Optimal Public Debt Adjustment with Heterogeneous Agents: the Role of Aggregate Risk

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Abstract

What is the role of aggregate risk in determining the speed at which a government repays the excess public debt contracted during a recession? I study optimal fiscal adjustments in a heterogeneous-agent model with capital and aggregate shocks. Excessive indebtedness puts downward pressure on wages by crowding out capital, with severe consequences for hand-to-mouth households. It also generates upward pressure on interest rates, leading to a trade-off between capital gains and a redistributive motive. These effects are relevant to the extent they are exacerbated during recessions. Numerically, I find the implied dynamics of public debt adjustment consistent with those observed in the US, where the optimal fiscal adjustment deviates from tax smoothing due to binding borrowing constraints and uncertainty about the future.

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1 Introduction

The Stability and Growth Pact aims to ensure the economic sustainability of the members of the European Union by, among other requirements, limiting the amount of debt and deficit a country can incur[1]. Many EU members fail to meet debt and deficit targets, thereby requiring debt reductions that are only fulfilled by a subset of them[2]. According to the Euopean Comission, these levels of public debt put countries at a "*high risk over the medium term*"; however, they also recognize that "*compliance with the debt reduction benchmark could imply a too demanding frontloaded fiscal effort that could risk to jeopardise economic growth*". This paper examines this trade-off and goes beyond aggregate efficiency gains by looking at the social costs of debt adjustments throughout the wealth distribution.

In simple economic environments, government debt has no social cost[3]. In more realistic ones with productive capital and households heterogeneity, large amounts of debt crowd out capital, putting upward pressure on the interest rate and downward pressure on wages. While the median household might not be impacted, hand-to-mouth households are affected as labor earnings are their unique source of income. Uncertainty about the future prevents the government from delaying the fiscal adjustment until the economy recovers from the shock — a recession may occur at any time in the future, further increasing public debt. Does aggregate uncertainty and household heterogeneity justify a tighter fiscal intervention?

This paper studies the role of aggregate risk in determining the speed at which a government pays back the debt incurred during bad times, focusing on an environment with households heterogeneity and incomplete markets. To understand the trade-offs faced by a planner, I derive the optimality conditions of a utilitarian Ramsey planner in both the cases with aggregate risk and without. Aggregate risk yields an optimal fiscal adjustment that depends on the comovements of public debt with five social costs. The relevant social costs are the crowding out of capital, a public debt penalty, the utility costs associated with hand-to-mouth agents, the gains from capital income, and a redistribution motive. I use numerical simulations to assess quantitatively the optimal fiscal policy. I find that the implied dynamics of debt are consistent with the data and that a departure from the tax-smoothing benchmark is required.

It is relevant to elaborate on the role of aggregate risk for optimal fiscal policy in this context. Recent work has emphasized that in an environment with household heterogeneity and under standard utility functions, the optimal fiscal policy without aggregate shocks is to increase government

[1] Countries whose governments exceed a ratio of debt-to-GDP of 60% and a deficit-to-GDP ratio above 3% are required to undergo a series of fiscal adjustments, with a minimum annual adjustment of at least 0.5% of GDP (Darvas and Zettelmeyer, 2023).

[2] See the 2023 report from the European Commission `https://eur-lex.europa.eu/legal-content/EN/TXT/?uri=CELEX:52023DC0631`. Due to the economic conditions in 2022, "*compliance with the debt reduction benchmark is not warranted*".

[3] Absent from household heterogeneity and incomplete markets, the tax smoothing result under complete markets of Lucas and Stokey (1983), whereby the optimal fiscal policy is to use government debt as an insurance mechanism against shocks and smooth labor income taxes, applies.

debt to satiate its demand[4]. Optimal policy aims to close the gap between the interest rate and the discount rate, leading to extremely high levels of government debt and labor income taxes. In the long run, as taxes approach 100%, the level of consumption tends to zero. This result is due to unreasonably strong anticipation effects about the path of future taxes. Anticipating higher future taxes, households decide to increase their savings today, which increases the supply of assets — Straub and Werning (2020) calls this effect the "anticipatory savings effects". Changing the utility function (or planner's preferences) decreases the anticipation effect, but another, perhaps more direct way, is to make the future uncertain; this justifies my approach[5]. Next, I discuss in more detail the three contributions of this paper.

To understand the role of aggregate uncertainty, I first consider a heterogeneous agent model without aggregate risk, essentially an Aiyagari (1994) economy. I study a utilitarian Ramsey planner under full commitment that controls the degree of fiscal adjustment (the relevant policy rate) when debt is not at the long-run target. Contrary to other papers, I abstract from capital income taxes[6], and relative to Aiyagari and McGrattan (1998), I do not optimize over the steady-state level of government debt. I focus on the dynamics away from the deterministic steady state where a planner aiming to pay back government debt will raise the labor income tax above the long-run level. I start by studying the trade-offs associated with paying back public debt quickly. To do so, I write the Lagrangian following Dávila and Schaab (2023) and derive the optimality conditions[7]. The interpretation of these equations reveals the crucial social costs associated with public debt: the crowding in/out of capital, the public debt penalty, and the utility cost for hand-to-mouth individuals.

First, the optimal policy in the representative agent benchmark targets a level of public debt such that its social cost — what I call the public debt penalty — equals the social cost of excess demand of assets. If public debt provides liquidity, there is no cost associated with the demand for assets, and the planner never repays its debt[8].

Second, when including heterogeneous agents, the optimal tax becomes the sum of different social costs of public debt weighted by the excess stock of government debt relative to the long-run value of public debt — the socially costlier public debt, the faster the repayment. Relative the the representative agent case, the planner also takes into account the utility cost specific to hand-to-mouth individuals. A hand-to-mouth household consumes her cash on hand, so if public debt or

[4] See Chien and Wen (2022), Auclert et al. (2023), Bayas-Erazo (2023) and LeGrand and Ragot (2023).

[5] Dealing with aggregate uncertainty introduces an additional difficulty related to the notion of equilibrium, namely that rational expectations are not tractable because the distribution of wealth, an object of infinite dimension, enters the state-space. Intuitively, households need to keep track of the whole distribution of wealth and income to predict tomorrow's prices.

[6] Capital income taxes is the topic of Straub and Werning (2020) and LeGrand and Ragot (2023). These authors revisit the Chamley (1986)-Judd (1985) result in a richer setting.

[7] I assume the existence and uniqueness of an interior solution and derive my results with a general time-separable utility function. In a similar setting, LeGrand and Ragot (2023) provide utility functions where the interior solution exists.

[8] A recent example in which public debt can provide liquidity is Angeletos et al. (2023a).

the speed of adjustment of debt increases, her post-tax labor income drops which has the direct effect of decreasing her utility levels.

Finally, the planner also considers that a higher level of debt exerts upward pressure on the interest rate, resulting in increased capital income for households with wealth; however, this effect disproportionately benefits wealthy households. The latter effect, a redistribution motive, is weighted by the planner against the former effect, capital income gains.

The main theoretical contribution of this paper is the introduction of aggregate shocks and the optimal fiscal policy formula. As for the case without aggregate shocks, I start by defining the utilitarian Ramsey planner's problem and derive the optimality conditions. The main difference is that I depart from rational expectations as I assume that households forecast the future paths of capital and government debt using consistent laws of motion[9]. On the one hand, households need these laws of motion to forecast the future path of labor income taxes, wages, and real interest rates; on the other hand, the planner understands that changes in, for instance, the policy rate, will affect these laws of motion, and in turn modify household's implementability conditions. Up to these considerations and a state-space that includes TFP, public debt, and capital, the optimality conditions are essentially the same as in the case without aggregate shocks. The optimal public debt adjustment, in the long run, is given by the covariance along the aggregate state[10] of the economy between the level of government debt and the sum of social costs of having government debt.

Different from the case without aggregate shocks, what matters now is whether these social costs increase whenever debt increases. In other words, if public debt and its social costs comove, the planner wants to avoid having high levels of public debt. High debt is managed through fiscal adjustments. The stronger the correlation, the higher will be the tax in order to avoid experiencing large social costs. In this augmented setting, the optimal fiscal adjustment depends on i) the correlation of debt with capital income gains of the median household; ii) the correlation of debt with crowding out of capital; iii) the correlation of debt with the redistribution motive; and iv) the correlation of debt with the utility costs for hand-to-mouth discussed above. Finally, a couple of terms related to bounded rationality also appear in the formula. They capture that a perturbation to the policy rate changes how beliefs about the future paths of aggregate variables are determined.

To study the quantitative implications of the optimal tax, I calibrate and numerically solve this one-asset heterogeneous agent model with aggregate risk at the optimal policy rate. I first study the consequences of a negative TFP shock. The response of macroeconomic aggregates is standard, where the negative shock causes public deficit absorbed by issuing more government debt. Under the optimal adjustment rate, the deleverage of pubic debt is front-loaded, as the bulk of the excess debt is repaid in the first four years after the negative shock. Instead, consumption needs more than

[9] These consistent laws of motion that are, on average, correct about the future path of capital and government debt, and depend on the macroeconomic aggregates, economic fundamentals, and the policy rate.

[10] Along the business cycle, there are periods of recession with low levels of TFP and capital, and high debt, and periods of boom with high TFP and capital, and low debt. The covariance computed along the distribution generated by these macro aggregates over the business cycle.

20 years to recover fully. This fast adjustment of public debt is achieved via a high labor income tax rate: a negative shock in productivity that generates a 1% drop in aggregate consumption raises the tax rate on impact by 1.25%. This tax increase is much larger than the one under the representative agent counterpart. Then, I compare the implied persistence of government debt to the one from US data, which I find to be quantitatively very similar. However, debt repayment is much more persistent in the representative agent model. The quantitative implication is that with heterogeneous agents and aggregate risk, the social costs of government debt make it too costly to disregard the issue of excess public debt. I also study the role of anticipations by considering a counterfactual in which individuals are mypoic about the future path of aggregate variables. In such case, the optimal tax triples, which indicates that the social costs associated to the anticipation effects about the future are sizable: if individuals do not respond today to future tax adjustments, the government can perform them without much social cost.

Layout. The rest of the paper is organized as follows. Section 2 discussed the related literature. Section 3 introduces the model, Section 4 derives optimal policy without aggregate shocks and Section 5 does the same with aggregate risk. Section 6 contains the quantitative assessment and Section 7 concludes. Proofs are provided in Appendix A.

2 Related literature

This paper contributes to the literature on optimal fiscal policy. The seminal contribution is Lucas and Stokey (1983) that derived a tax-smoothing result in which the government insures against unexpected shocks (e.g. an unexpected war that temporarily increases government spending) by increasing public debt and smooths labor income taxes. These results are derived in a setting with a representative agent and a government that has access to state-contingent debt. Aiyagari et al. (2002) show that the results are essentially the same if the government has only access to one-period bonds. I extend their model with household heterogeneity and incomplete markets and study the extent to which these new features yield a departure from tax smoothing. It is possible to relate these extra costs to the work by Angeletos et al. (2023a). They study optimal policy in an economy where public debt serves as collateral (public debt provides liquidity) but introduces a social cost from taxation. Albeit doing it in a setting without aggregate shocks, they find that at the optimum, the response of taxes to a negative government spending shock is front-loaded. Their results depend on the premium that households pay for debt, namely the difference between the interest rate and the discount rate. This premium is also present in my optimal policy rate.

In settings with heterogeneous agents arising from uninsurable labor productivity risk in the tradition of Bewley-Huggett-Aiyagari (Bewley, 1983; Huggett, 1993; Aiyagari, 1994), whilst much is known on the positive side of the consequences of public debt with household heterogeneity[11], the

[11] See, among many others Heathcote (2005), Auclert et al. (2018) and Ferriere and Navarro (2023).

literature has only recently put attention to optimal fiscal policy. The two exceptions are Aiyagari (1995) which assumes the existence of a Ramsey steady-state and shows that optimal capital taxes are positive and Aiyagari and McGrattan (1998) which computes the optimal level of public debt in the long run. I deviate from them in three dimensions. First, I study the role of aggregate shocks as justified above. Second, I focus on the dynamics away from the deterministic steady-state as I am interested in the speed of adjustment of debt, rather than in the optimal quantity of debt. Finally, I derive my results using a continuous time specification of the model because it allows me to characterize the role of hand-to-mouth households as Dávila and Schaab (2023) show.

Recently, several papers have studied the existence of such a Ramsey steady-state. Chien and Wen (2022) and Auclert et al. (2023) find that the Ramsey steady-state does not exit for separable CRRA preferences, LeGrand and Ragot (2023) shows existence with several preferences, and Bayas-Erazo (2023) use heterogeneous welfare weights as in Dávila and Schaab (2022) and LeGrand and Ragot (2023) to restore existence. By allowing for aggregate uncertainty, I restore existence and uniqueness, although I only verify this result numerically.

Related to the timing of taxes, Angeletos et al. (2023b) argue that a delay in fiscal adjustment is optimal because of the persistence of the boom generated: as the fiscal adjustment is delayed, debt returns to trend on its own and the required future tax hike vanishes. The presence of this phenomenon becomes impossible with aggregate uncertainty as along this hypothetical path of debt towards its pre-shock level, it might be that other negative shocks take place, which in turn will increase debt further. When introducing aggregate shocks, Bhandari et al. (2017b) finds that the optimal level of debt is close to zero and that the optimal policy for government debt displays slow mean reversion (a half-life of almost 250 years), although they derive these results in a setting where there are no binding borrowing constraints. In Bhandari et al. (2017a), they consider heterogeneous households and incomplete markets, but abstract from aggregate shocks and show that the results are qualitatively similar. Compared to these two papers, I undertake a different exercise as I consider a smaller set of policy instruments. This simplification allows me to analyze the case with aggregate shocks and heterogeneous households.

By adding aggregate risk in a model with household heterogeneity, I relate to the literature started with Krusell and Smith (1998). Thereafter, improvements have been proposed to the original approach. For instance, Ahn et al. (2018) extend standard linearization techniques to the heterogeneous agent context. However, when evaluating the welfare implications of a policy, one needs to perform higher-order approximations. Dávila and Schaab (2023) use sequence-space Hessians, generalizing the sequence-space Jacobians introduced by Auclert et al. (2021), which builds on Boppart et al. (2018), that are useful to perform positive analysis. My approach is to solve the model globally to capture all the non-linearities induced by borrowing constraints and aggregate risk.

3 Model

The model takes as its starting point the standard heterogeneous-agent productive economy framework first introduced by Aiyagari (1994). When analyzing the role of aggregate risk, I allow for shocks in the total factor productivity (TFP) level, otherwise I focus on a one time unanticipated shock. In sum it represents a departure from the textbook real business cycle setting (Kydland and Prescott, 1982; Romer, 2012) by including uninsurable labor income risk.

3.1 Households

The economy is populated by a continuum of households. Facing idiosyncratic earnings risk and, if relevant, aggregate uncertainty, households make consumption, labor supply and savings decisions across time. The idiosyncratic state of a household consists of its portfolio position, made up of liquid assets, and its earnings status.

Household preferences are defined over consumption and labor, given by

$$\max \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t, \ell_t) dt, \tag{1}$$

where c_t is the rate of consumption and ℓ_t is the rate of labor supply. Households can trade a liquid risk-free asset, in quantity a_t, whose return is denoted r_t and its position evolves according to

$$\dot{a}_t = r_t a_t + (1 - \tau_t) w_t z_t \ell_t - c_t, \tag{2}$$

where effective units of labor income are given by $z_t \ell_t$, and where z_t represents uninsurable labor earnings risk. Per effective unit of labor income wage is w_t and τ_t is a time-dependent labor income tax. Finally, individuals are subject to a borrowing constraint on their holding of assets, namely $a_t \geq \underline{a}$ with $\underline{a} \leq 0$.

Labor earnings risk. Households face uninsurable risk encoded in the state variable z_t, where log-labor productivity follows a continuous-time AR(1) processes, formally an Ornstein-Uhlenbeck process, given by

$$dlog(z_t) = -\theta_z \log(z_t) + \sigma_z dW_t, \tag{3}$$

where $\theta_z > 0$ is the persistence of the process, where σ_z denotes the volatility and where W_t is an individual specific Brownian motion.

Cross-sectional distribution. Since I abstract from permanent heterogeneity, each individual can be indexed by a pair of wealth and earnings (a, z), that forms a cross-sectional distribution denoted $g_t(a, z)$. In the presence of aggregate risk, such distribution is conditional on an aggregate state.

3.2 Firms

A representative firm produces the consumption good using capital K_t and labor L_t according to a Cobb-Douglas technology augmented by aggregate productivity,

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha},$$

where Z_t is total factor productivity (TFP), which is the exogenous source of aggregate productivity shocks. Whenever I abstract from aggregate uncertainty, I focus on potentially time-varying levels of TFP $\{Z_t\}_{t=0}^{\infty}$. Otherwise, I assume log TFP follows a continuous-time AR(1) processes[12].

The market for inputs is assumed to be competitive,

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \quad \text{and} \quad r_t = \alpha\frac{Y_t}{K_t} - \delta,$$

where δ is the depreciation rate per unit of capital.

3.3 Government

The government uses tax revenue and government debt to finance a constant amount of government spending $\bar{G}$. Government debt evolves according to

$$\dot{B}_t = r_t B_t + \bar{G} - \tau_t w_t L_t$$

where the government issues one-period risk-free bounds denoted B_t, and by a no-arbitrage condition its cost is the same as the one for capital. The functional form of the labor tax τ_t is chosen so that government debt mean-reverts to a level $\bar{B} = \mathbb{E}[B_t]$, and the adjustmento of public debt is governed by the policy rate τ^a,

$$\tau_t = \bar{\tau} + \tau_t^a(B_t - \bar{B}). \tag{4}$$

The base level $\bar{\tau}$ yields a balanced budget in the deterministic steady-state (with variables denoted with upper lines), which means $\bar{r}\bar{B} + \bar{B} = \bar{\tau}\bar{w}\bar{L}$. The term $\tau_t^a > 0$ captures the speed of adjustment of public debt, and one can interpret it as the extend of fiscal austerity over the business cycle. Since government spending is constant[13], a level of debt above the long-run level can be payed back in three ways: i) lower interest rates, ii) higher wages, and iii) larger adjustments of public debt.

[12] Precisely, I assume log-productivity is given by

$$d\log(Z_t) = -\theta_Z \log(Z_t) + \sigma_Z d\tilde{W}_t.$$

where $\tilde{W}_t$ is a Brownian motion common to all households.

[13] I assume a constant level of government debt to solely focus on the adjustment necessary when there is excess of debt. A version of this model including government spending shocks as an additional source of aggregate risk will exacerbate the negative consequences of debt if a negative TFP shock and a positive government spending shock happens to arrive at the same period.

3.4 Equilibrium

First, I define the notion of competitive equilibrium without aggregate shocks. Then, I include TFP shocks by departing from rational expectations about the way individuals form expectations about future aggregate variables.

Definition 1. ***(Competitive Equilibrium without Aggregate Shocks)*** *Given an initial distribution over assets and earnings level, $g_0(a,z)$, and given a path of the policy rate $\{\tau_t^a\}$ and a path for shocks $\{Z_t\}$, an equilibrium is defined as paths for prices $\{r_t, w_t\}$, aggregates $\{Y_t, L_t, C_t, B_t, K_t, A_t, I_t\}$, individual allocation rules $\{c_t(a,z), \ell_t(a,z), s_t(a,z)\}$, and a path of distributions $\{g_t(a,z)\}$ such that households optimize, firms optimize, and markets for goods and bonds clear, that is,*

$$Y_t = C_t + I_t + \bar{G} \tag{5}$$

$$A_t = K_t + B_t \tag{6}$$

where $C_t = \iint c_t(a,z)g_t(a,z)dadz$, and so on for the rest of the aggregates, and where $A_t = \iint ag_t(a,z)dadz$ denotes aggregate wealth held by households. Investment is given by $I_t = \delta K_t + \iint s_t(a,z)g_t(a,z)dadz$ where $s_t(a,z)$ is individual savings.

In order to tackle the issue of infinite dimensionality of the state-space with aggregate shocks, I assume a particular notion of bounded rationality, which is both useful to write the Ramsey problem, as well as for the numerical implementation. With aggregate uncertainty, all variables become a function of the aggregate state.

Assumption 2. ***(Bounded Rationality)*** *I assume that households forecast the future path of capital and government debt using the current value of TFP, capital and public debt*

$$\dot{K}_t = \mu_K(\Gamma_t) \quad \text{and} \quad \dot{B}_t = \mu_B(\Gamma_t) \tag{7}$$

where the aggregate state is given by $\Gamma_t = (Z_t, K_t, B_t)$. These functions minimize the L2 norm between the predicted path of K_t and B_t and the realized one.

Definition 3. ***(Competitive Equilibrium with Aggregate Shocks under Assumption 2)*** *Given an initial distribution over assets, earnings level and the aggregate state, $g_0(a,z,\Gamma)$, and given a policy rate τ^a, an equilibrium is defined as paths for prices $\{r_t(\Gamma), w_t(\Gamma)\}$, aggregates $\{Y_t(\Gamma), L_t(\Gamma), C_t(\Gamma), B_t(\Gamma), K_t(\Gamma), A_t(\Gamma), I_t(\Gamma)\}$, individual allocation rules $\{c_t(a,z,\Gamma), \ell_t(a,z,\Gamma), s_t(a,z,\Gamma)\}$, and a distribution $\{g_t(a,z,\Gamma)\}$ such that households optimize, firms optimize, markets for goods and bonds clear, and the process underlying the formation of beliefs is consistent with Assumption 2.*

3.5 Implementability

To conclude the description of the environment, since I focus on Ramsey planners that pick among implementable conditions, I state them.

First, a Ramsey planner must respect individual's optimal consumption-savings decision, which amounts to account for

$$\partial_c u_t(a,z,\Gamma) = \partial_a V_t(a,z,\Gamma) \tag{8}$$

where $u_t(a,z,\Gamma) = u(c_t(a,z,\Gamma),\ell_t(a,z,\Gamma))$ and marginal utility of consumption $\partial_c u_t(a,z,\Gamma) = \frac{\partial u_t(a,z,\Gamma)}{\partial c_t(a,z,\Gamma)}$. This first optimality condition equates marginal consumption to private marginal value of wealth. Second, a Ramsey planner must respect the intratemporal substitution between labor and consumption, namely

$$\partial_\ell u_t(a,z,\Gamma) = -(1-\tau_t(\Gamma))w_t(\Gamma)z\partial_c u_t(a,z,\Gamma) \tag{9}$$

equating the private marginal utility cost of labor, $\partial_\ell u_t(a,z,\Gamma) = \frac{\partial u_t(a,z,\Gamma)}{\partial \ell_t(a,z,\Gamma)}$, to the private marginal benefit of working, that is, the marginal utility of consuming the resources from working. Notice that now the labor income tax is now indexed by the aggregate state Γ. Third, a Ramsey planner needs to account for the evolution of private lifetime utility $V_t(a,z,\Gamma)$ which is given by a Bellman equation

$$\rho V_t(a,z,\Gamma) = u_t(a,z,\Gamma) + \mathbb{E}_t\left[\frac{dV_t(a,z,\Gamma)}{dt}\right] \tag{10}$$

or, more formally, a Hamilton-Jacobi-Bellman (HJB) equation in continuous time. The continuation value is $\mathbb{E}_t\left[\frac{dV_t(a,z,\Gamma)}{dt}\right] = \partial_t V_t(a,z,\Gamma) + \mathcal{A}_t V_t(a,z,\Gamma)$, where $\mathcal{A}_t$ is the infinitesimal generator[14] of the stochastic processes for (a,z,Γ). Fourth, the planner internalizes that changes in the policy rate affect the measure of individuals in each state of the economy. The evolution of the cross-sectional distribution is governed by a so-called Kolmogorov forward equation that obeys

$$\partial g_t(a,z,\Gamma) = \mathcal{A}_t^* g_t(a,z,\Gamma) \tag{11}$$

where $\mathcal{A}_t^*$ is the adjoint operator of $\mathcal{A}_t$, see Achdou et al. (2022) for a formal description of these objects. The perceived laws of motion defined in Assumption 2 are encoded in $\mathcal{A}_t$, which means that whenever the planner makes policy changes, she does not only affect the evolution of wealth, but also the beliefs about the evolution of capital and government debt. By the choice of the labor earnings risk process, I abstract form cyclicality of labor productivity shocks[15]. Finally, the planner needs to make sure that bonds market clears and the government budget constraint is satisfied.

[14] Formally, for any smooth function $f:\mathbb{R}^5 \to \mathbb{R}$ where the inputs are wealth, earnings and the aggregate state, the infinitesimal generator is

$$\begin{aligned}\mathcal{A}_t f_t(a,z,\Gamma) =& s_t(a,z,\Gamma)\partial_a f_t(a,z,\Gamma) - \theta_z z \partial_z f_t(a,z,\Gamma) + \frac{\sigma_z^2}{2}\partial_{zz} f_t(a,z,\Gamma) - \theta_Z Z \partial_Z f_t(a,z,\Gamma) + \frac{\sigma_Z^2}{2}\partial_{ZZ} f_t(a,z,\Gamma)\\ &+ \mu_K(\Gamma)\partial_K f_t(a,z,\Gamma) + \mu_B(\Gamma)\partial_B f_t(a,z,\Gamma).\end{aligned}$$

where $s_t(a,z,\Gamma)$ denotes savings. The adjoint operator, denoted $\mathcal{A}_t^*$ satisfies $< \mathcal{A}_t f_1, f_2 >=< f_1, \mathcal{A}_t^* f_2 >$ where $f_1:\mathbb{R}^5 \to \mathbb{R}$ and $f_2:\mathbb{R}^5 \to \mathbb{R}$ are 2 smooth functions and $< \cdot,\cdot >$ is the inner product.

[15] Bilbiie et al. (2023) study the quantitative role of countercyclical income risk in a business cycle model and Schaab (2020) measure its implication for macro uncertainty.

4 Optimal Policy without Aggregate Shocks

It will prove useful to begin by characterizing the optimal adjustment of public debt abstracting from aggregate shocks since the goal of the paper is to understand the role of aggregate risk. Therefore, in this section the state-space will be (a, z) and Assumption 2 is not relevant anymore. In Section 5 I introduce back aggregate shocks and derive optimal policy in such framework[16].

Ramsey problem with commitment. I shall start by defining the Ramsey planner without aggregate shocks. I derive optimal policy using the primal approach, in which the Ramsey planner chooses among implementable competitive equilibra, that is, those that arise from a decentralized economy.

Definition 4. ***(Ramsey Planner without Aggregate Shocks)*** *A Ramsey planner with infinite commitment horizon $[0, \infty)$ chooses allocations, a policy, and prices*[17]

$$\boldsymbol{X} = \left\{ c_t(a,z), \ell_t(a,z), V_t(a,z), g_t(a,z), \tau_t^a, r_t, w_t, K_t, B_t \right\}_{t=0}^{\infty}$$

as well as multipliers

$$\boldsymbol{M} = \left\{ \phi_t(a,z), \chi_t(a,z), \psi_t(a,z), \lambda_t(a,z), \mu_t, \theta_t \right\}_{t=0}^{\infty}$$

to maximize social welfare subject to implementability conditions

$$\rho V_t(a,z) = u_t(a,z) + \partial_t V_t(a,z) + \mathcal{A}_t V_t(a,z) \tag{HJB}$$

$$\partial_c u_t(a,z) = \partial_a V_t(a,z) \tag{FOC c}$$

$$\partial_\ell u_t(a,z) = -(1-\tau_t) w_t z \partial_a V_t(a,z) \tag{FOC ℓ}$$

$$\partial_t g_t(a,z) = \mathcal{A}_t^* g_t(a,z) \tag{KFE}$$

$$K_t + B_t = \iint a g_t(a,z) da dz \tag{McK}$$

$$\dot{B}_t = r_t B_t + \bar{G} - \tau_t w_t L_t \tag{GvtBC}$$

taking as given the initial cross-sectional distribution $g_0(a,z)$ and where $\tau_t = \bar{\tau} + \tau_t^a (B_t - \bar{B})$. It solves,

$$\mathcal{W} = \min_{\boldsymbol{X}} \max_{\boldsymbol{M}} \mathbb{E}_0 \left[L(g_0) \right] \tag{12}$$

[16] While I focus on infinite commitment horizon $[0, \infty)$, it is possible to study the case where the commitment horizon is $[0, \tau_1)$ with $\tau_1 \to 0$, that I refer as optimal policy under discretion, similarly as in Dávila and Schaab (2023).

[17] With respect to the definition of a competitive equilibrium (Definition 1), the Ramsey planner does not optimize directly over savings, aggregate savings and investment. The reason is that savings follow from the choice of consumption and labor, and goods market clearing is given by the Walras law.

where the expectation $\mathbb{E}_0$ is taken over the idiosyncratic, and $L(g_0)$ is the planner's Lagrangian given an initial cross-sectional distribution g_0:

$$
\begin{aligned}
L(g_0) = \int_0^\infty e^{-\rho t}\Bigg\{ \iint \Bigg\{ & \omega_t(a,z)u_t(a,z)g_t(a,z) \\
& + \phi_t(a,z)\Big[-\rho V_t(a,z) + u_t(a,z) + \partial_t V_t(a,z) + \mathcal{A}_t V_t(a,z)\Big] \\
& + \chi_t(a,z)\Big[\partial_c u_t(a,z) - \partial_a V_t(a,z)\Big] \\
& + \psi_t(a,z)\Big[\partial_\ell u_t(a,z) + (1-\tau_t)w_t z \partial_a V_t(a,z)\Big] \\
& + \lambda_t(a,z)\Big[-\partial_t g_t(a,z) + \mathcal{A}_t^* g_t(a,z)\Big]\Bigg\} dadz \\
& + \theta_t\Big[-\dot{B}_t + r_t B_t + \bar{G} - \tau_t w_t L_t\Big] \\
& + \mu_t\Big[\iint a g_t(a,z)dadz - K_t - B_t\Big]\Bigg\} dt
\end{aligned}
\tag{13}
$$

While the definition allow for a general specification of welfare weights, this paper focuses on the utilitarian case by setting the welfare weight $\omega(a,z) = 1$. There are six implementability conditions, two of them, the dynamics of the Bellman equation (HJB) and the dynamics of public debt, (GvtBC), are forward looking — in practice it means that they are ignored for the discretionary case since the planner cannot commit on the future path of variables. These implementability conditions can be interpreted as two promise keeping constraints, one about lifetime utility and the other about the level of government debt.

Proposition 5. ***(Necessary Optimality Conditions without Aggregate Shocks)*** *The first-order conditions that solve the Ramsey planner without aggregate shocks are*[18]

$$\partial_t \phi_t(a,z) = \mathcal{A}_t^* \phi_t(a,z) + \partial_a \chi_t(a,z) - \partial_a \psi_t(a,z)(1-\tau_t)w_t z \tag{14}$$

$$\rho\lambda_t(a,z) = u_t(a,z) + \mu_t a - \tau_t w_t \theta_t \ell_t(a,z) + \mathcal{A}_t \lambda_t(a,z) + \partial_t \lambda_t(a,z) \tag{15}$$

$$0 = \partial_c u_t(a,z) - \partial_a \lambda_t(a,z) - \tilde{\chi}_t(a,z) \tag{16}$$

[18] Auclert et al. (2023) show that there is no interior Ramsey steady-state, instead the planner wants as much government debt as possible to close the gap between r_t and ρ. Numerical simulations of this model (with the calibration in Section 6.1) show that the SWF is decreasing in the time-independent policy rate τ^a, suggesting that under a standard calibration, my model also fails to have an interior Ramsey steady-state. However, there exist preferences for which a Ramsey steady-state exists (LeGrand and Ragot, 2023). Here, I assume existence and uniqueness of the Ramsey allocation.

$$0 = \partial_\ell u_t(a,z) + (1-\tau_t) w_t \partial_a \lambda_t(a,z) - \tilde{\psi}_t(a,z) - \theta_t \tau_t w_t z \tag{17}$$

$$0 = \iint a \Big[\phi_t(a,z)\partial_a V_t(a,z) + \partial_a \lambda_t(a,z) g_t(a,z)\Big] dadz + \zeta_r^{HtM} + \theta_t B_t \tag{18}$$

$$\dot{\theta}_t = \tau_t^a w_t \iint z \Big[\ell_t(a,z)(\phi_t(a,z)\partial_a V_t(a,z) + \partial_a \lambda_t(a,z) g_t(a,z)) + \partial_a \psi_t(a,z) V_t(a,z)\Big] dadz$$

$$+ \tau_t^a \theta_t w_t L_t + \theta_t(\rho - r_t) + \mu_t + \zeta_B^{HtM} \tag{19}$$

$$0 = w_t(B_t - \bar{B}) \iint z \Big[\ell_t(a,z)(\phi_t(a,z)\partial_a V_t(a,z) + \partial_a \lambda_t(a,z) g_t(a,z)) + \partial_a \psi_t(a,z) V_t(a,z)\Big] dadz$$

$$+ \theta_t (B_t - \bar{B}) w_t L_t + \zeta_\tau^{HtM} \tag{20}$$

where

$$\tilde{\chi}_t(a,z) = -\frac{\chi_t(a,z)}{g_t(a,z)}\partial_{cc} u_t(a,z) - \frac{\psi_t(a,z)}{g_t(a,z)}\partial_{c\ell} u_t(a,z),$$

$$\tilde{\psi}_t(a,z) = -\frac{\psi_t(a,z)}{g_t(a,z)}\partial_{\ell\ell} u_t(a,z) - \frac{\chi_t(a,z)}{g_t(a,z)}\partial_{c\ell} u_t(a,z),$$

and where ζ_r^{HtM}*,* ζ_B^{HtM} *and* ζ_τ^{HtM} *are respectively defined in* (31)*,* (32) *and* (33)*; where the initial conditions for the forward-looking multipliers are* $\theta_0 = \phi_0(a,z) = 0$*; and where equations* (14)*-*(17) *hold in the interior space — a consequence of* (HJB)*,* (FOC c)*,* (FOC ℓ) *and* (KFE) *being defined only in the interior space.*

Equations (14) to (19) correspond to the optimality conditions for i) the value function, ii) the cross-sectional distribution, iii) consumption, iv) labor supply, v) the interest rate, and vi) government debt. Equation (20) corresponds to the optimality condition for the debt adjustment. The proof of this proposition is given in Appendix A.1. Before providing the main proposition of this section, namely the optimal policy rate for public debt adjustment, I begin by interpreting the necessary conditions the planner faces. This exercise features several similarities with Dávila and Schaab (2023), altought their focus is the optimal design of monetary policy, while mine is on optimal fiscal policy.

Before interpreting the optimality conditions, a remark is on order. For simplicity, I keep $\bar{B}$ constant and see it as the level of debt that yields a balanced budget constraint given by $\bar{\tau}$ in the deterministic steady-state. Others like LeGrand and Ragot (2023) choose the weights $\omega_t(a,z)$ such that the optimal deterministic steady-state level of debt is a calibrated value $\bar{B}$. Nonetheless, I optimize over $\{B_t\}_t$, so a Ramsey allocation can feature a persistently higher long-run level of debt $B_\infty > \bar{B}$ in case this is optimal. In turn, the steady-state labor income tax will be larger ($\tau_\infty = \bar{\tau} + \tau_\infty^a \log(B_\infty/\bar{B}) > \bar{\tau}$) and the planner would choose a level $\tau_\infty^a > 0$ such that the Ramsey allocation exists, that is, such that the government budget is balanced. I now proceed to interpret the seven optimality conditions. I will focus on the extra terms that are particular to the planner and distinguish this set of equations to the implementability conditions listed in Section 3.5.

I begin by equation (14), the one corresponding to the optimality conditions with respect to lifetime utility. The multiplier $\phi_t(a,z)$ captures the social marginal cost of an increase in lifetime utility. Whenever $\phi_t(a,z) > 0$, the planner associates a social cost — a penalty — to a positive perturbation of the lifetime utility of a households in state (a,z); likewise, if $\phi_t(a,z) < 0$, it has the interpretation of a reward. Therefore, I refer to it as a *distributional penalty*. Its evolution is governed by an equation that shares similarities with a standard Kolmogorov Forward equation (i.e. the equation that dictates the evolution of the cross-sectional distribution across time and at the steady-state) extended by the term "$\partial_a \chi_t(a,z) - \partial_a \psi_t(a,z)(1-\tau_t)w_t z$" — without this extra term, the distributional penalty will be equal to the cross-sectional distribution, see (KFE). Intuitively, it's like a life-cycle model where births increase the mass of individuals at a given state, and deaths decrease it, albeit here is about the rise or decline of distributional penalties and rewards. Whenever this extra term is positive, it increases the penalty (decreases the reward), and viceversa. These increases/declines are determined by the social marginal costs of increases in consumption and labor that I discuss below.

Now I study the optimality conditions related to optimally choosing the cross-sectional distribution, namely Equation (15). The multiplier $\lambda_t(a,z)$ represents the social shadow value of increasing the mass of individuals in the state (a,z). This optimality condition represents the *social lifetime value* of a household at state (a,z). First, observe that if $\mu_t a - \tau_t w_t \theta_t \ell_t(a,z) = 0$, then $\lambda_t(a,z)$ would be equal to the lifetime utility function given in Equation HJB. Whenever this term is non-zero, it acts as an extra reward that the planner perceives for increasing households' lifetime utility. An individual at this particular state has a quantity of assets a, which has an associated social gain (if $\mu_t > 0$) in the form of relaxing the market clearing condition for capital, since μ_t, which I discuss below, is the social marginal benefit of excess supply of assets. However, the social losses associated to decreasing the mass of individuals at state (a,z), causing a drop in tax revenue payed by these households, generates a loss that equals $\tau_t w_t \theta_t \ell_t(a,z)$, namely the tax revenue paid by household (a,z) expressed in social units using θ_t, the social marginal cost of government debt. Whenever government debt is socially costly ($\theta_t > 0$) we have $\tau_t w_t \theta_t \ell_t(a,z) > 0$ contributing in deceasing the social lifetime value of a household in state (a,z).

Next, I look at (16), the optimality condition arising from the choice of consumption. The multiplier $\chi_t(a,z)$ represents the social shadow value of relaxing households' consumption-savings decisions. When $\chi_t(a,z) > 0$,the planner perceives that households consume too much, or equivalently do not save enough. Indeed, a positive perturbation to consumption decreases social welfare, which is perceived by the planner as not being welfare improving. While a household equates marginal utility of consumption to the private marginal value of wealth, the planner equates marginal utility of consumption to $\partial_a \lambda_t(a,z) + \tilde{\chi}(a,z)$. The first term represents the social marginal value of wealth, while the second represents an extra shadow cost of changing consumption. It is related to both the risk-aversion towards consumption, as well as with the substitutability in the utility function between consumption and labor.

Optimally choosing labor supply yields (17). The multiplier $\chi_t(a,z)$ represents the social shadow value of relaxing households' intratemporal substitution between consumption and labor. When $\psi_t(a,z) > 0$, the planner perceives that the household works too much, or equivalently does not enjoy enough leisure. As for consumption, this equation is the planner's equivalent to the competitive equilibrium choice of labor. The only difference is that a new shadow cost appears: if individuals work less they pay fewer taxes, which, in case debt is socially costly, it will represent an extra shadow cost associated to a decrease in labor as fewer debt will be repaid.

I conclude by interpreting the three optimality conditions relating to aggregate objects: interest rates, government debt, and the policy rate. I shall start with (18), the optimality condition for the interest rate. The first two sources of welfare gains/losses are due to individuals obtaining more income from their saving (if $a > 0$) and the social valuation of the subsequent increase in wealth. However, the hand-to-mouth individuals, whose welfare losses associated to the increase in r are captured by ζ_r^{HtM}, are worse-off. Finally, the cost of financing government debt increases, which generates losses if public debt is socially costly.

Next, I consider (18), the optimality condition for government debt. The multiplier θ_t is the social marginal cost of government debt. Whenever $\theta_t > 0$, an increase in government debt (i.e. $\dot{B}_t > 0$) decreases welfare, that is why I refer to it as a *public debt penalty*. This public debt penalty will change for six different reasons. First, public debt needs to be payed by taxing labor income, which is a distortionary tax, and the social cost associated to it is captured by planner's cost of having a household with a before-tax labor income of $w_t z \ell_t(a,z)$. Second, the associated cost will depend on the social shadow value of relaxing/tightening the intratemporal substitution between consumption and labor — larger taxes decrease the private marginal gains from working. Third, the evolution of the public debt penalty depends on the current amount of debt paid today. These social costs are weighted by the policy rate τ_t^a since these social costs are about debt repayment, and the repayment of debt is done using this instrument. Fourth, the social cost of public debt only matters to the extend it is costly, namely the difference $\rho - r_t$. This is the premium that households pay in equilibrium for being able to self-insure using assets. Fifth, if debt generates a crowding out which decreases welfare, the public debt penalty will increase. Notice that μ_t is the social marginal cost of excess demand of assets. When $\mu_t > 0$, more assets are costly from a social point of view. And sixth, constrained individuals dislike more debt since, other things equal, their cash-on-hand decreases, rising the social cost of debt further.

Finally, Equation (18) corresponds to the optimality condition for the policy rate. This equation can be interpreted as a public debt adjustment condition. The first three motives of why having more debt may increase the social cost of debt exposed in the previous paragraph also shape the desirability of paying this debt and they are scaled up by the excess government debt, that is, by $B_t - \bar{B}$. Finally, the effect that this increase in the policy rate has on hand-to-mouth, essentially a decrease in their cash-on-hand, is also considered by the planner.

Optimal public debt adjustment. Having introduced the economic interpretation of the multipliers and necessary conditions, I provide the main proposition of this section, the formula for optimal public debt adjustment without aggregate shocks. This result follows from a sensible combination of the necessary conditions discussed above, and the proof is given in Appendix A.2.

Proposition 6. ***(Optimal Public Debt Adjustment without Aggregate Shocks)*** *Optimal public debt adjustment without aggregate shocks is given by*

$$\tau_t^a = \frac{B_t - \bar{B}}{\zeta_\tau^{HtM}} \left(\mu_t - \dot{\theta}_t - \frac{\rho - r_t}{B_t} \mathbb{E}_{g_t} \left[a \partial_c u_t(a,z) \left(1 + \Theta_t(a,z) \right) \right] + \zeta_B^{HtM} \right) \tag{21}$$

where $\Theta_t(a,z) = \frac{\phi_t(a,z)}{g_t(a,z)} - \frac{\tilde{\chi}_t(a,z)}{\partial_c u_t(a,z)}$*, and where* $\mathbb{E}_{g_t}$ *is the expectation operator defined using the cross-sectional distribution*[19].

The formula for optimal public debt adjustment without aggregate shocks shows how the planner weights the pros and cons of repaying the debt via distortionary labor income taxes. To shed light on all the mechanisms dictating the optimal policy rate, I rewrite the proposition as follows. At a high level, the formula for optimal tax can be written as

$$\tau_t^a = (B_t - \bar{B}) \times \text{"sum of social costs of govt debt"}.$$

Indeed, the higher the excess government debt relative to its long-run target $\bar{B}$, the larger the policy rate τ_t^a will be (assuming debt is socially costly in the first place, something that it's not straightforward in the RA case, see Proposition 7 bellow and neither in the HA case with standard preferences, see Auclert et al. (2023)). Alternatively, given a (positive) level of excess debt, the socially costlier holding government debt is the larger the policy rate, and hence the labor tax τ_t will be. In the next paragraphs, I provide more details on the components of the social costs of having government debt. Precisely, I will look at

$$\tau_t^a = \overbrace{\left(B_t - \bar{B}\right)}^{\text{excess debt}} \Bigg(\overbrace{\mu_t - \dot{\theta}_t}^{\text{contribution representative agent}} + \overbrace{(r_t - \rho)\frac{A_t}{B_t}\mathbb{E}_{g_t}\left[\partial_c u_t(a,z)\left(1 + \Theta_t(a,z)\right)\right]}^{\text{contribution incomplete markets}}$$
$$\underbrace{+ \frac{r_t - \rho}{B_t}\mathbb{C}ov_{g_t}\left[a,\, \partial_c u_t(a,z)\left(1 + \Theta_t(a,z)\right)\right] + \zeta_B^{\text{HtM}}\Bigg)\frac{1}{\zeta_\tau^{\text{HtM}}}}_{\text{inequality concerns}},$$

where the covariance operator is defined using $\mathbb{E}_{g_t}$. Now, I proceed to analyze the different social costs that shape the optimal tax rate, separating them into the contribution of the representative agent, the one coming from incomplete markets, and concluding with the redistribution concerns.

[19] Formally, the cross-sectional expectation operator is defined as $\mathbb{E}_{g_t}[x_t(a,z)] = \iint x_t(a,z) g_t(a,z)\, da dz$, with $x_t(a,z) : \mathbb{R}^2 \to \mathbb{R}$ for all t.

The first term, "$\mu_t - \dot{\theta}_t$", represents the contribution from a representative agent benchmark as it is these are the only two components that are relevant in such scenario. More precisely, the following proposition defines the target rule that a Ramsey planner follows in the representative agent limit of this framework.

Proposition 7. ***(Targeting Rule for Public Debt Adjustment without Aggregate Shocks with RA)*** *In the RA limit, optimal public debt adjustment is characterized by*

$$\dot{\theta}_t = \mu_t. \tag{22}$$

To understand this targeting rule, it is useful to consider the tax-smoothing setting à la Lucas and Stokey (1983). There, the planner wants to completely smooth taxes and public debt is costly, so $\theta_t = 0$. Once we introduce a supply side, an increase in public debt puts upward pressure in the demand of assets. If this is socially costly ($\mu_t > 0$), government debt acquires a positive social cost captured by θ_t. Under the necessary conditions for convergence, the law of motion of the public debt penalty is given by (22): as the cost of excess demand of assets decreases towards zero, the penalty converges towards a stationary, possibly non-zero level θ^{ss}. In particular note that is government debt puts upward pressure to the steady-state level of assets (i.e. $\mu_\infty > 0$), the public debt penalty tends towards infinity. In any case, if such a phenomenon is costly, namely is $\mu_t - \dot{\theta}_t > 0$, the planner will have the incentive to increase the tax rate τ_t^a to avoid the negative consequences associated with the presence of debt.

Next, I consider the contribution of incomplete markets into the optimal policy rate. This effect is due to capital income gains perceived by the median consumer, and which are only presence if markets are incomplete, that is, if households pay a premium for holding assets $r_t < \rho$. I start by looking at the case where $\Theta_t(a, z) = 0$. It becomes clear that this term is negatively contributing to decreasing the tax rate. Because the median household holds an amount A_t of assets (recall, there is a fixed mass one of consumers) an increase in the interest rate generated by having more debt will increase the capital income gains of this household. These gains, however, are weighted by the increasing cost of having more debt incurred by the planner, that is, it also depends negatively on the amount of debt. Finally, it is weighted by the differential between the market and the private time discounting rate since debt is only costly to the extent the market pays a lower interest rate than the private discount rate. Whenever $\Theta_t(a, z)$ is non-zero, this conclusion may be affected. This term asks the following three questions, from a social point of view: i) is the individual over- or under-consuming, ii) is the individual over- or under-working, and ii) does the individual have a too large, or too small, lifetime utility? Depending on the response to these three questions, the planner will over- or under-weight the private utility gains associated with an increase in the interest rate for the median household. For instance, if the planner perceives the individual is under-consuming, it will under-weight the utility gains generated by the rise in the interest rate, which in turn will require larger increases to reach a given social welfare level relative to the case where $\Theta_t(a,z) = 0$.

Finally, the planner also has inequality considerations when optimally setting the tax rate. These inequality concerns depend on the extend to which the capital gains are unequally distributed, the direct utility cost for hand-to-mouth households of increasing public debt and the utility cost for hand-to-mouth households of increasing the tax rate.

I start by the inequality concerns related to the capital gains and I first the case where $\Theta_t(a,z) = 0$ to abstract from the extra social considerations that the planner has about the effect of government debt on households. Since the marginal utility of consumption is decreasing in wealth, the covariance $\mathbb{C}ov_{g_t}\left[a,\, \partial_c u_t(a,z)\right] < 0$, and the larger in absolute terms this covariance is, the more unequal the decentralized economy is. Intuitively, since an economy in which there is more excess debt requires of a larger adjustment, whose negative consequences are specially carried on by the wealth-poor households, and the fact that the planner dislikes having unequal individuals makes her willing to repay the debt faster. As before, these conclusions are subject to the extent the parameter $\Theta_t(a,z)$ matters, in particular, if it comoves with wealth it will decrease the relevance of the redistribution motive, and vice versa if it negatively comoves. Finally, the redistribution motive is weighted by $(r_t - \rho)/B_t$ capturing that the effect that excess debt has over the cross-section of households only matters to the extent $r_t < \rho$.

Next, there is a utility cost of increasing the level of government debt for a constrained individual denoted ζ_B^{HtM}. Under mild conditions[20], this constant is positive, which means the planner aims to avoid large levels of government debt because it implies a decrease in the cash-on-hand to the hand-to-mouth households. Finally, the utility cost for hand-to-mouth of increasing the tax rate, ζ_τ^{HtM}, is positive under mild conditions. In sum, if a larger tax generates utility losses from hand-to-mouth, then the planner wants to set a lower tax rate.

5 Optimal Policy with Aggregate Shocks

While the previous section describes how a utilitarian Ramsey planner sets optimally public debt adjustment in a setting with heterogeneous agents but without aggregate risk, this section extends to the case with aggregate risk. The resulting optimal policy rate formula is the main theoretical contribution of the present paper.

Ramsey problem with commitment. In this section, I introduce back aggregate risk, so that Assumption 2 about the notion of bounded rationality is now relevant and the state-space is (a, z, Γ), where $\Gamma = (Z, K, B)$ is the aggregate state. I start by extending Definition 4 to the case with aggregate risk. Before that, a technical point is order. The Ramsey planner needs to set allocations

[20] These mild conditions include the negative effects that a higher labor tax has on constrained individuals' consumption that cannot be compensated by a larger increase in their incentives to provide more labor supply. Instead, if the utility losses coming from less consumption are greater than the gains from more incentives to work, then this effect pushes for a looser policy rate.

that are consistent with the beliefs of individuals, which in particular means that if for any given value $\tilde{B}$, at aggregate state $\Gamma = (Z, K, \tilde{B})$, the planner needs to choose a mapping $B_t : \mathbb{R}^3 \to \mathbb{R}$ such that $B_t(\Gamma) = \tilde{B}$, and the same applies for capital. Moreover, to soften notions, I usually drop the dependence on Γ for government debt and capital.

Definition 8. ***(Ramsey Planner with Aggregate Shocks)*** *A Ramsey planner with infinite commitment horizon $[0, \infty)$ chooses allocations, a policy, and prices*

$$\boldsymbol{X} = \Big\{c_t(a,z,\Gamma), \ell_t(a,z,\Gamma), V_t(a,z,\Gamma), g_t(a,z,\Gamma), \tau_t^a, r_t(\Gamma), w_t(\Gamma), K_t(\Gamma), B_t(\Gamma)\Big\}_{t=0}^{\infty}$$

as well as multipliers

$$\boldsymbol{M} = \Big\{\phi_t(a,z,\Gamma), \chi_t(a,z,\Gamma), \psi_t(a,z,\Gamma), \lambda_t(a,z,\Gamma), \mu_t(\Gamma), \theta_t(\Gamma)\Big\}_{t=0}^{\infty}$$

to maximize social welfare subject to implementability conditions

$$\rho V_t(a,z,\Gamma) = u_t(a,z,\Gamma) + \partial_t V_t(a,z,\Gamma) + \mathcal{A}_t V_t(a,z,\Gamma) \tag{HJB(Γ)}$$

$$\partial_c u_t(a,z,\Gamma) = \partial_a V_t(a,z,\Gamma) \tag{FOC c (Γ)}$$

$$\partial_\ell u_t(a,z,\Gamma) = -(1-\tau_t) w_t(\Gamma) z \partial_c u_t(a,z,\Gamma) \tag{FOC ℓ (Γ)}$$

$$\partial_t g_t(a,z,\Gamma) = \mathcal{A}_t^* g_t(a,z,\Gamma) \tag{KFE(Γ)}$$

$$K_t + B_t = \iint a g_t(a,z,\Gamma) da dz \tag{McK(Γ)}$$

$$\dot{B}_t = r_t(\Gamma) B_t + \bar{G} - \tau_t w_t(\Gamma) L_t(\Gamma) \tag{GvtBC(Γ)}$$

taking as given the initial cross-sectional distribution $g_0(a,z,\Gamma)$ and where $\tau_t = \bar{\tau} + \tau_t^a(B_t(\Gamma) - \bar{B})$. That is,

$$\mathcal{W} = \min_{\boldsymbol{X}} \max_{\boldsymbol{M}} \mathbb{E}_0 \Big[L(g_0)\Big]$$

where the expectation $\mathbb{E}_0$ is taken over the idiosyncratic and aggregate state, and $L(g_0)$ is the planner's Lagrangian given an initial cross-sectional distribution g_0:

$$\begin{aligned} L(g_0) = \int_0^\infty e^{-\rho t} \int \Bigg\{ \iint \Bigg\{ & \omega_t(a,z,\Gamma) u_t(a,z,\Gamma) g_t(a,z,\Gamma) \\ & + \phi_t(a,z,\Gamma) \Big[-\rho V_t(a,z,\Gamma) + u_t(a,z,\Gamma) + \partial_t V_t(a,z,\Gamma) + \mathcal{A}_t V_t(a,z,\Gamma) \Big] \\ & + \chi_t(a,z,\Gamma) \Big[\partial_c u_t(a,z,\Gamma) - \partial_a V_t(a,z,\Gamma) \Big] \\ & + \psi_t(a,z,\Gamma) \Big[\partial_\ell u_t(a,z,\Gamma) + (1-\tau_t) w_t(\Gamma) z \partial_a V_t(a,z,\Gamma) \Big] \end{aligned}$$

$$+ \lambda_t(a,z,\Gamma)\Big[- \partial_t g_t(a,z,\Gamma) + \mathcal{A}_t^* g_t(a,z,\Gamma)\Big]\Big\} dadz$$

$$+ \theta_t(\Gamma)\Big[- \dot{B}_t + r_t(\Gamma)B_t + \bar{G} - \tau_t w_t(\Gamma)L_t(\Gamma)\Big]$$

$$+ \mu_t(\Gamma)\Big[\iint a g_t(a,z,\Gamma) dadz - K_t - B_t\Big]\Big\} d\Gamma\, dt$$

As in the previous section, I focus on the utilitarian case by setting the welfare weight $\omega(a,z,\Gamma) = 1$. There are six implementability conditions, ordered as before: a Bellman equation, the first-order conditions for consumption and labor, the evolution of the cross-sectional distribution, the dynamics of debt, and market clearing. Notice that now, everything is indexed by the aggregate state Γ.

Proposition 9. ***(Necessary Optimality Conditions with Aggregate Shocks)*** *The first-order conditions that solve the Ramsey planner with aggregate shocks are*[21]

$$\partial_t \phi_t(a,z,\Gamma) = \mathcal{A}_t^* \phi_t(a,z,\Gamma) + \partial_a \chi_t(a,z,\Gamma) - \partial_a \psi_t(a,z,\Gamma)(1-\tau_t) w_t(\Gamma) z \tag{23}$$

$$\rho \lambda_t(a,z,\Gamma) = u_t(a,z,\Gamma) + \mu_t(\Gamma) a - \tau_t w_t(\Gamma)\theta_t(\Gamma)\ell_t(a,z,\Gamma) + \mathcal{A}_t \lambda_t(a,z,\Gamma) + \partial_t \lambda_t(a,z,\Gamma) \tag{24}$$

$$0 = \partial_c u_t(a,z,\Gamma) - \partial_a \lambda_t(a,z,\Gamma) - \tilde{\chi}_t(a,z,\Gamma) \tag{25}$$

$$0 = \partial_\ell u_t(a,z,\Gamma) + (1-\tau_t) w_t(\Gamma) \partial_a \lambda_t(a,z,\Gamma) - \tilde{\psi}_t(a,z,\Gamma) - \theta_t(\Gamma)\tau_t w_t(\Gamma) z \tag{26}$$

$$0 = \iint a \Big[\phi_t(a,z,\Gamma)\partial_a V_t(a,z,\Gamma) + \partial_a \lambda_t(a,z,\Gamma) g_t(a,z,\Gamma)\Big] dadz + \zeta_r^{HtM}(\Gamma) + \theta_t(\Gamma) B_t \tag{27}$$

$$\begin{aligned} \dot{\theta}_t(\Gamma) = {} & \tau_t^a w_t(\Gamma) \iint z \Big[\ell_t(a,z,\Gamma)(\phi_t(a,z,\Gamma)\partial_a V_t(a,z,\Gamma) + \partial_a \lambda_t(a,z,\Gamma) g_t(a,z,\Gamma)) \\ & + \partial_a \psi_t(a,z,\Gamma) V_t(a,z,\Gamma)\Big] dadz + \tau_t^a \theta_t(\Gamma) w_t(\Gamma) L_t(\Gamma) + \theta_t(\Gamma)(\rho - r_t(\Gamma)) \\ & + \mu_t(\Gamma) + \zeta_B^{HtM}(\Gamma) - \mathcal{E}_B(\Gamma) \end{aligned} \tag{28}$$

$$\begin{aligned} 0 = {} & w_t(\Gamma)(B_t - \bar{B}) \iint z \Big[\ell_t(a,z,\Gamma)(\phi_t(a,z,\Gamma)\partial_a V_t(a,z,\Gamma) + \partial_a \lambda_t(a,z,\Gamma) g_t(a,z,\Gamma)) \\ & + \partial_a \psi_t(a,z,\Gamma) V_t(a,z,\Gamma)\Big] dadz + \theta_t(\Gamma)(B_t - \bar{B}) w_t(\Gamma) L_t(\Gamma) + \zeta_\tau^{HtM} + \mathcal{E}_\tau \end{aligned} \tag{29}$$

[21] Footnote 18 surveys what we know about the existence and uniqueness of a Ramsey allocation in Aiyagari economies with heterogeneous but without aggregate shocks. In the economy with aggregate shocks and under the notion of bounded rationality used throughout this section, I verify numerically that for a wide range of values of τ^a, there exists a stationary equilibrium and the social welfare function has a unique maximum.

where

$$\tilde{\chi}_t(a,z,\Gamma) = -\frac{\chi_t(a,z,\Gamma)}{g_t(a,z,\Gamma)}\partial_{cc}u_t(a,z,\Gamma) - \frac{\psi_t(a,z,\Gamma)}{g_t(a,z,\Gamma)}\partial_{c\ell}u_t(a,z,\Gamma),$$

$$\tilde{\psi}_t(a,z,\Gamma) = -\frac{\psi_t(a,z,\Gamma)}{g_t(a,z,\Gamma)}\partial_{\ell\ell}u_t(a,z,\Gamma) - \frac{\chi_t(a,z,\Gamma)}{g_t(a,z,\Gamma)}\partial_{c\ell}u_t(a,z,\Gamma),$$

where $\mathcal{E}_B(\Gamma)$ *and* $\mathcal{E}_\tau$ *are defined respectively in* (34) *and* (35)*, and where* $\zeta_r^{HtM}(\Gamma)$*,* $\zeta_B^{HtM}(\Gamma)$ *and* ζ_τ^{HtM} *are respectively defined in* (36)*,* (37) *and* (38)*; where the initial conditions for the forward-looking multipliers are* $\theta_0(\Gamma) = \phi_0(a,z,\Gamma) = 0$*; and where equations* (23)*-*(26) *hold in the interior space — a consequence of* (HJB(Γ))*,* (FOC c (Γ))*,* (FOC ℓ (Γ)) *and* (KFE(Γ)) *being defined only in the interior space.*

Equations (23) to (28) correspond to the optimality conditions for i) the value function, ii) the cross-sectional distribution, iii) consumption, iv) labor supply, v) the interest rate, and vi) government debt. Equation (29) corresponds to the optimality condition for the debt adjustment. The intuitions for the necessary conditions are the same as in Section 4 except for the aggregate state Γ and the two functions $\mathcal{E}_B(\Gamma)$ and $\mathcal{E}_\tau$. These last two functions are a byproduct of the notion of bounded rationality, and they represent the social cost induced by changing the belief formation due to the planner's actions. I continue focusing on the utilitarian case by letting $\omega_t(a,z,\Gamma) = 1$.

Optimal long-run public debt adjustment. Having introduced the necessary conditions for optimality, now extend Proposition 6 to include aggregate risk. Hereafter, I focus on the long-run Ramsey allocation.

Proposition 10. ***(Optimal Public Debt Adjustment with Aggregate Shocks in the Long-Run)*** *Optimal public debt adjustment with aggregate shocks in the long-run is given by*

$$\tau^a = \frac{1}{\zeta_\tau^{HtM} + \mathcal{E}_\tau}\mathbb{C}ov_\Gamma\left[B,\ (\rho - r)\tilde{\theta} + \tilde{\mu} + \tilde{\zeta}_B^{HtM} - \tilde{\mathcal{E}}_B\right]$$

where variables with tildes denote variables normalized by the ergodic distribution of the aggregate state Γ*, with associated covariance operator* $\mathbb{C}ov_\Gamma[\cdot,\cdot]$[22]*.*

The formula for optimal public debt adjustment with aggregate shocks shows how the planner weighs the pros and cons of repaying the debt via distortionary labor income taxes. To shed light on all the mechanisms dictating the optimal policy rate, I rewrite the proposition as follows. At a high level, the formula for optimal tax can be written as

$$\tau_t^a = \mathbb{C}ov_\Gamma\Big(B,\ \text{“sum of social costs of govt debt”}\Big).$$

[22] Formally, a distribution g_Γ emerges over the aggregate state space (TFP, capital and government debt). This is because, in the stochastic steady state, TFP shocks are hitting the economy which change the value of capital and public debt. The resulting ergodic distribution is denoted g_Γ. Then, it is possible to define an expectation and a covariance operator.

Whenever the social costs of having more government debt comove with the actual level of debt, or in other words, when the social costs of having debt increase as debt increases, the optimal policy rate τ^a becomes larger. This is the main difference relative to the case without aggregate shocks, while without aggregate shocks what matters is the amount of excess debt, now what matters is whether debt is relatively more costly in those periods where debt is large.

As done in Section 4, to gain more insights about what determines the optimal policy rate, I decompose τ^a as follows, where I drop dependence on the state space for convenience

$$\tau^a = \frac{1}{\zeta_\tau^{\text{HtM}} + \mathcal{E}_\tau}\Bigg(\overbrace{\mathbb{C}ov_\Gamma\left[B,\, \tilde{\mu}\right] + \mathbb{C}ov_\Gamma\left[B,\, (r-\rho)\frac{A}{B}\mathbb{E}_{g|\Gamma}\left[u_c(c,\ell)\Big(1+\Theta\Big)\right]\right] - \mathbb{C}ov_\Gamma\left[B,\, \tilde{\mathcal{E}}_B\right]}^{\text{representative agent}}$$
$$\underbrace{+\,\mathbb{C}ov_\Gamma\left[B,\, \frac{r-\rho}{B}\mathbb{C}ov_{g|\Gamma}\left[a,\, u_c(c,\ell)\Big(1+\Theta\Big)\right]\right] + \mathbb{C}ov_\Gamma\left[B,\, \tilde{\zeta}_B^{\text{HtM}}\right]}_{\text{inequality concerns}}\Bigg)$$

where $g|\Gamma$ is the distribution over the idiosyncratic states at a given aggregate state Γ, and where $\mathbb{E}_{g|\Gamma}$ and $\mathbb{C}ov_{g|\Gamma}$ are, respectively, the expectation and covariance operator defined using this distribution[23].

The long-run optimal adjustment of public debt depends on various factors. As for the case without aggregate shocks, I shall start by the contribution from the representative agent[24].

Corollary 11. ***(Optimal Public Debt Adjustment in the RA limit)*** *Is given by*

$$\tau^a = \frac{1}{\mathcal{E}_\tau}\Bigg(\mathbb{C}ov_\Gamma\left[B,\, \tilde{\mu}\right] + \mathbb{C}ov_\Gamma\left[B,\, (r-\rho)\frac{A}{B}\Big(u_c(C,L)\Big(1+\Theta\Big)\Big)\right] - \mathbb{C}ov_\Gamma\left[B,\, \tilde{\mathcal{E}}_B\right]\Bigg).$$

Four terms are present in this formula. First, the social shadow cost of excess government debt, interpreted as the negative consequences of crowding out of capital due to excess government debt, matters. This formula shows that these costs matter to the extent they comove with the level of government debt, that is, it only matters if the crowding out is aggravated whenever the economy is in a period of high debt, that is, a recession and its aftermath.

Second, the optimal tax rate depends on the comovement of government debt and the capital income gains of the median household. A useful approximation of the first covariance term is

$$\frac{1}{\overline{B}}\mathbb{C}ov_\Gamma\left[B,\, (r-\rho)A\mathbb{E}_{g|\Gamma}\left[u_c(c,\ell)\Big(1+\Theta\Big)\right]\right]$$

[23] Formally, if $g:\mathbb{R}^5 \to \mathbb{R}$ is the stochastic steady-state distribution function over (a,z,Γ) and g_Γ is the stochastic steady-state distribution function of Γ, the distribution over the idiosyncratic space (a,z) conditional on an aggregate space Γ is $g_{\text{idio}|\Gamma}(a,z) = \frac{g(a,z,\Gamma)}{g_\Gamma(\Gamma)}$ for all Γ.

[24] I assume to be such that $\zeta_\tau^{\text{HtM}} + \mathcal{E}_\tau > 0$.

highlighting that the countercyclicality of capital income gains from changing the interest rate due to an increase in debt that contributes to increasing τ^a. In other words, if this particular social benefit of having more debt is greater whenever debt is high — equivalently, output is low since government debt is countercyclical — this term calls for a slower repayment of public debt.

Finally, the terms $\tilde{\mathcal{E}}_B$ and $\mathcal{E}_\tau$ appear. As said earlier, they represent the social cost of changing the perceived laws of motion due to adjustments by the Ramsey planner in both the level of public debt and the policy rate. Indeed, in the case where individuals do not form expectations about the future, namely individuals are myopic, these terms disappear. The following corollary gives the optimal policy in such scenario and show the quantitative relevance of these terms in the next section.

Corollary 12. ***(Optimal Public Debt Adjustment with Myopic Individuals)*** *Is given by*

$$\tau^a = \frac{1}{\zeta_\tau^{HtM}} \Bigg(\mathbb{C}ov_\Gamma \left[B,\, (r-\rho)\frac{A}{B}\mathbb{E}_{g|\Gamma}\left[u_c(c,\ell)\Big(1+\Theta\Big) \right] \right] + \mathbb{C}ov_\Gamma \left[B,\, \tilde{\mu} \right]$$
$$+ \mathbb{C}ov_\Gamma \left[B,\, \frac{r-\rho}{B}\mathbb{C}ov_{g|\Gamma}\left[a,\, u_c(c,\ell)\Big(1+\Theta\Big) \right] \right] + \mathbb{C}ov_\Gamma \left[B,\, \tilde{\zeta}_B^{HtM} \right] \Bigg)$$

Next, the optimal policy rate takes into account its effects over the cross-sectional distribution, namely the inequality concerns. First, it depends negatively, as is the case of Proposition 6, on the utility cost of increasing the policy rate for hand-to-mouth individuals.

Second, the same redistribution motive as in the case of no aggregate shock is present, albeit now what matters is how this redistribution motive comoves with the level of government debt. To see this consider the following approximation

$$\mathbb{C}ov_\Gamma \left[B,\, \frac{r-\rho}{B}\mathbb{C}ov_{g|\Gamma}\left[a,\, u_c(c,\ell)\Big(1+\Theta\Big) \right] \right] \approx \frac{1}{B}\mathbb{C}ov_\Gamma \left[B,\, (r-\rho)\mathbb{C}ov_{g|\Gamma}\left[a,\, u_c(c,\ell)\Big(1+\Theta\Big) \right] \right].$$

In an economy in which the redistribution motive increases in absolute value during bad times, something plausible since consumption over the business cycle for wealthy individuals fluctuates less relative to wealth-poor individuals, this covariance will tend to be negative. Of course, these conclusions might be affected by the comovement of the term $\Theta(a, z, \Gamma)$, whose effect remains a quantitative question.

Finally, the comovement of the utility cost for hand-to-mouth of increasing the level of government debt with the actual level of government debt also increases the optimal tax rate. The interpretation is straightforward, if the utility cost of an extra unit of government debt for these individuals is larger when there is more debt, the planner aims to avoid these situations, which in turn pushes her to have a faster adjustment of public debt.

Both the utility costs experienced by hand-to-mouth households and the redistributive motive are related to the inequality considerations. However, there is a major difference which is that the redistributive motive only matters to the extend the planner can commit to a future path for government debt. Indeed, the redistribution motive, as well as the capital income gains discussed above, emerge when public debt puts upward pressure to the interest rate.

Corollary 13. ***(Optimal Public Debt Adjustment with Discretion)*** *Is given by*

$$\tau^a = \frac{1}{\zeta_\tau^{HtM} + \mathcal{E}_\tau}\left(\mathbb{C}ov_\Gamma\left[B,\, \tilde{\mu}\right] + \mathbb{C}ov_\Gamma\left[B,\, \tilde{\zeta}_B^{HtM}\right] + +\mathbb{C}ov_\Gamma\left[B,\, \tilde{\mathcal{E}}_B\right]\right)$$

The relevant trade-offs reduces to the utility costs experienced by hand-to-mouth due to a lower post-tax labor income, the bounded rationality terms ($\tilde{\mathcal{E}}_B$ and $\mathcal{E}_\tau$) and the social cost of excess demand of assets. All these social costs have contemporaneous effects on households: either by decreasing the cash-on-hand of some individuals, or by crowding out capital, and hence, decreasing wages.

6 Quantitative Assessment

In this section, I numerically solve the Ramsey planner introduced in Definition 8, which is a planner that chooses the optimal speed of public debt adjustment in an environment with heterogeneous agents and aggregate risk. The aim of this section is first to quantify the policy rate found in Proposition 10, and second to explore its implications for the macroeconomic aggregates when the economy faces a negative shock. In this section, I use a slightly different tax function, which is an approximation of (4), the one that I have used to derive results. In the numerical application, I use

$$\tau_t = \bar{\tau} + \tau^a \log\left(\frac{B_t}{\overline{B}}\right) \approx \bar{\tau} + \frac{\tau^a}{\overline{B}}\left(B_t - \overline{B}\right) \tag{30}$$

so that both expressions are, using a first-order Taylor approximation, the same up to rescaling by the constant $\frac{1}{\overline{B}}$ the parameter that governs the speed of adjustment of public debt. The reason for this choice is that, while (4) is suitable for deriving an explicit tax formula, the log specification — which is also used in McKay and Reis (2016) — is better suited for numerical exercises.

6.1 Calibration

The goal of the calibration is to be consistent with state-of-the-art parametrizations of one-asset heterogeneous agent models with capital and bonds. It closely follows the "one-account heterogeneous-agent model with high-liquidity" of Auclert et al. (2018). Table 1 summarizes the calibration.

Households. I assume a CRRA utility separable utility function between consumption and labor of the form

$$u(c, \ell) = \frac{c^{1-\sigma}}{1-\sigma} - \Psi\frac{\ell^{1+\nu}}{1+\nu}$$

Description	Parameter	Value	Target
Preferences			
Discount rate (p.a.)	ρ	0.065	Riskfree rate 5%
Intertemporal elasticity	σ	1	Standard
Frisch elasticity labor supply	ν	1	Standard
Household			
Borrowing constraint	$\underline{a}$	0	Standard
Persistence earnings	θ_z	0.0943	Persistence log-wages
Volatility earnings	σ_z	0.2265	Volatility log-wages
Firms			
Capital share	α	0.3	Share labor 30%
Capital depreciation (p.a.)	δ	0.09	$K/Y = 2.26$
Persistence TFP	θ_Z	1	Autocorrelation TFP
Volatility TFP	σ_Z	0.007	Volatility TFP
Government			
Government spending	G	0.2	$G/Y = 20\%$
Government debt	B	0.7	$B/Y = 70\%$
Tax rate	τ	33%	Internally calibrated

Table 1. List of Calibrated Parameters

where in line with Kaplan et al. (2018) and Auclert et al. (2018), I assume an intertemporal elasticity $\sigma = 1$ (that is, a log-utility specification for consumption), and a Frisch elasticity of labor supply $\nu = 1$, which is in the bounds of Chetty (2012)'s estimate. I use the parameter Ψ from the disutility of labor to normalize steady-state output $Y = 1$. Next, I set the discount rate ρ to target a steady-state risk-free annual rate $r = 0.05$, which captures the average combined real return on capital and government bonds from 1969 to 2019. This requires $\rho = 0.0658$. As in Auclert et al. (2018), I assume households cannot borrow, a common assumption in one-asset models in which households can trade capital. I use Floden and Lindé (2001)'s estimates of the US wage process to calibrate the parameters of the idiosyncratic shock. In discrete time, the estimated persistence of the AR(1) process for wages is 0.91, whose continuous-time counterpart is $\theta_z = -\ln(0.91) = 0.0943$. The estimated standard deviation is $\varsigma^2 = 0.0426$, which in continuous time yields $\sigma_z = \sqrt{2\theta_z\varsigma/(1-e^{-2\theta_z})} = 0.2265$.

Firms. I normalize the steady-state TFP level $Z = 1$. I set the labor share $1 - \alpha = 0.706$. Then, to target a capital-to-income ratio of $K/Y = 2.26$, I set the annual rate of depreciation of capital to $\delta = 0.096$. For the evolution of TFP, I assume log-productivity is given by

$$d\log(Z_t) = -\theta_z \log(Z_t) + \sigma_Z d\tilde{W}_t$$

where $\tilde{W}_t$ is a Brownian motion. To target the quarterly autocorrelation and volatility of TFP observed in the data (Ahn et al., 2018), it requires choosing an annual persistence $\theta_Z = 1$ and annual volatility $\sigma_Z = 0.007$.

Fiscal policy. At the steady-state, for government spending to GDP ratio, I target $G/Y = 0.2$; and for debt to GDP ratio, I target $B/Y = 0.7$. These are standard values and are those used in Auclert et al. (2018) and in Campos et al. (2024). The resulting marginal labor income tax is $\tau = 0.33$.

Untargeted moments. The steady-state of the model manages to match correctly the assets to pre-tax labor income $\frac{A}{wL} = 4.2$ form Kaplan and Violante (2022) and the share of hand-to-mouth households that is equal to 30% (Kaplan et al., 2018).

6.2 Numerical Implementation

I solve the model numerically using the finite-difference method developed by Achdou et al. (2022). To handle the high dimension state-space, I use the sparse grids tools developed by Schaab and Zhang (2022). As Assumption 2 naturally suggests, I solve the stochastic steady state using the Krusell and Smith (1998) algorithm. The algorithm produces a Den Haan (2010) error between the perceived laws of motion and the simulated ones — the preferred accuracy metric in Ahn et al. (2018) — of around 0.2, which is approximately the same as a baseline Krusell-Smith economy. In Appendix B, I provide further details on the algorithm.

6.3 Optimal Public Debt Adjustment

The optimal tax rate in the model with heterogeneous agents and aggregate risk is $\tau^a = 5.85$[25]. Under the tax function in (30), the interpretation of the optimal tax is as follows: if the government debt is 1% above the long-run target of 70% for debt-to-GDP, the labor income tax is 41% ($0.33 + 5.85 \ln(0.71/0.7) = 0.41$). Instead, if public debt is at the long-run level, labor income tax equals 33%. This result can be interpreted as the model suggesting a high level of fiscal austerity whenever public debt is high. During the first periods of high debt, the economy is still in a recession, and yet optimal policy suggests that a large adjustment is needed. This value might seem unrealistically high if being compared with the debt-to-GDP ratios seen nowadays, however, note that in my simulations at the optimal policy, debt never reaches large levels given the large fiscal adjustment[26].

[25] I search for the optimal policy rate τ^a and check that it is an interior solution.

[26] Real fiscal systems have other instruments to conduct large fiscal adjustments. For instance, from 2008-2014, all four southern countries in Europe (Greece, Italy, Portugal, and Spain) increased their VAT taxes as one of the ways of conducting the fiscal adjustment needed. These countries also introduced reforms in the labor market, the public sector, the social security system, etc. By concentrating all the adjustments into one tool, I generate a disproportionally large optimal labor income tax, but it allows me to identify the trade-offs determining the optimal response. Increasing the number of instruments will plausibly dampen the optimal labor income tax rate, and it is left for future research.

I compare the model with estimates from the data computed by Auclert and Rognlie (2018). They estimate $dB_t = \rho_B(dB_t + dG_t) + \varepsilon_t$, that is, a process for first differences of government debt B_t. Their process also includes changes in government spending dG_t and a shock ε_t. In the data, they find an annual persistence of $\rho_B = 0.93$; whereas in my model, I find an estimated persistence with simulated data at the optimum equal to $\rho_G^{sim} = 0.91$. My model suggests the government should pay back its debt slightly faster than what it has done in the past years, or equivalently, that government debt should be slightly less persistent. Reassuringly, the implied dynamics at the optimal policy rate are very similar to those estimated from US data. These authors acknowledge the uncertainty around this point estimate. Nonetheless, if I compare the implied persistence ρ_G^{sim} with Galí et al. (2007)'s estimates of annual persistence of government debt transformed into first differences, I obtain an annualized persistence of 0.903, which is again, very close to the optimal one.

To put these numbers into perspective, I solve the representative agent limit of the model in which aggregate uncertainty is present. The optimal fiscal adjustment is substantially lower as it equals $\tau^{a|RA} = 0.54$. This means that if the government debt is 1% above the long-run target of 70% for debt-to-GDP, the labor income tax is 33.7% ($0.33 + 0.54 \ln(0.71/0.7) = 0.337$), only 0.7 percentage points above the long-run tax. This small increase needs to be compared to the 8 percentage point increase that such excess debt generates in the full model that also includes heterogeneous agents. The annual persistence of debt estimated from simulated data increases until reaching $\rho_G^{sim|RA} = 0.98$. Without binding borrowing constraints, government debt is more persistent, more than the estimates from the data and the estimates at the optimal policy when including household heterogeneity.

To illustrate the role of anticipations about the future and find the optimal policy according to Corollary 12, I solve for optimal policy by setting the perceived laws of motion $\mu_K(\Gamma) = 0$ and $\mu_B(\Gamma) = 0$. With this specification, individuals expect the same prices in the next period. Once the next period arrives, prices change excursively due to changes in the TFP. The optimal policy becomes $\tau^a = 19.95$, namely an almost complete stabilization of government debt. Whenever a negative shock induces a public deficit and an increase in debt, the government heavily taxes individuals to bring it back to the long-run level. The reason is that increasing taxes is socially costly: individuals anticipate them and save in the present to insure themselves against tax hikes in the future. However, the planner perceives it as socially desirable to have stable debt, which requires rapidly addressing public deficits with sizable taxes. Under myopic households, the first social cost disappears, making the optimal tax substantially larger than the one with individuals who can forecast the future.

6.4 Optimal Response to a Negative TFP Shock

In Figure 1, I illustrate the IRFs of a negative TFP shock in the model with heterogeneous agents and aggregate shocks (HA-AR) and in the model with a representative agents and aggregate shocks

(RA-AR). Notice that in this exercise aggregate risk is relevant, each IRF is the average across 100 Monte Carlo simulations where I only negatively shock the innovation at period 0. The size of the IRFs is normalized by the size of the (negative) shock.

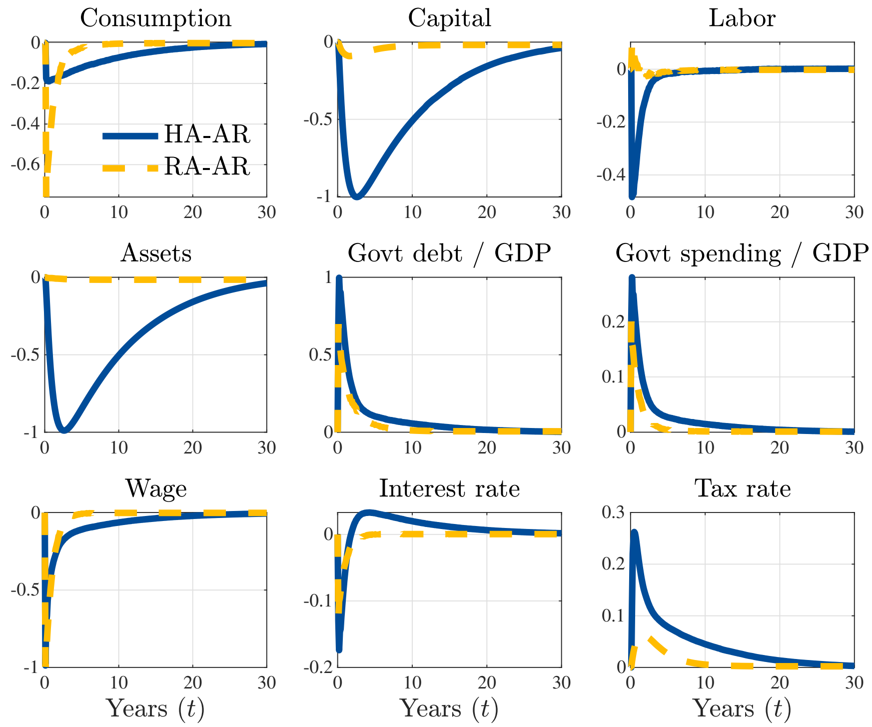

Figure 1. Simulated IRFs in the stochastic steady state for a negative TFP shock.

I start by the response in the full model, namely with heterogeneous agents and aggregate risk (HA-AR). The response of macroeconomic aggregates is standard, the shock generates a decrease in consumption, capital, and effective hours of labor. While labor hours quickly return to their normal path, the impact of capital lasts for longer. In turn, this generates a decrease in output. As mentioned earlier, I keep the level of government spending constant as I want to identify the negative consequences of public debt, avoiding a response of government spending as a by-product of the other shocks that would reassemble a government spending shock. The negative shock causes the government spending to GDP ratio to increase. Because government spending is fixed, the negative shock yields an increase in public deficit absorbed by issuing more government debt. On impact, the interest rate decreases due to the low marginal productivity of capital, however, quickly after, it increases as the excess demand for assets pushes the price upwards.

Does the government need to increase taxes in response to a negative shock? Under the optimal adjustment rate, the deleverage of pubic debt is front-loaded as the bulk of the excess debt is repaid in the first four years after the negative shock. Instead, consumption needs more than 20 years to fully recover. This fast adjustment of public debt is achieved via a high labor income tax rate: a decrease in productivity that generates a 1% drop in aggregate consumption raises the tax rate on impact by 1.25%. If the repayment was slower (meaning a lower policy rate), we would see a hump-shaped response to public debt.In this scenario, during the recession the tax would not be able to stop the upward trend of debt, and it would require several periods for debt to start decreasing. The shock fades out after 20 years, and after that public debt and the other aggregates keep fluctuating around their mean. In particular, government debt will keep fluctuating over a level $\bar{B}$[27].

The optimal response with a representative agent is markedly different. First, while the government does increase the tax in response to an increase in public debt, the adjustment is much more smoother. Indeed, labor income taxes do not deviate as much from the stochastic steady-state value after a negative TFP shock. In turn, this generates a slower convergence of public debt with respect to the HA-AR case, which can be better appreciated by looking at the persistence of debt denoted $\rho_G^{sim|RA}$ discussed in the previous subsection. Compared with the case with heterogeneous agents, the crowding out of capital is reduced by one order of magnitude, assets drop by much less, but the convergence is much slower, and consumption suffers from a more important drop. In sum, these effects indicate the role of the utility losses suffered by hand-to-mouth agents as well as the redistributive motive in Proposition 10 relative to Corollary 11.

6.5 Discussion

Lucas and Stokey (1983)'s result about tax-smoothing — the government uses public debt as an insurance mechanism against unexpected shocks and smooths labor income taxes — fails to apply here. As Figure 1 suggests, a utilitarian Ramsey planner temporarily disregards tax smoothing to pay back a large amount of government debt. My Ramsey planner's problem, since it does not include the choice of long-run government debt, fails in being fully comparable with Lucas and Stokey (1983) and Aiyagari et al. (2002). Nonetheless, the "Tax rate" panel in Figure 1 suggests that the combination of aggregate risk and heterogeneous agents disproves the tax-smoothing result as a sizeable fiscal response is optimal, and this is particular to the model where this two characteristics are present.

The joint determination of optimal labor income taxes and public debt has also been studied by Bhandari et al. (2017b). In an economy with aggregate risk and heterogeneous agents but no binding borrowing constraints, that is, without an endogenous distribution over wealth and income, they find that in the long run, the government pays back its debt. However, it does so fairly slowly, with a half-life of the repayment of at least 250 years, depending on the version of the model considered. It suggests that what matters — what makes the effect quantitatively large —

[27] My computational procedure does not allow me to find the optimal deterministic steady-state level of public debt around which debt fluctuates, but it only informs about the speed of size of the fiscal adjustment. Alternatively, I could have used an "inverse optimal approach at the steady-state" LeGrand and Ragot (2023) in which I choose the social weights ($\omega(a, z, \Gamma)$ in Section 5) that deliver $\bar{B}$ as the optimal steady-state level of debt. Nevertheless, this approach is unsatisfactory because the implied path of public debt with aggregate shocks may not be the optimal one, perhaps in the presence of aggregate risk, it is optimal for debt to fluctuate around a different value of $\bar{B}$.

is that heterogeneous agents face aggregate risk and some of them cannot insure themselves by saving more, as in my model there are binding borrowing constraints. Indeed, when agents do not face these borrowing onstraints the fiscal adjustment is much smaller as depicted in Figure 1.

About fiscal adjustments, Angeletos et al. (2023b) argue that a delay in paying back debt is optimal because of the persistence of the boom generated; fiscal adjustment is delayed, debt returns to trend on its own and the required future tax hike vanishes. Their result is derived in a model with uninsurable income risk but without aggregate risk. Once aggregate risk is present, by delaying the fiscal adjustment, the economy may suffer from another negative shock while the level of public debt remains high. Hence, it is no longer possible to wait for debt to be repaid on its own. Again, this points out the importance of aggregate risk.

7 Conclusion

This paper studies the role of aggregate risk in determining the optimal fiscal adjustment whenever there is excess of public debt using a heterogeneous-agent model with capital and aggregate shocks. The trade-offs shaping the optimal policy rate consist on the comovements of public debt with: (i) the presence of crowding out of capital, (ii) the public debt penalty, (iii) the utility costs for hand-to-mouths, (iv) the gains from capital income, and (v) a redistribution motive. To study their importance, I calibrate and numerically solve the model. I find the implied dynamics of public debt adjustment at the optimal policy to be consistent with those observed in the US. I finish by discussing some of the literature on optimal fiscal policy and argue that the combination of binding constraints and aggregate risk invalidates the desirability of tax-smoothing: excess public debt becomes too socially costly.

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APPENDIX

A Proofs

A.1 Proof of Proposition 5

I recall the Lagrangian (13), and I drop both time indices and the dependence on the state-space of the economy, namely (a, z) for convenience. Now, I use the index to denote derivatives. It writes,

$$
\begin{aligned}
L = \int_0^\infty e^{-\rho t} \Bigg\{ \iint \Bigg\{ & u(c,\ell) g \\
& + \phi \Big[-\rho V + u(c,\ell) + V_t + \mathcal{A} V \Big] \\
& + \chi \Big[u_c(c,\ell) - V_a \Big] \\
& + \psi \Big[u_\ell(c,\ell) + (1-\tau) w z V_a \Big] \\
& + \lambda \Big[-g_t + \mathcal{A}^* g \Big] \Bigg\} dadz \\
& + \theta \Big[-B_t + rB + G - \tau w L \Big] \\
& + \mu \Big[\iint agdadz - K - B \Big] \Bigg\} dt
\end{aligned}
$$

where $\tau = \bar{\tau} + \tau^a (B - \bar{B})$. I will first abstract from boundary conditions, and treat them formally later using a "discretize-optimize" approach as in Dávila and Schaab (2023) González et al. (2024). Integrating by parts, and setting $\phi(0) = \theta(0) = 0$ we get for all (a, z)

$$
\begin{aligned}
\int_0^\infty e^{-\rho t} \phi V_t dt &= \rho \int_0^\infty e^{-\rho t} \phi V dt - \int_0^\infty e^{-\rho t} \phi_t V dt \\
\int_0^\infty e^{-\rho t} \lambda g_t dt &= \rho \int_0^\infty e^{-\rho t} \lambda g dt - \int_0^\infty e^{-\rho t} \lambda_t g dt \\
\int_0^\infty e^{-\rho t} \theta B_t dt &= \rho \int_0^\infty e^{-\rho t} \theta B dt - \int_0^\infty e^{-\rho t} \theta_t B dt
\end{aligned}
$$

Using the definition of the adjoint operator — for any smooth functions f, g, the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ satisfies $< \mathcal{A} f, g > = < f, \mathcal{A}^* g >$, where $< \cdot, \cdot >$ is the inner product — we get

$$
\int \lambda \mathcal{A}^* g dadz = \int \mathcal{A} \lambda g dadz \quad \text{and} \quad \int \mathcal{A} V \phi dadz = \int V \mathcal{A}^* \phi dadz
$$

Finally, integrating by parts yields for all t

$$\int \chi V_a dadz = -\int \chi_a V dadz \quad \text{and} \quad \int (1-\tau)wz\psi V_a dadz = -\int (1-\tau)wz\psi_a V dadz$$

where I have abstracted, as mentioned earlier, from boundary conditions. Thus, the Lagrangian rewrites

$$\begin{aligned}
L = \int_0^\infty e^{-\rho t} \Bigg\{ \iint \Bigg\{ & \Big[u(c,\ell) + \mu a\Big] g \\
& + \phi u(c,\ell) + V\mathcal{A}^*\phi - \phi_t V \\
& + \chi u_c(c,\ell) + \chi_a V \\
& + \psi u_\ell(c,\ell) - \psi_a (1-\tau) wzV \\
& + \mathcal{A}\lambda g + \lambda_t g - \rho\lambda g \Bigg\} dadz \\
& + \theta \Big[(r-\rho)B + G - \tau w L \Big] + \theta_t B \\
& + \mu \Big[-K - B \Big] \Bigg\} dt
\end{aligned}$$

Employing techniques from calculus or variations, consider a perturbation indexed by $\alpha \in \mathbb{R}$, that is, perturb a candidate solution (for example) of the function c in the direction $c + \alpha h_c$ and so on for the other variables. We have

$$\begin{aligned}
L = \int_0^\infty e^{-\rho t} \Bigg\{ \iint \Bigg\{ & \Big[u(c+\alpha h_c, \ell + \alpha h_\ell) + \mu a\Big] \Big(g + \alpha h_g\Big) \\
& + \phi u(c+\alpha h_c, \ell+\alpha h_\ell) + \Big(V + \alpha h_V\Big) \mathcal{A}^*(\alpha)\phi - \phi_t \Big(V + \alpha h_V\Big) \\
& + \chi u_c(c+\alpha h_c, \ell+\alpha h_\ell) + \chi_a \Big(V + \alpha h_V\Big) \\
& + \psi u_\ell(c+\alpha h_c, \ell+\alpha h_\ell) - \psi_a \Big(1 - \bar{\tau} - \Big(\tau^a + \alpha h_\tau\Big)\Big(B + \alpha h_B - \bar{B}\Big)\Big)\Big(w + \alpha h_w\Big) z \Big(V + \alpha h_V\Big) \\
& + \mathcal{A}(\alpha)\lambda\Big(g + \alpha h_g\Big) + \lambda_t (g + \alpha h_g\Big) - \rho\lambda(g + \alpha h_g\Big) \Bigg\} dadz \\
& + \theta \Bigg[\Big(r + \alpha h_r - \rho\Big)(B + \alpha h_B\Big) + G - \Big(\bar{\tau} + \Big(\tau^a + \alpha h_\tau\Big)\Big(B + \alpha h_B - \bar{B}\Big)\Big)\Big(w + \alpha h_w\Big) \\
& \quad \Big(\iint \Big(\ell + \alpha h_\ell\Big) z \Big(g + \alpha h_g\Big) dadz \Big) \Bigg] + \theta_t \Big(B + \alpha h_B\Big)
\end{aligned}$$

$$+\mu\Big[-K-\alpha h_K - B - \alpha h_B\Big]\Big\}dt$$

Differentiating and taking the limit as $\alpha \to 0$

$$0 = L_\alpha = \int_0^\infty e^{-\rho t}\Big\{\iint\Big\{\Big[u_c(c,\ell)h_c + u_\ell(c,\ell)h_\ell\Big]g + \Big[u(c,\ell)+\mu a\Big]h_g$$

$$+\phi\Big[u_c(c,\ell)h_c + u_\ell(c,\ell)h_\ell\Big] + \Big[\frac{d}{d\alpha}\mathcal{A}^*(0)V + \mathcal{A}^* h_V\Big]\phi - \phi_t h_V$$

$$+\chi u_{cc}(c,\ell)h_c + \chi u_{c\ell}(c,\ell)h_\ell + \chi_a h_V$$

$$+\psi u_{\ell\ell}(c,\ell)h_\ell + \psi u_{\ell c}(c,\ell)h_c - \psi_a(1-\tau)wzh_V - \psi_a\Big((B-\bar{B})h_\tau + \tau^a h_B\Big)wzV$$

$$-\psi_a(1-\tau)h_w zV$$

$$+\frac{d}{d\alpha}\mathcal{A}(0)\lambda g + \mathcal{A}\lambda h_g + \lambda_t h_g - \rho\lambda h_g\Big\}\Big\}dadz$$

$$+\theta\Big[h_r B + (r-\rho)h_B - \Big((B-\bar{B})h_\tau + \tau^a h_B\Big)wL - \tau w\Big(\iint z(gh_\ell + \ell h_g)dadz\Big)\Big] + \theta_t h_B$$

$$+\mu\Big[-h_K - h_B\Big]\Big\}dt$$

where, for smooth functions f, g, we have

$$\iint \frac{d}{d\alpha}\mathcal{A}(0)fgdadz = \iint\Big[ah_r - \Big((B-\bar{B})h_\tau + \tau^a h_B\Big)w\ell + (1-\tau)h_w z\ell + (1-\tau)wh_\ell - h_c\Big]f_a g dadz$$

and it is important to notice that h_τ, h_B, and h_K only depend on the period while the rest of the perturbations are also state-dependent. Now I proceed to apply the fundamental lemma of calculus of variation.

Consumption. Regrouping the terms that are multiplied by h_c

$$0 = u_c(c,\ell)g + \phi u_c(c,\ell) - \phi V_a + \chi u_{cc}(c,\ell) + \psi u_{\ell c}(c,\ell) - \lambda_a g$$

and dividing by g and using the FOC of the competitive equilibrium for consumption

$$0 = u_c(c,\ell) - \lambda_a - \tilde{\chi}; \quad \text{with} \quad \tilde{\chi} = -\frac{\chi}{g}u_{cc}(c,\ell) - \frac{\psi}{g}u_{c\ell}(c,\ell)$$

Labor supply. Regrouping the terms that are multiplied by h_ℓ

$$0 = u_\ell(c,\ell)g + \phi u_\ell(c,\ell) + \phi(1-\tau)wzV_a + \chi u_{c\ell}(c,\ell) + \psi u_{\ell\ell}(c,\ell) + (1-\tau)w\lambda_a g - \theta\tau wzg$$

and dividing by g and using the FOC of the competitive equilibrium for the supply of labor

$$0 = u_\ell(c,\ell) + (1-\tau)w\lambda_a - \tilde{\psi} - \theta\tau wz; \quad \text{with} \quad \tilde{\psi} = -\frac{\psi}{g}u_{\ell\ell}(c,\ell) - \frac{\chi}{g}u_{c\ell}(c,\ell)$$

Lifetime utility. Regrouping the terms that are multiplied by h_V

$$\phi_t = \mathcal{A}^*\phi + \chi_a - \psi_a(1-\tau)wz$$

Cross-sectional distribution. Regrouping the terms that are multiplied by h_g

$$\rho\lambda = u(c,\ell) + \mu a - \tau w\theta\ell + \mathcal{A}\lambda + \lambda_t$$

Interest Rate. Regrouping the terms that are multiplied by h_r

$$0 = \iint a\Big[\phi V_a + \lambda_a g\Big] dadz + \theta B$$

Government debt. Regrouping the terms that are multiplied by h_B

$$0 = -\tau^a w \iint \ell z(\phi V_a + \lambda_a g) dadz - \tau^a w \iint \psi_a zV dadz - \theta\tau^a wL + \tau(r-\rho) - \theta_t - \mu$$

and finally,

$$\theta_t = \tau^a w \iint \Big[\ell z(\phi V_a + \lambda_a g) + \psi_a zV\Big] dadz + \tau^a\theta wL + \theta(\rho - r) + \mu$$

Speed of adjustment. Regrouping the terms that are multiplied by h_τ

$$0 = w(B-\bar{B}) \iint \Big[\ell z(\phi V_a + \lambda_a g) + \psi_a zV\Big] dadz + \theta(B-\bar{B})wL$$

Accounting for boundary conditions, a "discretize-optimize" approach. The goal now is to formally account for the boundary conditions, that is the fact that the planner cannot freely set consumption for the contained individuals. Instead, she understands that she can only indirectly affect it by changing the cash-on-hand via the parameters that depend on it.

Now (a,z) are discretized in a matrix in a scattered way, so that row i of the grid corresponds to an individual with wealth a_i and productivity z_i; with grid size I, and step sizes da and dz. I denote for convenience $dx = dadz$. Grid point $i = 1$ corresponds to the constrained individual, which I assume to be at $(\underline{a}, \underline{z})$, it's straightforward to extend to multiple values of z. The other continuous variable, time, take now the form of a grid denoted $\{t_n\}_n$, that has size N and time step dt. For convenience, I index a variable $x_t(a,z)$ by x_{in}, and the discretized Lagrangian writes

$$
\begin{aligned}
L(g_0) = \sum_{n=0}^{N-1} e^{-\rho t_n} \Bigg\{ & \sum_{i=1}^{I} u(c_{in}, \ell_{in}) g_{in} dx \\
& - \sum_{i>1} \frac{\phi_{in} - \phi_{in-1}}{dt} V_{in} dx + \sum_{i>1} \phi_{in} \Big[u(c_{in}, \ell_{in}) + \mathcal{A}_{in} V_{in} \Big] dx \\
& + \sum_{i>1} \chi_{in} \Big[u_c(c_{in}, \ell_{in}) - \Big(\partial_a V \Big)_{in} \Big] dx \\
& + \sum_{i>1} \psi_{in} \Big[u_\ell(c_{in}, \ell_{in}) + (1-\tau) wz \Big(\partial_a V \Big)_{in} \Big] dx \\
& + \sum_{i>1} \frac{\lambda_{in} - \lambda_{in-1}}{dt} g_{in} dx - \sum_{i>1} \rho \lambda_{in} g_{in} dx + \sum_{i>1} \lambda_{in} \mathcal{A}^*_{in} g_{in} dx \\
& - \mu_n \Big[K_n + B_n \Big] + \sum_{i=1}^{I} \mu_n a_i g_{in} dx \\
& + \frac{\theta_n - \theta_{n-1}}{dt} B_n + \theta_n \Big[(r_n - \rho) B_n + G_n - \tau w_n \sum_{i=1}^{I} \ell_{in} z_i \Big] \Bigg\} dt.
\end{aligned}
$$

As pointed out in Achdou et al. (2022), the domain of differentiation for the variable wealth a of the KFE, the HJB, and therefore of the FOC $u_c(c,\ell) = \partial_a V$, is $(\underline{a}, \infty)$. This is why the sum of i's starts at $\bar{I}+1$, and not at $i=1$. Hence, the necessary conditions for consumption, labor supply, lifetime utility, and cross-sectional distribution only hold in the interior. Having discretized the state-space, now I optimize over r_n, government debt B_n, and the policy rate τ_n^a. To do so, notice that consumption for the constrained individual writes

$$
c_{1n} = r_n a_1 + (1 - \bar{\tau} - \tau_n^a (B_n - \bar{B})) w_n z_1 \ell_{1n}
$$

and the supply of labor is given by the intra-temporal substitution of consumption and labor.

Now I can finally provide the necessary conditions including the terms of the borrowing constraint. Formally, I take the FOC directly using the discretized Lagrangian and later take the limit to the continuous state-space

Interest Rate. We have

$$
0 = \iint a \Big[\phi V_a + \lambda_a g \Big] dadz + \zeta_r^{\text{HtM}} + \theta B
$$

where

$$
\zeta_r^{\text{HtM}} = \Big(\underline{a} + (1-\tau) wz \underline{\ell}_r \Big) u_c(\underline{c}, \underline{\ell}) \underline{g} dadz + (1-\tau) wz \underline{\ell}_r u_\ell(\underline{c}, \underline{\ell}) \underline{g} dadz \tag{31}
$$

where $\underline{c}$ (resp. $\underline{\ell}$ with derivative with respect to r denoted $\underline{\ell}_r$) denote consumption (resp. labor supply) of the constrained individual, and where $\underline{g}dadz$ is the measure of individuals at the Dirac mass point $(\underline{a}, \underline{z})$.

Government debt. We have

$$\theta_t = \tau^a w \iint z\Big[\ell(\phi V_a + \lambda_a g) + \psi_a V\Big] dadz + \tau^a \theta w L + \theta(\rho - r) + \mu + \zeta_B^{\mathrm{HtM}}$$

where

$$\zeta_B^{\mathrm{HtM}} = \Big(\tau^a w \underline{z}\underline{\ell} - (1-\tau) w \underline{z}\underline{\ell}_B\Big) u_c(\underline{c}, \underline{\ell})\underline{g}dadz - (1-\tau) w \underline{z}\underline{\ell}_B u_\ell(\underline{c}, \underline{\ell})\underline{g}dadz \tag{32}$$

Speed of adjustment. And finally, we have

$$0 = w(B - \bar{B}) \iint z\Big[\ell(\phi V_a + \lambda_a g) + \psi_a V\Big] dadz + \theta(B - \bar{B}) w L + \zeta_\tau^{\mathrm{HtM}}$$

where

$$\zeta_\tau^{\mathrm{HtM}} = \Big((B - \bar{B}) w \underline{z}\underline{\ell} - (1-\tau) w \underline{z}\underline{\ell}_\tau\Big) u_c(\underline{c}, \underline{\ell})\underline{g}dadz - (1-\tau) w \underline{z}\underline{\ell}_\tau u_\ell(\underline{c}, \underline{\ell})\underline{g}dadz \tag{33}$$

which concludes the proof.

A.2 Proof of Proposition 6 and 7

Using

$$\theta_t = \tau^a \Big[w \iint z\Big[\ell(\phi V_a + \lambda_a g) + \psi_a V\Big] dadz + \theta w L\Big] + \theta(\rho - r) + \mu + \zeta_B^{\mathrm{HtM}}$$

$$0 = (B - \bar{B}) \Big[w \iint z\Big[\ell(\phi V_a + \lambda_a g) + \psi_a V\Big] dadz + \theta w L\Big] + \zeta_\tau^{\mathrm{HtM}}$$

we get

$$\theta_t = -\tau^a \left[\frac{\zeta_\tau^{\mathrm{HtM}}}{B - \bar{B}}\right] + \theta(\rho - r) + \mu + \zeta_B^{\mathrm{HtM}}$$

and in the RA limit it reduces to $\theta_t = \theta(\rho - r) + \mu$ since $\zeta_\tau^{\mathrm{HtM}} = \zeta_B^{\mathrm{HtM}} = 0$ as there is no hand-to-mouth individuals at equilibrium. Furthermore, since $\rho = r$, in the RA limit we obtain $\theta_t = \mu$. Back in the general case, rearranging,

$$\tau^a = \frac{B - \bar{B}}{\zeta_\tau^{\mathrm{HtM}}}\Big(-\theta_t + \theta(\rho - r) + \mu + \zeta_B^{\mathrm{HtM}}\Big).$$

We can also use $0 = \iint a\Big[\phi V_a + \lambda_a g\Big] dadz + \theta B$, to get

$$\tau^a = \frac{B - \bar{B}}{\zeta_\tau^{\text{HtM}}}\Bigg(- \theta_t - \frac{\rho - r}{B} \iint a\Big[\phi V_a + \lambda_a g\Big] dadz + \mu + \zeta_B^{\text{HtM}}\Bigg)$$

Moreover, since $0 = u_c(c, \ell) - \lambda_a - \tilde{\chi}$, then

$$\tau^a = \frac{B - \bar{B}}{\zeta_\tau^{\text{HtM}}}\Bigg(- \theta_t - \frac{\rho - r}{B} \iint a\Big[\phi u_c(c, \ell) + (u_c(c, \ell) - \tilde{\chi})g\Big] dadz + \mu + \zeta_B^{\text{HtM}}\Bigg)$$

$$\tau^a = \frac{B - \bar{B}}{\zeta_\tau^{\text{HtM}}}\Bigg(- \theta_t - \frac{\rho - r}{B} \iint a u_c(c, \ell)\Big[1 + \Theta\Big] g dadz + \mu + \zeta_B^{\text{HtM}}\Bigg)$$

where $\Theta = \frac{\phi}{g} - \frac{\tilde{\chi}}{u_c(c,\ell)}$.

A.3 Proof of Proposition 9

Because this proof mimics the one for Proposition 5 but extends the aggregate state space, all the derivations until using calculus of variation to derive the necessary conditions are exactly the same. Therefore, I start by recalling the differentiated Lagrangian

$$\begin{aligned}
0 = L_\alpha = \int_0^\infty e^{-\rho t}\Bigg\{ \iiint \Bigg\{\Bigg\{ & \Big[u_c(c, \ell)h_c + u_\ell(c, \ell)h_\ell\Big] g + \Big[u(c, \ell) + \mu a\Big] h_g \\
& + \phi\Big[u_c(c, \ell)h_c + u_\ell(c, \ell)h_\ell\Big] + \Big[\frac{d}{d\alpha}\mathcal{A}^*(0)V + \mathcal{A}^* h_V\Big]\phi - \phi_t h_V \\
& + \chi u_{cc}(c, \ell)h_c + \chi u_{c\ell}(c, \ell)h_\ell + \chi_a h_V \\
& + \psi u_{\ell\ell}(c, \ell)h_\ell + \psi u_{\ell c}(c, \ell)h_c - \psi_a(1 - \tau)wzh_V - \psi_a\Big((B - \bar{B})h_\tau + \tau^a h_B\Big)wzV \\
& \quad - \psi_a(1 - \tau)h_w zV \\
& + \frac{d}{d\alpha}\mathcal{A}(0)\lambda g + \mathcal{A}\lambda h_g + \lambda_t h_g - \rho\lambda h_g \Bigg\}\Bigg\} dadz \\
& + \theta\Bigg[h_r B + (r - \rho)h_B - \Big((B - \bar{B})h_\tau + \tau^a h_B\Big) wL - \tau w\Big(\iint z(gh_\ell + \ell h_g)dadz\Big)\Bigg] + \theta_t h_B \\
& + \mu\Big[- h_K - h_B\Big]\Bigg\} d\Gamma\Bigg\} dt
\end{aligned}$$

where, and here comes the main difference, for smooth functions f, g, we have

$$\iiint \frac{d}{d\alpha}\mathcal{A}(0) f g dadz = \iiint \Big[ah_r - \Big((B-\bar{B})h_\tau + \tau^a h_B\Big)w\ell + (1-\tau)h_w z\ell + (1-\tau)wh_\ell - h_c\Big] f_a g dadzd\Gamma$$

$$+ \iiint \Big[h_K \partial_K \mu_K(\Gamma) + h_B \partial_B \mu_K(\Gamma) + h_\tau \partial_\tau \mu_K(\Gamma)\Big] f_K g dadzd\Gamma$$

$$+ \iiint \Big[h_B \partial_B \mu_B(\Gamma) + h_K \partial_K \mu_B(\Gamma) + h_\tau \partial_\tau \mu_B(\Gamma)\Big] f_B g dadzd\Gamma.$$

The last two lines of the latter expression show only the first-order condition of government debt and the speed of adjustment. The two conditions that change in a meaningful weight (that is, something extra apart from including an extra variable in the state-space) are:

Government debt with Aggregate Shocks. Regrouping the terms that are multiplied by h_B

$$0 = -\tau^a w \iint \ell z(\phi V_a + \lambda_a g) dadz - \tau^a w \iint \psi_a z V dadz - \theta\tau^a wL + \tau(r-\rho) - \theta_t - \mu + \mathcal{E}_B$$

where

$$\mathcal{E}_B(\Gamma) = \iint \partial_B \mu_B(\Gamma)\phi V_B dadz + \iint \partial_B \mu_K(\Gamma)\phi V_K dadz$$
$$+ \iint \partial_B \mu_B(\Gamma)\lambda_B g dadz + \iint \partial_B \mu_K(\Gamma)\lambda_K g dadz \quad (34)$$

and finally,

$$\theta_t = \tau^a w \iint \Big[\ell z(\phi V_a + \lambda_a g) + \psi_a z V\Big] dadz + \tau^a \theta wL + \theta(\rho - r) + \mu - \mathcal{E}_B - \mathcal{E}_B$$

Speed of adjustment with Aggregate Shocks. Regrouping the terms multiplied by h_τ yields

$$0 = \int w(B-\bar{B}) \iint \Big[\ell z(\phi V_a + \lambda_a g) + \psi_a z V\Big] dadzd\Gamma + \int \theta(B-\bar{B})wLd\Gamma + \mathcal{E}_\tau$$

where

$$\mathcal{E}_\tau = \iiint \partial_\tau \mu_B(\Gamma)\phi V_B dadzd\Gamma + \iiint \partial_\tau \mu_K(\Gamma)\phi V_K dadzd\Gamma$$
$$+ \iiint \partial_\tau \mu_B(\Gamma)\lambda_B g dadzd\Gamma + \iiint \partial_\tau \mu_K(\Gamma)\lambda_K g dadzd\Gamma \quad (35)$$

Accounting for boundary conditions, a "discretize-optimize" approach. Now (a, z, Γ) are discretized in a matrix in a scattered way, so that row i of the grid corresponds to an individual with wealth a_i, productivity z_i, in a world with aggregate state Γ_i. The grid for $\bar{\Gamma}$ has size I, and step sizes da, dz and $d\Gamma$. I denote for convenience $dx = dadzd\Gamma$. Many different i lead to the same aggregate state, I introduce a mapping $\phi(i)$ that maps each row i to a subset $\{1, \ldots, J\}$ of aggregate states so that there is no $j \neq j'$ such that $\Gamma_j = \Gamma_{j'}$. I denote the inverse of this function by $\Phi(j)$ and it equals all the i's that have aggregate state j associated with them. Finally, $\Phi(j|H)$ is the subset of i's associated with Γ_j that is hand-to-mouth, and its complement, the subset of unconstrained households, is denoted $\Phi(j|R)$. I assume that the first $\bar{I}$ entries of the grid correspond to hand-t-mouth agents, and the rest to unconstrained. The time grid, denoted $\{t_n\}_n$, has size N and time step dt. Indexing a variable $x_t(a, z, \Gamma)$ by x_{in}, the Lagrangian becomes

$$
\begin{aligned}
L(g_0) = \sum_{n=0}^{N-1} e^{-\rho t_n} \Bigg\{ \sum_i u(c_{in}) g_{in} dx - \sum_{i>\bar{I}} \frac{\phi_{in} - \phi_{in-1}}{dt} V_{in} dx + \sum_i \phi_{in} \Big[u(c_{in}, \ell_{in}) + \mathcal{A}_{in} V_{in} \Big] dx \\
+ \sum_{i>\bar{I}} \chi_{in} \Big[u_c(c_{in}, \ell_{in}) - \Big(\partial_a V\Big)_{in} \Big] dx \\
+ \sum_{i>\bar{I}} \psi_{in} \Big[u_\ell(c_{in}, \ell_{in}) - (1-\tau_n) w_{\phi(i)n} z_i \Big(\partial_a V\Big)_{in} \Big] dx \\
+ \sum_{i>\bar{I}} \frac{\lambda_{in} - \lambda_{in-1}}{dt} g_{in} dx - \sum_{i>\bar{I}} \rho \lambda_{in} g_{in} dx + \sum_{i>\bar{I}} \lambda_{in} \mathcal{A}^*_{in} g_{in} dx \\
- \sum_j \mu_{jn} \Big[K_{jn} + B_{jn} \Big] d\Gamma + \sum_i \mu_{\phi(i)n} a_i g_{in} dx \\
+ \sum_j \frac{\theta_{jn} - \theta_{jn-1}}{dt} B_{jn} d\Gamma \\
+ \sum_j \theta_{jn} \Big[(r_{jn} - \rho) B_{jn} + G_{jn} - \tau_n w_{jn} L \Big] d\Gamma \Bigg\} dt.
\end{aligned}
$$

Interest Rate. We have

$$0 = \iint a \Big[\phi V_a + \lambda_a g \Big] dadz + \zeta_r^{\mathrm{HtM}} + \theta B$$

where

$$\zeta_r^{\mathrm{HtM}} = \Big(\underline{a} + (1-\tau) wz \underline{\ell}_r \Big) u_c(\underline{c}, \underline{\ell}) \underline{g} dadz + (1-\tau) wz \underline{\ell}_r u_\ell(\underline{c}, \underline{\ell}) \underline{g} dadz \tag{36}$$

where $\underline{c}$ (resp. $\underline{\ell}$ with derivative with respect to r denoted $\underline{\ell}_r$) denote consumption (resp. labor supply) of the constrained individual, and where $\underline{g} dadz$ is the measure of individuals at the Dirac mass point $(\underline{a}, \underline{z})$.

Government debt. We have

$$\theta_t = \tau^a w \iint z \Big[\ell(\phi V_a + \lambda_a g) + \psi_a V \Big] dadz + \tau^a \theta w L + \theta(\rho - r) + \mu + \zeta_B^{\mathrm{HtM}} - \mathcal{E}_B$$

where

$$\zeta_B^{\text{HtM}} = \Big(\tau^a w\underline{z}\underline{\ell} - (1-\tau)w\underline{z}\underline{\ell}_B\Big)u_c(\underline{c},\underline{\ell})\underline{g}dadz - (1-\tau)w\underline{z}\underline{\ell}_B u_\ell(\underline{c},\underline{\ell})\underline{g}dadz \tag{37}$$

Speed of adjustment. And finally, we have

$$0 = \int w(B-\bar{B})\iint z\Big[\ell(\phi V_a + \lambda_a g) + \psi_a V\Big]dadzd\Gamma + \int \theta(B-\bar{B})wLd\Gamma + \zeta_\tau^{\text{HtM}} + \mathcal{E}_\tau$$

where

$$\zeta_\tau^{\text{HtM}} = \int \Big((B-\bar{B})w\underline{z}\underline{\ell} - (1-\tau)w\underline{z}\underline{\ell}_\tau\Big)u_c(\underline{c},\underline{\ell})\underline{g}dadzd\Gamma - \int (1-\tau)w\underline{z}\underline{\ell}_\tau u_\ell(\underline{c},\underline{\ell})\underline{g}dadzd\Gamma \tag{38}$$

which concludes the proof.

A.4 Proof of Proposition 10 and Corollaries 12, 11 and 13

Using

$$\theta_t = \tau^a w \iint z\Big[\ell(\phi V_a + \lambda_a g) + \psi_a V\Big]dadz + \tau^a\theta wL + \theta(\rho - r) + \mu + \zeta_B^{\text{HtM}} - \mathcal{E}_B$$

$$0 = \int (B-\bar{B})w\Big[\iint z\Big[\ell(\phi V_a + \lambda_a g) + \psi_a V\Big]dadz + \theta L\Big]d\Gamma + \zeta_\tau^{\text{HtM}} + \mathcal{E}_\tau$$

where the first line rewrites as

$$\frac{1}{\tau^a}\Big(\theta_t - \theta(\rho - r) - \mu - \zeta_B^{\text{HtM}} + \mathcal{E}_B\Big) = w\Big[\iint z\Big[\ell(\phi V_a + \lambda_a g) + \psi_a V\Big]dadz + \theta L\Big]$$

we get

$$0 = \int (B-\bar{B})\frac{1}{\tau^a}\Big(\theta_t - \theta(\rho - r) - \mu - \zeta_B^{\text{HtM}} + \mathcal{E}_B\Big)d\Gamma + \zeta_\tau^{\text{HtM}} + \mathcal{E}_\tau$$

which yields

$$\tau^a = \frac{1}{\zeta_\tau^{\text{HtM}} + \mathcal{E}_\tau}\int (B-\bar{B})\Big(-\theta_t + \theta(\rho - r) + \mu + \zeta_B^{\text{HtM}} - \mathcal{E}_B\Big)d\Gamma.$$

Using

$$\theta = -\frac{1}{B}\iint au_c(c,\ell)\Big[1+\Theta\Big]gdadz$$

we finally get

$$\tau^a = \frac{1}{\zeta_\tau^{\text{HtM}} + \mathcal{E}_\tau}\int (B-\bar{B})\Big(-\theta_t - \frac{\rho - r}{B}\iint au_c(c,\ell)\Big[1+\Theta\Big]gdadz + \mu + \zeta_B^{\text{HtM}} - \mathcal{E}_B\Big)d\Gamma,$$

where in the long-run solution, $\theta_t = 0$. In the main text, I use the decomposition

$$\frac{1}{g_\Gamma(\Gamma)} \iint a u_c(c,\ell)\Big[1+\Theta\Big] g dadz = \mathbb{E}_{g|\Gamma}\Big[a u_c(c,\ell)\Big[1+\Theta\Big]\Big],$$

where $g_\Gamma(\Gamma) = \iint g(a,z,\Gamma)dadz$ and the expectation is taken over the pdf g conditional on an aggregate state Γ. Then, I decompose

$$\mathbb{E}_{g|\Gamma}\Big[a u_c(c,\ell)\Big[1+\Theta\Big]\Big] = A\mathbb{E}_{g|\Gamma}\Big[u_c(c,\ell)\Big[1+\Theta\Big]\Big] + \mathbb{C}ov_{g|\Gamma}\Big[a,\, u_c(c,\ell)\Big[1+\Theta\Big]\Big].$$

For Corollary 12, I set $\mathcal{E}_B = \mathcal{E}_\tau = 0$. For Corollary 11, I use $g(a,z|\Gamma) \to \delta(a-A)\delta(z-z^{RA})g_\Gamma(\Gamma)$, where z^{RA} is the single level of labor productivity in the RA case and $\delta(\cdot)$ is the delta function. As a consequence, the covariance in the latter expression equals zero and the expectation tends to $u_c(C,L)(1+\Theta)$. Corollary 13 follows from setting $\theta = 0$, which is, as discussed in Dávila and Schaab (2023), the continuous time counterpart of optimal discretion policy.

B Numerical Implementation

B.1 Step 1: Deterministic steady state

The set of optimality conditions in the deterministic competitive equilibrium are

$$\begin{aligned}
\rho v(a,z) &= u(c,\ell) + \partial_a v(a,z)s(a,z) + \partial_z v(a,z)\theta_z + \partial_{zz} v(a,z)\frac{\sigma_z^2}{2}; \quad a \geq \underline{a} \\
c &= (\partial_c u)^{-1}\Big(\partial_a V\Big) \;\&\; \ell = (\partial_\ell u)^{-1}\Big((1-\tau)wz\partial_a V\Big) \\
0 &= \mathcal{A}^* g(a,z) \\
w &= (1-\alpha)K^\alpha L^{-\alpha} \\
r &= \alpha K^{\alpha-1} L^{1-\alpha} - \delta \\
K + B &= \int_0^\infty a g(a,z) dadz = A
\end{aligned}$$

where $s(a,z) = (1-\tau)wz\ell + ra - c$ is the optimal saving policy function and the HJB is only satisfied in the interior space. As detailed in Achdou et al. (2022), this system of partial differetial equations can be discretized as

$$\begin{aligned}
\rho \mathbf{v} &= u(\mathbf{c},\mathbf{l}) + \mathbf{A}\mathbf{v}; \quad \mathbf{c} = (u_c)^{-1}\Big(D_a \mathbf{V}\Big) \;\&\; \mathbf{l} = (u_\ell)^{-1}\Big((1-\tau)w\mathbf{z}D_a\mathbf{V}\Big) \\
0 &= \mathbf{A}'\mathbf{g} \\
\mathbf{p} &= [w,r] = F(\mathbf{g})
\end{aligned}$$

where $\mathbf{v}$ is the discrete counterpart of v, and likewise for $\mathbf{c}$, $\mathbf{l}$ and $\mathbf{g}$. The finite-difference matrix D_a discretizes the derivative with respect to wealth a. The discretization of the infinitessimal generator is denoted $\mathbf{A}$, and the one of its adjoint corresponds to the transpose. The vector of prices $\mathbf{p}$ solves a fix point problem that ensures market clearing for capital and labor holds given by $F(\cdot)$. I use the deterministic steady-state to calibrate the model as described in Section 6.1 and to initialize the algorithm for the stochastic steady-state explained below.

B.2 Step 2: Stochastic Steady State

The set of optimality conditions in the stochastic competitive equilibrium are

$$\begin{aligned}
\rho v(a,z,\Gamma) &= u(c,\ell) + \partial_a v(a,z,\Gamma)s(a,z,\Gamma) + \partial_z v(a,z,\Gamma)\theta_z + \partial_{zz} v(a,z,\Gamma)\frac{\sigma_z^2}{2} \\
&\quad + \partial_\Gamma v(a,z,\Gamma)\mu(\Gamma); \quad a \geq \underline{a} \\
0 &= \mathcal{A}^* g(a,z,\Gamma) \\
w &= (1-\alpha)K^\alpha L^{-\alpha} \\
r &= \alpha K^{\alpha-1} L^{1-\alpha} - \delta \\
K + B &= \int_0^\infty a g(a,z,\Gamma) da dz = A; \forall \Gamma
\end{aligned}$$

and this can be discretized as in Step 1, albeit now the state space includes Γ and the perceived laws of motion summarized into $\mu(\Gamma)$.

The algorithm follows the one of Achdou et al. (2022) extended as in Fernández-Villaverde et al. (2023) to include aggregate risk as in Krusell and Smith (1998) and it is based on the finite difference method (LeVeque, 2007). To speed-up computations, I use sparse grids, instead of dense grids (used in Achdou et al. (2022) and Fernández-Villaverde et al. (2023)) with the tools developed by Schaab and Zhang (2022).

Algorithm: 1) Choose a PLM parametrized by weights $(\beta_i^\ell)_i$. 2) Given μ, solve the HJB. 3) Simulate a realization of the TFP sequence $\{Z_t\}_t$. 4) Given the simulated path of TFP $\{Z_t\}_t$, solve the KF equation and obtain $g_t(a,z)$. Use it to get aggregate capital K_t and government debt B_t. 5) Run an OLS regression[28] where the regressor is the simulated path $\{\Gamma_t\}_t$ and the response variable is dK_t and dB_t. With this, I update the weights $\beta^{\ell+1} = \beta^\ell$. If the weights have converged, I stop, otherwise I go back to step 2.

[28] Fernández-Villaverde et al. (2023) uses a neural network. Since the quasi-aggregation result holds in this model, there is no need for using a more perceived linear function than a liner one.

B.3 Step 3: Find Optimal Policy Rate τ^a

The previous step computes the stochastic steady-state. Using Monte Carlo simulations, I compute welfare $\mathcal{W}$. Then, I search for the optimal τ^a, that is, the one that maximizes $\mathcal{W}$ subject to the implementability conditions.

Proof of the student condition of the author

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Economics for the Common Good

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March 6, 2021

Dear Sergi Barcons Llorach,

My colleagues and I are very happy that you have chosen to apply to our Master 2 Economic Theory and Econometrics program. This letter follows up on the message you received from the President of Université Toulouse Capitole. We look forward to the prospect of you joining our program.

Our program: The Master Economic Theory and Econometrics (ETE) is in practice the first year of TSE's Ph.D. program. Formally, in France, the Ph.D. starts after the Master. At TSE, the year after the Master is a training year where you have two semesters of classes, as it is the practice in US programs. The subsequent years are devoted to research.

Scholarship: During the Master year, you will receive a fellowship of 1 000€ per month, from September until June. Teaching and research assistance are not allowed. You can do research assistance in the summer.

During the second year of your training (the first year of the Ph.D. program in the French system), you will receive a fellowship of 1 200 € per month for 12 months. You will be able to teach, which could increase your monthly revenue by 150 €.

During the next three years (year two, three and four of the Ph.D. program in the French system), you will have a fellowship of 1 400 € per month. You will be able to teach, which could increase your monthly revenue by 150 €.

The year after, which is the job market year, you will have a fellowship of 1 400 € per month from September until April. Teaching is not allowed. You can do research assistance after these eight months.

Funding is guaranteed conditional on good standing in the program and is not cumulative with other funding. The funding for the last year is subject to some conditions, in particular, being on the market.

Fees: For the Master year, the fees are decomposed in two parts: 243 € to the University and 5 550€ to TSE. You have a waive for the TSE's part (5 500€). Unfortunately, by law, we cannot give you a waiver for the second part (243 €) that you must pay. For the

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first year of the Ph.D., the fees are 1 220 € (it is not possible to have a waive), and then around 380 € for the subsequent years. These fees include health insurance.

Commitment: Funding and the fees' waiver of the Master' year subject to the commitment that you will do your Ph.D. at TSE and that you will not apply elsewhere.

Living in Toulouse: Toulouse is a nice town which is not very expensive. Students often share flats and they pay between 400€ and 500€ per month. Those who chose to rent alone pay between 500€ and 700€ per month. There are rooms at the University which cost around 280€ per month; their number is limited. Note that you will be able to receive public housing subsidies, usually between 100 and 200 € per month, depending on the type and location of the flat.

Deadline to make a decision about the funding: The deadline for your response is March 11.

If you have any questions about our program, please contact me at nour.meddahi@tse-fr.eu. Additional information appears on TSE's web site at www.tse-fr.eu. Other queries may be addressed to our graduate coordinator, Ludmila Nomalavan-Stephan, at Ludmila.namolavan@tse-fr.eu.

Sincerely yours,

Nour Meddahi
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