_{\underline{s}}(M^k)}{r(\underline{s})} \\
=&\sum_{\underline{b}\in\underline{B}} \sum_{i=1}^{k} Sh_{\underline{b}(h)}(B^k \cup S^k, v^{M^k}) + \sum_{\underline{s}\in\underline{S}} \sum_{j=1}^{k} Sh_{\underline{s}(g)}(B^k \cup S^k, v^{M^k}) = v^{M^k}(B^k \cup S^k)
\end{aligned}$$

for every k, where the last equality holds because the Shapley value is efficient. Therefore, the equality also holds when $k \to +\infty$, that is, for $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{w}^*)$. Hence, we have shown that the payoff vector $\lim_{k\to+\infty} \psi^\lambda(M^k)$ is feasible. We also need to show that $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{w}^*)$ satisfies the constraints (22), that is, it is pairwise stable. For each $\underline{b} \in \underline{B}$ and each $\underline{s} \in \underline{S}$,

$$\begin{aligned}
&y^*_{\underline{b}(h)} + z^*_{\underline{s}(g)} + w^*_{\underline{b}(h)\underline{s}(g)} \\
=&\lim_{k\to+\infty} \frac{\psi^\lambda_{\underline{b}}(M^k)}{r(\underline{b})} + \lim_{k\to+\infty} \frac{\psi^\lambda_{\underline{s}}(M^k)}{r(\underline{s})} \\
=&\mathbb{E}[\underline{u}_{\underline{b}}(\mathcal{M}^\xi)] + \mathbb{E}[\underline{p}_{\underline{s}}(\mathcal{M}^\xi)] \\
=&\sum_{\mathcal{M}\in T(\underline{B},\underline{S})} \lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M}) \underline{u}_{\underline{b}}(\mathcal{M}) + \sum_{\mathcal{M}\in T(\underline{B},\underline{S})} \lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M}) \underline{p}_{\underline{s}}(\mathcal{M}) \\
=&\sum_{\mathcal{M}\in T(\underline{B},\underline{S})} \lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M}) [\underline{u}_{\underline{b}}(\mathcal{M}) + \underline{p}_{\underline{s}}(\mathcal{M})] \\
\geq&\sum_{\mathcal{M}\in T(\underline{B},\underline{S})} \lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M}) a_{\underline{bs}} = a_{\underline{bs}},
\end{aligned}$$

where the first equality follows the definition of $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{w}^*)$, the second from equation (21), the third from equation (20), the fifth from $\underline{u}_{\underline{b}}(\mathcal{M}) + \underline{p}_{\underline{s}}(\mathcal{M}) \geq a_{\underline{bs}}$ for all $\mathcal{M} \in T(\underline{B}, \underline{S}, \mathbf{r})$ because $(\underline{\mathbf{u}}(\mathcal{M}), \underline{\mathbf{p}}(\mathcal{M}))$ is the CE payoff vector of any game in $\mathcal{M}$, and the sixth holds because $(\lim_{k\to+\infty} P(\tilde{M}^k \in \mathcal{M}))_{\mathcal{M}\in T(\underline{B},\underline{S},\mathbf{r})}$ is a probability distribution. Therefore, we have proven equation (22).

We note that individual rationality means that $\psi^\lambda_{\underline{b}}(M^k)/r(\underline{b}) \geq 0$ for all $\underline{b} \in \underline{B}$ and all k, and $\psi^\lambda_{\underline{s}}(M^k)/r(\underline{s}) \geq 0$ for all $\underline{s} \in \underline{S}$ and all k. It holds because a semivalue can be represented as an expected marginal contribution, and a buyer's or seller's marginal contribution is always greater than zero in any multiple-partners game.

Therefore, we have shown that the limit of the Shapley value (hence, the limit of every semivalue with $\lambda(\{0,1\}) = 0$) of any replicated multiple-partners game is in the set of stable outcomes of the limit of the replicated game. Moreover, consider an outcome that satisfies equal treatment of equals and equal treatment of partnerships so that we can write the constraints of the dual problem as in (22). Then, by inspecting the dual problem, it is easy to check that the outcome satisfies (22) for some $k \geq K$ if it satisfies the constraint for K. Since the set of CE satisfies the equal treatment properties if $k \geq K$ (Corollary 3), and it is the subset of the set of stable outcomes that satisfy the equal treatment properties (see Remark 2), then the set of CE is the same for all $k \geq K$.

Finally, we claim that there is a matching that sustains the payoff vector, i.e., the payoff vector is feasible. Indeed, let x^* be an integer solution to the primal problem, which is also a feasible matching for M^K. It is well-known such a solution can be found by the simplex method. Moreover, $y^*_{\underline{b}(h)} + z^*_{\underline{s}(g)} > a_{\underline{bs}}$ implies $x^*_{\underline{b}(h)\underline{s}(g)} = 0$ for every buyer and every seller by the complementary slackness theorem (See Vanderbei, 2021). Thus $(\mathbf{y}^*, \mathbf{z}^*)$ is compatible with x^*; hence it is a feasible payoff vector for M^K. □

Remark 5. *A proviso in Theorem 1 is that the semivalues under consideration are those identified by* $\lambda \in \Delta([0,1])$ *such that* $\lambda(\{0,1\}) = 0$*. This does not rule out the possibility that a semivalue with* $\lambda(\{0,1\}) > 0$ *converges to the same limit, for some games. For instance, consider an asymmetric glove market (Example 1). The semivalue with* $\lambda(\{1\}) = 1$ *coincides with the CE outcome for every k-fold replica.*

The first implication of Theorem 1 is that, although the different semivalues generally prescribe different payoff vectors for every multiple-partners game, many of them (all except possibly those semivalues with $\lambda(\{0,1\}) > 0$) converge to the same payoff vector when we replicate the game. This class of semivalues includes the Shapley value.

The second implication of the theorem is that the limit payoff vector is a CE payoff vector of a sufficiently large replica. We recall (Corollary 4) that the set of CE is constant once the market has been replicated $K = r^{\max} + 1$ times. Hence, $\mathcal{CE}(M^K) = \mathcal{CE}(M^k)$ for all $k \geq K$. Moreover, the set $\mathcal{CE}(M^K)$ coincides with the set of stable outcomes satisfying equal treatment of equals and equal treatment of partnerships.

We end the section with a short discussion on the type of convergence of the semivalues. In the framework of replicated multiple-partners games, Sotomayor (2019) proved that the set of stable outcomes shrinks finitely to the set of CE payoff vectors, which in turn shrinks finitely to the set of stable outcomes satisfying equal treatment of equals and equal treatment of partnerships. By contrast, the convergence of semivalues to a CE payoff vector, as stated in Theorem 1, occurs differently. Although the limit of a semivalue is a CE payoff vector, the semivalue of any finite replica may not be. For example, consider an asymmetric glove market. On the one hand, any semivalue allocates a strictly positive payoff to each player from the long side since, for every finite replica, there is always a non-negligible chance that this player joins a coalition with a majority of players from the short side. On the other hand, the unique CE outcome prescribes a zero payoff to players from the long side.

The example of an asymmetric glove market does not preclude the possibility of finding, for each game, a distinct semivalue that finitely converges to a competitive equilibrium outcome. We illustrate the convergence through an example. Consider the assignment game $M = \langle B, S, \mathbf{a}, \mathbf{r} \rangle$, where $B = \{b_1, b_2\}$, $S = \{s_1, s_2\}$, $r(b_1) = r(b_2) = r(s_1) = r(s_2) = 1$, $a_{b_1 s_1} = 1, a_{b_1 s_2} = 3$, and $a_{b_2 s_1} = a_{b_2 s_2} = 2$. The limit of the semivalues for this game is the CE payoff vector $(\frac{11}{8}, \frac{9}{8}, \frac{7}{8}, \frac{13}{8}) = (1.375, 1.125, 0.875, 1.625)$. We compute some semivalues determined by a distribution over $[0, 1]$ given by a Beta distribution, characterized by two parameters $\alpha, \beta > 0$, and those given by a Dirac probability measure concentrated on a point $q \in [0, 1]$. In the table, the semivalue corresponding to $\alpha = \beta = 1$ is the Shapley value, and the one with $q = 0.5$ is the Banzhaf value.

Parameters	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
Beta distr $\alpha=1$, $\beta=1$	(1.417, 1.083, 0.917, 1.583)	(1.402, 1.098, 0.902 ,1.598)	(1.396, 1.104, 0.896, 1.604)	(1.392, 1.108, 0.892, 1.608)	(1.390, 1.110, 0.890, 1.610)	(1.388, 1.112, 0.888, 1.612)
Beta distr $\alpha=0.5$, $\beta=0.5$	(1.438, 1.063, 0.938, 1.563)	(1.424, 1.076, 0.924, 1.576)	(1.417, 1.083, 0.917, 1.583)	(1.412, 1.088, 0.912, 1.588)	(1.409, 1.091, 0.901, 1.591)	(1.406, 1.094, 0.906, 1.594)
Dirac measure $q=0.5$	(1.375 ,1.125, 0.875, 1.625)	(1.375 ,1.125, 0.875, 1.625)	(1.375 ,1.125, 0.875, 1.625)	(1.375 ,1.125, 0.875, 1.625)	(1.375 ,1.125, 0.875, 1.625)	(1.375 ,1.125, 0.875, 1.625)
Dirac measure $q=0.4$	(1.152, 0.992, 0.752, 1.392)	(1.250, 1.050, 0.802, 1.498)	(1.280, 1.067, 0.819, 1.529)	(1.295, 1.076, 0.827, 1.544)	(1.305, 1.082, 0.833, 1.554)	(1.312 ,1.086, 0.837, 1.561)
Dirac measure $q=0$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$

We notice that the Banzhaf value ($q = 0.5$) attains the limit from the beginning and stays constant after that. On the other hand, the semivalue with $q = 0$, which is outside the subclass that we study, does not converge to a CE payoff vector. The other semivalues converge to the CE payoff vector $(1.375, 1.125, 0.875, 1.625)$ but do not reach it.

6 Conclusion

The classic solution concepts for the multiple-partners game, as for matching games in general, are stability and competitive equilibrium. Single-valued solutions concepts, such as the Shapley value, are not well-studied. In this paper, we have contributed to a better understanding of the behavior of the Shapley value and many other semivalues in the multiple-partners game. We have shown that, when the game is replicated, they all converge to the same competitive equilibrium payoff vector.

Sotomayor's (2019) analysis of the replicated multiple-partners game concluded that the sets of stable allocations, competitive equilibrium allocations, and stable allocations that satisfy equal treatment of equals and equal treatment of partnerships converge finitely to the same set. By contrast, the convergence of the Shapley value and the other semivalues is generally not finite.

Appendix

Proof of Proposition 1. Take any CE outcome of $\underline{M}$ and denote $p_{\underline{s}(g)}$ the price set by the seller $\underline{s}(g)$ of type $\underline{s}$ for her objects in that outcome. First, we show the equal treatment of equals for the sellers, that is, for any type $\underline{s} \in \underline{S}$, the prices $p_{\underline{s}(g)}$ and $p_{\underline{s}(g')}$ set by two equal sellers of type $\underline{s}$ are the same if $y^{\min} > r^{\max}$. We prove this property by contradiction.

Let $\underline{s}$ be a seller's type such that $p_{\underline{s}(g)} \neq p_{\underline{s}(g')}$, for some g and g'. Denote $p_{\underline{s},\min} := \min\{p_{\underline{s}(g)} \mid g = 1, \ldots, y(\underline{s})\}$ and split the set $S_{\underline{s}}$ in two subsets, L and F, with $L := \{\underline{s}(g) \mid p_{\underline{s}(g)} = p_{\underline{s},\min}\}$ and $F := \{\underline{s}(g) \mid p_{\underline{s}(g)} > p_{\underline{s},\min}\}$. Both sets are non-empty.

Denote $\mathbf{x}$ the matching in this CE outcome and take a seller $\underline{s}(g) \in F$. Given that $p_{\underline{s}(g)} > p_{\underline{s},\min} \geq 0$, $\underline{s}(g)$ sells all her objects in the CE outcome; hence, the set $C_{\underline{s}(g)}(x)$ of partners of $\underline{s}(g)$ contains $r(\underline{s})$ buyers. Moreover, it must hold that $L \subseteq C_{\underline{b}(h)}(x)$ for all those buyers $\underline{b}(h) \in C_{\underline{s}(g)}(x)$ because otherwise $\underline{b}(h)$ would have an incentive to swap the object $\underline{s}(g)$ with some identical object owned by a seller in $L \setminus C_{\underline{b}(h)}(x)$. Since the quota of the sellers in L is also $r(\underline{s})$, each of them sells all her objects in the CE outcome to the buyers in $C_{\underline{s}(g)}(x)$.

We now claim that it must also be the case that $F \subseteq C_{\underline{b}(h)}(x)$ for all $\underline{b}(h) \in C_{\underline{s}(g)}(x)$. If not, there is at least one $\underline{s}(g') \in F$ who does not sell one of her objects to a buyer in $C_{\underline{s}(g)}(x)$. Given that $p_{\underline{s}(g')} > 0$, the agent $\underline{s}(g')$ sells all her objects, which implies that there is a buyer $\underline{b}'(h') \in C_{\underline{s}(g')}(x)$ such that $\underline{b}'(h') \notin C_{\underline{s}(g)}(x)$. However, it is the case that $\underline{b}'(h')$ has an incentive to swap $\underline{s}(g')$ with an object owned by a seller in L that he does not buy, since we have shown that the sellers in L sell all their objects to the buyers in $C_{\underline{s}(g')}(x)$. This is a contradiction, hence, $F \subseteq C_{\underline{b}(h)}(x)$ for all $\underline{b}(h) \in C_{\underline{s}(g)}(x)$.

Hence, we have proved that $S_{\underline{s}} = L \cup F \subseteq C_{\underline{b}(h)}(x)$ for all $\underline{b}(h) \in C_{\underline{s}(g)}(x)$. Since $S_{\underline{s}}$ has at least $y^{\min}$ elements and $C_{\underline{b}(h)}(x)$ has at most $r^{\max}$ elements, $S_{\underline{s}} \subseteq C_{\underline{b}(h)}(x)$ can only happen if $y^{\min} \leq r^{\max}$, which proves our contradiction. Therefore, a CE outcome satisfies equal treatment of equals among sellers for $y^{\min} > r^{\max}$.

Second, we show the equal treatment of partnerships among buyers (the property holds among sellers by the definition of a CE). Hence, we must show that $u_{\underline{b}(h)\underline{s}(g)} = u_{\underline{b}(h)\underline{s}'(g')}$ for all $\underline{b} \in \underline{B}$, $h = 1, \ldots, y(\underline{b})$, and $\underline{s}(g), \underline{s}'(g') \in C_{\underline{b}(h)}(x)$. Suppose otherwise, that is, $u_{\underline{b}(h)\underline{s}(g)} > u_{\underline{b}(h)\underline{s}'(g')}$ for some $\underline{s}(g), \underline{s}'(g') \in C_{\underline{b}(h)}(x)$. There exists $\underline{s}(g^*) \notin C_{\underline{b}(h)}(x)$ because $y^{\min} > r^{\max}$, there are at least $y^{\min}$ sellers of type $\underline{s}$, and the set $C_{\underline{b}(h)}(x)$ has at most $r(\underline{b}) \leq r^{\max}$ elements. Then, buyer $\underline{b}(h)$ has an incentive to swap the object owned by $\underline{s}'(g')$ with an object owned by $\underline{s}(g^*)$ because $u_{\underline{b}(h)\underline{s}(g^*)} = a_{\underline{b}\underline{s}} - p_{\underline{s}(g^*)} = a_{\underline{b}\underline{s}} - p_{\underline{s}(g)} = u_{\underline{b}(h)\underline{s}(g)} > u_{\underline{b}(h)\underline{s}'(g')}$, where we have used the property $p_{\underline{s}(g^*)} = p_{\underline{s}(g)}$ that we have proved above. Therefore, a CE outcome satisfies equal treatment of partnerships if $y^{\min} > r^{\max}$.

Finally, once we have proven the equal treatment of partnerships among buyers, we can use a similar argument to prove that two equal buyers (not necessarily of the same type) attain the same payoff vector at a CE outcome. Therefore, a CE outcome satisfies equal treatment of equals if $y^{\min} > r^{\max}$.

□

Proof of Proposition 2. Given Remark 2, the characteristics of the CE outcomes of the sellers and the buyers are similar in a large multiple-partners game with types (in particular, as it is the case for the sellers by definition, the CE payoff vectors satisfy equal treatment of partnerships among the buyers). Therefore, we can assume, without loss of generality, that $\sum_{\underline{b} \in \underline{B}} y(\underline{b}) r(\underline{b}) \leq \sum_{\underline{s} \in \underline{S}} y(\underline{s}) r(\underline{s})$ because the proof of the other case is symmetric. Moreover, to prove the proposition, it suffices to show that $(\underline{\mathbf{u}}, \underline{\mathbf{p}}) \in \mathbb{R}^{\underline{B}} \times \mathbb{R}^{\underline{S}}$ is a CE payoff vector of $\underline{M}'$ if it is a CE payoff vector of $\underline{M}$.

Denote $\hat{B}$ and $\hat{S}$ the sets of buyers and sellers of the one-to-one assignment game connected with the game $\underline{M}$ and $\hat{\mathbf{x}} \in \mathcal{A}(\hat{B}, \hat{S}, \mathbf{r})$ the matching constructed based on the CE matching compatible with $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ (see section 2.4). Hence, an element in $\hat{B}$ is $(\underline{b}(h), l)$, where $\underline{b}$ is the type of the buyer, $h \in \{1, ..., y(\underline{b})\}$ is a buyer of type $\underline{b}$, and $l \in \{1, ..., r(\underline{b})\}$ indicates one of the agents of the buyer $\underline{b}(h)$ in the assignment game. Similarly, $(\underline{s}(g), f)$ is an element of $\hat{S}$. Given $\sum_{\underline{b} \in \underline{B}} y(\underline{b}) r(\underline{b}) \leq \sum_{\underline{s} \in \underline{S}} y(\underline{s}) r(\underline{s})$, the sets $\hat{B}$ and $\hat{S}$ satisfy that $|\hat{B}| \leq |\hat{S}|$.

We first prove by contradiction that there exists a feasible matching $\mathbf{x}$ for $\underline{M}$ compatible with $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ such that every agent in $\hat{B}$ acquires an object in $\hat{S}$. Suppose that some agent $(\underline{b}(h), l) \in \hat{B}$ does not acquire any object in $\hat{\mathbf{x}}$. Since $|\hat{B}| \leq |\hat{S}|$, there exists an unsold object $(\underline{s}(g), f) \in \hat{S}$.

Then we claim that we can construct a new matching $\bar{x}$ by adding a partnership between $(\underline{b}(h), l)$ and $(\underline{s}(g), f)$ to $\hat{x}$. First, assume that $a_{\underline{bs}} > 0$. Then, it must be the case that $r(\underline{s}) > 1$ and that an object $(\underline{s}(g), f')$, with $f' \neq f$, is acquired by an agent $(\underline{b}(h), l')$, with $l' \neq l$, because otherwise $(\underline{b}(h), l)$ would acquire $(\underline{s}(g), f)$ at a price in the interval $(0, a_{\underline{bs}})$. However, by equal treatment of partnerships, $\underline{u}_{\underline{b}} = 0$ (because $\underline{b}(h)$ does not form all his partnerships), hence $\underline{p}_{\underline{s}} = a_{\underline{bs}} - \underline{u}_{\underline{b}} > 0$, which contradicts the existence of an unsold type-$\underline{s}$ object $(\underline{s}(g), f)$. Second, if $a_{\underline{bs}} = 0$, the existence of such a matching also holds because the set of matchings compatible with a competitive equilibrium is upper hemi-continuous with respect to the worth matrix $\mathbf{a}$.[22]

Now, take $\mathbf{x}$ compatible with $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ such that every agent in $\hat{B}$ acquires an object in $\hat{S}$, and let $\hat{x}$ be the one-to-one matching between the set of agents $\hat{B}$ and the set of objects $\hat{S}$ of $\underline{M}$ induced by $\mathbf{x}$. By Proposition 1 and the observation that every buyer's

[22] To see this, notice that the set of matchings compatible with a competitive equilibrium is a finite subset of the set of solutions to a linear programming problem. Then, it follows from Berge's theorem of maximum (see, e.g., Kreps, 2013) that the set of solutions, and, hence, the set of matchings compatible with a competitive equilibrium, is upper hemi-continuous with respect to the worth matrix $\mathbf{a}$.

quota is full, we can define the following correspondence $\varphi : \hat{B} \rightsquigarrow \hat{S}$:

$$\varphi(\underline{b}(h), l) := \bigcup_{\substack{\underline{s}: \underline{u}_{\underline{b}} + \underline{p}_{\underline{s}} = a_{\underline{bs}} \\ \text{for some } \underline{b} \in \underline{B}}} \{(\underline{s}(g), f)) \mid g = 1, \ldots, y(\underline{s}) \text{ and } f = 1, \ldots, r(\underline{s})\} \tag{23}$$

for all $(\underline{b}(h), l) \in \hat{B}$. That is, $\varphi(\underline{b}(h), l)$ is the set of all the objects $(\underline{s}(g), f)$ the agent $(\underline{b}(h), l)$ could acquire to obtain his equilibrium utility $\underline{u}_{\underline{b}}$, given the equilibrium price $\underline{p}_{\underline{s}}$ and the worth $a_{\underline{bs}}$ of the partnership.

We can consider the one-to-one matching $\hat{x}$ as a function from $\hat{B}$ to $\hat{S}$ that assigns different objects to different agents $(\underline{b}(h), l), (\underline{b}'(h'), l') \in \hat{B}$. Hence, by Hall's theorem (Lemma 1), we have $|P| \leq |\varphi(P)|$ for all $P \in 2^{\hat{B}} \setminus \{\varnothing\}$. This implies, in particular,

$$\sum_{\underline{b} \in \underline{P}} r(\underline{b}) y(\underline{b}) \leq \sum_{\substack{\underline{s}: \underline{u}_{\underline{b}} + \underline{p}_{\underline{s}} = a_{\underline{bs}} \\ \text{for some } \underline{b} \in \underline{P}}} r(\underline{s}) y(\underline{s}) \tag{24}$$

for all $\underline{P} \in 2^{\underline{B}} \setminus \{\varnothing\}$. We note that the right-hand side of the equation (24) corresponds to the sum over a certain subset $G \subseteq \underline{S}$.

By the definition of the equivalence relation $\sim$ (Definition 10), Condition (24) also holds for the game $\underline{M}'$. That is,

$$\sum_{\underline{b} \in \underline{P}} r(\underline{b}) y'(\underline{b}) \leq \sum_{\substack{\underline{s}: \underline{u}_{\underline{b}} + \underline{p}_{\underline{s}} = a_{\underline{bs}} \\ \text{for some } \underline{b} \in \underline{P}}} r(\underline{s}) y'(\underline{s}) \tag{25}$$

for all $\underline{P} \in 2^{\underline{B}} \setminus \{\varnothing\}$.

Following equation (23), define a correspondence φ' between the set of agents $\hat{B}'$ and the set of objects $\hat{S}'$ of $\underline{M}'$ by

$$\varphi'(\underline{b}(h), l) := \bigcup_{\substack{\underline{s}: \underline{u}_{\underline{b}} + \underline{p}_{\underline{s}} = a_{\underline{bs}} \\ \text{for some } \underline{b} \in \underline{B}}} \{(\underline{s}(g), f)) \mid g = 1, \ldots, y'(\underline{s}) \text{ and } f = 1, \ldots, r(\underline{s})\}$$

for all $(\underline{b}(h), l) \in \hat{B}'$. The correspondence φ' is well-defined because the matrix of worth $\mathbf{a}$ in the game $\underline{M}'$ is the same as in $\underline{M}$. Moreover, φ' satisfies $|P| \leq |\varphi'(P)|$ for all $P \in 2^{\hat{B}'} \setminus \{\varnothing\}$ because of Condition (25).

Using Hall's theorem again, there is a one-to-one matching $\hat{x}'$ between $\hat{B}'$ and $\hat{S}'$. Moreover, each buyer $(\underline{b}(h), l) \in \hat{B}'$ maximizes his utility, given the vector of prices, since he obtains the same utility as in the market $\underline{M}$ and the prices are the same. However, $\hat{x}'$ may not correspond to a feasible matching of $\underline{M}'$ because we cannot rule out the possibility that multiple agents of the same buyer are assigned to multiple objects owned by the same seller via $\hat{x}'$, which is not allowed in the model.

We now use $\hat{x}'$ to construct a new matching that does not assign multiple agents of the same buyer to objects owned by the same seller. Suppose that there exists a buyer $\underline{b}(h)$ and a seller $\underline{s}(g)$ such that $\hat{x}'_{(\underline{b}(h),l)(\underline{s}(g),f)} = 1$ and $\hat{x}'_{(\underline{b}(h),l')(\underline{s}(g),f')} = 1$, with $l \neq l'$ and $f \neq f'$ (otherwise, we are done). Then, there exists a buyer $\underline{b}(h') \neq \underline{b}(h)$ such that $\hat{x}'$ does not assign any agent of $\underline{b}(h')$ to an object owned by the seller $\underline{s}(g)$. Such a buyer exists because the number of objects owned by $\underline{s}(g)$ is $r(\underline{s})$ and $y'^{\min} > r^{\max}$. Then, take the first agent $(\underline{b}(h'), 1)$ of the buyer $\underline{b}(h')$. Let $(\underline{b}'(h'), 1)$ be assigned to an object $(\underline{s}'(j), q)$ via $\hat{x}'$. Notice that $\underline{s}' \neq \underline{s}$. Then, we can swap the assigned object to $(b(h), l')$ with the assigned object to $(\underline{b}(h'), 1)$. Moreover, since the buyers $\underline{b}(h)$ and $\underline{b}(h')$ are identical, their worth vector is the same, and their payoffs are identical under the CE payoff $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$. Hence, the CE payoff obtained by all the players is compatible with the new matching.

We continue this swapping procedure until there does not exist a pair of agents of the same buyer who are assigned to objects owned by the same seller. Denote by $\hat{x}''$ this new function. It corresponds to a feasible matching of $\underline{M}'$. Moreover, given the price vector, each buyer maximizes his utility, and the prices are zero if a seller does not sell all her objects. Therefore, $(\underline{u}, \underline{p})$ is a CE payoff vector of $\underline{M}'$. □

Proof of Proposition 3. We prove the result by contradiction. Let $(\underline{\mathbf{u}}, \underline{\mathbf{p}}; \mathbf{x})$ and $(\underline{\mathbf{u}}', \underline{\mathbf{p}}'; \mathbf{x}')$ be two distinct CE outcomes of the game $\underline{M}$, where we use that $y^{\min} > r^{\max}$ to simplify so that $\underline{\mathbf{u}}, \underline{\mathbf{u}}' \in \mathbb{R}^{\underline{B}}$ and $\underline{\mathbf{p}}, \underline{\mathbf{p}}' \in \mathbb{R}^{\underline{S}}$. Let $\hat{x} \in \mathcal{A}(\hat{B}, \hat{S}, \mathbf{r})$ be a one-to-one matching constructed from $\mathbf{x}$ and $\hat{M} = \langle \hat{B}, \hat{S}, \hat{\mathbf{a}} \rangle$ be the simple game of $\underline{M}$, as defined in section

2.4.

Suppose that $\underline{u}_{\underline{b}} > \underline{u}'_{\underline{b}}$ for some $\underline{b} \in \underline{B}$. Given that $\underline{u}_{\underline{b}} > 0$, any agent $(\underline{b}(h), l)$ of the type-$\underline{b}$ buyer is matched with an object $(\underline{s}(g), f)$ owned by some type-$\underline{s}$ seller under $\hat{x}$. Thus $\underline{u}_{\underline{b}} + \underline{p}_{\underline{s}} = a_{\underline{bs}}$, for any such seller type $\underline{s}$. Moreover, it must be the case $\underline{u}'_{\underline{b}} \geq a_{\underline{bs}} - \underline{p}'_{\underline{s}}$ because $(\underline{\mathbf{u}}', \underline{\mathbf{p}}'; \mathbf{x}')$ is a CE outcome. Hence, $\underline{p}'_{\underline{s}} \geq a_{\underline{bs}} - \underline{u}'_{\underline{b}} > a_{\underline{bs}} - \underline{u}_{\underline{b}} = \underline{p}_{\underline{s}}$.

Notice that we have constructed a one-to-one function from $O = \{(\underline{b}(h), l) \in \hat{B} \mid \underline{u}_{\underline{b}} > \underline{u}'_{\underline{b}}\}$ to $R = \{(\underline{s}(g), f) \in \hat{S} \mid \underline{p}_{\underline{s}} < \underline{p}'_{\underline{s}}\}$ (because $\hat{x}$ is one-to-one). This implies that $|O| \leq |R|$.

A symmetric argument allows to construct a one-to-one function from R to O (via x'), which implies that $|O| \geq |R|$. Therefore, $|O| = |R|$.

Since all the agents of all the buyers of a type $\underline{b}$ have the same utility level $u_{\underline{b}}$ and all the objects of all the sellers of a type $\underline{s}$ post the same price $p_{\underline{s}}$, it is the case that $O = \bigcup_{\underline{b} \in H} B_{\underline{b}}$ and $R = \bigcup_{\underline{s} \in G} S_{\underline{s}}$ for some $H \in 2^{\underline{B}} \setminus \{\varnothing\}$ and some $G \in 2^{\underline{B}} \setminus \{\varnothing\}$. However, this implies that $\sum_{\underline{b} \in H} y(\underline{b}) r(\underline{b}) = |O| = |R| = \sum_{\underline{s} \in G} y(\underline{s}) r(\underline{s})$, which is in contradiction with the definition of an uneven game (see condition (5) of Definition 8). □

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