Updating under Imprecise Information

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January, 2024

Abstract

This paper models an agent that ranks actions with uncertain payoffs after observing a signal which may have been generated by multiple objective information structures. Assuming that the agent's preferences conform to the multiple priors model (Gilboa and Schmeidler (1989)), we show that a simple behavioral axiom characterizes a generalization of Bayesian updating. Intuitively, our axiom requires that whenever all possible sources of information agree that an action with uncertain payoffs is more "likely" to be better than one with certain payoffs, the agent prefers the former. We also provide axiomatizations for various special cases. Additionally, we explore a scenario where a signal's informational content is purely subjective. We analyze the presence of a subjective set of information structures under full Bayesian updating for two extreme cases: (i) no ex-ante state ambiguity, and (ii) no signal imprecision.

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1 Introduction

Individuals often acquire information before making a choice. However, in many situations, they are unable to completely determine the acquired information's source. Since the same information can have different meanings depending on its source, individuals may have a hard time processing it if they are unsure about its true source. Here, we aim to provide a theory of how agents process such information.

To illustrate this, we provide an example inspired by early research conducted during the COVID-19 pandemic.[1] Consider a policymaker who must choose to adopt a costly novel security measure. They are aware that a study has shown this measure reduces the risk of an accident. However, she is unsure if the data employed in the study comes from a reliable source. If the data was collected properly, the results suggest the measure should be adopted. However, if the data is unreliable or of poor quality, the results are meaningless. With no additional evidence about the data quality, both cases are plausible. Then, how should the policymaker process such information?

This paper aims to provide a behavioral foundation for belief updating under unspecified information source. We take as primitives and outcome space X, an objective state space Ω, an objective signal space S, and a set of information structures $\mathcal{L} \subseteq \{\ell : \Omega \to \Delta(S)\}$, where $\Delta(S)$ is the set of probability measures over S. We refer to $\mathcal{L}$ as an imprecise information structure. The idea behind our definition is that $\mathcal{L}$ is the set of identifiable information sources. The agent is assumed to have an ex-ante preference $\succeq_0$ and signal-conditional preferences $(\succeq_s)_{s\in S}$ over actions that have uncertain payoffs. We interpret $\succeq_s$ to be the agent's preferences conditional on observing signal s generated by some $\ell \in \mathcal{L}$. We assume that ex-ante $\succeq_0$ and ex-post $\succeq_s$ conforms to the Gilboa and Schmeidler (1989) maxmin expected utility model (MEU).

Our theory is built on the premise that if all identifiable information structures recommend choosing an action with uncertain payoffs over one with deterministic payoffs, the agent should select the former. Imposing this condition as an axiom yields a representation for the ex-post preferences (Theorem 2.1), and consequently, an updating rule. For any action $f : \Omega \to X$, the utility of f after observing s is given by:

$$U_s(f) = \min_{q\in\rho(\mathcal{M}_0,\mathcal{L},s)} \int u(f)dq$$

[1]During the early days of the pandemic, over 66 clinical prediction models were proposed in peer-reviewed literature on the effect of a mask mandate. However, further research revealed that all these models suffered a high risk of bias due to concerns surrounding the data quality, statistical analysis, and reporting. See Collins et al. (2020) for a discussion.

Here, $\mathcal{M}_0$ represents the set of priors, and $\rho(\mathcal{M}_0, \mathcal{L}, s)$ is a subset of the convex hull of all posteriors generated by "point-wise" Bayesian updating. We term this updating rule as the generalized Bayesian updating (GBU) rule.

Because of the normative nature of our axiom, it can be viewed as a benchmark for any new updating theory that challenges the Bayesian approach within the MEU framework. Essentially, any updating rule that goes against it needs a good reason, backed by an example, to show why it's a reasonable choice.

In our context, signals are not considered payoff-relevant; that is, an action's payoff does not depend on the realized signal. This departure from the typical axiomatic literature on belief updating, where signals are often modeled as subsets of Ω, is a distinct feature of our framework. Unlike the conventional approach, our setting does not assume signals to be subsets of Ω. This assumption allows us to disentangle the agent's attitude towards imprecise information from their attitude towards uncertainty. If signals were modeled as subsets of Ω, this separation would not be feasible. Additionally, our approach helps us test our axioms in a laboratory setting, as our framework serves as a theoretical representation of experiments conducted in Epstein and Halevy (2022) and Shishkin et al. (2021).

Our updating rule generalizes both *full Bayesian updating* (FBU) and *maximum likelihood updating* (MLU), both of which are widely popular. However, the conceptual rationale behind why an agent may adopt either is not entirely clear. FBU updates each subjective prior belief under each objectively-given possible information structure. Thus, while the agent can have a subjective view of the uncertainty about the state space, she does not have a subjective view of the uncertainty about the information source. Meanwhile, MLU utilizes the realized signal to *jointly* discriminate among priors and information structures; specifically, it evaluates each information structure using the prior that maximizes the likelihood of the observed signal.

A more conservative or intermediate approach can involve evaluating each information structure using *all* priors. That is, an agent may prefer considering an information structure that has a reasonable likelihood according to all priors than that with maximal likelihood according to a single prior and minimal likelihood according to another. We establish the axiomatic foundations for such a rule in Theorem 4.1.

Psychology research has consistently demonstrated that when updating information, individuals can exhibit susceptibility to certain biases. Many studies reveal a tendency to selectively consider information structures aligned with their pre-existing beliefs, a phenomenon commonly referred to as confirmation bias.[2] In our framework, given a signal, the probability that the signal originated in each state induces a likelihood ranking over states. If the agent can articulate a

[2]See Rabin and Schrag (1999) for a review of evidence in psychology.

likelihood ranking over states and is prone to confirmation bias, they may limit their consideration to information structures aligned with the agent's pre-existing likelihood ranking. GBU allows for such updating; its axiomatic foundations are described in Theorem 4.2.

Our analysis hinges on the assumption that imprecise information is objective. However, in numerous situations, information may be private or lack a reliable description. Then, assuming the observability of $\mathcal{L}$ is not appropriate. Further, its nature should be inferred from behavior. In Section 5, assuming a FBU rule, we provide necessary and sufficient conditions on the ex-ante and ex-post preferences for them to be related by a subjective set of information structures $\mathcal{L}$ in scenarios with a single prior belief or when $\mathcal{L}$ is a singleton.

The paper proceeds as follows: This introduction concludes with a literature review. Axioms and the implied updating rule are described in Sections 2.2 and 2.3, respectively. Section 3 contains a detailed discussion of the model properties. Special cases of the GBU rule are investigated in Section 4. Section 5 delves into the consideration of subjective information structures. Concluding remarks are presented in Section 6. All proofs are provided in Appendix A.

1.1 Related Literature

In recent years, there has been a growing interest in belief updating under ambiguity. Papers in this line of research may be classified into two categories. Some allow the payoff-relevant state space to be perceived ambiguous, but information is assumed precise. Some other papers further incorporates objective but imprecise information into their analyses.

Gul and Pesendorfer (2021) presents a theory of updating for the Choquet Expected Utility model (Schmeidler (1989)). Their updating rule is based on the requirement that a random variable that resolves gradually is evaluated by backwards induction. On the other hand, Suleymanov (2018), Cheng (2022), Tang (2022), and Kovach (2023) offer distinct axiomatic updating rules for the Maxmin Expected Utility model. The motivation for their work stems from the descriptive shortcomings of the Maximum Likelihood Updating and Full Bayesian updating rules, axiomatized by Gilboa and Schmeidler (1993) and Pacheco Pires (2002), respectively. In all these papers, information is considered payoff-relevant, and the information itself is precise in the sense that it is an event. In contrast, our work focuses on imprecise information that is not necessarily payoff-relevant.

Like us, Dominiak et al. (2021) model imprecise information as sets of probability measures that contain the "true" distribution. They provide an updating rule for the Subjective Expected Utility model that selects the posterior that minimizes the distance between her prior and the set she is provided with. Their updating rule has a somewhat similar flavor to ours due to the fact that they allow the distance to be subjective.

Jaffray (1989), Ahn (2008), Dumav and Stinchcombe (2013), Olszewski (2007), Gajdos et al. (2008), and Riedel et al. (2018) consider settings in which the objects of choice are "merged" with imprecise information. More specifically, Jaffray considers a preference over belief functions. Gajdos et al. (2008) and Riedel et al. (2018) assume the agent can choose the set of data-generating processes that contains the true law. The rest of the authors study preferences over sets of lotteries. All of these assumptions imply that imprecise information is payoff-relevant.

There are also papers particularly focusing on learning under imprecise information. Epstein and Schneider (2007,8,10) provide a non-axiomatic updating rule for imprecise information (therein referred to as ambiguous signals). They introduce a thought experiment that highlights its importance. In a dynamic setting, they provide conditions under which their updating rule delivers convergence in beliefs after repeated sampling. Reshidi et al. (2022) further investigate when beliefs converge under a more general data-generating process. Like Epstein and Schneider (2007), Lanzani (2023) also studies a learning problem. However, unlike Epstein and Schneider, he assumes robust control preferences. He shows how a miss-pecification concern can lead to different preferences under uncertainty arising in the limit.

In recent years, we also have experimental study on how individuals react to imprecise information. Building on Epstein and Schneider's thought experiment, Epstein and Halevy (2022) provides a definition of aversion towards signal ambiguity and tests it in an experimental setting. The key difference between our work and theirs is that they study the attitude towards the information as opposed to *how* to process it, which is our focus. Shishkin and Ortoleva (2023) tests if new information dilates the set of prior beliefs. Such a dilation is a significant implication of full Bayesian updating.

Finally, outside the updating literature but within the imprecise information literature, Wang et al. (2023) axiomatize a selection criterion for imprecise information. They provide conditions under which a choice function over "theories" always selects the ones that pass a likelihood ratio test.

2 General Model

2.1 Preliminary Definitions

We consider a set Ω of states of the world, set S of signals, and set X of consequences. We assume Ω and S are finite, and X is a compact and convex subset of a linear space. This is the case in Anscombe and Aumann (1963), where X is the set of all lotteries over some finite set of prizes. Let n denote the cardinality of Ω.

Each choice alternative is an action, or called *act*, that yields a state-dependent outcome. Formally, an act is a function $f : \Omega \to X$. The set of all acts is denoted by $\mathcal{F}$. We write x for the constant act f such that $f(\omega) = x$ for all $\omega \in \Omega$. Given $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, we use $\alpha f + (1 - \alpha)g$ to denote the act in $\mathcal{F}$ which equals $\alpha f(\omega) + (1 - \alpha)g(\omega)$ in state ω.

An *information structure* is a function $\ell : \Omega \to \Delta(S)$ where $\Delta(S)$ is the set of all probability measures over S. We only consider information structures that satisfy the following property:

$$\forall\ s \in S, \text{ there exists } \omega \in \Omega \text{ such that } \ell(s|\omega) > 0.$$

As shown later, this assumption is made purely for exposition purposes. An *imprecise information structure* is a finite set of such information structures. If the set only contains a single information structure, we refer to it as a *precise information structure*. Generic imprecise information structures are denoted by $\mathcal{L}$.

Let $\Delta(\Omega)$ denote the set of all probability measures on Ω. Given a probability measure $q \in \Delta(\Omega)$, an information structure ℓ, and a signal s, we use $BU(q, \ell, s)$ to denote the probability measure given by Bayesian updating:

$$BU(q, \ell, s)(\omega) = \frac{\ell(s|\omega)q(\omega)}{\sum_{\omega} \ell(s|\omega)q(\omega)}$$

whenever it exists. Furthermore, for any set of measures $\mathcal{M} \subseteq \Delta(\Omega)$, imprecise information structure $\mathcal{L}$, and signal $s \in S$, $BU(\mathcal{M}, \mathcal{L}, s)$ denotes the set of all posteriors generated by *point-wise Bayesian updating*:

$$BU(\mathcal{M}, \mathcal{L}, s) = \{BU(q, \ell, s) \mid q \in \mathcal{M}, \ell \in \mathcal{L}\}.$$

Finally, for any set $C \subseteq \Delta(\Omega)$, $ch(C)$ denotes its convex hull.

The following example illustrates how an experiment regarding signal ambiguity in the literature can fit within our framework.

Example 2.1. *Consider the following experimental setting in Epstein and Halevy (2022). Consider an urn, called a payoff urn, containing 100 balls. A fair coin was flipped; if the result was heads, the urn contains 75 red and 25 black balls. If the result was tails, the urn contains 25 red and 75 black. A decision maker must choose a color to bet on. They can sample a ball from a signal urn before making their decision. The signal urn can be either identical to the payoff urn, or constructed by adding 1000 extra red and black balls each into the payoff urn.*

This experimental setting can be formulated by our model as follows. The state of nature is the composition of the payoff urn: $\Omega = \{(75, 25), (25, 75)\}$. *A bet on* *a color is an act. For instance, betting 1 dollar on red yields a lottery that wins 1 dollar with probability 0.75 in the first state, and a lottery that wins 1 dollar with probability 0.25 in the second state. By sampling a ball from a signal urn, two possible signal realizations can happen: "red" or "black". Consider $S = \{R, B\}$. If the signal urn is identical to the payoff urn, the information structure, say ℓ_1, is such that $\ell_1(R|(75, 25)) = \frac{3}{4}$ and $\ell_1(R|(25, 75)) = \frac{1}{4}$. If the signal urn is constructed by adding extra balls in the payoff urn, the information structure, say ℓ_2, is such that $\ell_2(R|(75, 25)) = \frac{1075}{2100}$ and $\ell_2(R|(25, 75)) = \frac{1025}{2100}$. Since no additional clue about the signal's composition is available, the signal is imprecise and can be represented by $\mathcal{L} = \{\ell_1, \ell_2\}$.*

2.2 Axioms and Representation

Our primitive is a family of preferences over acts $(\succeq_0, (\succeq_s)_{s\in S})$ and an imprecise information structure $\mathcal{L}$. We impose two axioms on $(\succeq_0, (\succeq_s)_{s\in S})$, of which the first is that the preferences admit a MEU representation.

A utility function $U : \mathcal{F} \to \mathbb{R}$ is *MEU* if there exists an affine function $u : X \to \mathbb{R}$, and a closed and convex set of probability measures $\mathcal{M}$ over Ω such that

$$U(f) = \min_{q\in\mathcal{M}} \int_\Omega u(f)dq. \tag{1}$$

Consider that $(\mathcal{M}, u)$ represents $\succeq$ if the utility function given by (1) represents $\succeq$.

MEU Utility $\succeq_s$ admits representation by $(\mathcal{M}_s, u)$ for all $s \in S\cup\{0\}$. Moreover, each $q_0 \in \mathcal{M}_0$ has full support.

This axiom is not stated in terms of behavior, which is presumably the only observable. However, its behavioral foundations are widely known. Further, it imposes that all priors have full support; our model says nothing about updating zero-probability events.

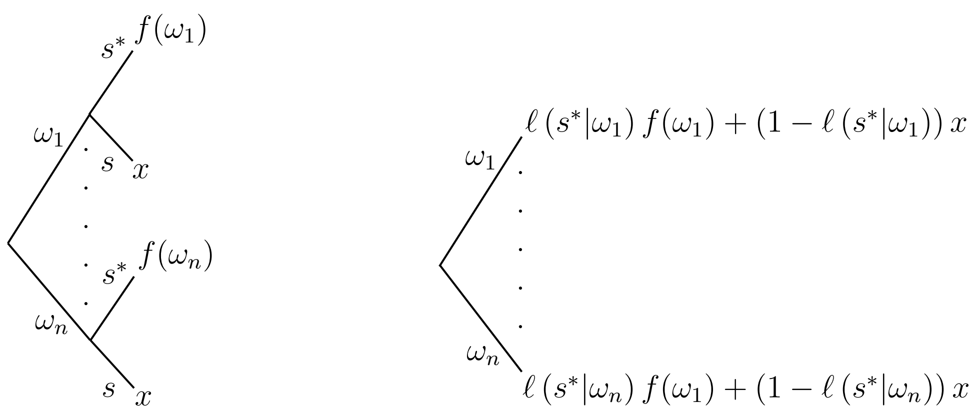
Our main axiom is based on reasoning in contingent planning. Suppose that the agent can either choose a constant act x whatsoever, or set a contingent plan where if a particular signal s^* is realized then switch to choose act f instead x. If all the identifiable information structures suggest the agent to take the plan, then she should actually choose f over x when s^* is realized. However, since the signals are not payoff-relevant, there seems no straightforward way to formulate this condition. We begin by demonstrating that our primitives are sufficiently rich to identify this condition if the agent satisfies *Reduction*: she is indifferent between a two-stage lottery and its equivalent one-stage counterpart.

Consider an act f and a constant act x, along with an information structure $\ell \in \mathcal{L}$. Further, extend the ex-ante preference to acts in which the signals are payoff-relevant: $\{F : \Omega \times S \to X\}$. Suppose the agent is told that ℓ is the true information structure, and is asked to choose between the constant act x and *signal act*:

$$F(\omega, s) = \begin{cases} f(\omega) & \text{if } s = s^* \\ x & \text{if } s \neq s^* \end{cases}.$$

Figure 1(a) visually represents this signal act. Notably, given a state of the world, there is no ambiguity about the probability of observing s–it is precisely $\ell(\cdot|\omega) \in \Delta(S)$. As both acts yield identical payoffs when ℓ generates $s \neq s^*$, the agent only needs to compare them under the assumption that s^* occurred. Consequently, they will prefer F to x if they believe that, conditional on observing s^*, they will prefer f to x.

For a given state ω, the signal act F yields the same expected payoffs as a lottery that pays $f(\omega)$ with probability $\ell(s^*|\omega)$ and x with probability $1-\ell(s^*|\omega)$. Therefore, under Reduction, the agent is indifferent between F and an act that pays $\ell(s^*|\omega)f(\omega) + (1-\ell(s^*|\omega))x$ for each state ω. Figure 1(b) illustrates such an act.



(a) Signal Act

(b) AA Act

Figure 1

Thus,

$$F \succeq_0 x \iff \begin{bmatrix} \ell(s^*|\omega_1)f(\omega_1) + (1-\ell(s^*|\omega_1))x \\ \vdots \\ \ell(s^*|\omega_n)f(\omega_n) + (1-\ell(s^*|\omega_n))x \end{bmatrix} \succeq_0 x$$

which means that we can identify when the agent expects preferring f over x after observing s if they knew ℓ is the true source of information.

To introduce our main axiom, we establish some notation. For each $f, x \in \mathcal{F}$, $\ell \in \mathcal{L}$, and $s \in S$, let $f_x^{\ell,s}$ denote the act that equals $\ell(s|\omega)f(\omega) + (1 - \ell(s|\omega))x$ in state ω.

Total Information Agreement For any $f, x \in \mathcal{F}$ and $s \in S$,

$$f_x^{\ell,s} \succeq_0 x \text{ for all } \ell \in \mathcal{L} \implies f \succeq_s x.$$

Given our discussion, the Total Information Agreement asserts that if, under all information structures, the act f is deemed better than the constant act x after observing signal s, the agent should prefer f over x after observing s.

2.3 Representation: Generalized Bayesian Updating

Our first result is an axiomatization of the GBU described in the introduction.

Theorem 2.1. *Let $(\succeq_0, (\succeq_s)_{s \in S})$ be a family of preferences over $\mathcal{F}$ that satisfies MEU utility and $\mathcal{L}$ an imprecise information structure. Then, $(\succeq_0, (\succeq_s)_{s \in S})$ satisfies the Total Information Agreement if and only if $\mathcal{M}_s \subseteq ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ for all $s \in S$.*

Theorem 2.1 only provides the conditions for the set of posteriors to be a subset of $ch(BU(\mathcal{M}_0, \mathcal{L}, s))$. We now provide the necessary axiom required to replace "$\subseteq$" with "$=$".

Default to Certainty For all $f, x \in \mathcal{F}$ and $s \in S$,

$$x \succeq_0 f_x^{\ell,s} \text{ for some } \ell \in \mathcal{L} \implies x \succeq_s f.$$

To interpret Default to Certainty, note that the Total Information Agreement imposes constraints on the agent's behavior in a highly specific scenario. It becomes relevant only when every conceivable source of information unanimously indicates a preference for an uncertain payoff act over a certain one. Nevertheless, this constraint will not have an impact when different information sources lead to divergent preferences based on the generated signal. For instance, an agent may favor an act f over a constant act x if they knew ℓ is the source of information; however, their preference may reverse if they knew ℓ' is the source. Default to Certainty asserts that in such situations where preferences depend on the source of information, the agent consistently opts for the constant act.[3][4]

[3] Faro and Lefort (2019) uses a similar axiom to characterize FBU in a precise information context.

[4] Observe that Total Information Agreement and Default to Certainty together imply that the ex-post preference is completely determined by the ex-ante preference: $f \succeq_s x$ if and only if $f_x^{\ell,s} \succeq_0 x$ for all $\ell \in \mathcal{L}$. This further implies that the ex-post preferences are forced to follow MEU even if we did not presume it.

Proposition 2.1. *Suppose* $(\succeq_0, (\succeq_s)_{s\in S})$ *and* $\mathcal{L}$ *satisfy the axioms of Theorem 2.1. Then,* $(\succeq_0, (\succeq_s)_{s\in S})$ *satisfies Default to Certainty if and only if* $\mathcal{M}_s = ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ *for all* $s \in S$*.*

3 Discussion

The relationship between Theorem 2.1 and the standard Bayesian model deserves special attention. The latter arises when we enhance MEU Utility by requiring that belief sets are singletons and assume $\mathcal{L}$ as a singleton. This adjustment has a significant implication for interpreting the Total Information Agreement. Specifically, under the SEU axioms, Bayesian updating is characterized by consequentialism and dynamic consistency (see Ghirardato (2002)). Consequentialism requires that when an event is learned, the value of an act does not depend on what it yields outside of the event. On the other hand, dynamic consistency requires that if two acts agree outside of an event, the preference between these acts do not change when this event is learned. Given that signals in our context are not payoff-relevant, Consequentialism cannot be directly assumed. Yet, it is implicitly used in our reasoning for Total Information Agreement. To see this, let F be the signal act described in the discussion surrounding Total Information Agreement. Note that $F \succsim_0 x$ implies $F \succsim_s x$ under the classic notion of dynamic consistency. Then we can argue by Consequentialism that $F \sim_s f$ since the signal act F agrees with f when s^* is realized. In fact, Total Information Agreement reduces to Dynamic Consistency plus Consequentialism when ex-ante and ex-post preferences adhere to SEU, and the realized signal is the occurence of an event.

Theorem 2.1 delivers a representation, and thus, an updating rule for imprecise information. Formally, *GBU* is a function $\rho : (\mathcal{M}, \mathcal{L}, s) \mapsto \mathcal{M}'$ such that $\mathcal{M}' \subseteq ch(BU(\mathcal{M}, \mathcal{L}, s))$.

Because the GBU rule imposes little structure on the set of posteriors for the conditional preferences, the model can accommodate diverse behavior. However, some may view the model as "too general" as it permits the posteriors to not be generated by the Bayesian updating of any feasible information source. Indeed, it may be the case that:

$$\rho(\mathcal{M}_0, \mathcal{L}, s) \subseteq ch(BU(\mathcal{M}_0, \mathcal{L}, s)) \setminus BU(\mathcal{M}_0, \mathcal{L}, s). \tag{2}$$

One reason to allow (2) is to nest the case in which the agent has possibly non-singleton beliefs over $\mathcal{L}$. To illustrate, consider Example 2.1. The prior belief is a singleton belief μ that is uniform. Suppose that the signal realization is "Red". Then, $BU(\mu, \ell_1, R)((75, 25)) = \frac{3}{4}$, and $BU(\mu, \ell_2, R)((75, 25)) = \frac{1075}{2100}$. If the agent subjectively believes that with probability λ, the true information structure is ℓ_1, the actual posterior belief is neither $\frac{3}{4}$ nor $\frac{1075}{2100}$, but $\lambda\frac{3}{4} + (1-\lambda)\frac{1075}{2100}$. In general, an agent may embrace a set Λ of probability measures over $\mathcal{L}$. Then, the agent's set of posteriors is given by following:

$$\rho(\mathcal{M}_0, \mathcal{L}, s) = \{p \in \Delta(\Omega) | p = \sum_{\ell \in \mathcal{L}} \lambda(\ell) BU(q, \ell, s), q \in \mathcal{M}_0, \lambda \in \Lambda\},$$

which may not contain $BU(q, \ell, s)$ for any $q \in \mathcal{M}_0$ and $\ell \in \mathcal{L}$.

Another potential concern is that the GBU allows the agent to use different ℓ's depending on the signal. If one follows a "maximum likelihood" type of rule, the ℓ's that have the maximal likelihood of generating s may differ than the ones generating s'. Similarly, an agent who only considers information structures that "confirm" their own beliefs may consider different information structures depending on the signal's realization. We further investigate these types of rules in Section 4. In the next section, we discuss the necessary conditions needed to rule out this feature of the model.

Next, we discuss the uniqueness properties of the model. Because we assumed that the preferences conform to the MEU model, $(\mathcal{M}_0, (\mathcal{M}_s)_{s \in S})$ are unique.[5] This does not mean that the information structures used to construct the posterior set are unique. In general, there is no hope for an identification result. As the following example shows, the Bayesian updating of a set of priors $\mathcal{M}_0$ and two different subsets of a given $\mathcal{L}$ may lead to the same set of posteriors.

Example 3.1. *Consider a binary state space $\Omega = \{\omega_1, \omega_2\}$ and binary signal space $S = \{s_1, s_2\}$. The prior belief q_0 is uniform, $q_0 = (1/2, 1/2)$. Let $\mathcal{M}_i$ be the set of posteriors given signal s_i. Since we only have two states, we can identify a belief by its assessment on ω_1. Suppose that $\mathcal{M}_1 = [5/8, 7/8]$ and $\mathcal{M}_2 = [1/8, 3/8]$.*

Consider the following two distinct sets of information structures:

$$\mathcal{L}_1 = \{\ell : \ell(s_1|\omega_1) \in [5/8, 7/8];\ \ell(s_1|\omega_2) = \ell(s_2|\omega_1)\},\ \textit{and}$$
$$\mathcal{L}_2 = \{\ell : k \in [5/8, 7/8];\ \ell(s_1|\omega_1) = 4k(1-k);\ \ell(s_1|\omega_2) = 4(1-k)^2\}.$$

Then, under FBU, these two sets of information structures both induce $\mathcal{M}_1$ and $\mathcal{M}_2$. That is, given the unconditional and conditional preferences, from which we can identify q_0, $\mathcal{M}_1$, and $\mathcal{M}_2$, we cannot distinguish if the set of information structures adopted by the agent is $\mathcal{L}_1$ or $\mathcal{L}_2$. Hence, if $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, there is no hope for identifying the set of information structures used.

We discuss the impossibility of providing a measure of aversion to updating imprecise information in the current framework. The main reason is that the current framework does not allow the agent to choose between facing imprecise and precise information. Indeed, any measure of aversion towards a phenomenon

[5] See the uniqueness result in Gilboa and Schmeidler (1989).

requires a comparison between the phenomenon and an object that does not suffer from the phenomenon.[6] Therefore, to provide such a measure, we would need a different setting. One possibility would be pairs of menus of acts and imprecise information structures. Although our analysis may be adapted to such a setting, as most of the experimental literature does not consider menus, we feel that it would obscure the message of the paper.

3.1 Consistent Updating

As we just discussed, our model allows the agent to use different information structures for different signals. The following axiom ensures that we avoid the case that ℓ plays a role after observing s' if it is ignored after observing s.

Consistency Across Signals For any $f, g, x, x' \in \mathcal{F}$ and $s, s' \in S$,

$$\text{if } x \succeq_0 f_x^{\ell,s} \text{ and } f \succeq_s x, \text{ then}$$
$$g_{x'}^{\ell,s'} \succeq_0 x' \text{ and } g \succeq_{s'} x' \implies g_{x'}^{\ell',s'} \succeq_0 x' \text{ for some } \ell' \in \mathcal{L},$$

where $g_{x'}^{\ell',s'} \succeq_0 x'$ holds with indifference (strictness) if $g \succeq_{s'} x'$ holds with indifference (strictness).

Recall that from the discussion of Total Information Agreement, $x \succeq_0 f_x^{\ell,s}$ and $f \succeq_s x$ reveal that the agent is ignoring the possibility that ℓ generated s. Therefore, any behavior that considering ℓ which may lead to after observing another signal s', such as $g_{x'}^{\ell,s'} \succeq_0 x$ and $g \succeq_s x'$, has to be rationalizable by another information structure ℓ' ($g_{x'}^{\ell',s'} \succeq_0 x'$).

To state the result, we need some notations. For any closed and convex set of priors $\mathcal{M}$, $\mathcal{E}(\mathcal{M})$ denotes all of its extreme points.

Proposition 3.1. *Suppose $(\succeq_0, (\succeq_s)_{s\in S})$ and $\mathcal{L}$ satisfy the axioms of Theorem 2.1. Then, $(\succeq_0, (\succeq_s)_{s\in S})$ satisfies Consistency Across Signals if and only if for all $s, s' \in S$ and $\ell \in \mathcal{L}$*

$$BU(q_0, \ell, s) \notin \mathcal{M}_s \ \forall\, q_0 \in \mathcal{M}_0 \implies BU(q_0', \ell, s') \notin \mathcal{E}(\mathcal{M}_{s'}) \ \forall\, q_0' \in \mathcal{M}_0.$$

Proposition 3.1 states that if ℓ is ignored after observing s, it cannot be an extreme point of $\mathcal{M}_{s'}$. Note that only the extreme points of $\mathcal{M}_{s'}$ matter for behaviors in the MEU model. Specifically,

$$\min_{q\in\mathcal{M}_s} \int_\Omega u(f)dq = \min_{q\in\mathcal{E}(\mathcal{M}_s)} \int_\Omega u(f)dq$$

for all f. Therefore, our axiom ensures that if the agent does not consider ℓ after s, they never consider it whenever it could affect their preferences.

[6] For example, to check if an agent is averse to ambiguity, one needs to observe a choice between acts and lotteries.

4 Specializations

Here, we discuss several special cases of the GBU. We separate the discussion into two classes motivated by statistics and behavioral biases.

4.1 Statistical GBU

From a statistical perspective, it is natural to only consider pairs (q, ℓ) that pass some statistical test. We model a test as a function $\phi : (\mathcal{M}, \mathcal{L}, s) \mapsto (\mathcal{M}', \mathcal{L}') \subseteq \mathcal{M} \times \mathcal{L}$. We refer to such rules as statistical GBU rules (SGBU):

$$\rho(\mathcal{M}_0, \mathcal{L}, s) = ch(BU(\phi(\mathcal{M}_0, \mathcal{L}, s), s)).$$

Clearly, FBU (3) and MLU (4) are special cases of SGBU:

$$\phi(\mathcal{M}_0, \mathcal{L}, s) = (\mathcal{M}_0, \mathcal{L}) \tag{3}$$

$$\phi(\mathcal{M}_0, \mathcal{L}, s) = \{(q_0, \ell) | (q_0, \ell) \in \arg \max_{(q_0, \ell) \in \mathcal{M}_0 \times \mathcal{L}} \int_\Omega \ell(s|\omega) dq_0\}. \tag{4}$$

In the introduction, we discussed how MLU uses the signals to *jointly* discriminate among priors and information structures. Each information structure is evaluated according to the prior which maximizes the likelihood of the observed signal. We propose the following conservative approach: evaluate each information structure using *all* priors. The following special case of SGBU, referred to as maximum robust likelihood updating (MRLU), formalizes this idea:

$$\phi_{MRLU}(\mathcal{M}_0, \mathcal{L}, s) = \{(q_0, \ell) | q_0 \in \mathcal{M}_0, \ell \in \arg \max_{\ell \in \mathcal{L}} \min_{q \in \mathcal{M}} \int_\Omega \ell(s|\omega) dq\}). \tag{5}$$

The axioms that characterize (5) are strengthenings of Total Information Agreement and Default to Certainty. They basically require the agent to only consider information structures they believe are the most likely to generate the observed signal. We now show how this can be identified in our framework if the agent satisfies Reduction.

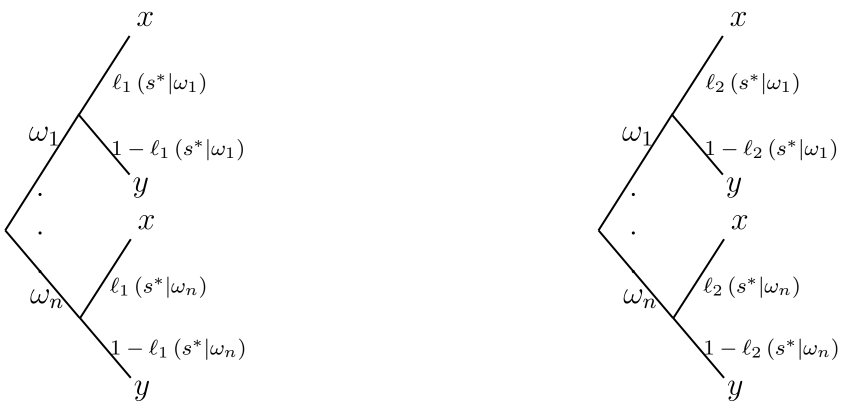
Consider a signal $s^* \in S$ and two constant acts x, y such that $x \succ y$. Consider an extension of $\succeq_0$ to pairs of signal acts and information structures (F, ℓ). The idea is that conditional on each state ω, $\ell(\cdot|\omega)$ describes the probability law on S.

Suppose we ask the agent to choose between (F, ℓ_1) and (F, ℓ_2), where

$$F(\omega, s) = \begin{cases} x & s = s^* \\ y & s \neq s^*. \end{cases}$$

Figures 2 (a) and 2 (b) illustrate (F, ℓ_1) and (F, ℓ_2), respectively. The agent will choose (F, ℓ_2) over (F, ℓ_1) if and only if they think ℓ_2 is more likely to generate

s^* than ℓ_1. Further, by an identical argument to the one in the motivation of Total Information Agreement, $(F, \ell_i) \sim_0 x_y^{\ell_i, s^*}$ for $i = 1, 2$. Hence, for a given x, y, we can identify which information structures the agent believes are more likely to generate a given signal.



(a) Signal Act 1 (b) Signal Act 2

Figure 2

To state the next axiom, we need some notation. Let $\geq_{x,y}^s$ be the binary relation over $\mathcal{L}$ such that $\ell \geq_{x,y}^s \ell'$ if $x_y^{\ell,s} \succeq_0 x_y^{\ell',s}$, and let

$$\arg\max(\mathcal{L}, \geq_{x,y}^s) = \{\ell \in \mathcal{L} | \ell \geq_{x,y}^s \ell' \text{ for all } \ell' \in \mathcal{L}\}.$$

Likelihood Information Agreement For any $x, y, z, f \in \mathcal{F}$ such that $x \succ_0 y$ and $s \in S$,

$$f_z^{\ell,s} \succeq_0 z \text{ for all } \ell \in \arg\max(\mathcal{L}, \geq_{x,y}^s) \implies f \succeq_s z.$$

The likelihood information agreement (LIA) guarantees that the set of posteriors is a subset of the set of the convex hull of the set of posteriors generated by the point-wise Bayesian updating of (5). The following axiom strengthens set contention to equality.

Likelihood Default to Certainty For any $x, y, z, f \in \mathcal{F}$ such that $x \succ_0 y$ and $s \in S$,

$$z \succeq_0 f_z^{\ell,s} \text{ for some } \ell \in \arg\max(\mathcal{L}, \geq_{x,y}^s) \implies z \succeq_s f.$$

The interpretation of both axioms is analogous to the one of Total Information Agreement and Default to Certainty.

Theorem 4.1. *Let $(\succeq_0, (\succeq_s)_{s\in S})$ be a family of preferences over $\mathcal{F}$ that satisfies MEU utility and $\mathcal{L}$ an imprecise information structure. Then, $(\succeq_0, (\succeq_s)_{s\in S})$ satisfies LIA if and only if*

$$\mathcal{M}_s \subseteq ch(BU(\phi_{MRLU}(\mathcal{M}_0, \mathcal{L}, s), s)) \tag{6}$$

for all $s \in S$. Moreover, the set contention in (6) is replaced with equality if and only if $(\succeq_0, (\succeq_s)_{s\in S})$ also satisfies the Likelihood Default to Certainty.

Note that for the case in which the agent holds a single prior q_0, Theorem 4.1 delivers a characterization of a version of the MLU rule. Specifically, the agent will only consider the information structures $\ell \in \mathcal{L}$ such that:

$$\ell \in \arg\max_{\ell' \in \mathcal{L}} \int_\Omega \ell(s|\omega) dq_0.$$

Hence, our result can be viewed as an imprecise information counterpart of the Gilboa and Schmeidler (1993) result for precise information.

4.2 Behavioral GBU

One of the more robust findings in the empirical literature on updating is that people tend to only update information that confirms their prior beliefs. This phenomenon is called confirmatory bias (Rabin and Schrag (1999)). Here, we argue that our model is well-suited to accommodate it. Because confirmatory bias and state space ambiguity do not share a conceptual link, we focus on the case in which $\mathcal{M}_0 = \{q_0\}$.

When the agent's ex-ante prior beliefs can be described by a single probability distribution, we can recover the agent's probabilistic ranking among the states of the world. Indeed, for any $x, y \in \mathcal{F}$ such that $x \succ_0 y$, one can recover the ranking by observing the agent's preferences over the following acts:

$$f_\omega(\omega') = \begin{cases} x & \omega' = \omega \\ y & \omega' \neq \omega \end{cases}.$$

Let $\succeq^*$ be the binary relation over Ω induced by the ranking.

Intuitively, we can use $\succeq^*$ to define whether an information structure ℓ provides information consistent with the agents prior: For a given signal s, $\ell(s|\omega)$ is the likelihood that it was generated by state ω. Hence, the information will confirm the agent's ex-ante beliefs if the order among the $\ell(s|\omega)$'s is the same as their ranking among states. Formally, we say that $\ell(s|\cdot)$ *s-confirms* $\succeq^*$ if

$$\omega \succeq^* \omega' \iff \ell(s|\omega) \geq \ell(s|\omega').$$

An agent who suffers from confirmatory bias will only consider information that confirms their beliefs. The following axiom captures this.

Confirmatory Information Agreement For any $f, x \in \mathcal{F}$,

$$f_x^{\ell,s} \succeq_0 x \text{ for all } \ell \in \mathcal{L} \text{ that } s\text{-confirms } \succeq^* \implies f \succeq_s x.$$

The Confirmatory Information Agreement guarantees that the set of posteriors is a subset of the convex hull of the posteriors generated by point-wise Bayesian updating of the information structures that confirm the agent's beliefs (whenever it is non-empty). The following axiom strengthens set contention to equality.

Confirmatory Default to Certainty For any $f, x \in \mathcal{F}$ such that $x \succ_0 y$ and $s \in S$,

$$x \succeq_0 f_x^{\ell,s} \text{ for some } \ell \in \mathcal{L} \text{ that } s\text{-confirms } \succeq^* \implies x \succeq_s f.$$

Theorem 4.2. *Let $(\succeq_0, (\succeq_s)_{s\in S})$ be a family of preferences over $\mathcal{F}$ that satisfies MEU utility and $\mathcal{L}$ an imprecise information structure. Assume $\mathcal{M}_0$ is a singleton; for each $s \in S$, there exists some $\ell \in \mathcal{L}$ that s-confirms $\succeq^*$. Then, $(\succeq_0, (\succeq_s)_{s\in S})$ satisfies the Confirmatory Information Agreement if and only if*

$$\mathcal{M}_s \subseteq ch(BU(\mathcal{M}_0, \{\ell \in \mathcal{L} | \ell \ s\text{-confirms } \succeq^*\}, s)) \tag{7}$$

for all $s \in S$. Moreover, the set contention in (7) is replaced with equality if and only if $(\succeq_0, (\succeq_s)_{s\in S})$ also satisfies the Confirmatory Default to Certainty.

5 Subjective Information Structures

We have assumed that $\mathcal{L}$ is both observable and objective. This is not always an appropriate assumption. For example, an agent may receive a signal without a clear description of its content and may construct $\mathcal{L}$ by themself. In such cases, $\mathcal{L}$ becomes purely subjective and can only be inferred from observable behavior. Here, we study when the ex-ante and ex-post preferences are consistent with the existence of a *subjective* $\mathcal{L}$.

As in the previous analysis, we consider a family of preferences over acts $(\succeq_0, (\succeq_s)_{s\in S})$ that admit a MEU representation. We establish the necessary and sufficient conditions on behavior for these preferences to be linked by the FBU of a subjective $\mathcal{L}$ under two specific scenarios: (i) no ex-ante state ambiguity, and (ii) no signal imprecision. The general case poses a significant challenge and is left for future research.

To introduce our axioms, we need some preliminaries. For any MEU preference $\succeq$ represented by $(\mathcal{M}, u)$, let $\bar{\succeq}$ denote the preference over acts represented by

$$\max_{q\in\mathcal{M}} \int_\Omega u(f)dq.$$

Observe that for any acts f, g and constant act x satisfying $x = \alpha f + (1-\alpha)g$ for some $\alpha \in (0,1)$, we have

$$f \bar{\succeq} x \text{ if and only if } x \succeq g.$$

Thus, $\bar{\succeq}$ can be understood as a conjugate of $\succeq$ and can be recovered from $\succeq$.

5.1 No ex-ante state ambiguity

First, we consider the situation where the state space is ex-ante unambiguous, meaning that the agent possesses a single prior belief, and thus, the ex-ante preference $\succeq_0$ is SEU with belief q_0.

We begin by considering the case in which the ex-post preference $\succeq_s$ are all SEU with belief p_s. A necessary and sufficient condition for the existence of a likelihood function ℓ such $p_s = BU(q_0, \ell, s)$ for all $s \in S$ is that q_0 lies in the convex hull of $\{p_s | s \in S\}$. This property is equivalent to an adaptation of dynamic consistency to our setting: For any act f and constant act x, if $f \succeq_s x$ for all $s \in S$, then $f \succeq_0 x$. Intuitively, if f is preferred to x under any signal realization, then f is also preferred to x ex-ante.

In scenarios where ex-post preferences can be MEU, the above dominance property is insufficient to determine the existence of $\mathcal{L}$ such that $\mathcal{M}_s = BU(q_0, \mathcal{L}, s)$. Our main finding in this section demonstrates that the following stronger dominance property is a necessary and sufficient condition for this relationship.

P1 For any $f, x \in \mathcal{F}$,

$$f \bar{\succeq}_s x \text{ for some } s \in S \text{ and } f \succeq_{s'} x \text{ for all } s' \neq s \implies f \succeq_0 x.$$

If, $f \bar{\succ}_s x$ or $f \succ_{s'} x$ as well for some $s' \neq s$, then $f \succ_0 x$.

To see why P1 is stronger, observe that $f \succeq x$ implies $f \bar{\succeq} x$, but the converse fails. However, under SEU, $\succeq = \bar{\succeq}$ makes P1 equivalent to dynamic consistency.

Proposition 5.1. *Let $(\succeq_0, (\succeq_s)_{s \in S})$ be a family of preferences over $\mathcal{F}$. Assume (u, q_0) represents $\succeq_0$ and $(u, \mathcal{M}_s)$ represents $\succeq_s$ for all $s \in S$. Then, $(\succeq_0, (\succeq_s)_{s \in S})$ satisfies P1 if and only if there exists $\mathcal{L}$ such that $\mathcal{M}_s = BU(q_0, \mathcal{L}, s)$ for all s.*

To aid the intuition for the result, we describe the necessity of P1. Given $\succeq$, $f \bar{\succeq}_s x$ requires the existence of a belief p_s in $\mathcal{M}_s$ under which the utility of f is no less than the utility of x. Under FBU, $p_s = BU(q_0, \ell, s)$ for some $\ell \in \mathcal{L}$. Moreover, for any other signal s', the Bayesian posterior of q_0 and ℓ, say $p_{s'}$, also lies in $\mathcal{M}_{s'}$. Thus, $f \succeq_{s'} x$ implies that the utility of f under $p_{s'}$ is no less than the utility of x. Since the prior q_0 must lie in the convex hull of the posteriors $\{p_s | s \in S\}$, we obtain $f \succeq_0 x$.

5.2 No signal imprecision

We now consider the case in which the ex-ante and ex-post preferences are MEU, and focus on the existence of a precise information structure.

To state our axioms, we introduce some notation. For any $x, y, z \in X$ and states ω, ω', let $[x, \omega | y, \omega' | z]$ denote the act that yields x in state ω, y in state ω', and z in any other state.

P2 For any $f \in \mathcal{F}$, state ω and outcomes$(y^0_{\omega'})_{\omega' \in \Omega \setminus \{\omega\}}, (y^s_{\omega'})_{\omega' \in \Omega \setminus \{\omega\}}$ such that $f(\omega) = \sum_{\omega' \neq \omega} \frac{y^s_{\omega'}}{|\Omega|-1} = \sum_{\omega' \neq \omega} \frac{y^0_{\omega'}}{|\Omega|-1}$ for all $s \in S$,

$$\begin{cases} x \succeq_s [f(\omega'), \omega' | y^s_{\omega'}, \omega | x]; \\ x \bar{\succeq}_s [f(\omega'), \omega' | y^s_{\omega'}, \omega | x] \text{ if } f(\omega') \succeq_s x \end{cases} \quad \text{for all } s \in S \text{ and } \omega' \neq \omega$$

$$\Longrightarrow \begin{cases} x \succeq_0 [f(\omega''), \omega'' | y^s_{\omega''}, \omega | x]; \\ x \bar{\succeq}_0 [f(\omega''), \omega' | y^s_{\omega''}, \omega | x] \text{ if } f(\omega'') \succeq_0 x \end{cases} \quad \text{for some } \omega''.$$

To understand P2, consider utility acts (i.e., $X \subset \mathbb{R}$ and $u(x) = x$) for simplicity. Suppose that $|\Omega| = 3$. Thus, an act is an element of $\mathbb{R}^3$. Suppose that, for instance, we have for all $s \in S$,

$$x \succeq_s (x - a_1, x, x + b^s_1) \text{ and } x \succeq_s (x, x - a_2, x + b^s_2),$$

where $a_1, a_2 > 0$ and $b^s_1 + b^s_2 = k$ for all s. We can interpret these rankings as follows. Given a constant act x, we lower its payoff in ω_1 by a_1, and we increase its payoff in ω_3 by b^s_1 as compensation. However, the compensation is not large enough to fully compensate for the loss in ω_1. Thus, the first ranking follows. Similarly, if we lower the payoff in ω_2 by a_2, we increase the payoff in ω_3 by b^s_2 as compensation. The second-ranking suggests that this compensation is not large enough. As $b^s_1 + b^s_2 = k$ for all s, we can say that k is too small as a total stake for compensation for any signal realization. Then, P2 says that k is also too small from the ex-ante perspective. We can never split k into b^0_1 and b^0_2 such that

$$(x - a_1, x, x + b^0_1) \succeq_0 x \text{ and } (x, x - a_2, x + b^0_2) \succeq_0 x.$$

In terms of the MEU model, P2 captures the following "convex hull" implication of a single subjective information structure. Consider a state ω. As we show in the proof, if $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ for all $s \in S$, then for all $\omega' \neq \omega$,

$$\sum_s \ell(s|\omega) \max_{p \in \mathcal{M}_s} \frac{p(\omega')}{p(\omega)} = \sum_s \ell(s|\omega) \times \frac{\ell(s|\omega')}{\ell(s|\omega)} \times \max_{q \in \mathcal{M}_0} \frac{q(\omega')}{q(\omega)} = \max_{q \in \mathcal{M}_0} \frac{q(\omega')}{q(\omega)}.$$

This means that the vector $(\max_{q\in\mathcal{M}_0}\frac{q(\omega')}{q(\omega)})_{\omega'\neq\omega}$ lies in the convex hull of the set $\{(\max_{p\in\mathcal{M}_{s_1}}\frac{p(\omega')}{p(\omega)})_{\omega'\neq\omega},\dots,\max_{p\in\mathcal{M}_{s_{|S|}}}\frac{p(\omega')}{p(\omega)})_{\omega'\neq\omega}\}$. An implication is that for all reals k and $(r_{\omega'})_{\omega'\neq\omega}$,

$$\left[\sum_{\omega'\neq\omega} r_{\omega'}\max_{p\in\mathcal{M}_s}\frac{p(\omega')}{p(\omega)}\geq k\ \forall\ s\in S\right]\Rightarrow\sum_{\omega'\neq\omega} r_{\omega'}\times\max_{q\in\mathcal{M}_0}\frac{q(\omega')}{q(\omega)}\geq k.$$

P2 follows from this dominance property. Specifically, the necessity of P2 follows from the fact that for any MEU preference $\succeq$ represented by $(\mathcal{M},u)$,

$$\begin{cases} x\succeq[f(\omega'),\omega'|y_{\omega'},\omega|x]; \\ x\bar{\succeq}[f(\omega'),\omega'|y_j,\omega|x] \text{ if } f(\omega')\succeq x \end{cases}$$

if and only if

$$[u(x)-u(f(\omega'))]\max_{p\in\mathcal{M}}\frac{p(\omega')}{p(\omega)}\geq u(y_{\omega'})-u(x).$$

Hence, P2 allows us to find an information structure ℓ such that

$$\max_{p\in\mathcal{M}_s}\frac{p(\omega')}{p(\omega)}=\frac{\ell(s|\omega')}{\ell(s|\omega)}\times\max_{q\in\mathcal{M}_0}\frac{q(\omega')}{q(\omega)}$$

for all ω,ω' and $s\in S$. Yet, this is not enough to establish $\mathcal{M}_s=BU(\mathcal{M}_0,\ell,s)$. By Proposition 2.1, we need that $f_x^{\ell,s}\succeq_0 x\Leftrightarrow f\succeq_s x$. This implication is captured by the following axiom.

P3 For any state ω, acts f,g,x such that $f(\omega)=g(\omega)$, and outcomes $(y_{\omega'})_{\omega'\in\Omega\setminus\{\omega\}}$, the following two statements are true:

(i) If for all $\omega'\neq\omega$,

$$\begin{cases} [y_{\omega'},\omega'|g(\omega'),\omega|x]\bar{\succeq}_s x; \\ [y_{\omega'},\omega'|g(\omega'),\omega|x]\succeq_s x \text{ if } x\succeq_s y_{\omega'} \end{cases} \quad\text{and}\quad \begin{cases} x\succeq_0[y_{\omega'},\omega'|f(\omega'),\omega|x]; \\ x\bar{\succeq}_0[y_{\omega'},\omega'|f(\omega'),\omega|x] \text{ if } y_{\omega'}\succeq_0 x \end{cases}$$

then, $x\succeq_0 g$ implies $x\succeq_s f$.

(ii) If for all $\omega'\neq\omega$,

$$\begin{cases} x\succeq_s[y_{\omega'},\omega'|g(\omega'),\omega|x]; \\ x\bar{\succeq}_s[y_{\omega'},\omega'|g(\omega'),\omega|x] \text{ if } y_{\omega'}\succeq_s x \end{cases} \quad\text{and}\quad \begin{cases} [y_{\omega'},\omega'|f(\omega'),\omega|x]\bar{\succeq}_0 x; \\ [y_{\omega'},\omega'|f(\omega'),\omega|x]\succeq_0 x \text{ if } x\succeq_0 y_{\omega'} \end{cases}$$

then, $g\succeq_0 x$ implies $f\succeq_s x$.

To illustrate P3, consider utility acts and assume $|\Omega| = 3$. We set $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ where $f_3 = g_3$. Suppose that

$$(a_1, x, g_1) \succeq_s x \text{ and } (x, a_2, g_2) \succeq_s x$$

where $x > a_1$ and $x > a_2$. The former suggests that g_1, under signal s, is good enough in the following sense: if we lower the payoff of a constant act x in ω_1 to a_1 and replace its payoff in ω_3 by g_1 as a compensation, we improve the act. The latter ranking also suggests that g_2 is good enough under signal realization s in a similar sense. Suppose also that

$$x \succeq_0 (a_1, x, f_1) \text{ and } x \succeq_0 (x, a_2, f_2).$$

Again, they suggest that f_1 and f_2 are not good enough from ex-ante perspective. Now P3 requires that we cannot conversely have g worse than x ex-ante but f better than x ex-post.

The existence of a precise information structure under MEU preferences and full-Bayesian updating is characterized by P2 and P3, as shown in the following proposition.

Proposition 5.2. *Let $(\succeq_0, (\succeq_s)_{s \in S})$ be a family of preferences over $\mathcal{F}$ that satisfies MEU Utility. Then, $(\succeq_0, (\succeq_s)_{s \in \mathcal{S}})$ satisfies P2 and P3 if and only if there exists ℓ such that $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ for all $s \in \mathcal{S}$.*

This proposition illustrates the challenge in identifying subjective information from $(\succeq_0, (\succeq_s) s \in S)$. Indeed, P2 and P3 are not straightforward axioms. Each axiom governs, in terms of the ex-ante preferences, the extent to which an agent is willing to transfer utility across states for different signal realizations. The more general case where $\mathcal{L}$ is allowed to not be a singleton also requires an understanding of how discipline the transfers. Imposing these regularities is particularly challenging due to the non-payoff-relevance of information. Indeed, taking as primitive an ex-ante preference $\succeq_0$ over signal acts $F : \Omega \times S \to X$ can simplify the analysis. In this framework, $\succeq_s$ can be the preference induced by $\succeq_0$ over acts that yield the same payoff whenever s is not realized. However, such complexities cannot be avoided while attempting to understand the behavioral implications of imprecise information in non-payoff-relevant contexts. Moreover, our analysis suggests that there is no simple way to capture the trade-offs.

6 Concluding Remarks

We posit a theory of updating under imprecise information that generalizes FBU and MLU. Although both these rules are widely popular, a conceptual reason to adopt either is missing.[7]

The ability to accommodate different attitudes towards updating is particularly significant in certain applications. For instance, Beauchêne et al. (2019) shows that under FBU, a sender can extract the full surplus from a receiver in a Bayesian persuasion style game.[8] However, once we allow for a more general updating rule, such a result may not hold. This opens the door for a richer theory of persuasion under imprecise information.

Our framework was inspired by the experimental literature, where observations are typically limited to ex-ante and conditional on signal preferences. Here, signals are often considered to be payoff irrelevant and information structures are commonly employed. We hope that the constructive nature of our axioms offers some guidance on how to test behavior under imprecise information.

Lastly, our analysis has been normative, with the Total Information Agreement presented as an attractive property that delivers a generalization of Bayesian updating. Consequently, it serves as a test for any updating theory that challenges Bayesianism within the MEU framework. Essentially, any updating rule that contravenes the Total Information Agreement must be supported by an example demonstrating its unreasonableness. We view this aspect of the paper as a separate contribution as it can offer valuable guidance for future research.

[7] One can draw a parallel with the three-color Ellsberg paradox. As formalized by Gajdos et al. (2008), the agent has no reason to consider *all* the objectively possible probabilistic beliefs (priors). The same intuition applies to our setting: the agent has no reason to consider all objectively possible information structures.

[8] An equally striking result holds in a cheap talk game with FBU. See Kellner and Le Quement (2018).

Appendix A Proofs

The necessity of the axioms is obvious in each of our representation theorems. Therefore, we only prove sufficiency.

Throughout, it is assumed that $\succeq_s$ is represented by $(u, \mathcal{M}_s)$ for all $s \in S \cup \{0\}$. Finally, each proof of sufficiency makes use of the following lemma.

Lemma A.1. *Assume $\succeq$ and $\succeq'$ admit MEU representations $(u, \mathcal{M})$, $(u, \mathcal{M}')$ respectively such that $\mathcal{M} \not\subseteq \mathcal{M}'$. Then there exists an act f and a constant act x such that $f \sim x$ and $f \succ' x$.*

The proof follows from an identical argument from the uniqueness result in Gilboa and Schmeidler (1989).

A.1 Proof of Theorem 2.1

Let $(\mathcal{M}_s)_{s \in S \cup \{0\}}$ be the sets of probability measures described by MEU Utility. Suppose $(\succeq_s)_{s \in S \cup \{0\}}$ satisfies Total Information Agreement (TIA) and assume by way of contradiction that $\mathcal{M}_s \not\subseteq ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ for some $s \in S$.

By Lemma A.1, there exists an act f and a constant act x such that

$$u(x) < \int_\Omega u(f) dBU(q', \ell, s) \text{ for all } (q', \ell) \in \mathcal{M}_0 \times \mathcal{L}.$$

Fix $(q, \ell) \in \mathcal{M}_0 \times \mathcal{L}$, then

$$\begin{aligned}
u(x) &< \int_\Omega u(f) dBU(q, \ell, s) \\
\left(\int_\Omega \ell(s|\omega) dq\right) u(x) &< \int_\Omega u(f) \ell(s|\omega) dq \\
\left(\int_\Omega \ell(s|\omega) dq\right) u(x) + \left(1 - \int_\Omega \ell(s|\omega) dq\right) u(x) &< \int_\Omega u(f) \ell(s|\omega) dq + \left(1 - \int_\Omega \ell(s|\omega) dq\right) u(x) \\
u(x) &< \int_\Omega u(\ell(s|\omega) f(\omega) + (1 - \ell(s|\omega)) x) dq.
\end{aligned}$$

Thus, $u(x) < \int_\Omega u(\ell(s|\omega) f(\omega) + (1 - \ell(s|\omega) x)) dq$ for all $q \in \mathcal{M}_0$ and $\ell \in \mathcal{L}$. Therefore, $u(x) < \min_{q \in \mathcal{M}_0} \int_\Omega u(\ell(s|\omega) f(\omega) + (1 - \ell(s|\omega) x)) dq$ for all $\ell \in \mathcal{L}$. Hence, $f_x^{\ell, s} \succ_0 x$ for all $\ell \in \mathcal{L}$ and $x \succeq_s f$, a contradiction of TIA.

A.2 Proof of Proposition 2.1

Given Theorem 2.1, we only need to show that if $(\succeq, (\succeq_s)_{s \in S})$ also satisfies Default to Certainty (DTC), then $ch(BU(\mathcal{M}_0, \mathcal{L}, s)) \subseteq \mathcal{M}_s$.

Fix $s \in S$ and assume by way of contradiction there exists $p^* \in ch(BU(\mathcal{M}_0, \mathcal{L}, s))$ such that $p^* \notin \mathcal{M}_s$. By Lemma A.1, there exists f and x such that $\int_\Omega u(f)p^* < \min_{q' \in \mathcal{M}_s} \int_\Omega u(f)dq' = u(x)$.

Since $p^* \in ch(BU(\mathcal{M}_0, \mathcal{L}, s))$, then there exist $(q_1, \ell_1), ..., (q_n, \ell_n) \in \mathcal{M}_0 \times \mathcal{L}$ such that $p^* = \sum_i \alpha_i BU(q_i, \ell_i, s)$ and

$$\int_\Omega u(f)p^* = \sum_i \alpha_i \int_\Omega u(f)dBU(q_i, \ell_i, s).$$

Hence, there exists i such that $\int_\Omega u(f)dBU(q_i, \ell_i, s) \leq \int_\Omega u(f)dp^*$. Let $q = q_i$ and $\ell = \ell_i$. Then,

$$\int_\Omega u(f)dBU(q, \ell, s) < u(x)$$

$$\int_\Omega u(f)\ell(s|\omega)dq + \left(1 - \int_\Omega \ell(s|\omega)dq\right)u(x) < u(x)\left(\int_\Omega \ell(s|\omega)dq\right) + \left(1 - \int_\Omega \ell(s|\omega)dq\right)u(x)$$

$$\int_\Omega u(f(\omega)\ell(s|\omega) + (1 - \ell(s|\omega))x)dq < u(x).$$

Hence, $\min_{q \in \mathcal{M}_0} \int_\Omega u(f(\omega)\ell(s|\omega) + (1 - \ell(s|\omega))x)dq < u(x)$. This implies that there exists $\ell \in \mathcal{L}$ such that $x \succ_0 f_x^{\ell,s}$. Thus, by DTC, $x \succ_s f$, a contradiction.

A.3 Proof of Proposition 3.1

By assumption, $BU(\mathcal{M}_0, \ell, s) \cap \mathcal{M}_s = \emptyset$. Thus, by a hyperplane separating argument, there exists f such that

$$\int_\Omega u(f)dq_s < c < \int_\Omega u(f)dBU(q_0, \ell, s)$$

for all $q_0 \in \mathcal{M}_0$ and $q_s \in \mathcal{M}_s$. Let x be such that $x \sim_s f$. Then $f_x^{l,s} \succ_0 x$ and $x \succeq_s f$.

Next, assume that there exists $\ell \in \mathcal{L}$, $s' \in S$ and $q_0' \in \mathcal{M}_0$ such that $BU(q_0', \ell, s') \in \mathcal{E}(\mathcal{M}_{s'})$. Observe that $BU(q_0', \ell, s') \in \mathcal{E}(\mathcal{M}_{s'})$ implies there exists an act g such that

$$\min_{q \in \mathcal{M}_{s'}} \int_\Omega u(g)dq = \int_\Omega u(g)dBU(q_0', \ell, s') < \int_\Omega u(g)dq \text{ for all } q \in \mathcal{M}_{s'} \backslash \{BU(q_0', \ell, s')\}.$$

Let y be the constant act such that $u(y) = \int_\Omega u(g)dBU(q_0', \ell, s')$. Then, $g_x^{\ell,s} \sim_0 x$ and $g \sim_s x$. Moreover, $g_x^{\ell',s} \succ_0$ for all $\ell' \in \mathcal{L}$. Hence, for Consistency Across Signals to hold, either $BU(\mathcal{M}_0, \ell, s) \cap \mathcal{M}_s \neq \emptyset$ or $BU(q_0', \ell, s') \notin \mathcal{E}(\mathcal{M}_{s'})$ for all q_0'.

A.4 Proof of Theorem 4.1

Suppose $(\succeq_0, (\succeq_s)_{s\in S})$ satisfies Likelihood Information Agreement (LIA) and assume by way of contradiction that

$$\mathcal{M}_s \not\subseteq ch(\{BU(q,\ell,s)|q\in\mathcal{M}_0, \ell\in\arg\max_{\ell\in\mathcal{L}}\max_{q\in\mathcal{M}_0}\int_\Omega \ell(s|\omega)dq\})$$

for some $s\in S$.

Let $\mathcal{L}_s^{**} = \arg\max_{\ell\in\mathcal{L}}\min_{q\in\mathcal{M}_0}\int_\Omega \ell(s|\omega)dq$. By Lemma A.1, there exists f and x such that

$$u(x) = \min_{q\in ch(BU(\mathcal{M}_0,\mathcal{L}_s^{**},s))}\int_\Omega u(f)dq \le \min_{(q,\ell)\in\mathcal{M}_0\times\mathcal{L}_s^{**}}\int_\Omega u(f)dBU(q,\ell,s)$$

Hence, for all $q\in\mathcal{M}_0$ and $\ell\in\mathcal{L}_s^{**}$

$$u(x) < \int_\Omega u(f)dBU(q,\ell,s)$$
$$u(x) < \int_\Omega u(f)\ell(s|\omega)dq + u(p)(1-\int_\Omega \ell(s|\omega)dq).$$

Thus,

$$u(x) < \min_{q\in\mathcal{M}_0}\int_\Omega u(f)\ell(s|\omega)dq + u(x)(1-\int_\Omega \ell(s|\omega)dq).$$

If we can show that $\ell\in\arg\max(\mathcal{L},\geq^s_{x,y})$ implies $\ell\in\mathcal{L}_s^{**}$ for any x,y such that $u(x)>u(y)$, we will have a contradiction of LIA. To see that $\ell\in\max(\mathcal{L},\geq^s_{x,y})$ implies $\ell\in\mathcal{L}_s^{**}$ note that $\ell\geq^s_{x,y}\ell'$ if and only if

$$\min_{q\in\mathcal{M}_0}[u(x)\int_\Omega \ell(s|\omega)dq + u(y)(1-\int_\Omega \ell(s|\omega)dq)] \ge \min_{q\in\mathcal{M}_0}[u(x)\int_\Omega \ell'(s|\omega)dq + u(y)(1-\int_\Omega \ell'(s|\omega)dq)]$$

which holds if and only if

$$\min_{q\in\mathcal{M}_0}\int_\Omega \ell(s|\omega)dq \ge \min_{q\in\mathcal{M}_0}\int_\Omega \ell'(s|\omega)dq.$$

Hence, $\ell\in\max(\mathcal{L},\geq^s_{x,y})$ implies $\ell\in\arg\max_{\ell\in\mathcal{L}}\min_{q\in\mathcal{M}_0}\int_\Omega \ell(s|\omega)dq = \mathcal{L}_s^{**}$.

Next, we prove that Likelihood Default to Certainty (LDC) implies equality of the sets described in the Theorem.

Let $ch(\{BU(q,\ell,s)|\ell\in\arg\max_{\ell\in\mathcal{L}}\min_{q\in\mathcal{M}_0}\int_\Omega \ell(s|\omega)dq \text{ and } q\in\mathcal{M}_0\})\equiv\mathcal{M}_s^{MM}$. We only need to show that $\mathcal{M}_s^{MM}\subseteq\mathcal{M}_s$.

Assume by way of contradiction that $\mathcal{M}_s^{MM} \not\subseteq \mathcal{M}_s$. Then, by Lemma A.1, there exists f and x such that

$$\min_{q\in\mathcal{M}_s^{MM}} \int_\Omega u(f)dq < \min_{q\in\mathcal{M}_s} \int_\Omega u(f)dq = u(x).$$

Fix $q' \in \arg\min_{q\in\mathcal{M}_s^{MM}} \int_\Omega u(f)dq$. Then, by an analogous argument as in the proof of Proposition 2.1, there exists $(q,\ell) \in \mathcal{M}\times\mathcal{L}^{**}$ such that $\int_\Omega u(f)dBU(q,\ell,s) \leq \int_\Omega u(f)dq'$. Hence,

$$\begin{aligned}
\int_\Omega u(f)dBU(q,\ell,s) &< u(x) \\
\int_\Omega \ell(s|\omega)u(f)dq + (1-\int_\Omega \ell(s|\omega)dq)u(x) &< u(x) \\
\min_{q\in\mathcal{M}} \int_\Omega \ell(s|\omega)u(f)dq + (1-\int_\Omega \ell(s|\omega)dq)u(x) &< u(x).
\end{aligned}$$

Thus, $x \succ_0 f_x^{\ell_i,s}$. Moreover, $\ell \in \arg\max_{\ell\in\mathcal{L}} \min_{q\in\mathcal{M}_0} \int_\Omega \ell(s|\omega)dq$, thus, $\ell \geq_{x,y}^s l'$ for all $\ell' \in \mathcal{L}$ and x, y sych that $u(x) > u(y)$. By LDC, $x \succ_s f$, a contradiction.

A.5 Proof of Theorem 4.2

Suppose $(\succeq_s)_{s\in S\cup\{0\}}$ the satisfies Confirmatory Information agreement (CIA) and assume by way of contradiction $\mathcal{M}_s \not\subseteq ch(BU(\mathcal{M}_0, \{\ell\in\mathcal{L}|\ell \text{ confirms } \succeq^*\}, s)))$. Then, by Lemma A.1, there exists f and x such that

$$u(x) = \min_{q\in\mathcal{M}_s} \int_\Omega u(f)dq < \int_\Omega u(f)dBU(q,\ell,s) \text{ for all } (q,\ell) \in \mathcal{M}_0 \times \{\ell\in\mathcal{L}|\ell \text{ confirms } \succeq^*\}.$$

An identical argument as in Theorem 2.1 shows that $f_x^{\ell,s} \succ_0 x$ for all $\ell \in \{\ell \in \mathcal{L}|\ell \text{ confirms } \succeq^*\}$. This contradicts CIA as $x \sim_s f$.

Next, we show that $\mathcal{M}_s^{CM} \equiv ch(BU(\mathcal{M}_0, \{\ell\in\mathcal{L}|\ell \text{ confirms } \succeq^*\}, s))) \subseteq \mathcal{M}_s$ under Confirmatory Default to Certainty (CDC). Assume by way of contradiction that this is not the case. Then, by Lemma A.1, there exists f, x such that

$$\min_{p\in\mathcal{M}_s^{CM}} \int_\Omega u(f)dp < \min_{q\in\mathcal{M}_s} \int_\Omega u(f)dp = u(x).$$

Fix $p \in \arg\min_{p\in\mathcal{M}_s^{CM}}$. Then an analogous argument as in the proof of Proposition 2.1, establishes that there exists $(q,\ell) \in \mathcal{M}_0 \times \{\ell\in\mathcal{L}|\ell \text{ confirms } \succeq^*\}$ such that $\int_\Omega u(f)dBU(q,\ell,s) \leq \int_\Omega u(f)dp$. Hence,

$$\int_\Omega u(f)dBU(q,\ell,s) < u(x)$$
$$\int_\Omega \ell(s|\omega)u(f)dq + (1 - \int_\Omega \ell(s|\omega)dq)u(x) < u(x)$$
$$\min_{q\in\mathcal{M}_0} \int_\Omega \ell(s|\omega)u(f)dq + (1 - \int_\Omega \ell(s|\omega)dq)u(x) < u(x).$$

Thus, $x \succ_0 f_x^{\ell,s}$. Moreover, $\ell \in \{\ell \in \mathcal{L} | \ell \text{ confirms } \succeq^*\}$. Therefore, CDC, $x \succ_s f$, a contradiction.

A.6 Proof of Proposition 5.1

We first prove the necessity of statement 1. Take any $s \in S$. Suppose that $f \bar{\succeq}_s x$ and $f \succeq_{s'} x$ for all $s' \neq s$. Since $f \bar{\succeq}_s x$, there exists $p_s \in \mathcal{M}_s$ such that $p_s \cdot (u \circ f) \geq u(x)$. Since $\mathcal{M}_s = BU(q_0, \mathcal{L}, s)$, there exists $\ell \in \mathcal{L}$ such that $p_s = BU(q_0, \ell, s)$. Let $p_{s'} = BU(q_0, \ell, s')$ for all $s' \neq s$. Since $\mathcal{M}_{s'} = BU(q_0, \mathcal{L}, s')$, $p_{s'} \in \mathcal{M}_{s'}$. Now for all $s' \neq s$, because $f \succeq_{s'} x$, we have $p_{s'} \cdot (u \circ f) \geq u(x)$. Since q_0 lies in the convex hull of $p_s, p_{s'}, ...$, we have $q_0 \cdot (u \circ f) \geq u(x)$. In addition, if $f \bar{\succ}_s x$ or $f \succ_{s'} x$ for some $s' \neq s$, we obtain $q_0 \cdot (u \circ f) > u(x)$. This proves the necessity of statement 1.

Next, we prove the sufficiency of statement 1. The following claim will be useful.

Claim 1. *Suppose that $A_1, \cdots, A_I$ are closed and convex sets in $\mathbb{R}^N$. Then*

$$ri(ch(\cup_i A_i)) \subset \left\{ v \in \mathbb{R}^N : \exists w_i \in A_i,\ \lambda_i > 0\ \forall\ i \text{ s.t. } \sum_i \lambda_i w_i = v,\ \sum_i \lambda_i = 1 \right\}.$$

Proof. Suppose $v \in ri(ch(\cup_i A_i))$. Take any $v' = \sum_i \lambda_i w_i$ with $w_i \in A_i$, $\lambda_i > 0$, and $\sum_i \lambda_i = 1$. Since $v \in ri(ch(\cup_i A_i))$, there exists $v'' \in ch(\cup_i A_i)$ and $k \in (0,1)$ such that $v = kv' + (1-k)v''$. Since A_i is convex for all i, v'' can be expressed as $v'' = \sum_i \lambda'_i w'_i$ with $w'_i \in A_i$, $\lambda'_i \geq 0$, and $\sum_i \lambda'_i = 1$. Thus,

$$v = kv' + (1-k)v'' = \sum_i (k\lambda_i + (1-k)\lambda'_i) \frac{k\lambda_i w_i + (1-k)\lambda'_i w'_i}{k\lambda_i + (1-k)\lambda'_i}.$$

Since $\lambda_i > 0$, $k\lambda_i + (1-k)\lambda'_i > 0$. Since A_i is convex, $\frac{k\lambda_i w_i + (1-k)\lambda'_i w'_i}{k\lambda_i + (1-k)\lambda'_i} \in A_i$. Therefore v belongs to the set on the right-hand side. □

We will also use the following separating hyperplane theorem: Two non-empty convex sets A and B can be separated properly if and only if their relative interiors do not intersect. Here, proper separation means that there is a hyperplane H such that A and B lie in opposite closed half-spaces with respect to H, and at least one of the sets A, B is not contained in H.

Suppose that statement 2 fails. There exists a signal realization s and $q_s \in \mathcal{M}_s$ such that there exist no $\in \mathcal{M}_{s'}$ for each $s' \neq s$ such that q_0 equals a convex combination of $q_s, q_{s'}, \cdots$. We want to establish a violation of statement 1. Consider two cases: $q_s = q_0$ and $q_s \neq q_0$.

Assume $q_s = q_0$. By the claim, $q_0 \notin ri(ch(\cup_{s' \neq s}\mathcal{M}_{s'}))$. By the aforementioned separating hyperplane theorem, there exist a vector $(v_\omega)_{\omega \in \Omega} \equiv v$ and a real r such that $q_0 \cdot v = r \leq q \cdot v$ for all $q \in ch(\cup_{s' \neq s}\mathcal{M}_{s'})$ where the inequality holds strictly for some q. Note that we are free to take a positive linear transformation on v and r, which means that we can choose v and r such that r and v_ω all lie in the range of u. Therefore, there exist an act f and a constant act x such that $q_0 \cdot (u \circ f) = u(x) \leq q \cdot (u \circ f)$ for all $q \in ch(\cup_{s' \neq s}\mathcal{M}_{s'})$ where the inequality holds strictly for some q. Now we have $f \succeq_{s'} x$ for all $s' \neq s$ and $f \succ_{s'} x$ for some $s' \neq s$. Since $q_0 \cdot (u \circ f) = u(x)$, $f \sim_0 x$. Since $q_s = q_0 \in \mathcal{M}_s$ by assumption, $f \bar{\succeq} x$. Hence statement 1 fails.

Assume instead $q_s \neq q_0$. Consider the convex set

$$\{q \in \Delta(S) : \exists \lambda > 1 \ s.t. \ q - q_s = \lambda(q_0 - q_s)\} \equiv C.$$

By the claim, C is disjoint with $ri(ch(\cup_{s' \neq s}\mathcal{M}_{s'}))$. By a separating hyperplane theorem, there exist an act f and an constant act x such that $q_0 \cdot (u \circ f) = u(x) > q \cdot (u \circ f)$ for all $q \in C$, and $q_0 \cdot (u \circ f) = u(x) \leq q \cdot (u \circ f)$ for all $q \in ch(\cup_{s' \neq s}\mathcal{M}_{s'})$. It follows that $f \sim_0 x$ and $f \succeq_{s'} x$ for all $s' \neq s$. Notice that C and $\{q_s\}$ are also separated by f and x. So $q_s \cdot (u \circ f) > u(x)$, implying $f \bar{\succ}_s x$. Hence statement 1 fails.

A.7 Proof of Proposition 5.2

A.7.1 Preliminaries

For ease of exposition, we enumerate the states $\Omega = \{\omega_1, ..., \omega_n\}$ and write P2 and P3 using this notation.

P2 Take any f and fix ω_i. Let $y_j^0, y_j^s \in X$ be such that $f(\omega_i) = \sum_{j \neq i} \frac{y_j^s}{n-1} =$

$\sum_{j \neq i} \frac{y_j^0}{n-1}$ for all $s \in S$. Suppose that for all $s \in S$ and all $j \neq i$

$$\begin{cases} x \succeq_s [f(\omega_j), \omega_j | y_j^s, \omega_i | x]; \\ x \bar{\succeq}_s [f(\omega_j), \omega_j | y_{\omega_j}^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_s x \end{cases} \quad \text{for all } s \in S \text{ and } j \neq i$$

$$\implies \begin{cases} x \succeq_0 [f(\omega_j), \omega_j | y_{\omega_j}^s, \omega_i | x]; \\ x \bar{\succeq}_0 [f(\omega_j), \omega_j | y_{\omega_j}^s, \omega_i | x] \text{ if } f(\omega_j) \succeq_0 x \end{cases} \quad \text{for some } j \neq i$$

P3 Take any f, g and fix ω_i. Suppose that $f(\omega_i) = g(\omega_i)$. Take any $y_j \in X$ for each $j \in \{1, ..., n\} \backslash \{i\}$. Then the following two statements are true:

(i) For any $x \in X$ if for all $j \neq i$,

$$\begin{cases} [y_{\omega_j}, \omega_j | g(\omega_j), \omega_i | x] \bar{\succeq}_s x; \\ [y_{\omega_j}, \omega_j | g(\omega_j), \omega_i | x] \succeq_s x \text{ if } x \succeq_s y_{\omega_j} \end{cases} \quad \text{and} \quad \begin{cases} x \succeq_0 [y_{\omega_j}, \omega_j | f(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_0 [y_{\omega_j}, \omega_j | f(\omega_j), \omega_i | x] \text{ if } y_{\omega_j} \succeq_0 x \end{cases}$$

then $x \succeq_0 g$ implies $x \succeq_s f$.

(ii) For any $x \in X$ if for all $j \neq i$,

$$\begin{cases} x \succeq_s [y_{\omega_j}, \omega_j | g(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_s [y_{\omega_j}, \omega_j | g(\omega_j), \omega_i | x] \text{ if } y_{\omega_j} \succeq_s x \end{cases} \quad \text{and} \quad \begin{cases} [y_{\omega_j}, \omega_j | f(\omega_j), \omega_i | x] \bar{\succeq}_0 x; \\ [y_{\omega_j}, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x \text{ if } x \succeq_0 y_{\omega_j} \end{cases}$$

then $g \succeq_0 x$ implies $f \succeq_s x$.

Next, we write the implications of the preferences expressed in P2 and P3 in terms of the MEU representation: Take any MEU preference $\succeq$ represented by $(\mathcal{M}, u)$. Take any $x, y, z \in X$ and any distinct states $\omega_i, \omega_j \in \Omega$. We have

$$\begin{aligned} & x \succeq [y, \omega_j | z, \omega_i | x] \\ & \Leftrightarrow \exists p \in \mathcal{M},\ u(x) \geq p(\omega_j) u(y) + p(\omega_i) u(z) + [1 - p(\omega_i) - p(\omega_j)] u(x) \\ & \Leftrightarrow \exists p \in \mathcal{M},\ [u(x) - u(y)] \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)] \\ & \Leftarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)]. \end{aligned}$$

When $u(x) - u(y) \geq 0$,

$$x \succeq [y, \omega_j | z, \omega_i | x] \Leftrightarrow [u(x) - u(y)] \max_{p \in \mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)].$$

Similarly, we have

$$
\begin{aligned}
&x\bar{\succeq}[y,\omega_j|z,\omega_i|x] \\
&\Leftrightarrow \forall p \in \mathcal{M},\ u(x) \geq p(\omega_j)u(y) + p(\omega_i)u(z) + [1 - p(\omega_i) - p(\omega_j)]u(x) \\
&\Leftrightarrow \forall p \in \mathcal{M},\ [u(x) - u(y)]\frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)] \\
&\Rightarrow [u(x) - u(y)] \max_{p\in\mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)].
\end{aligned}
$$

When $u(x) - u(y) \leq 0$,

$$x\bar{\succeq}[y,\omega_j|z,\omega_i|x] \Leftrightarrow [u(x) - u(y)] \max_{p\in\mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq [u(z) - u(x)].$$

Consequently,

$$\begin{cases} x \succeq [y,\omega_j|z,\omega_i|x]; \\ x\bar{\succeq}[y,\omega_j|z,\omega_i|x] \text{ if } y \succeq x \end{cases} \Leftrightarrow [u(x) - u(y)] \max_{p\in\mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \geq u(z) - u(x),$$

and

$$\begin{cases} [y,\omega_j|z,\omega_i|x]\bar{\succeq}x; \\ [y,\omega_j|z,\omega_i|x] \succeq x \text{ if } x \succeq y \end{cases} \Leftrightarrow [u(x) - u(y)] \max_{p\in\mathcal{M}} \frac{p(\omega_j)}{p(\omega_i)} \leq u(z) - u(x).$$

A.7.2 Necessity

Now we prove the necessity of P2 and P3. Consider P2 first. For any belief $q \in \Delta(\Omega)$, if $p \in \Delta(\Omega)$ is the Bayesian updating of q given signal s and likelihood function ℓ, then for any states ω_i, ω_j,

$$\frac{p(\omega_j)}{p(\omega_i)} = \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \times \frac{q(\omega_j)}{q(\omega_i)}.$$

Hence $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ implies

$$\max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \times \max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}.$$

Fixing any i, we have

$$\sum_{s\in S} \ell(s|\omega_i) \max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = \sum_{s\in S} \ell(s|\omega_i) \times \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \times \max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} = \max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}$$

for any $j \neq i$. Hence the vector $\left(\max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}\right)_{j\neq i} \in \mathbb{R}^{|\Omega|-1}$ is in the convex hull of $\left\{\left(\max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)}\right)_{j\neq i} : s \in \mathcal{S}\right\}$.

For all $s \in \mathcal{S}$ and all $j \neq i$, because

$$\begin{cases} x \succeq_s [f(\omega_j), \omega_j|y_j^s, \omega_i|x]; \\ x\bar{\succeq}_s[f(\omega_j), \omega_j|y_j^s, \omega_i|x] \text{ if } f(\omega_j) \succeq_s x \end{cases},$$

we have

$$[u(x) - u(f(\omega_j))] \max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} \geq u(y_j^s) - u(x).$$

Thus, for all $s \in \mathcal{S}$,

$$\sum_{j\neq i}[u(x) - u(f(\omega_j))] \max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} \geq \sum_{j\neq i}[u(y_j^s) - u(x)].$$

Because $\sum_{j\neq i} \frac{y_j^s}{|\Omega|-1} = \sum_{j\neq i} \frac{y_j^0}{|\Omega|-1}$, we have $\sum_{j\neq i} u(y_j^s) = \sum_{j\neq i} u(y_j^0)$. Consequently,

$$\sum_{j\neq i}[u(x) - u(f(\omega_j))] \max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \geq \sum_{j\neq i}[u(y_j^0) - u(x)].$$

This implies that

$$\exists j \neq i,\ [u(x) - u(f(\omega_j))] \max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \geq u(y_j^0) - u(x).$$

Therefore,

$$\exists j \neq i,\ \begin{cases} x \succeq_0 [f(\omega_j), \omega_j|y_j^s, \omega_i|x]; \\ x\bar{\succeq}_0[f(\omega_j), \omega_j|y_j^s, \omega_i|x] \text{ if } f(\omega_j) \succeq_0 x. \end{cases}$$

We have established P2.

Now we check P3. By Proposition 2.1, $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$ implies that $f_x^{\ell,s} \succeq_0 x \Leftrightarrow f \succeq_s x$. Fix any i. Suppose that

$$\begin{cases} [y_j, \omega_j|g(\omega_j), \omega_i|x]\bar{\succeq}_s x; \\ [y_j, \omega_j|g(\omega_j), \omega_i|x] \succeq_s x \text{ if } x \succeq_s y_j \end{cases} \quad \text{and} \quad \begin{cases} x \succeq_0 [y_j, \omega_j|f(\omega_j), \omega_i|x]; \\ x\bar{\succeq}_0[y_j, \omega_j|f(\omega_j), \omega_i|x] \text{ if } y_j \succeq_0 x \end{cases}$$

for all $j \neq i$. Then

$$u(g(\omega_j)) - u(x) \geq [u(x) - u(y_j)] \max_{p\in\mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = [u(x) - u(y_j)] \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \max_{q\in\mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)}$$
$$\geq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}[u(f(\omega_j)) - u(x)].$$

Hence

$$u(g(\omega_j)) \geq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} u(f(\omega_j)) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] u(x) \; \forall \; j \neq i.$$

Moreover, $f(\omega_i) = g(\omega_i)$ by assumption. Thus, g state-by-state dominates the act $\left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|}$. Note that

$$\ell(s|\omega_i) \left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|} + (1 - \ell(s|\omega_i))x = f_x^{\ell,s}.$$

Hence

$$x \succeq_0 g \Rightarrow x \succeq_0 \left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|} \Rightarrow x \succeq_0 f_x^{\ell,s} \Rightarrow x \succeq_s f.$$

Suppose instead

$$\begin{cases} x \succeq_s [y_j, \omega_j | g(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_s [y_j, \omega_j | g(\omega_j), \omega_i | x] \text{ if } y_j \succeq_s x \end{cases} \quad \text{and} \quad \begin{cases} [y_j, \omega_j | f(\omega_j), \omega_i | x] \bar{\succeq}_0 x; \\ [y_j, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x \text{ if } x \succeq_0 y_j \end{cases}$$

for all $j \neq i$. Then

$$\begin{aligned} u(g(\omega_j)) - u(x) &\leq [u(x) - u(y_j)] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_i)} = [u(x) - u(y_j)] \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_i)} \\ &\leq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} [u(f(\omega_j)) - u(x)]. \end{aligned}$$

Hence

$$u(g(\omega_j)) \leq \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} u(f(\omega_j)) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] u(x) \; \forall \; j \neq i.$$

Thus, the act $\left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|}$ state-by-state dominates g. We have

$$g \succeq_0 x \Rightarrow \left(\frac{\ell(s|\omega_j)}{\ell(s|\omega_i)} f(\omega_j) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_i)}\right] x\right)_{j=1}^{|\Omega|} \succeq_0 x \Rightarrow f \succeq_s x.$$

We have established P3.

A.7.3 Sufficiency

Let $|\Omega| = N$. For each $j = 1, \cdots, N-1$, let

$$\phi_j^0 = \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_N)} \text{ and } \phi_j^s = \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_N)} \; \forall \; s \in \mathcal{S}.$$

Let $\phi^0 = (\phi^0_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$ and $\phi^s = (\phi^s_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$.

Claim that P2 implies that ϕ^0 lies in the relative interior of the convex hull of $\{\phi^s : s \in \mathcal{S}\}$. If not, then by a separating hyperplane theorem, there exists $r \in \mathbb{R}^{N-1}$ and $k \in \mathbb{R}$ such that (i) $\phi^s \cdot r \geq k$ for all $s \in \mathcal{S}$, (ii) $\phi^s \cdot r > k$ for some $s \in \mathcal{S}$, and (iii) $\phi^0 \cdot r \leq k$.

For any $s \in \mathcal{S}$, take $(k^s_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$ such that (i) $\phi^s_j r_j \geq k^s_j$ for all $j = 1, \cdots, N-1$, and (ii) $\sum_{j=1}^{N-1} k^s_j = k$.

Take $(k^0_j)_{j=1}^{N-1} \in \mathbb{R}^{N-1}$ such that (i) $\phi^0_j r_j \leq k^0_j$ for all $j = 1, \cdots, N-1$, and (ii) $\sum_{j=1}^{N-1} k^0_j = k$.

Fix $x \in X$. Consider any $s \in \mathcal{S}$. Let f be an act such that $u(x) - u(f(\omega_j)) = r_j$ for all $j = 1, \cdots, N-1$. Let $y^s_j \in X$ satisfying $u(y^s_j) - u(x) = k^s_j$ for all $j = 1, \cdots, N-1$. We have

$$\begin{aligned}
&\forall\, j \in \{1, \cdots, N-1\},\ \phi^s_j r_j \geq k^s_j \\
\Leftrightarrow &\forall\, j \in \{1, \cdots, N-1\},\ [u(x) - u(f(\omega_j))] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_N)} \geq u(y^s_j) - u(x) \\
\Rightarrow &\forall\, j \in \{1, \cdots, N-1\}, \begin{cases} x \succeq_s [f(\omega_j), \omega_j | y^s_j, \omega_N | x]; \\ x \bar{\succeq}_s [f(\omega_j), \omega_j | y^s_j, \omega_N | x] \text{ if } f(\omega_j) \succeq_s x. \end{cases}
\end{aligned}$$

Let $y^0_j \in X$ satisfying $u(y^0_j) - u(x) = k^0_j$ for all $j = 1, \cdots, N-1$. We have

$$\begin{aligned}
&\forall\, j \in \{1, \cdots, N-1\},\ \phi^0_j r_j \leq k^0_j \\
\Leftrightarrow &\forall\, j \in \{1, \cdots, N-1\},\ [u(x) - u(f(\omega_j))] \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_N)} \leq u(y^0_j) - u(x) \\
\Rightarrow &\forall\, j \in \{1, \cdots, N-1\}, \begin{cases} [f(\omega_j), \omega_j | y^0_j, \omega_N | x] \bar{\succeq}_0 x; \\ [f(\omega_j), \omega_j | y^0_j, \omega_N | x] \succeq_0 x \text{ if } x \succeq_0 f(\omega_j). \end{cases}
\end{aligned}$$

We have found a violation of P2. Therefore, we must have ϕ^0 in the relative interior of the convex hull of $\{\phi^s : s \in \mathcal{S}\}$.

Let $(\lambda_s)_{s \in \mathcal{S}}$ be such that (i) $\lambda_s \in (0,1)$ for all $s \in \mathcal{S}$, (ii) $\sum_{s \in \mathcal{S}} \lambda_s = 1$, and (iii) $\phi^0 = \sum_{s \in \mathcal{S}} \lambda_s \phi^s$. Let $\ell(s|\omega_N) = \lambda_s$ and $\ell(s|\omega_j) = \lambda_s \phi^s_j / \phi^0_j$ for each $s \in \mathcal{S}$ and each $j \in \{1, \cdots, N-1\}$. Observe that $\ell(s|\omega) > 0$ and $\sum_{s \in \mathcal{S}} \ell(s|\omega) = 1$ for all $s \in \mathcal{S}$ and $\omega \in \Omega$.

Now we verify that $\mathcal{M}_s = BU(\mathcal{M}_0, \ell, s)$. By Proposition 2.1, it is equivalent to show $f^{\ell,s}_x \succeq_0 x \Leftrightarrow f \succeq_s x$. Take any act f and constant act x. Let g be such that

$$g(\omega) := \frac{\ell(s|\omega)}{\ell(s|\omega_N)} f(\omega) + \left[1 - \frac{\ell(s|\omega)}{\ell(s|\omega_N)}\right] x \ \forall\, \omega \in \Omega.$$

For every $j < N$, pick y_j such that

$$u(x) = \frac{u(y_j) + \frac{1}{\phi_j^0} u(f(\omega_j))}{1 + \frac{1}{\phi_j^0}}.$$

Then we have

$$\frac{u(f(\omega_j)) - u(x)}{u(x) - u(y_j)} = \phi_0^j$$

and

$$\frac{\frac{\ell(s|\omega_j)}{\ell(s|\omega_N)} u(f(\omega_j)) + \left[1 - \frac{\ell(s|\omega_j)}{\ell(s|\omega_N)}\right] u(x) - u(x)}{u(x) - u(y_j)} = \frac{\ell(s|\omega_j)}{\ell(s|\omega_N)} \times \phi_j^0 = \phi_j^s.$$

Thus,

$$u(f(\omega_j)) - u(x) = [u(x) - u(y_j)] \max_{q \in \mathcal{M}_0} \frac{q(\omega_j)}{q(\omega_N)}$$

and

$$u(g(\omega_j)) - u(x) = [u(x) - u(y_j)] \max_{p \in \mathcal{M}_s} \frac{p(\omega_j)}{p(\omega_N)}.$$

Therefore, we have

$$\begin{cases} [y_j, \omega_j | g(\omega_j), \omega_i | x] \bar{\succeq}_s x; \\ [y_j, \omega_j | g(\omega_j), \omega_i | x] \succeq_s x \text{ if } x \succeq_s y_j \end{cases} \quad \text{and} \quad \begin{cases} x \succeq_0 [y_j, \omega_j | f(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_0 [y_j, \omega_j | f(\omega_j), \omega_i | x] \text{ if } y_j \succeq_0 x \end{cases}.$$

We also have

$$\begin{cases} x \succeq_s [y_j, \omega_j | g(\omega_j), \omega_i | x]; \\ x \bar{\succeq}_s [y_j, \omega_j | g(\omega_j), \omega_i | x] \text{ if } y_j \succeq_s x \end{cases} \quad \text{and} \quad \begin{cases} [y_j, \omega_j | f(\omega_j), \omega_i | x] \bar{\succeq}_0 x; \\ [y_j, \omega_j | f(\omega_j), \omega_i | x] \succeq_0 x \text{ if } x \succeq_0 y_j \end{cases}.$$

By P3, we have $g \succeq_0 x$ if and only if $f \succeq_s x$. Since $\ell(s|\omega_N) g + (1 - \ell(s|\omega_N)) x = f_x^{\ell,s}$, we obtain $f_x^{\ell,s} \succeq_0 x \Leftrightarrow f \succeq_s x$. This completes the proof.

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