# Researcher–Entrepreneur Relationship and Performance of Innovative Startups*

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**Abstract**

Many innovative startups are joint ventures between researchers and entrepreneurs, combining research and development (R&D) with commercialization in the product market. Government policies, such as grants, subsidies, and patent license fees, are Pigouvian subsidies that incentivize R&D activities by narrowing the gap between social and private returns on innovation. However, R&D subsidies may strengthen the researcher's bargaining power in the researcher–entrepreneur relationship, leading to lower research effort and unbalanced equity allocation, which can jeopardize startup performance. Our findings suggest that a policy portfolio is needed to address both external market failure and internal researcher–entrepreneur friction. Conventional Pigouvian subsidies to innovation must be supplemented with policies on startup governance and the entrepreneur labor market.

**Keywords:** innovative startup, entrepreneurship, innovation policy, entrepreneurial finance

**JEL codes:** L26, M13, O30

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# 1 Introduction

Innovative startups are increasingly important in driving economic development. Studies have shown that investment in innovative startups results in more patents and radical innovations than the same expenditure on traditional corporate research and development (R&D) (Kortum and Lerner, 2000; Acs and Audretsch, 1988). The social returns of innovation in the form of new knowledge and technology spillovers often exceed private returns (Mansfield et al., 1977; Jaffe, 1986; Griliches, 1992; Jones and Williams, 1998; Bloom et al., 2013), so innovation is under-provided in laissez-faire (Arrow, 1972). To address this market failure, governments have established various innovation policies, such as patent systems, research grants, startup subsidies, and venture investments with public funding, to incentivize R&D activities and entrepreneurship.[1] License fees from the intellectual property (IP) market and subsidies from various programs compensate researchers for their efforts in discovering new knowledge and developing innovative technologies.

In the modern knowledge-based economy, there is growing interest in exploring the practical and commercial implications of research output from researchers, particularly university professors. Across the globe, universities have gradually become the center of national innovation systems (Nelson, 1993; Etzkowitz, 2004). The emergence of incubators and industrial conurbations surrounding universities has become a hallmark of innovation and entrepreneurial hubs, as seen in the biotechnology industry around Route 128 and the semiconductor industry in Silicon Valley (Bania et al., 1993; Zucker et al., 1998).

Governments and universities launch numerous programs to provide financial and other support to universities for the purpose of cultivating university spinoffs. Once a professor develops IPs with commercial value, the university knowledge transfer office or entrepreneurship center will provide assistance in filing patents and establishing startups. The professor usually will not serve as the CEO of the startup but will instead hire a professional entrepreneur with business expertise to run the startup. Hence, most innovative startups are joint ventures between a researcher and an entrepreneur, combining R&D and commercialization in the product market.

In an innovative startup, the researcher usually holds a dominant position and controls the contract terms offered to the entrepreneur.[2] The asymmetric bargaining power between researchers and entrepreneur lies in the nature of their roles in the startups. The researcher is often the IP holder and the designated recipient of grants, subsidies, and support from the university. The researcher's input to the startup is usually the most scarce and defining resource, while talents with business expertise are relatively easy to hire from the labor market.

Furthermore, the number of researchers in a particular field is sticky in the short run, while entrepreneurs can more easily switch fields according to external market conditions. On the one

[1]Popular examples include the Small Business Investment Company of the United States (Howell, 2017), the European Investment Fund (Munari and Toschi, 2021), and the Labour Sponsored Venture Capital Corporation of Canada (Cumming and MacIntosh, 2006).

[2]Researchers holding stronger bargaining power is common in Silicon Valley. See `www.businessinsider.com/pitching-silicon-valley-investors-research-funding-science-vibe-data-2022-9`.

hand, when the market related to a research area is booming, many entrepreneurs compete for researchers as partners.[3] On the other hand, when the research area experiences a downturn, the poor performance of a startup affects entrepreneurs more than researchers. Researchers can maintain their recognition as scholars and earn income from academic jobs. However, the entrepreneur's reputation and income are determined by the startup's performance. These factors all contribute to the stronger bargaining power of researchers. Therefore, the main model of this paper assumes that the researcher has dominant bargaining power and acts as the principal in the contracting relationship, which reflects features of early-stage innovative startups, especially those founded by academic entrepreneurs in high-tech sectors (Azoulay et al., 2017).

In this paper, we analyze how innovation subsidies affect the researcher–entrepreneur relationship and startup performance. We construct a model of a joint venture with a researcher (he) and an entrepreneur (she), who both contribute to the development of a technological innovation that generates profit from the product market and creates valuable knowledge in the process of R&D. The researcher is more productive in contributing to the discovery of new knowledge, while the entrepreneur is more capable of increasing product-market profit using her business expertise. The researcher and the entrepreneur also differ in their bargaining power in the internal labor market of the joint venture, with the researcher holding the dominant position and determining the contract terms of the entrepreneur.

Conventional wisdom holds that the government can incentivize innovation by compensating for the value of knowledge discovery. This is true if we treat the startup as a joint entity. However, the internal researcher–entrepreneur relationship affects how different parties respond to the subsidies strategically. After considering the friction within the startup, we show that full Pigouvian subsidies cannot restore the first-best outcome and may even exacerbate the effort distortion within the startup, which can lead to lower startup performance than in laissez-faire.

There are many cases in which innovation support programs amplify internal conflicts between the researcher and the entrepreneur in high-tech startups. For example, Kno,[4] an education technology startup, received U.S. Department of Education funds contingent on innovation development. The government funds triggered a disagreement among the co-founders about their strategies and IP management. The leading researcher, Osman Rashid, with a dominant position in the startup, prioritized attracting funds from the government and through technology licensing and placed new technology development as the main strategy. However, the co-founder, Babur Habib, wanted to develop new products and features for Kno's platform. This conflict in strategies jeopardized the performance of Kno, which was acquired by Intel at a low price in 2013.[5] As another example, Eolas was founded by Michael Doyle, who developed a patent for web browser technology that

[3] One example is the entrepreneur Elon Musk courting and persuading Tom Mueller, a highly regarded propulsion engineer, to leave TRW Inc. to co-found SpaceX. See www.businessinsider.com/elon-musk-spacex-tom-mueller-race-mars-walter-isaacson-biography-2023-9ManysimilarcaseshaveoccurredinSiliconValleyandotherentrepreneurialhubs.

[4] See www.wsj.com/articles/BL-DGB-30701.

[5] See om.co/gigaom/how-much-kno-sold-for-why-it-failed/.

enabled interactive content. [6] Michael Doyle wanted to focus on licensing the patent to other companies in the industry, while the co-founder, David Martin, wanted to develop products from the technology. The conflict between Doyle and Martin arose over the management of the patent and the direction of the company, which led to internal tension and impacted the performance of Eolas. Other examples include the bankruptcy of Better Place in 2013,[7] A123 Systems in 2012,[8] and Solyndra in 2011.[9] These vivid examples indicate that innovation policies such as subsidy and patent licensing may instigate internal conflicts in startups and harm their performance.

Intuitively, innovative startups face two types of distortions: First, under-provision of effort due to positive externalities of innovation and internal labor market frictions rooted in asymmetric bargaining power between the researcher and entrepreneur. Most existing compensation schemes for innovation only address external market failure but ignore the internal organization of startup teams. This is similar to Barnett (1980) and Boyer and Laffont (1999), who notes that the Pigouvian tax alone cannot fix the problem of negative externalities in the presence of monopoly market power.

How can policymakers fix both internal and external distortions? We show that policymakers can design an optimal innovation policy portfolio that includes the entrepreneur's outside-option value as an additional policy tool. In practice, researchers often have higher outside-option values than entrepreneurs, as they can maintain their recognition as scholars and earn income from academic jobs. However, the entrepreneur usually has worse outside options. Policymakers can establish a favorable environment for serial entrepreneurs by collaborating with incubators in innovation and technology hubs to provide additional income sources for entrepreneurs, such as paid coaching services given by serial entrepreneurs in incubators and entrepreneurial universities. Incubators and other institutes in technology hubs can also facilitate the mediated search and re-matching of serial entrepreneurs with researchers to set up new ventures, which encourages dynamism in the labor market for professional startup managers and serial entrepreneurs.

The paper is organized as follows. Section 2 lists the related literature. Section 3 presents the main model and the main result of Pigouvian subsidy backfiring. Section 4 discusses how policymaker can fix the problem and extends the main model. Section 5 concludes the paper and offers policy implications. All proofs are provided in the Appendix.

## 2 Related Literature

The core contribution of this paper is to characterize how the internal organization of tech startups interacts with innovation subsidies and affects startup performance. We contribute to the literature on incentive issues of innovation and entrepreneurship. This literature consists of two streams. One

[6] See Eolas Technologies Inc, `en.wikipedia.org/wiki/Eolas`.

[7] `www.theguardian.com/environment/2013/mar/05/better-place-wrong-electric-car-startup`.

[8] `ceramics.org/ceramic-tech-today/about-the-a123-demise-and-bankruptcy-tech-failure-or-business-failure/`.

[9] `www.washingtonpost.com/politics/specialreports/solyndra-scandal/`.

focuses on the positive externality of innovation and entrepreneurship, and the other studies the contract design of innovative activities given the internal labor market frictions in organizations.

### 2.1 Public Intervention in Innovation and Entrepreneurship

Public intervention plays an increasingly important role in stimulating innovation and entrepreneurship in the modern knowledge-based economy. Subsidies, tax credits, IP markets, and patent policies are widely adopted innovation policies. In the literature, there are extensive studies on the effectiveness of these policy tools (Bloom et al., 2019). For example, Wright (1983) analyzes the different incentive implications of patents, prizes, and research contracts. He points out that researchers often have an informational advantage over public administrators, so decentralized research incentives in the form of patents work better than centralized alternatives such as research prizes and contracts.

Many later studies on public funding of innovation incorporate realistic intervention constraints, most notably asymmetric information. Some theoretical works consider the funding design problem in the moral hazard framework, such as Jensen and Thursby (2001) and Lerner and Malmendier (2010), among others. Hellmann and Thiele (2019) categorizes public funding support into two categories: subsidizing entrepreneurs of founding ventures or investors' funding startups and compares them from the lens of entrepreneurial tacit knowledge accumulation. Entrepreneurs' tacit knowledge must be accumulated through past entrepreneurial activities and are necessary for future success and the development of the entrepreneurial ecosystem. Subsidies for investment funding could encourage tolerance of early failures and, thus, the accumulation of entrepreneurial experience. This long-term impact of funding subsidies outperforms the direct subsidy of founding ventures, which has only a short-term effect.

Many empirical works find that the effectiveness of innovation subsidies needs to be carefully evaluated and reviewed. Table 1 summarizes the results of empirical papers evaluating the effect of innovation policy. Many researchers find that R&D subsidies or enhancing patent protection have limited or even perverse effects on innovation (Wallsten, 2000; Lach, 2002; Hujer and Radić, 2005; Merito et al., 2007). For example, Denes et al. (2020) demonstrate that subsidizing through investors in innovative startups is ineffective. Pahnke et al. (2015) show that direct funding has a negative impact on patenting and no impact on product innovation among early-stage medical device companies. Stevenson et al. (2021) find that despite a possible positive effect on subsequent chances in obtaining venture capital investment, public direct funding eventually has a negative effect on firms' financial performance because grants may reduce ventures' ability to effectively use their resources. Ayoub et al. (2017) also find a negative effect of public direct support on university spinoffs.

The reasons for the negative effect include misaligned interests of stakeholders of these programs (Pahnke et al., 2015), corruption and bureaucrats deteriorating program effectiveness (Lerner, 2012), and governments' multiple goals in designing these programs (Lanahan and Feldman, 2015).

Table 1: Literature on Evaluating the Effect of Innovation Policy

| Articles | Policy tool | Outcome variable | Results | Country |
|---|---|---|---|---|
| Mansfield (1986) | Stronger patent protection | Patent | No effect | United States |
| Levin et al. (1987) | Stronger patent protection | Patent | Negative | United States |
| Lerner (2000) | R&D subsidy | Employment, sales | Positive | United States |
| Wallsten (2000) | R&D subsidy | Employment, R&D expenditure | No effect | United States |
| Busom Piquer (2000) | R&D subsidy | Employment, R&D expenditure | Positive | Spain |
| Cohen et al. (2000a) | Stronger patent protection | Patent | Heteogeneous | United States |
| Sakakibara and Branstetter (1999) | Stronger patent protection | Innovation activity | Positive | Japan |
| Lach (2002) | R&D subsidy | Investment | No effect | Israel |
| Almus and Czarnitzki (2003) | R&D subsidy | R&D expenditure | Positive | Eastern Germany |
| Hujer and Radić (2005) | R&D subsidy, tax credit, loan | Patent | No effect | Germany |
| González et al. (2005) | R&D subsidy | R&D expenditure | Positive | Spain |
| Görg and Strobl (2007) | R&D subsidy | R&D expenditure | Heterogeneous | Ireland |
| Merito et al. (2007) | R&D subsidy | Employment, sales, patent | No effect | Italy |
| Hussinger (2008) | R&D subsidy | R&D expenditure | Positive | Germany |
| Azoulay et al. (2011) | R&D subsidy | Patent | Positive | United States |
| Takalo et al. (2013) | R&D subsidy | Welfare | Positive | Finland |
| Pahnke et al. (2015) | R&D subsidy | Patent and product innovation | Negative | United States |
| Budish et al. (2015) | Stronger patent protection | R&D expenditures, welfare | Positive | United States |
| Lanahan and Feldman (2015) | R&D subsidy | Startup performance | Heterogeneous | United States |
| Bronzini and Piselli (2016) | R&D subsidy | Patent | Positive | Italy |
| Igami (2017) | Stronger patent protection | R&D expenditure | Negative | US, Asia, Europ, Brazil |
| Ayoub et al. (2017) | R&D subsidy (via university) | Revenue, firm size, investment return | Negative | Germany |
| Howell (2017) | R&D subsidy | Patent, R&D expenditure, revenue | Positive | United States |
| Kong (2020) | R&D subsidy | Patent | Negative | United States |
| Azoulay et al. (2019) | R&D subsidy | Patent | Positive | United States |
| Denes et al. (2020) | Tax credit | Startup performance, investment return | No effect | United States |
| Lanahan et al. (2021) | R&D subsidy | Employment | Negative | United States |
| Stevenson et al. (2021) | R&D subsidy | Subsequent investment, revenue, firm size | Ambiguous | United States |

For example, government support often targets a specific technology area. Directing research and entrepreneurial investment into the targeted area is possibly a higher priority than individual startup performance. This priority determines the screening of funded startups, requirements and funding terms, and the government's interim and ex post evaluation of startups. These screening criteria and requirement terms may not be aligned with startup performance, which can result in a negative effect in empirical studies.

Regarding the effect of patent regulation, Mazzoleni and Nelson (1998a) and Mazzoleni and Nelson (1998b) survey the concerns about the conducive role of a strong patent system in improving social welfare. Mansfield (1986), Levin et al. (1987) and Cohen et al. (2000b) show that patents were not an essential part of firms' incentives for investing in R&D. Budish et al. (2015) note that firms' investment in innovation is not socially optimal under the patent system. Acemoglu and Akcigit (2012) show that full patent protection is not socially optimal. Igami (2017) shows that strong patent protection reduces innovation incentives of incumbents and hence could be detrimental industry-wide.

In addition to the two main public interventions to boost innovation, governments often offer corporate tax credits for R&D expenditure. The literature on this topic has also progressed over time, incorporating information asymmetry in the design of the optimal corporate tax (Akcigit and Liu, 2016; Makris and Pavan, 2021; Akcigit et al., 2022). The empirical examination of corporate tax incentives also advances. Appelt et al. (2016), Edler et al. (2016), and Moser (2005) analyze the coordination of tax incentives and other policy tools such as patent regulation. Others empirically evaluate the effectiveness of tax incentives in promoting corporate innovation and call for a cautious review of the intervention's effectiveness and its heterogeneity resulting depending on firm characteristics (Lokshin and Mohnen, 2012; Acemoglu and Akcigit, 2012).

All papers reviewed above do not consider the joint production feature of innovation. We contribute to this literature by studying how the internal labor market shapes the intervention's effectiveness under different policy environments. We provide a novel explanation of why R&D subsidy programs fail.

### 2.2 Optimal Innovation Contract

Our paper is also related to studies of internal labor market frictions in the organizational economics literature. Following Holmstrom (1989), many researchers have studied the optimal contract design problem of innovators under moral hazard or private information (e.g., Aghion and Bolton 1992; Aghion and Tirole 1997; Bergemann and Hege 1998; Hörner and Samuelson 2013; Shan 2017). Kerr et al. (2014) and Kerr and Nanda (2015) survey the literature that applies organizational economics to innovation and entrepreneurship.

The widely adopted moral hazard framework leads to high-powered incentive design, i.e., payment contingent on research output proxying for knowledge value. Manso (2011) highlights the pitfalls of the high-powered incentive design derived from moral hazard. This perverse effect receives empirical support from He and Tian (2013) and Azoulay et al. (2011).

Most of this literature centers on startups contracting with external funders such as venture capitalists and the innovation incentive for R&D of established firms. Hellmann and Thiele (2015) is one pioneering work explicitly focusing on contracting within founding teams. Hellmann and Thiele (2015) embeds the standard team incentive problem in Holmstrom (1982) into a multitask environment and imposes incompleteness of contracts to examine an endogenous choice between upfront versus delayed contracting between founders. They show how the stage of founders committing to each other determines startup ownership, incentives, and performance. Our work abstracts from the informational and contractual frictions widely examined in this literature to concentrate on the interaction of bargaining power asymmetry with the Pigouvian subsidy of the innovation externality and highlight the subsidy's backfiring effect without any further friction except the bargaining power asymmetry.

The optimal contract framework under asymmetric information usually abstracts from the externality of innovative products. Our work closes the gap of examining the crucial feature of innovation, knowledge externality, in the negligent internal market structure of producing innovation and its externality. Policies often call for public efforts to collaborate with private markets, such as the private-public partnership in boosting innovation and entrepreneurship in many technology hubs worldwide. The growing popularity of private-public partnerships has attracted more scholarly attention to this topic. Bai et al. (2021) discuss the role of public policies in entrepreneurship and the private equity industry. The most closely related paper to ours is Lach et al. (2021). It studies the incentive design of public funding in innovation and entrepreneurship in the framework of private information. Gao et al. (2022) show that the optimal investment in innovative startups depends on the structure of the product market and the amount of resources.

# 3 Main Model

## 3.1 Setup

A researcher (he) and an entrepreneur (she) jointly engage in a startup that develops and commercializes an innovative technology, generating both profit from the product market and value from new knowledge. The researcher is the principal who has more bargaining power in the joint venture. The entrepreneur is the agent.[10]

The two players differ in their abilities: the researcher has expertise in conducting research and contributes more to the value of knowledge, while the entrepreneur is more capable of generating profit from the product market than knowledge spillovers. Let $e$ denote the research effort exerted by the principal. Let $x$ denote the business effort by the agent. These two efforts jointly determine the product market profit and the value of new knowledge. Let $\pi(e, x)$ denote the product market profit and $g(e, x)$ denote the value of knowledge. Both the production of profit and value of new

[10] We analyze the case of the entrepreneur being the principal in Section 4.4.

knowledge follow the Cobb-Douglas form:

$$\pi(e, x) = A_\pi e^\alpha x^{(1-\alpha)}, \quad (1)$$

$$g(e, x) = A_g e^\beta x^{(1-\beta)}. \quad (2)$$

We restrict $e > 1$ and $x > 1$. Assume that $\beta > \alpha$, which reflects that the research effort plays a more essential role in generating knowledge spillovers than product market profits.

Both the principal and agent incur a constant marginal cost of effort, denoted by $c_e$ and $c_x$, respectively. Their utility functions are increasing and concave over their earnings. For simplicity, assume that the principal's utility function is $V(\cdot) = \nu \ln(\cdot)$ and that of the agent is $U(\cdot) = \ln(\cdot)$. The parameter $\nu$ reflects that the principal and agent may have different marginal values of wealth.

The timing of the game is as follows.

1. The principal offers a contract to the agent that specifies the transfer, $w$, contingent on the startup's output $\pi$ and $g$.
2. The agent decides whether to accept the contract. If the agent accepts the contract, she participates in the joint production with the principal. If not, she obtains an outside option value $\underline{u} = 0$.[11]
3. The principal chooses his effort level $e$, and the agent chooses her effort level $x$.
4. The product market profits and knowledge value are realized. The agent obtains the transfer as specified in the contract.

We assume that the outputs of the joint venture, $\pi$ and $g$, are common knowledge. Because there are one-to-one mappings from the effort combinations to the outputs, there is no information asymmetry.[12] The principal can specify the transfer at each level of income that just compensates for the agent's effort costs. The performance-based transfer can also be regarded as the agent's bonus, which is common in the financial contracting design literature and reality.[13]

The contracting relationship in our model can also be regarded as an internal labor market. The unequal bargaining power in the model captures the difference in market power between suppliers of two different inputs. The principal can be regarded as a monopsony of his input, whereas the agent faces more competition on the input she provides. Our setting can also be viewed as a rent capture behavior by the more powerful side and be applied to other scenarios such as the relationship between a power investor and a founder. Figure 1 illustrates the model.

[11] As is common in the contracting literature, assume that under the circumstances with the agent indifferent between actions, she acts in favor of the principal. We later consider the case of a positive outside option ($\underline{u} > 0$) in Section 4.3.

[12] This setting allows us to focus on the role of asymmetric bargaining power in the joint venture. We introduce asymmetric information into the model in Section 4.5.

[13] In the literature, it is common to assume that the equity share rule depends on the startup's performance (Kaplan and Strömberg, 2003). The agent's equity share is the ratio of the transfer to the startup's total income.

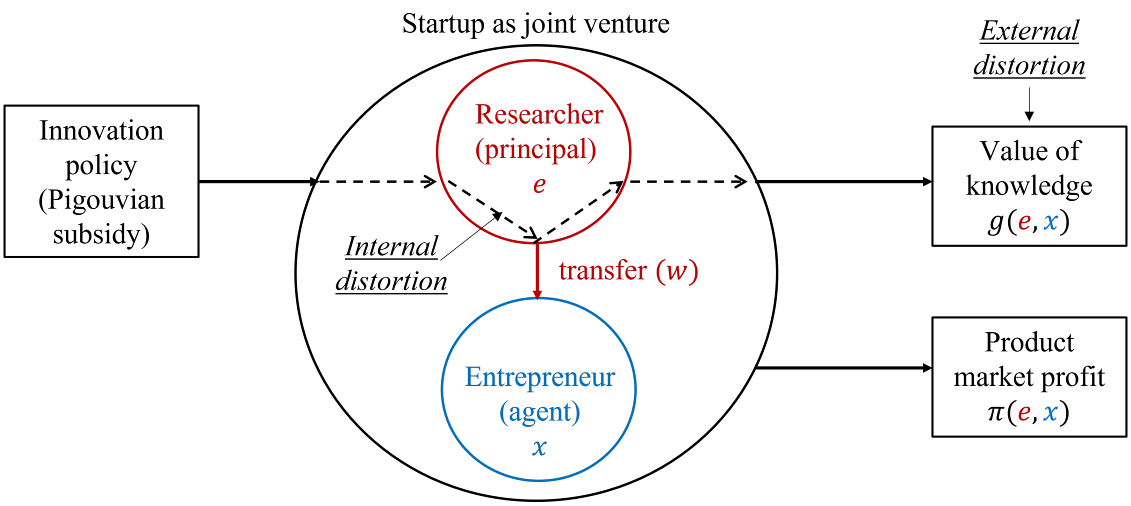



Figure 1: Illustration of the Model

### 3.2 First-best Allocation

Let $Y$ denote the sum of the startup's profit generated in the product market and the value of knowledge, $Y \doteq \pi(e,x) + g(e,x)$. Let $w$ denote the transfer received by the agent. The first-best allocation maximizes social welfare, i.e., the total payoff of both players:

$$SW(e,x,w) \doteq V(Y-w) - c_e e + U(w) - c_x x.$$

**Lemma 1.** *The first-best allocation, $(w^*, e^*, x^*)$, is characterized by the following first-order conditions (FOCs):*

$$w^* = \frac{1}{\nu+1} Y, \tag{3}$$

$$\frac{c_e e^*}{1+\nu} = \alpha + (\beta-\alpha)\frac{t}{1+t}, \tag{4}$$

$$\frac{c_x x^*}{1+\nu} = (1-\alpha) - (\beta-\alpha)\frac{t}{1+t}, \tag{5}$$

*where* $t^* \doteq \frac{A_g}{A_\pi}\left(\frac{e^*}{x^*}\right)^{(\beta-\alpha)}$.

To avoid triviality, we focus on the scenarios where the agent earns a nonnegative net utility under the first-best allocation. condition (3) implies that the relative marginal utility determines how the researcher and the entrepreneur share the startup's total value. The effort provision depends on the marginal effort cost and its marginal product represented by the right-hand sides of (5) and (4). An effort's marginal product depends on its contribution to the two payoff components and their relative value. $t$ is the ratio of knowledge value to the product market profit.

### 3.3 Equilibrium in laissez-faire

In laissez-faire, there is no IP protection or other subsidies for knowledge. The startup only earns income from the product market profit, $\pi(e,x) = A_\pi e^{(1-\alpha)} x^\alpha$. The principal offers a contract as a transfer, $w$, contingent on the profit. The optimization problem of the principal is

$$\max_{\{w,e\}} \{V(\pi(e,\hat{x}) - w) - c_e e\} \tag{6}$$

$$\text{s.t. } \hat{x} = \arg\max_x \{U(w) - c_x x, \underline{u}\}. \tag{7}$$

The Lagrangian is

$$\mathcal{L} = V(\pi(e,x) - w) - c_e e + \lambda [U(w) - c_x x].$$

Let $(\hat{w}_0, \hat{e}_0, \hat{x}_0)$ denote the solution to problem (6), where the subscript "0" indicates that the startup can appropriate zero income from the value of knowledge.

**Lemma 2.** *The equilibrium in laissez-faire is a unique tuple, $(\hat{w}_0, \hat{e}_0, \hat{x}_0)$, characterized by FOCs:*

$$\hat{w}_0 = \frac{\hat{\lambda}_0}{\nu + \hat{\lambda}_0}\pi, \tag{8}$$

$$\frac{c_e}{\hat{\lambda}_0 + \nu} = \frac{\alpha}{\hat{e}_0}, \tag{9}$$

$$\frac{\hat{\lambda}_0 c_x}{\hat{\lambda}_0 + \nu} = \frac{(1-\alpha)}{\hat{x}_0}, \tag{10}$$

$$\ln\left(\frac{\hat{\lambda}_0}{\nu + \hat{\lambda}_0}\pi\right) = c_x \hat{x}_0, \tag{11}$$

*where $\hat{\lambda}_0$ is the Lagrangian multiplier at optimum.*

Define the agent's equity share as the proportion of the agent's wage to the startup's total income. Under laissez-faire, the agent's equity share and the research effort are below the first-best levels. The formal statement is as follows:

**Lemma 3.** *In the laissez-faire market equilibrium,*

*(a) the agent's equity share is below the first-best level, i.e., $\frac{\hat{w}_0}{\hat{\pi}_0} < \frac{w^*}{Y^*}$;*

*(b) the researcher's effort is below the first-best level, i.e., $\hat{e}_0 < e^*$.*

Without informational noise or uncertainty, the principal can perfectly infer the agent's effort from the profit. The transfer is determined by the participation constraint (7). The principal only needs to compensate for the agent's effort cost given $\underline{u} = 0$.[14] Hence, the agent earns a zero payoff

[14] An alternative interpretation is that the labor market of entrepreneurs is perfectly competitive, and the principal with full bargaining power is the monopsony employer.

in laissez-faire but has a positive net payoff in the first-best allocation.[15] As a result, the laissez-faire outcome features underprovision of the research effort and the principal obtaining too much equity share.

### 3.4 Full Internalization by Pigouvian Subsidy

We now assume that the startup can appropriate the value of knowledge. The compensation can be in the form of grants, subsidies, or license fees from the IP market. We abstract from the specific form of policy intervention and consider the Pigouvian subsidy that fully compensates for the positive externalities of innovation. The startup's total income is

$$Y(e,x) = \pi(e,x) + g(e,x) = A_\pi e^\alpha x^{(1-\alpha)} + A_g e^\beta x^{(1-\beta)}$$

The principal's problem is

$$\max_{(w,e)} \left\{ V\left(Y\left(e,\hat{x}\right) - w\right) - c_e e \right\}, \tag{12}$$

$$\text{s.t. } \hat{x} = \arg\max_x \{ \left(U(w) - c_x x\right), \underline{u} \}, \tag{13}$$

The Lagrangian of the problem is

$$\mathcal{L} = V\left(Y - w\right) - c_e e + \lambda \left[U\left(w\right) - c_x x.\right]$$

where $\lambda$ is the Lagrangian multiplier that represents the shadow value of the agent's utility from the perspective of the principal.

Let $(\hat{w}_1, \hat{e}_1, \hat{x}_1)$ denote the equilibrium under full internalization by the Pigouvian subsidy, where the subscript "1" indicates that the startup can appropriate the full value of knowledge.

**Lemma 4.** *The equilibrium under the full Pigouvian subsidy is a unique tuple* $(\hat{w}_1, \hat{e}_1, \hat{x}_1)$ *that can be characterized by the following FOCs:*

$$\hat{w}_1 = \frac{\hat{\lambda}_1}{\nu + \hat{\lambda}_1} \hat{Y}_1, \tag{14}$$

$$\frac{c_e}{\hat{\lambda}_1 + \nu} = \left\{ \alpha + (\beta - \alpha) \frac{\hat{t}_1}{1 + \hat{t}_1} \right\} \frac{1}{\hat{e}_1}, \tag{15}$$

$$\frac{\hat{\lambda}_1 c_x}{\hat{\lambda}_1 + \nu} = \left\{ (1 - \alpha) - (\beta - \alpha) \frac{\hat{t}_1}{1 + \hat{t}_1} \right\} \frac{1}{\hat{x}_1}, \tag{16}$$

$$\ln\left( \frac{\hat{\lambda}_1}{\nu + \hat{\lambda}_1} \hat{Y}_1 \right) = c_x \hat{x}_1, \tag{17}$$

*where* $\hat{\lambda}_1$ *denotes the Lagrangian multiplier at optimum, and* $\hat{t}_1 \doteq \frac{A_g}{A_\pi} \left( \frac{\hat{e}_1}{\hat{x}_1} \right)^{(\beta - \alpha)}$.

[15] If there is asymmetric information, the agent will earn an information rent.

Condition (14) indicates that the agent obtains $\frac{\hat{\lambda}_1}{\nu+\hat{\lambda}_1}$ share of the total income of the startup. We have the following result:

**Lemma 5.** *Under the Pigouvian subsidy, $\hat{\lambda}_1 < 1$, and the equilibrium equity share of the agent is below the first-best level, i.e., $\frac{\hat{w}_1}{\hat{Y}_1} < \frac{w^*}{Y^*}$.*

The Lagrangian multiplier ($\hat{\lambda}_1$) affects the research effort and the business effort via conditions (15) and (16). The right-hand sides of these two conditions are the marginal benefit of research effort and of business effort, respectively.[16] The left-hand sides of (15) and (16) are the adjusted marginal cost of the corresponding effort, and the adjustment accounts for the allocation of the equity share. Because $\hat{\lambda}_1 < 1$, it is more appealing to employ the business effort rather than the research effort under the Pigouvian subsidy. In other words, a large equity share held by the researcher increases his marginal effort cost and induces him to substitute his effort for the entrepreneur's effort. Here is the formal result.

**Proposition 1.** *The first-best effort levels cannot be restored by the full Pigouvian subsidy for the value of knowledge:*

*(a) the principal's equilibrium effort provision is below the first-best level, i.e., $\hat{e}_1 < e^*$;*

*(b) the agent's equilibrium effort provision is above the first-best level, i.e., $\hat{x}_1 > x^*$.*

### 3.5 Pigouvian Subsidy Backfire

Proposition 1 shows that full internalization of the value of knowledge cannot restore the first-best allocation because the Pigouvian subsidy aggravates the effort distortion problem within the startup. More importantly, we find that the distortion can lead to an outcome even worse than that in laissez-faire.

**Proposition 2.** *The Pigouvian subsidy reduces the equilibrium effort of the researcher compared to laissez-faire, i.e., $\hat{e}_1 < \hat{e}_0$, given that*

$$\frac{b_g}{\beta} - \frac{b_\pi}{\alpha} > \ln\left(\frac{\beta}{\alpha}\right) + \frac{1-\beta}{\beta} - \frac{1-\alpha}{\alpha}, \tag{18}$$

$$(\beta-\alpha)\left[\frac{b_\pi}{(1-\alpha)\alpha} - \frac{1}{1-\alpha} - \frac{2}{\nu((1-\alpha)+(1-\beta))}\right] < 1, \tag{19}$$

*where $b_\pi \doteq \ln\left(A_\pi\left(\frac{\alpha}{c_e}\right)^\alpha\left(\frac{1-\alpha}{c_x}\right)^{1-\alpha}\right)$ and $b_g \doteq \ln\left(A_g\left(\frac{\beta}{c_e}\right)^\beta\left(\frac{1-\beta}{c_x}\right)^{1-\beta}\right)$.*

[16] The marginal benefit of effort exhibits decreasing marginal returns and complementarity between the two efforts. These two features are from the assumptions of production functions (1) and (2). The marginal benefit of research effort increases in the ratio of knowledge value to product profit denoted as $t$, whereas the marginal benefit of business decreases in it. Specifically, greater productivity to generate knowledge ($A_g$) increases the marginal benefit of the research effort and decreases that of the business effort. The opposite is true for $A_\pi$.

Proposition 2 presents the striking result that the Pigouvian subsidy can lead to unintended consequences after considering the researcher–entrepreneur relationship. The purpose of compensating for knowledge spillovers is to encourage the researcher to exert more effort, but the opposite occurs in equilibrium. Intuitively, the Pigouvian subsidy has a **stimulating effect** that increases the demand for research effort because it corrects the ratio of the marginal benefits of two efforts represented by the right-hand sides of (15) and (16). The social value of knowledge is internalized, so it demands more research effort. However, the unequal bargaining power within the startup allows the principal to save his effort and exploit the agent after receiving the subsidy. As a result, the subsidy also imposes a **replacement effect** that reduces the research effort and increases the business effect in equilibrium.

Because the researcher's effort is more critical, the startup's performance in R&D and the product market can both worsen after receiving the Pigouvian subsidy. This result contrasts with conventional wisdom because it implies that protecting IP or subsidizing innovation can harm the performance of innovative startups and social welfare.

**Corollary 1.** *Given conditions (18) and (19), social welfare is reduced by the Pigouvian subsidy,*

$$SW(\hat{e}_1, \hat{x}_1, \hat{w}_1) < SW(\hat{e}_0, \hat{x}_0, \hat{w}_0).$$

This perverse welfare effect occurs under conditions (18) and (19). A necessary condition for (18) is $A_g > A_\pi$, which means that the startup is more productive in generating valuable knowledge than profit from the product market. Intuitively, the Pigouvian subsidy generates more value for a startup with higher productivity of knowledge, so a less effective effort bundle with less research effort and more business effort can generate a level of knowledge value that satisfies the participation constraint. In this sense, (18) ensures a sufficiently large replacement effect.

Condition (19) requires that $\beta - \alpha$ is not too large, and $\alpha$ is close to $1 - \alpha$. In other words, efforts from the researcher are both important in generating commercial profits and knowledge ($\alpha$ is not small, and $\beta - \alpha$ is small), and the efforts from the researcher and entrepreneur have strong synergy ($\alpha$ close to $1 - \alpha$). This is an essential feature of tech startups that develop innovative products and services. For example, ChatGPT developed by OpenAI is a technological breakthrough and also lays the foundation for many commercial applications. Inputs from researchers and entrepreneurs are both essential in contributing to its vast knowledge spillovers and commercial value. If $\beta$ is much larger than $\alpha$, R&D's contribution to knowledge generation and profit differ a lot, and thus the effect of the Pigouvian subsidy to stimulate research efforts will be large. To sum up, condition (18) and condition (19) ensure a large replacement effect substituting $e$ with $x$ and a small direct effect of stimulating $e$.

Based on Proposition 2, we have other results.

**Corollary 2.** *When $\hat{e}_1 < \hat{e}_0$, the Pigouvian subsidy decreases the agent's equity share.*

Corollary 2 states that the effort distortion problem also results in the agent obtaining a lower equity share of the startup. The replacement effect of the Pigouvian subsidy decreases the necessary equity share to incentivize the agent at each level of effort. Consequently, the agent works more but receives less transfer than in the laissez-faire equilibrium.

**Corollary 3.** *If $\hat{e}_1 < \hat{e}_0$ under $A_g$, then $\hat{e}_1 < \hat{e}_0$ holds for all $A_g^{'} > A_g$.*

Corollary 3 indicates that the effort distortion by the Pigouvian subsidy is more likely to occur in industries with higher productivity in knowledge. Many innovation policies focus on high-tech industries that generate valuable knowledge, but these industries are also more likely to exhibit the perverse welfare effect after receiving the subsidy.

# 4 Extension and Discussion

We have shown that the conventional Pigouvian subsidy ignores the internal organization of innovative startups and can backfire. How can policymakers correct the effort distortion with the imbalanced researcher–entrepreneur relationship? We analyze three policy tools in Sections 4.1 to 4.3. Sections 4.4 and 4.5 extend the main model.

## 4.1 Optimal Rate of Compensation

Suppose that the social planner can choose the proportion of the value of knowledge that the startup can appropriate. Let $\mu \in [0,1]$ denote the rate of compensation. The startup's income is

$$Y_\mu (e,x) \doteq A_\pi e^\alpha x^{(1-\alpha)} + \mu A_g e^\beta x^{(1-\beta)}.$$

In the main model, we study the case of full compensation with $\mu = 1$. The social planner chooses the compensation rate $\mu$ to maximize social welfare:

$$\max_{\mu} \left\{ V\left(\frac{\nu}{\nu+1}\tilde{Y}\left(\tilde{e}_\mu,\tilde{x}_\mu\right)\right) + U\left(\frac{1}{\nu+1}\tilde{Y}\left(\tilde{e}_\mu,\tilde{x}_\mu\right)\right) - c_e\tilde{e}_\mu - c_x\tilde{x}_\mu \right\}, \tag{20}$$

$$\text{s.t. } \tilde{e}_\mu = \arg\max_{e_\mu} \left\{ V\left(\frac{\nu}{\nu+\tilde{\lambda}_\mu}\tilde{Y}\left(e_\mu,\tilde{x}_\mu\right)\right) - c_e e_\mu \right\}, \tag{21}$$

$$c_x\tilde{x}_\mu = U\left(\frac{\tilde{\lambda}_\mu}{\nu+\tilde{\lambda}_\mu}\tilde{Y}\left(\tilde{e}_\mu,\tilde{x}_\mu\right)\right). \tag{22}$$

The objective function (20) is the net social surplus, where the payoff split between players is based on the first-best allocation in Lemma 1. Constraint (20) represents the principal's best response to the compensation rate $\mu$. The agent's equity share $\left(\frac{\lambda}{\nu+\lambda}\right)$ is derived from the FOC of the principal's problem similar to (87). Define the solution to (20) as the second-best allocation. We obtain the following results:

**Proposition 3.** *The second-best allocation features zero subsidy ($\mu^* = 0$) given conditions (18), (19), and*

$$\ln A_\pi + \alpha \ln\left(\frac{\nu}{c_e}\right) + (1-2\alpha)\ln\left(\frac{1-\alpha}{c_x}\right) \leq (1-\alpha) + \frac{(1-\alpha)^2}{c_x}, \tag{23}$$

$$\frac{1-\alpha}{\alpha(\beta-\alpha)} \leq 2. \tag{24}$$

Proposition 3 characterizes the conditions under which no subsidy for innovation is best for society. The subsidy has two effects in opposite directions. It could be welfare-improving as an additional income source. However, it allows the researcher to utilize his advantageous bargaining power to substitute his effort with the agent's effort. If the latter effect is stronger, the subsidy reduces R&D. The startup is less productive in delivering knowledge and profit. The Pigouvian subsidy reduces total welfare relative to the laissez-faire equilibrium.

Conditions (18) and (19) indicate that the subsidy reduces equilibrium R&D, and the replacement effect dominates the stimulating effect. The gap between equilibrium R&D and its first-best level depends on $\alpha$, which is the contribution of research effort to product market performance. A larger $\alpha$ implies a greater R&D gap and higher inefficiency.

Condition (23) indicates a large $A_\pi$. The replacement effect induced by the subsidy distorts the effort more severely in commercial production than in knowledge generation. Given the high productivity in the product market represented by a large $A_\pi$, the welfare loss in the product market is large and dominates.

Both conditions (23) and (24) indicate a large $\alpha$. Hence, the inefficiency of the subsidy is so strong that even at a very small level, the stimulating effect cannot improve welfare, and hence the social planner's optimum is laissez-faire if the power asymmetry within the startup team cannot be corrected.

**Proposition 4.** *The second-best allocation features partial subsidy ($\mu^* \in (0,1)$) given the following conditions:*

$$c_x\left(1 - \frac{(\beta-\alpha)\alpha}{1-\alpha}\right) \geq (\beta-\alpha)\alpha, \tag{25}$$

$$\ln\left((1-\alpha)A_\pi\left(\frac{\nu c_x}{c_e}\right)^\alpha + (1-\beta)A_g\left(\frac{\nu c_x}{c_e}\right)^\beta\right) \leq c_x, \tag{26}$$

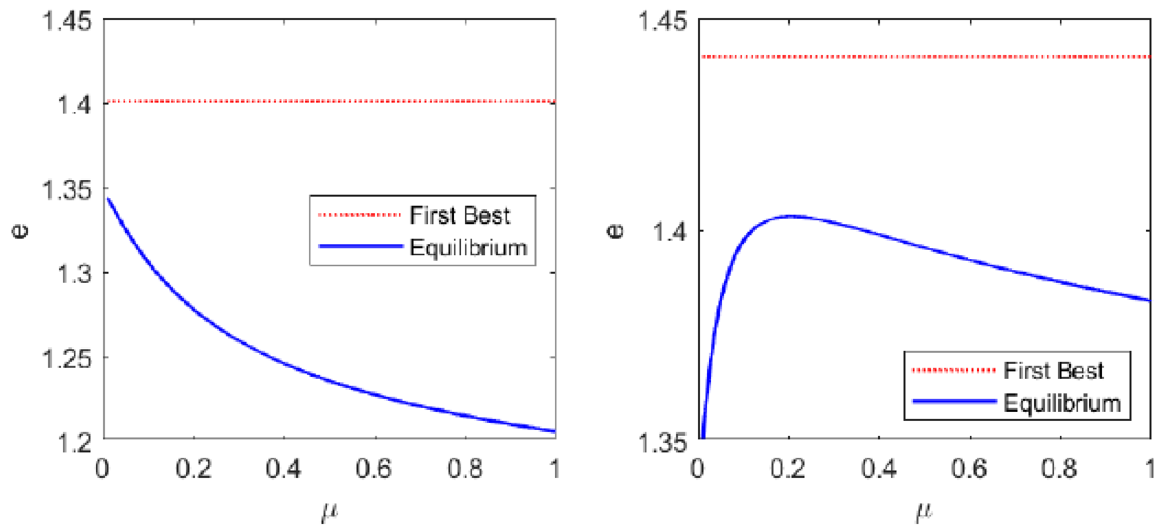
$$\ln\left(\frac{A_\pi(2\nu)^\alpha(1-\alpha)}{c_e^\alpha c_x^{1-2\alpha}}\right) - (1-\alpha)(2-\alpha) \geq \alpha \ln\left(\frac{(1-\alpha)^2}{\alpha} + b_2\right) + \alpha b_2, \tag{27}$$

*where* $b_2 \doteq \sqrt{\frac{1-\alpha}{\alpha}\left(\frac{(1-\alpha)^3}{\alpha} - \frac{4c_x^2}{\beta-\alpha}\right)}$.[17]

According to Proposition 4, partial subsidy maximizes social welfare. condition (27) requires

[17] We assume a regularity for the sufficient condition to hold is $(\beta-\alpha)\alpha < 1-\alpha$.

the subsidy to improve welfare at a small subsidy rate, i.e., the marginal welfare at $\mu$ approaching zero is positive. It holds under small product-market productivity, i.e., small $A_\pi$ and low $\alpha$, which restrains the size of the replacement effect such that it is below the stimulating effect at a small level of subsidy.




Note: The parameters for the left-panel are $((1-\beta),(1-\alpha),A_\pi,A_g,c_x,c_e,\nu) = (0.4, 0.6, 20, 5, 1.58, 1, 1.7)$. The parameters for the right-panel are $((1-\beta),(1-\alpha),A_\pi,A_g,c_x,c_e,\nu) = (0.4, 0.7, \frac{20}{1.3}, \frac{5}{1.3}, \frac{1.28}{1.2}, 1, 2)$

Figure 2: How Subsidy Rate Affects Research Effort, $e_\mu$

Conditions (25) and (26) indicate a moderate level of $A_g$ and similar importance of R&D in generating two incomes. They result in the negative marginal effect of the subsidy on total welfare as subsidy rate $\mu$ approaches one. The negative net effect shows that the replacement effect dominates the stimulating effect with a large subsidy but is dominated by stimulating at small subsidy. An intermediate subsidy restricts the principal's ability to exploit the agent while still enabling the stimulating effect to play a role, and thus balance two forces and reach social optimum. Figure 2 depicts the equilibrium research effort with respect to the compensation rate. The left and right panels illustrate Propositions 3 and 4, respectively.

The ultimate impact of startup subsidy on the research effort is the net effect of two forces, the demand enhancement of $e$ and the relaxed supply of $x$. The two forces are in opposite directions. The former force drives $e$ up, as the compensation in the value of knowledge depends more on the research effort than the business effort. The latter force tends to replace $e$ with $x$, which is called the replacement effect. The demand enhancement effect is larger at small levels of subsidy than at high levels of subsidy due to the decreasing return feature. However, the relaxation of supply for $x$ is determined by the productivity of knowledge generation ($A_g$). In the right panel of Figure 2, the stimulating effect dominates the replacement effect under a small subsidy and improves welfare. As the subsidy rate increases, the demand enhancement goes below the relaxed supply effect, revealed as the net effect of the subsidy on research effort becoming negative, as shown in Proposition 4. The left panel depicts Proposition 3 when the stimulating effect cannot outweigh the replacement effect, so the net effect is negative at a very small subsidy. Hence, the laissez-faire equilibrium coincides with the second best, which is in stark contrast to the conventional wisdom.

Note that, as shown in Figure 2, adjusting the compensation rate of the Pigouvian subsidy cannot fully solve the problem. The first-best efforts can never be restored with any level of subsidy.

**Proposition 5.** *There does not exist* $\mu \in [0,1]$ *such that the solution to the problem (20) yields the first-best effort levels,* $(e^*, x^*)$.

### 4.2 Multidimensional Compensation Scheme

One natural remedy to the dual problem of externalities and internal friction is to make the compensation scheme richer. Specifically, consider a multidimensional compensation scheme that can target individual players in the startup or specific outputs. An example of multidimensional compensation schemes includes a startup subsidy together with innovation awards and grants given to individual researchers. Let $(\mu, \tau_{e,\pi}, \tau_{e,g}, \tau_{x,\pi}, \tau_{x,g})$ denote a menu of subsidies for the startup and individuals. Each element in the vector is a compensation rate. Subscripts "$e$" and "$x$" represent the researcher and the entrepreneur, respectively. Subscripts "$\pi$" and "$g$" represent the product market profit and the value of knowledge. For example, $\tau_{x,g}$ denotes the compensation rate received by the entrepreneur based on the value of knowledge.

Given a compensation scheme $(\mu, \tau_{e,\pi}, \tau_{e,g}, \tau_{x,\pi}, \tau_{x,g})$ and effort levels $(e, x)$, the researcher receives a total subsidy $T_e \doteq \tau_{e,\pi}\pi(e,x) + \tau_{e,g} g(e,x)$, and the entrepreneur receives $T_x \doteq \tau_{x,\pi}\pi(e,x) + \tau_{x,g} g(e,x)$. The startup's profit is $\tilde{Y}_s(e,x) \doteq A_\pi e^\alpha x^{(1-\alpha)} + \mu A_g e^\beta x^{(1-\beta)}$. The social planner's problem is

$$(28)\qquad \begin{aligned} &\max_{(\mu, \tau_{e,\pi}, \tau_{e,g}, \tau_{x,\pi}, \tau_{x,g})} V\left(\frac{\nu}{\nu+1}\tilde{Y}(\tilde{e},x)\right) - c_e\tilde{e} + U\left(\frac{1}{\nu+1}\tilde{Y}(\tilde{e},x)\right) - c_x x, \\ &\text{s.t. } \tilde{e} = \arg\max_{e,w}\left\{V\left(\tilde{Y}_s(e,x) + T_e - w\right) - c_e e\right\}, \\ &c_x x = U(w + T_x). \end{aligned}$$

We find that the multidimensional subsidy still cannot restore the first-best allocation:

**Proposition 6.** *There does not exist a subsidy scheme* $(\mu, \tau_{e,\pi}, \tau_{e,g}, \tau_{x,\pi}, \tau_{x,g}) \in [0,1]^5$ *that yields an equilibrium with the first-best effort levels,* $(e^*, x^*)$.

This result demonstrates the robustness of our main results. Because all subsidies and incomes are public information, the principal can write contract terms based on the subsidies. As long as the principal holds superior bargaining power, he can exploit his power to substitute his efforts with the agent's, and the replacement effect distorts equilibrium from the first-best.[18] This result also indicates that fixing the internal labor market structure is essential to restore the first best.

### 4.3 Increasing Entrepreneur's Outside Option Value

The weak position of the entrepreneur in the startup is represented by her low outside option value. Increasing the entrepreneur's outside option value can be a potential policy instrument to intervene in the researcher–entrepreneur relationship within the startup. We now examine the effect of changing the value of the outside option, $\underline{u}$. We find that increasing $\underline{u}$ mitigates the effort distortion problem.

Let $u_x^* = U(e^*, x^*)$ denote the agent's utility in the first-best allocation derived in Lemma 1. With the full Pigouvian subsidy, the principal's problem is

$$\max_{(w,e)} \{V\left(Y\left(e, \hat{x}\right) - w\right) - c_e e\} \text{ s.t. } \hat{x} = \arg\max\{\left(U\left(w\right) - c_x x\right), \underline{u}\}.$$

Let $\hat{e}_1(\underline{u})$ and $\hat{x}_1(\underline{u})$ denote the equilibrium efforts of the principal and the agent under the full subsidy when the agent's outside option is $\underline{u}$. We have the following result:

**Proposition 7.** *Under the full Pigouvian subsidy,*

*(a) When $0 < \underline{u} < u_x^*$, the equilibrium exhibits the following properties: (i) $\hat{\lambda}_1(\underline{u}) < 1$; (ii) the research effort is below the first-best level, i.e., $\hat{e}_1(\underline{u}) < e^*$; (iii) the business effort is above the first-best level, i.e., $\hat{x}_1(\underline{u}) > x^*$.*

*(b) As $\underline{u}$ increases in $[\underline{u}, u_x^*]$, the equilibrium effort distortion decreases, i.e., $\frac{\partial(e^* - \hat{e}_1(\underline{u}))}{\partial \underline{u}} < 0$ and $\frac{\partial(\hat{x}_1(\underline{u}) - x^*)}{\partial \underline{u}} < 0$. When $\underline{u} = u_x^*$, the equilibrium is the first-best allocation.*

Proposition 7-(a) shows that the results in Propositions 1 and 2 are robust as long as the agent has an outside option value below the first-best payoff ($0 < \underline{u} < u_x^*$). Given $\underline{u} < u_x^*$, the principal can still benefit from replacing his own effort with the agent's relative to the first-best allocation. Proposition 7-(b) demonstrates that the outside option value is an effective tool to mitigate the effort distortion problem. As $\underline{u}$ increases, the power asymmetry in the startup team reduces, i.e., viewing the startup as an internal labor market, the market powers of the researcher and the entrepreneur become close. When the internal labor market friction of the startup is eliminated, the Pigouvian subsidy can restore the first-best allocation.

### 4.4 Entrepreneur as Principal

What if the entrepreneur has greater bargaining power and serves as the principal? In this case, the entrepreneur offers a take-it-or-leave-it contract to the researcher. She can exploit the researcher by

[18] Note that our analysis assumes away information frictions. In reality, the social planner faces the problem of asymmetric information, which may further compromise the effectiveness of complex subsidy programs.

replacing her business effort with the researcher's effort. The researcher, who mainly contributes to the value of knowledge, works more but obtains less equity share. In this case, the external funding compensating for knowledge spillovers may also amplify the internal organizational frictions, which leads to effort distortion and production inefficiency.

One example is the recent internal conflict at OpenAI.[19] Sam Altman, CEO of OpenAI and an experienced entrepreneur, disagreed dramatically with the chief scientist, Ilya Sutskever, on the company's future strategies. The internal conflict led to a restructuring of OpenAI's board and the layoff of many staff. This conflict had a significant and possibly prolonged negative impact on the startup performance and the AI industry with adverse social impact.

The entrepreneur solves the following problem in the laissez-faire market:

$$\max_{\{w,e\}} \{V(\pi(e,x) - w) - c_x x\},$$
$$\text{s.t. } e = \arg\max\{(U(w(\pi)) - c_e e),\ \underline{u}\}. \tag{29}$$

Denote the equilibrium effort level as $\{\check{e}_0, \check{x}_0\}$.

With a full subsidy of the value of knowledge, the entrepreneur's problem is

$$\max_{\{w,e\}} \{V(Y(e,x) - w) - c_e e\},$$
$$\text{s.t. } e = \arg\max\{(U(w(Y)) - c_e e), \underline{u}\}.$$

where $Y(e,x) = (e,x) + g(e,x) = A_\pi e^\alpha x^{(1-\alpha)} + A_g e^\beta x^{(1-\beta)}$. Denote the equilibrium effort levels under the Pigouvian subsidy as $\{\check{e}_1, \check{x}_1\}$.

We compare the equilibrium effort levels in these two cases and find the following result.

**Proposition 8.** *In the scenario with the entrepreneur being the principal,*

*(a) the researcher's equilibrium effort is higher after receiving the full subsidy, i.e.,* $\check{e}_1 > \check{e}_0$*;*

*(b) the entrepreneur's equilibrium effort is lower after receiving the full subsidy, i.e.,* $\check{x}_1 \leq \check{x}_0$*, if* $A_\pi \geq c_e$*.*

*(c) the subsidy reduces the agent's equity shares when* $\ln\left(\frac{\alpha A_\pi}{c_e}\right) \leq \alpha$ *and* $\frac{\beta(1-\alpha)}{\alpha(1-\beta)} \leq \left(\frac{A_g}{A_\pi}\right)^{\frac{1}{1-\beta}}$*.*

The Pigouvian subsidy still yields the stimulating effect and the replacement effect, but these two effects are in the same direction when the entrepreneur holds the dominant position. The stimulating effect encourages the researcher to exert more effort to generate valuable knowledge and receive compensation, while the replacement effect allows the entrepreneur to replace her effort with the researcher's effort. Hence, both effects lead to an increase in the research effort. On the other hand, these two effects influence business effort in opposite directions. If the research effort's

[19] See `contxto.com/en/artificial-intelligence/openai-leadership-crisis-the-timeline-analysis/`.

marginal cost is not high, indicated by condition $A_\pi \geq c_e$, the replacement effect dominates the stimulating effect, and the business effort decreases.[20]

Proposition 8 shows that the Pigouvian subsidy can distort the effort choices when the entrepreneur has the dominant position. The subsidy enables the principal to incentivize a given level of research effort by relinquishing fewer equity shares. The researcher works more but obtains a smaller share of the startup, which makes tech startups less attractive to researchers in the long run.

**Corollary 4.** *If* $A_\pi \geq c_e$, $\ln\left(\frac{\alpha A_\pi}{c_e}\right) \leq \alpha$ *and* $\frac{\beta(1-\alpha)}{\alpha(1-\beta)} \leq \left(\frac{A_g}{A_\pi}\right)^{\frac{1}{1-\beta}}$, *social welfare is reduced by the Pigouvian subsidy, i.e.,*

$$SW(\check{e}_1, \check{x}_1, \check{w}_1) < SW(\check{e}_0, \check{x}_0, \check{w}_0).$$

This perverse welfare effect occurs when the replacement effect is large. Intuitively, the Pigouvian subsidy generates more value for a startup with higher productivity of knowledge, so a less effective effort bundle with less research effort and more business effort can generate a level of knowledge value that satisfies the participation constraint.

The corollary above implies that the essential problem is asymmetric bargaining power. In the previous section, having the correct level of outside options is the best. If the entrepreneur has superior bargaining power, there is still efficiency loss.

In summary, this section examines the policy bundles with the Pigouvian subsidy and other policies improving the entrepreneur's market power over the researcher within the internal labor market. In the extreme, with the entrepreneur as the principal, the Pigouvian subsidy can increase the research effort from the level of the laissez-faire equilibrium. In contrast, the entrepreneur's effort may decrease, and the distribution of equity shares becomes more unbalanced. The underprovision of entrepreneurial effort and the overprovision of research effort lead to distortion in resource utilization and reduced marginal product under the Pigouvian subsidy relative to the laissez-faire equilibrium.

### 4.5 Information Friction

Our main model assumes complete and symmetric information to focus on how the startup's internal organization jeopardizes the good intention of the Pigouvian subsidy. In this section, we consider an extension of the main model with hidden action, which is commonly studied in the entrepreneurial finance literature. Because the agent's effort is usually not contractible, most entrepreneurial finance contracts are based on coarse signals of startup performance called milestones, such as patent approval, development of the product prototype, and having the first corporate clients.

For tech startups, achieving these milestones requires input from both the researcher and the entrepreneur. Following the main model, let $e$ denote the research effort and $x$ denote the business

[20] The condition $A_\pi \geq c_e$ is not strong considering that $A_\pi$ should be large to ensure that income is positive,i.e., $\ln(A_\pi r^\alpha) > 0$.

effort. However, they are not contractible. For simplicity, we assume that the cost function of the researcher takes the quadratic form $\frac{c_e e^2}{2}$ to ensure that the net utility of the principal is concave. We keep the cost function of business effort as $c_x x$.

We consider two scenarios. First, in laissez-faire, the startup only earns income from the product market. Suppose that the commercialization milestone has a binary outcome of success ($s$) or failure ($f$). This outcome is commonly observable and verifiable. The researcher and entrepreneur jointly determine the probability of success:

$$p(e, x) = A_p e^{\rho} x^{(1-\rho)}.$$

Here, $A_p$ represents the efficiency of the startup team and ensures that the probabilities are below one. Achieving the commercialization milestone yields a profit $\pi_s$ to the startup; otherwise, the startup obtains $\pi_f < \pi_s$.

Second, from a social welfare perspective, the startup generates value from both the product market and innovation. Suppose that the startup now faces a milestone that combines the goals of commercialization and knowledge discovery. The outcome is either success ($s$) or failure ($f$), and the success rate is

$$q(e, x) = A_q e^{\sigma} x^{(1-\sigma)},$$

where $A_q$ is an efficiency parameter similar to $A_p$. Successfully achieving the milestone generates a social value $Y_s$, and failure leads to a social value $Y_f < Y_s$. Because the social value milestone includes innovation and knowledge discovery, it depends more on the research effort than the commercialization milestone, we assume $\sigma > \rho$.

In the main model, the principal can infer the agent's effort from observing the startup performance, which is a one-to-one mapping of both players' effort. However, the binary outcome of achieving the milestone does not provide sufficient information on the agent's effort level due to uncertainty. Thus, the principal's contracting problem is subject to the hidden action issue. The agent chooses her business effort level based on the incentive compatibility (IC) constraint that maximizes her net utility.

The timing of the game is the same as the main model. The principal offers the contract specifying the transfer to the agent contingent on whether the milestone is achieved. If the agent accepts the contract, the principal chooses $e$, and then the agent chooses $x$. Thereafter, the startup performance is realized, and the agent obtains payment as specified in the contract. If the agent rejects the contract, the agent leaves with the reservation utility $\underline{u} = 0$.

Due to the informativeness of startup performance, the contracting model here features the ex ante hidden action.[21] Following ex ante hidden action literature(Innes, 1990; Sappington, 1983), we consider a limited liability assumption together with incentive compatibility constraint and

[21] Ex ante hidden action means that the agent takes action prior to the realization of states. The effort only alters the success probabilities, and the principal cannot infer the agent's effort from the performance realization, which leads to the agent's IC constraint being imposed on the principal's problem.

participation constraint.[22]

Let $(\widehat{w}_{s,0}, \widehat{w}_{f,0}, \widehat{e}_0, \widehat{x}_0)$ denote the laissez-faire equilibrium,[23] where the subscript "0" represents the zero subsidy. The equilibrium can be solved by the principal's problem as follows:

$$\max_{(w_{s,0}, w_{f,0}, e)} \left\{ \mathbb{E}\left[V\left(e, w_{s,0}, w_{f,0}\right)\right] - \frac{c_e e^2}{2} \right\}, \tag{30}$$

$$\text{s.t. } \check{x}_0 = \arg\max_x \left\{ \mathbb{E}\left[U\left(x, w_{s,0}, w_{f,0}\right)\right] - x c_x \right\}, \tag{31}$$

$$U\left(w_{s,0}\right) \geq \underline{u} \text{ and } U\left(w_{f,0}\right) \geq \underline{u}, \tag{32}$$

$$\mathbb{E}\left[U\left(x, w_{\xi,0}, w_{f,0}\right)\right] - x c_x \geq \underline{u}, \tag{33}$$

$$\text{where } \mathbb{E}\left[U\left(x, w_{s,0}, w_{f,0}\right)\right] \doteq p\left(e,x\right) U\left(w_{s,0}\right) + \left[1 - p\left(e,x\right)\right] U\left(w_{f,0}\right),$$

$$\mathbb{E}\left[V\left(e, w_{s,0}, w_{f,0}\right)\right] \doteq p\left(e,x\right) V\left(\pi_s - w_{s,0}\right) + \left[1 - p\left(e,x\right)\right] V\left(\pi_f - w_{f,0}\right).$$

A full characterization of the laissez-faire equilibrium is in the Appendix.[24]

Let $(\widehat{w}_{s,1}, \widehat{w}_{f,1}, \widehat{e}_1, \widehat{x}_1)$ denote the equilibrium under the full Pigouvian subsidy, and the subscript "1" represents the full Pigouvian subsidy. The principal's problem is

$$\max_{(w_{s,1}, w_{f,1}, e)} \left\{ \mathbb{E}\left[V\left(e, w_{s,1}, w_{f,1}\right)\right] - \frac{c_e e^2}{2} \right\},$$

$$\text{s.t. } \check{x}_1 = \arg\max_x \left\{ \mathbb{E}\left[U\left(x, w_{s,1}, w_{f,1}\right)\right] - x c_x \right\}, \tag{34}$$

$$U\left(w_{s,1}\right) \geq \underline{u} \text{ and } U\left(w_{f,1}\right) \geq \underline{u}. \tag{35}$$

$$\mathbb{E}\left[U\left(x, w_{\xi,0}, w_{f,0}\right)\right] - x c_x \geq \underline{u}, \tag{36}$$

$$\text{where } \mathbb{E}\left[U\left(x, w_{s,1}, w_{f,1}\right)\right] \doteq q\left(e,x\right) U\left(w_{s,1}\right) + \left[1 - q\left(e,x\right)\right] U\left(w_{f,1}\right),$$

$$\mathbb{E}\left[V\left(e, w_{s,1}, w_{f,1}\right)\right] \doteq q(e,x) V\left(Y_s - w_{s,1}\right) + \left[1 - q(e,x)\right] V\left(Y_f - w_{f,1}\right).$$

A full characterization of the equilibrium under the Pigouvian subsidy is in the Appendix.

Comparing these two equilibria, we obtain the following result:

**Proposition 9.** *There exists a parameter range such that the Pigouvian subsidy reduces research effort, i.e.,* $\widehat{e}_1 < \widehat{e}_0$.

Proposition 9 indicates that the perverse welfare effect of the Pigouvian subsidy is robust to the presence of uncertainty and asymmetric information. We characterize the condition of $\check{e}_1 < \check{e}_0$ in the Appendix. Note that the intuition behind the conditions is not as clear as the main model because of the coexistence of three problems: positive externalities, asymmetric bargaining power in the

[22] The limited liability assumption is usually modeled as the nonnegative payment the agent obtains at each state. Because, in our case, the agent's utility takes a log-utility functional form, a zero transfer at any state makes the ex post utility and the ex ante utility approach negative infinity. Thus, we assume, for simplicity, that at each state, the agent's wage generates her nonnegative utility.

[23] Efforts are exerted ex ante before the outcome state is realized, so they are not state-dependent.

[24] The full characterization is derived from FOC and the binding constraint (32) under the failing state.

researcher–entrepreneur relationship, and information friction. However, the major driving force for the subsidy to backfire is the principal's exploitation of superior bargaining power. Although the agent can gain positive net utility through information rent if the startup is successful and achieves a milestone, their level of effort is still inefficiently high due to the replacement effect.

## 5 Conclusion and Policy Implications

We analyze how subsidies for innovation affect the performance of tech startups under an unbalanced researcher–entrepreneur relationship. Pigouvian subsidies tend to strengthen the researcher's dominant position and exaggerate the effort distortion problem, leading to unintended perverse welfare effects. Many existing innovation policies that focus on compensating for R&D activities only address the issue of positive externalities but ignore the internal frictions within startup teams.

Our study offers several policy implications. First, policymakers should supplement innovation subsidies with policies that can affect the internal structure of tech startups. They can use various tools to improve the entrepreneur's outside option value and weaken the researcher's dominant position in the startup. For example, policymakers increase the mobility of the labor market of startup managers by holding events and creating platforms. Governments, incubators, and university entrepreneurship centers can provide entrepreneurs with temporary jobs, such as tutors and managers. Additionally, it is important for a society to embrace a culture of tolerating entrepreneurs' failure.

Second, our study demonstrates that the effectiveness of innovation policies depends on the relationship between economic agents. Policymakers should consider how policy intervention changes the relationship among relevant parties beyond the direct impacts of innovation policies. This echoes the literature on the national innovation system, which emphasizes the university-industry-government relationship determines innovation outcomes of an economy (Nelson, 1993; Etzkowitz and Leydesdorff, 2000; Etzkowitz, 2004).

Last, our results also shed light on other public interventions of externalities, such as the pursuit of ESG (environmental, social, and governance) goals. Many economic environments are subject to twofold market failures and cannot be fixed by a single policy tool. For example, Pigouvian taxes have side effects when firms have market power (Buchanan, 1969; Barnett, 1980; Boyer and Laffont, 1999). Ganapati et al. (2020) and Fabra and Reguant (2014) find that most emission costs have been passed on to end users via higher electricity prices. Environmental regulation could exaggerate market power despite reducing pollution (Cropper and Oates, 1992; Fowlie et al., 2016). Thus, policymakers should consider the relationship between relevant parties and coordinate multiple policy tools to address these complex problems.

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# Appendix

**Proof of Lemma 1.**

The conditions are directly obtained from the FOCs of the planner's program:

$$\max_{e,x,w} SW(e,x,w) = V\left(Y - w\right) - c_e e + U\left(w\right) - c_x x.$$

$\square$

**Proof of Lemma 2.**

The Lagrangian is

$$\mathcal{L} = \ln\left(\pi\left(e,x\right) - w\right) - c_e e + \lambda\left[\ln(w) - c_x x\right].$$

The FOC of the wage transfer is

$$\frac{\nu}{\pi - w} = \frac{\lambda}{w} \Rightarrow \hat{w}_0 = \frac{\hat{\lambda}_0}{\nu + \hat{\lambda}_0}\pi.$$

Combining the FOCs of $w$, $e$ and $x$ with the binding IR, we obtain the FOCs in the lemma. $\square$

**Proof of Lemma 5.**

From (14) and (3), the equilibrium equity shares are $\frac{\hat{w}_1}{\hat{Y}_1} = \frac{\hat{\lambda}_1}{\nu+\hat{\lambda}_1}$ and $\frac{w^*}{Y^*} = \frac{1}{1+\nu}$. Therefore, the inequality, $\frac{\hat{w}_1}{\hat{Y}_1} < \frac{w^*}{Y^*}$ is equivalent to $\hat{\lambda}_1 < 1$.

First, we prove $\hat{\lambda}_1 \neq 1$ by contradiction. Suppose not, i.e., $\hat{\lambda}_1 = 1$. Then the agent's equilibrium equity share is equal to the first-best level, and the FOCs are the same too. The agent obtains positive net utility in equilibrium, contradicting equation (17).

Second, suppose $\hat{\lambda}_1 > 1$. Let $F(e;x,\lambda) \doteq \frac{c_e e}{\hat{\lambda}_1+\nu} - \left[\alpha + (\beta - \alpha)\frac{t}{1+t}\right]$, where $t \doteq \frac{A_g}{A_\pi}\left(\frac{x}{e}\right)^{(1-\beta)-(1-\alpha)}$. $F(e;x,\lambda)$ decreases in $e$.

Because the right-hand side of equation (78) is the same as equation (4) and $F(e;x,\lambda)$ decreases in $e$,

$$\hat{\lambda}_1 > 1 \Rightarrow \hat{e}_1 > e^*.$$

Similarly, comparing (16) and (5),

$$\hat{\lambda}_1 > 1 \Rightarrow \hat{x}_1 < x^*.$$

Construct a hypothetical equity share $\tilde{\epsilon}$, such that $U\left(\tilde{\epsilon}Y^*\right) = c_x x^*$, where $Y^* \doteq Y(e^*, x^*)$. Let $\hat{\epsilon} \doteq \frac{\hat{\lambda}_1}{\hat{\lambda}_1+\nu}$. Now the allocation $(\tilde{\epsilon}Y^*, e^*, x^*)$ is feasible of the principal's problem (12). The optimality of the principal's problem at $(\hat{w}_1, \hat{e}_1, \hat{x}_1)$ implies his net utility is larger at $(\hat{w}_1, \hat{e}_1, \hat{x}_1)$

than $(\tilde{\epsilon}Y^*, e^*, x^*)$, i.e.,

$$\begin{aligned} V\left(\left(1-\tilde{\epsilon}\right)Y^*\right) - c_e e^* &< V\left(\left(1-\hat{\epsilon}_1\right)\hat{Y}_1\right) - c_e \hat{e}_1, \\ \left(1-\tilde{\epsilon}\right)Y^* &< \left(1-\hat{\epsilon}_1\right)\hat{Y}_1, \\ \frac{(1-\tilde{\epsilon})}{(1-\hat{\epsilon}_1)} &< \frac{\hat{Y}_1}{Y^*}. \end{aligned}$$

The second inequality is because $\hat{e}_1 > e^*$.

From $\hat{x}_1 < x^*$ and the agent's net utility is zero at $(\tilde{\epsilon}Y^*, e^*, x^*)$ and $(\hat{w}_1, \hat{e}_1, \hat{x}_1)$,

$$\begin{aligned} U\left(\tilde{\epsilon}Y^*\right) - c_x x^* &= U\left(\left(\hat{\epsilon}_1\right)\hat{Y}_1\right) - c_x \hat{x}_1, \\ U\left(\tilde{\epsilon}Y^*\right) &> U\left(\left(\hat{\epsilon}_1\right)\hat{Y}_1\right), \\ \tilde{\epsilon}Y^* &> \hat{\epsilon}_1\hat{Y}_1 \\ \frac{\tilde{\epsilon}}{\hat{\epsilon}_1} &> \frac{\hat{Y}_1}{Y^*}. \end{aligned}$$

Therefore,

$$\frac{\tilde{\epsilon}}{\hat{\epsilon}_1} > \frac{1-\tilde{\epsilon}}{1-\hat{\epsilon}_1}, \Rightarrow \hat{\epsilon}_1 < \tilde{\epsilon}.$$

From the construction of $\tilde{\epsilon}$,

$$\begin{aligned} U\left(\tilde{\epsilon}Y^*\right) - c_x x^* &= 0, \\ U\left(\frac{1}{1+\nu}Y^*\right) - c_x x^* &> 0, \end{aligned}$$

(37)
$$\Rightarrow \tilde{\epsilon} < \frac{1}{1+\nu}.$$

Combine (37) and (37), $\hat{\epsilon}_1 < \frac{1}{1+\nu}$, contradicting $\hat{\lambda}_1 > 1$. □

### Proof of Lemma 3

**(a)** Consider a hypothetical planner's optimization program where the only income source is product profit:

(38)
$$\max_{e,x,w} V\left(\pi(e,x) - w\right) - c_e e + \left[U\left(w\right) - c_x x\right].$$

The FOC of $w$ implies the agent's equity share in this hypothetical planner's program is $\frac{1}{\nu+1}$. By Lemma 2, the agent's equity share in laissez-faire equilibrium is $\frac{\hat{\lambda}_0}{\nu+\hat{\lambda}_0}$. Because $\hat{\lambda}_0 < 1$ in Lemma 5, $\frac{\hat{\lambda}_0}{\nu+\hat{\lambda}_0} < \frac{1}{\nu+1}$.

**(b)** The proof consists of two steps.

**Step 1:** $\hat{e}_0 < e_0^{**}$,where $e_0^{**}$ denotes the solution of (38).

The FOC of $e$ in (38) is

$$\frac{(1+\nu)\,\alpha}{e_0^{**}} = c_e \Rightarrow \frac{c_e e_0^{**}}{1+\nu} = \alpha \Rightarrow e_0^{**} = \frac{(1+\nu)\,\alpha}{c_e}.$$

Given that $\hat{\lambda}_0 < 1$ from part (a), equation (9) implies

$$\hat{e}_0 = \frac{\alpha\left(\hat{\lambda}_0 + \nu\right)}{c_e} < e_0^{**}.$$

**Step 2:** $e_0^{**} < e^*$, where $e^*$ denotes the first-best effort. From FOC of $e^{**}$,

$$\frac{c_e e^*}{1+\nu} = \alpha + (\beta-\alpha)\,\frac{t}{1+t} > \alpha = \frac{c_e e_0^{**}}{1+\nu}.$$

The last inequality is from $t \doteq \frac{A_g}{A_\pi}\left(\frac{x}{e}\right)^{(1-\beta)-(1-\alpha)} > 0$ and $(1-\alpha) > (1-\beta)$. Combining step 1 and 2, $e^* > e_0^{**} > \hat{e}_0$. □

**Proof of Proposition 1.**

**(a)** Let $t \doteq \frac{A_g}{A_\pi} r^{(\beta-\alpha)}$ denote the ratio of knowledge value to product profit. From the FOCs from the Pigouvian subsidy problem (15) and (16) and the FOCs for the first-best allocation,

$$\frac{c_e}{c_x} = \frac{\alpha + (\beta-\alpha)\frac{\hat{t}_1}{1+\hat{t}_1}}{\hat{t}_1\left[(1-\alpha) - (\beta-\alpha)\frac{\hat{t}_1}{1+\hat{t}_1}\right]}, \tag{39}$$

$$\frac{c_e}{c_x} = \frac{\alpha + (\beta-\alpha)\frac{t^*}{1+t^*}}{t^*\left[(1-\alpha) - (\beta-\alpha)\frac{t^*}{1+t^*}\right]}. \tag{40}$$

Comparing (39) and (40), $\hat{\lambda}_1 < 1$. Because $\frac{\alpha+(\beta-\alpha)\frac{t}{1+t}}{r\left[(1-\alpha)-(\beta-\alpha)\frac{t}{1+t}\right]}$ decreases in $r$,

$$\frac{\alpha + (\beta-\alpha)\frac{\hat{t}_1}{1+\hat{t}_1}}{\hat{r}_1(\underline{u})\left[(1-\alpha) - (\beta-\alpha)\frac{\hat{t}_1}{1+\hat{t}_1}\right]} > \frac{\alpha + (\beta-\alpha)\frac{t^*}{1+t^*}}{r^*\left[(1-\alpha) - (\beta-\alpha)\frac{t^*}{1+t^*}\right]}$$

and $\hat{r}_1 < r^*$.

Comparing the right-hand side of FOC for $\hat{e}_1$ to $e^*$,

$$\begin{aligned} \alpha + (\beta-\alpha)\frac{\hat{t}_1}{1+\hat{t}_1} &< \alpha + (\beta-\alpha)\frac{t^*}{1+t^*}, \\ \Leftrightarrow \frac{c_e \hat{e}_1}{\hat{\lambda}_1 + \nu} &< \frac{c_e e^*}{1+\nu} \\ \Rightarrow \hat{e}_1 &< e^*, \end{aligned}$$

where the first inequality is from $\hat{r}_1 < r^*$; the second is from FOCs (15) and (4); and the third is from $\hat{\lambda}_1 < 1$.

**(b)** Similar to part (a),

$$
\begin{aligned}
(1-\alpha)-(\beta-\alpha)\frac{\hat{t}_1}{1+\hat{t}_1} &> (1-\alpha)-(\beta-\alpha)\frac{t^*}{1+t^*}, \\
\Leftrightarrow \frac{\hat{\lambda}_1 c_x}{\hat{\lambda}_1+\nu}\hat{x}_1 &< \frac{c_x x^*}{1+\nu}, \\
\Rightarrow \hat{e}_1 &> e^*,
\end{aligned}
$$

where the first inequality is from $\hat{r}_1 < r^*$; the second is from FOCs (16) and (5); and the third is from $\hat{\lambda}_1 < 1$. □

**Proof of Proposition 2.**

**Step 1:** To show $\tilde{e} < \hat{e}_0$.

Let $(\tilde{e}, \tilde{x})$ denote the solution of the following hypothetical contracting problem,

$$
\begin{aligned}
&\max_{(w,e)} \left\{ V\left(A_g e^{\beta} \hat{x}^{(1-\beta)} - w\right) - c_e e \right\}, \\
&\text{s.t. } \hat{x} = \arg\max_x \{(U(w) - c_x x), \underline{u}\},
\end{aligned}
$$

The FOCs are

$$
\begin{cases}
\frac{c_e e}{\lambda+\nu} = \beta \\
\frac{\lambda c_x x}{\lambda+\nu} = (1-\beta) \\
\ln\left(\frac{\lambda}{\lambda+\nu}\right) + \ln\left(A_g e^{\beta} x^{(1-\beta)}\right) = c_x x.
\end{cases} \tag{41}
$$

**(i)** Transform (41) as

$$
\begin{aligned}
\ln\left(\frac{\lambda}{\lambda+\nu}\right) + \ln\left(A_g e^{\beta} x^{(1-\beta)}\right) &= \ln\left(\frac{\lambda}{\lambda+\nu} e^{\beta} x^{(1-\beta)}\right) + \ln A_g, \\
\Leftrightarrow \frac{\lambda}{\lambda+\nu} A_g e^{\beta} x^{(1-\beta)} &= \exp\left(c_x x\right).
\end{aligned} \tag{42}
$$

Represent the solution of equation (41) $\tilde{e}$ and $\tilde{x}$ by $\tilde{\lambda}$,

$$
\begin{aligned}
\tilde{e} &= \frac{\beta\left(\tilde{\lambda}+\nu\right)}{c_e}, \\
\tilde{x} &= \frac{(1-\beta)}{c_x}\left(1+\frac{\nu}{\tilde{\lambda}}\right).
\end{aligned}
$$

Plug the equations above back into (42) and define

$$F(z_\lambda) \doteq \beta \ln(z_\lambda) + b_g - (1-\beta)\left(1 + \frac{\nu}{z_\lambda}\right), \tag{43}$$

where $_g = \ln\left(A_g \left(\frac{\beta}{c_e}\right)^{\beta} \left(\frac{(1-\beta)}{c_x}\right)^{(1-\beta)}\right)$ and $\tilde{\lambda}$ is the solution of $F(z_\lambda) = 0$.

From (9), (10) and (11),

$$\alpha \ln \hat{\lambda}_0 + b_\pi = (1-\alpha)\left(1 + \frac{\nu}{\hat{\lambda}_0}\right), \tag{44}$$
$$\text{where } b_\pi \doteq \ln\left(A_\pi \left(\frac{\alpha}{c_e}\right)^{\alpha} \left(\frac{(1-\alpha)}{c_x}\right)^{(1-\alpha)}\right).$$

where $\hat{\lambda}_0$ is the solution of $\lambda$ in the laissez-faire equilibrium.

**(ii)** Transforming the comparison of $\tilde{e}$ and $\hat{e}_0$ into the comparison of $\tilde{\lambda}$ and $\hat{\lambda}_0$,

$$\begin{aligned} &\tilde{e} < \hat{e}_0, \\ \Leftrightarrow &\frac{\beta\left(\tilde{\lambda} + \nu\right)}{c_e} < \frac{\alpha\left(\hat{\lambda}_0 + \nu\right)}{c_e}, \\ \Leftrightarrow &\tilde{\lambda} < \frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\nu. \end{aligned} \tag{45}$$

Hence, the proof to show $\tilde{e} < \hat{e}_0$ is equivalent to the proof of (45).

**(iii)** This step is to prove $F\left(\frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\right) > 0$, where $F(\cdot)$ is defined in(43), if and only if $\tilde{\lambda} < \frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\nu$.

Because $F'(z_\lambda) = \frac{\beta}{z_\lambda} + \frac{(1-\beta)\nu}{z_\lambda^2} > 0$, $F(z_\lambda)$ is monotonically increasing in $z_\lambda$.

Because $F(\tilde{\lambda}) = 0$ from the definition equation (43),

$$\begin{aligned} &\frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\nu > \tilde{\lambda} \\ \Leftrightarrow &F\left(\frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\nu\right) > 0. \end{aligned}$$

**(iv)** We find a lower bound of $F\left(\frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\right)$ and she that it is positive under conditions (18) and (19).

The lower bound is $F\left(\frac{\alpha\hat{\lambda}_0}{\beta}\right) + \frac{\nu((1-\beta)-(1-\alpha))}{\beta} F'\left(\frac{\alpha\hat{\lambda}_0}{\beta}\right)$.

Because $(1-\alpha) > (1-\beta)$, $\frac{(1-\beta)-(1-\alpha)}{\beta} < 0$.

Because $F''(z_\lambda) = -\frac{\beta}{z_\lambda^2} - \frac{2(1-\beta)\nu}{z_\lambda^3} < 0$ and $F'(z_\lambda) > 0$,

$$F'\left(\frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\nu\right) > F'(x) > F'\left(\frac{\alpha\hat{\lambda}_0}{\beta}\right),$$
$$\forall x \in \left(\frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\nu,\ \frac{\alpha\hat{\lambda}_0}{\beta}\right)$$
$$\begin{aligned}\Rightarrow F\left(\frac{\alpha\hat{\lambda}_0}{\beta}\right) - F\left(\frac{\alpha\hat{\lambda}_0}{\beta} - \frac{(\beta-\alpha)}{\beta}\nu\right) &= \int_{\frac{\alpha\hat{\lambda}_0}{\beta}-\frac{(\beta-\alpha)}{\beta}}^{\frac{\alpha\hat{\lambda}_0}{\beta}} F'(x)dx,\\ &< \int_{\frac{\alpha\hat{\lambda}_0}{\beta}-\frac{(\beta-\alpha)}{\beta}}^{\frac{\alpha\hat{\lambda}_0}{\beta}} F'\left(\frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\right)dx,\\ &= \frac{(\beta-\alpha)}{\beta}\nu F'\left(\frac{\alpha\hat{\lambda}_0}{\beta} - \frac{(\beta-\alpha)}{\beta}\nu\right).\end{aligned}$$

Hence,

$$F\left(\frac{\alpha\hat{\lambda}_0}{\beta} + \frac{(1-\beta)-(1-\alpha)}{\beta}\nu\right) > F\left(\frac{\alpha\hat{\lambda}_0}{\beta}\right) - \frac{(\beta-\alpha)}{\beta}F'\left(\frac{\alpha\hat{\lambda}_0}{\beta} - \frac{(\beta-\alpha)}{\beta}\nu\right),$$

where $F(\cdot)$ is defined in equation (43) and $\hat{\lambda}_0$ is the laissez-faire equilibrium $\lambda$.

**(v)**

$$F\left(\frac{\alpha\hat{\lambda}_0}{\beta}\right) - \frac{(\beta-\alpha)}{\beta}\nu F'\left(\frac{\alpha\hat{\lambda}_0}{\beta} - \frac{(\beta-\alpha)}{\beta}\nu\right) > 0,$$
(46)
$$\Leftrightarrow F\left(\frac{\alpha\hat{\lambda}_0}{\beta}\right) > \frac{(\beta-\alpha)}{\beta}\nu F'\left(\frac{\alpha\hat{\lambda}_0}{\beta} - \frac{(\beta-\alpha)}{\beta}\nu\right),$$

where $F(x)$ is defined in (43).

Expand both sides of inequality (46), we have

$$\begin{aligned}F\left(\frac{\alpha\hat{\lambda}_0}{\beta}\right) &= \beta\ln\left(\frac{\alpha\hat{\lambda}_0}{\beta}\right) + b_g - (1-\beta)\left(1 + \frac{\nu}{\frac{\alpha\hat{\lambda}_0}{\beta}}\right)\\ &= \beta\ln\left(\alpha\hat{\lambda}_0\right) + b_g - \beta\ln\beta - (1-\beta)\left(1 + \frac{\nu\beta}{\alpha\hat{\lambda}_0}\right),\end{aligned}$$
$$\begin{aligned}\frac{(\beta-\alpha)}{\beta}\nu F'\left(\frac{\alpha\hat{\lambda}_0}{\beta} - \frac{(\beta-\alpha)}{\beta}\right) &= \frac{(\beta-\alpha)\,\nu}{\beta\left(\frac{\alpha\hat{\lambda}_0}{\beta} - \frac{(\beta-\alpha)}{\beta}\nu\right)}\left(\beta + \frac{(1-\beta)\nu}{\frac{\alpha\hat{\lambda}_0}{\beta} - \frac{(\beta-\alpha)}{\beta}\nu}\right)\\ &= \frac{(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0 - (\beta-\alpha)\,\nu}\left(\beta + \frac{(1-\beta)\nu\beta}{\alpha\hat{\lambda}_0 - (\beta-\alpha)\,\nu}\right).\end{aligned}$$

Therefore, (46) is equivalent to

$$\beta\ln\left(\alpha\hat{\lambda}_0\right)+b_g-\beta\ln\beta-(1-\beta)\left(1+\frac{\nu\beta}{\alpha\hat{\lambda}_0}\right),$$
$$>\frac{(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\left(\beta+\frac{(1-\beta)\nu\beta}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\right),$$
$$\ln\left(\alpha\hat{\lambda}_0\right)+\frac{b_g}{\beta}-\ln\beta-(1-\beta)\left(\frac{1}{\beta}+\frac{\nu}{\alpha\hat{\lambda}_0}\right),$$
$$>\frac{(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\left(1+\frac{(1-\beta)\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\right). \tag{47}$$

The next step proves conditions (18) and (19) are sufficient for (47).

**(vi)** This step transforms the left-hand side of (47).

Because $\alpha\ln\hat{\lambda}_0=(1-\alpha)\left(1+\frac{\nu}{\hat{\lambda}_0}\right)-b_\pi$, from (44),

$$\begin{aligned}&\ln\left(\alpha\hat{\lambda}_0\right)+\frac{b_g}{\beta}-\ln\beta-(1-\beta)\left(\frac{1}{\beta}+\frac{\nu}{\alpha\hat{\lambda}_0}\right)\\&=\ln\hat{\lambda}_0+\frac{b_g}{\beta}-\ln\left(\frac{\beta}{\alpha}\right)-(1-\beta)\left(\frac{1}{\beta}+\frac{\nu}{\alpha\hat{\lambda}_0}\right)\\&=\frac{(1-\alpha)}{\alpha}\left(1+\frac{\nu}{\hat{\lambda}_0}\right)-\frac{b_\pi}{\alpha}+\ln\left(\frac{\alpha}{\beta}\right)+\frac{b_g-(1-\beta)}{\beta}+\frac{\nu(1-\beta)}{\alpha\hat{\lambda}_0}.\end{aligned}$$

Therefore, inequality (47) is equivalent to

$$\frac{(1-\alpha)}{\alpha}\left(1+\frac{\nu}{\hat{\lambda}_0}\right)-\frac{b_\pi}{\alpha}+\ln\left(\frac{\alpha}{\beta}\right)+\frac{b_g-(1-\beta)}{\beta}+\frac{\nu(1-\beta)}{\alpha\hat{\lambda}_0}$$
$$>\frac{(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\left(1+\frac{(1-\beta)\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\right).$$

**(vii)** To show the transformed version of inequality (47) holds.

Because $\alpha\hat{\lambda}_0\in(0,1)$,

$$\frac{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu+(1-\beta)\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}=\frac{\alpha\hat{\lambda}_0-(1-\alpha)\nu+2(1-\beta)\nu}{\alpha\hat{\lambda}_0-(1-\alpha)\nu+(1-\beta)\nu}<2.$$

Therefore, the right-hand side of (47) is below $\frac{2(\beta-\alpha)\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\nu}$, i.e.,

$$\frac{2\,(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}>\frac{(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\left(1+\frac{(1-\beta)\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\right).$$

Let $x_0$ denote the intersection of $(1-\alpha)\left(1+\frac{\nu}{x}\right)-b_\pi$ at x-axis, thus,

$$
\begin{aligned}
(1-\alpha)\left(1+\frac{\nu}{x_0}\right)-b_\pi &= 0,\\
(1-\alpha)\left(1+\frac{\nu}{\hat{\lambda}_0}\right)-b_\pi &= \alpha\ln\hat{\lambda}_0,\\
\alpha\ln 1 &= 0,
\end{aligned}
$$

where the second equation is because $\alpha\ln\hat{\lambda}_0$ intersects $(1-\alpha)\left(1+\frac{\nu}{\hat{\lambda}_0}\right)-b_\pi$ at $\hat{\lambda}_0$; and the last is because $\alpha\ln x$ intersects $x$-axis at 1.

Because $\alpha\ln x$ increases in $x$ and $(1-\alpha)\left(1+\frac{\nu}{x}\right)-b_\pi$ decreases in $x$, $\hat{\lambda}_0\in(x_0,1)$.

Because $\frac{\nu}{x_0}=\frac{b_\pi}{(1-\alpha)}-1$, $\frac{\nu}{\hat{\lambda}_0}<\frac{b_\pi}{(1-\alpha)}-1$.

Condition (19) leads to

$$
\begin{aligned}
&(\beta-\alpha)\left[\frac{b_\pi}{(1-\alpha)\alpha}-\frac{1}{(1-\alpha)}-\frac{2}{\nu((1-\alpha)+(1-\beta))}\right]<1,\\
\Leftrightarrow\, &1-\frac{2\,(\beta-\alpha)}{\nu((1-\alpha)+(1-\beta))}>\frac{(\beta-\alpha)\,b_\pi}{(1-\alpha)\alpha}-\frac{(\beta-\alpha)}{(1-\alpha)}>\frac{(\beta-\alpha)}{\alpha}\frac{\nu}{\hat{\lambda}_0}\\
\Leftrightarrow\, &1-\frac{(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0}>\frac{2\,(\beta-\alpha)}{((1-\alpha)+(1-\beta))},\\
\Leftrightarrow\, &\frac{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0}>\frac{2\,(\beta-\alpha)}{((1-\alpha)+(1-\beta))},\\
\Leftrightarrow\, &\frac{\nu\,((1-\alpha)+(1-\beta))}{\alpha\hat{\lambda}_0}>\frac{2\,(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}.
\end{aligned}
\tag{48}
$$

Thus, condition (19) ensures that

$$
\frac{\nu\,((1-\alpha)+(1-\beta))}{\alpha\hat{\lambda}_0}>\frac{2\,(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}
$$

Combining inequality (48) and $\frac{\alpha\hat{\lambda}_0-(\beta-\alpha)\nu+(1-\beta)\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\nu}<2$,

$$
\frac{\nu\,((1-\alpha)+(1-\beta))}{\alpha\hat{\lambda}_0}>\frac{(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\left(1+\frac{(1-\beta)\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\right)
$$

**(viii)** condition (18) leads to

$$
\begin{aligned}
&\frac{b_g}{\beta}-\frac{b_\pi}{\alpha}>\ln\left(\frac{\beta}{\alpha}\right)+\frac{(1-\beta)}{\beta}-\frac{(1-\alpha)}{\alpha},\\
\Leftrightarrow\, &\frac{(1-\alpha)}{\alpha}-\frac{b_\pi}{\alpha}+\ln\left(\frac{\alpha}{\beta}\right)+\frac{b_g-(1-\beta)}{\beta}>0,
\end{aligned}
\tag{49}
$$

Combining inequalities (49) and (48),

$$\frac{(1-\alpha)}{\alpha}\left(1+\frac{\nu}{\hat{\lambda}_0}\right)-\frac{b_\pi}{\alpha}+\ln\left(\frac{\alpha}{\beta}\right)+\frac{b_g-(1-\beta)}{\beta}+\frac{\nu(1-\beta)}{\alpha\hat{\lambda}_0}$$
$$>\frac{(\beta-\alpha)\,\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\left(1+\frac{(1-\beta)\nu}{\alpha\hat{\lambda}_0-(\beta-\alpha)\,\nu}\right).$$

Hence, condition (18) ensures that

$$\frac{(1-\alpha)}{\alpha}-\frac{b_\pi}{\alpha}+\ln\left(\frac{\alpha}{\beta}\right)+\frac{b_g-(1-\beta)}{\beta}>0.$$

Therefore, $\tilde{e}<\hat{e}_0$.

**Step 2:** To show $\tilde{e}>\hat{e}_1$, where $\tilde{e}$ is the hypothetical equilibrium in (41).

Define $D_e(r)$ as the right-hand side of equation (15), where $r\doteq\frac{e}{x}$. Let $t=\frac{A_g}{A_\pi}\left(\frac{x}{e}\right)^{(1-\beta)-(1-\alpha)}$, and $T(r)\doteq\frac{t}{1+t}$,

$$\begin{aligned}D_e(r)&=\alpha+((\beta-\alpha))\frac{t}{1+t}\\&=\beta-((\beta-\alpha))T(r)\\\Rightarrow&D_e(r)\in(\alpha,\beta).\end{aligned}$$

Similarly define $D_x(r)$ from equation (16) and $D_x(r)\in((1-\beta),(1-\alpha))$.

(50)
$$\begin{cases}\frac{c_e e}{\lambda+\nu}=\beta-(\beta-\alpha)T(r)\\\frac{\lambda c_x x}{\lambda+\nu}=(1-\beta)+(\beta-\alpha)T(r)\\\ln\left(\frac{\lambda}{\lambda+\nu}\right)+\ln Y(e,x)=c_x x.\end{cases}$$

Rearranging equation (41), $(\tilde{\lambda},\tilde{r})$ is the unique solution o

(51)
$$\begin{cases}\frac{c_e r}{c_x}=\frac{\lambda\beta}{1-\beta}\\\exp\left((1-\beta)\left(1+\frac{\nu}{\lambda}\right)\right)=\frac{(1-\beta)}{c_x}r^\beta A_g.\end{cases}$$

Rearranging equation (50), $(\hat{\lambda}_1,\hat{r}_1)$ is the unique solution of the following equations,

(52)
$$\frac{c_e r}{c_x}=\frac{\lambda\left(\beta-((\beta-\alpha))T(r)\right)}{(1-\beta)+((\beta-\alpha))T(r)}$$

(53)
$$\exp\left(((1-\beta)+(\beta-\alpha))T(r))\left(1+\frac{\nu}{\lambda}\right)\right)=\frac{(1-\beta)+((\beta-\alpha))T(r)}{c_x}r^\beta\left(A_g+A_\pi r^{(1-\beta)-(1-\alpha)}\right).$$

Define $\tilde{F}(r)$ and $\hat{F}(r)$ by the functions characterizing $\lambda$ in the first equation of (51) and equation

(52). Similarly define $\tilde{G}(r)$ and $\hat{G}(r)$ from the second equation of (51) and equation (53).

$$\hat{F}(r) = \frac{c_e r}{c_x}\frac{((1-\beta)+((\beta-\alpha))T(r))}{\beta-((\beta-\alpha))T(r)} > \frac{c_e r}{c_x}\left(\frac{1-\beta}{\beta}\right) = \tilde{F}(r)$$
$$\tilde{G}(r) = \nu\left[\frac{\ln\left(\frac{(1-\beta)}{c_x}r^\beta A_g\right)}{1-\beta} - 1\right]^{-1}$$
$$\hat{G}(r) = \nu\left[\frac{\ln\left(\frac{(1-\beta)+(\beta-\alpha)T(r)}{c_x}\left(A_g r^\beta + A_\pi r^\alpha\right)\right)}{[(1-\beta)+(\beta-\alpha))T(r)]} - 1\right]^{-1} < \tilde{G}(r),$$

where the last inequality is proved as follows. Let $R_e(r) \doteq \beta - \frac{t(r)(\beta-\alpha)}{1+t(r)}$, where $t(r) \doteq \frac{A_g}{A_\pi}r^{\beta-\alpha}$. First, define $g(R) \doteq \frac{(1-R)A_g r^\beta}{c_x \exp(1-R)}$ where $R \in [R_e(r), \beta]$ with $R_e(r)$ defined above.

Take the derivative with regard to $R$,

$$g^{'}(R) \propto \frac{d\left(\frac{\ln(1-R)}{1-R}\right)}{dR} \propto \frac{1-R}{1-R} + \frac{\ln(1-R)}{(1-R)^2} > 0.$$

Thus, $g$ increases in $R$. Let $\hat{H}(r) \doteq \frac{(1-R_e(r))\left[A_g r^\beta + A_\pi r^\alpha\right]}{c_x \exp(1-R_e(r))}$ and $\tilde{H}(r) \doteq \frac{(1-\beta)A_g r^\beta}{c_x \exp(1-\beta)}$.

Because $R_e(r) < \beta, \quad \forall r$,

$$\tilde{H}(r) < \frac{(1-R_e(r))A_g r^\beta}{c_x \exp(1-R_e(r))} < \hat{H}(r).$$

Let $z \doteq \exp\left(1+\frac{\nu}{\lambda}\right)$, then $\lambda = \frac{\nu}{\ln z - 1}$. From the definitions of $\tilde{G}$ and $\hat{G}$, they are the inverse functions of $z = \tilde{H}(r)$ and $z = \hat{H}(r)$, i.e., $\tilde{G}^{-1}(\cdot) = \frac{\nu}{\ln\left(\tilde{H}(r)\right)-1}$ and $\hat{G}^{-1}(\cdot) = \frac{\nu}{\ln\left(\hat{H}(r)\right)-1}$.

Hence, for each given $\lambda$, $\tilde{G}(\lambda) > \hat{G}(\lambda)$. Because $\tilde{F}(r)$ and $\hat{F}(r)$ increase in $r$ and $\tilde{G}(r)$ and $\hat{G}(r)$ decrease at $r$, $\hat{\lambda}_1 < \tilde{\lambda}$. Therefore,

$$\hat{e}_1 = (\beta - T(\hat{r}_1)(\beta-\alpha))\frac{\hat{\lambda}_1+\nu}{c_e} < \beta\frac{\hat{\lambda}_1+\nu}{c_e} < \beta\frac{\tilde{\lambda}_1+\nu}{c_e} = \tilde{e}.$$

Combining Steps 1 and 2, $\hat{e}_1 < \tilde{e} < \hat{e}_0$. □

**Proof of Corollary 2.**

We first prove that when $\hat{e}_1 < \hat{e}_0$, $\hat{\lambda}_1 < \hat{\lambda}_0$. Then, $\hat{\lambda}_1 < \hat{\lambda}_0$ implies $\hat{w}_1 < \hat{w}$.

From FOCs of two market equilibria, equations (9) and (15) imply

$$\frac{c_e\hat{e}_0}{\hat{\lambda}_0+\nu} = \alpha < \alpha + (\beta-\alpha)\frac{\hat{t}_1}{1+\hat{t}_1} = \frac{c_e\hat{e}_1}{\hat{\lambda}_1+\nu}.$$

Rearrange the inequality above,

$$\frac{\hat{\lambda}_1 + \nu}{\hat{\lambda}_0 + \nu} < \frac{\hat{e}_1}{\hat{e}_0} < 1.$$

Hence, $\hat{\lambda}_1 < \hat{\lambda}_0$.

According to equations (8) and (17), the equity share in laissez-faire equilibrium ($\hat{\epsilon}_0$) and the equity share with the full Pigouvian subsidy ($\hat{\epsilon}_1$) satisfy

$$\hat{\epsilon}_1 \doteq \frac{\hat{w}_1}{\hat{Y}_1} = \frac{\hat{\lambda}_1}{\nu + \hat{\lambda}_1} < \frac{\hat{\lambda}_0}{\nu + \hat{\lambda}_0} = \hat{\epsilon}.$$

□

**Proof of Corollary 1.**

From Proposition 2, conditions (18) and (19) ensure $\hat{e}_1 < \hat{e}_0$. Then, in the proof of Corollary 2, $\hat{\lambda}_1 < \hat{\lambda}_0$ and $\hat{e}_1 < \hat{e}_0$. Let $\hat{\epsilon}_1 \doteq \frac{\hat{w}_1}{\hat{Y}_1}$ and $\hat{\epsilon}_0 \doteq \frac{\hat{w}_0}{\hat{\pi}_0}$, then $\hat{\epsilon}_1 < \hat{\epsilon}_0$.

The social welfare function is

$$SW\left(e, x, \epsilon\right) = \left(1+\nu\right)\ln Y + \nu \ln\left(1-\epsilon\right) + \ln \epsilon - ec_e - xc_x.$$

From the FOCs,

$$\begin{aligned} SW_\epsilon(\hat{e}_0, \hat{x}_0, \hat{\epsilon}_0) &= \frac{\nu}{1-\epsilon} + \frac{1}{\epsilon} > 0,\ \forall \epsilon \in [\hat{\epsilon}_1, \hat{\epsilon}_0] \\ SW_{\epsilon\epsilon}(\hat{e}_0, \hat{x}_0, \epsilon) &\leq 0, \forall \epsilon \in \left[\hat{\epsilon}_1, \hat{\epsilon}_0\right], \end{aligned}$$

where inequality is from $\hat{\epsilon}_0 < \frac{1}{1+\nu}$. Then, from $\hat{\epsilon}_1 < \hat{\epsilon}_0$,

$$SW(\hat{e}_0, \hat{x}_0, \hat{\epsilon}_1) < SW(\hat{e}_0, \hat{x}_0, \hat{\epsilon}_0). \tag{54}$$

Similarly, let $t \doteq \frac{A_g}{A_\pi}\left(\frac{e}{x}\right)^{\beta-\alpha}$

$$\begin{aligned} SW_e(\hat{e}_0, \hat{x}_0, \hat{\epsilon}_0) &= \frac{(1+\nu)}{\hat{e}_0}\left(\beta - \frac{\beta-\alpha}{1+t\left(\hat{e}_0, \hat{x}_0\right)}\right) - c_e \\ &\propto \beta - \frac{\beta-\alpha}{1+t\left(\hat{e}_0, \hat{x}_0\right)} - \frac{c_e \hat{e}_0}{1+\nu} \\ &= \beta - \frac{\beta-\alpha}{1+t\left(\hat{e}_0, \hat{x}_0\right)} - \frac{\alpha\left(\nu + \hat{\lambda}_0\right)}{1+\nu} \\ &> \beta - \frac{\beta-\alpha}{1+t\left(\hat{e}_0, \hat{x}_0\right)} - \alpha \\ &= (\beta-\alpha)\left(1 - \frac{1}{1+t\left(\hat{e}_0, \hat{x}_0\right)}\right) > 0, \end{aligned}$$

where the inequality is because $\hat{\lambda}_0 < 1$.

Then for $\forall e \in [\hat{e}_1, \hat{e}_0]$,

$$
\begin{aligned}
SW_e\left(e, \hat{x}_0, \hat{\epsilon}_0\right) &\propto \beta - \frac{\beta-\alpha}{1+\frac{A_g e^{\beta-\alpha}}{A_\pi \hat{x}_0^{\beta-\alpha}}} - \frac{c_e e}{1+\nu} \\
&\geq \beta - \frac{\beta-\alpha}{1+t\left(\hat{e}_0, \hat{x}_0\right)} - \frac{c_e \hat{e}_0}{1+\nu} > 0 \\
SW_{ee}\left(e, \hat{x}_0, \epsilon\right) &\leq 0,
\end{aligned}
$$

where the last inequality of second-order derivative is from the regularity assumption that social welfare function is concave at each variable. Therefore, from $\hat{e}_1 < \hat{e}_0$,

$$
SW\left(\hat{e}_1, \hat{x}_0, \hat{\epsilon}_1\right) < SW\left(\hat{e}_0, \hat{x}_0, \hat{\epsilon}_1\right). \tag{55}
$$

Take the first-order derivative of social welfare function with regard to $x$,

$$
\begin{aligned}
SW_x\left(\hat{e}_1, \hat{x}_1, \hat{\epsilon}_1\right) &= \frac{(1+\nu)}{\hat{x}_1}\left(1-\beta+\frac{\beta-\alpha}{1+t\left(\hat{e}_1, \hat{x}_1\right)}\right) - c_x \\
&\propto 1-\beta+\frac{\beta-\alpha}{1+t\left(\hat{e}_1, \hat{x}_1\right)} - \frac{c_x \hat{x}_1}{1+\nu} \\
&= 1-\beta+\frac{\beta-\alpha}{1+t\left(\hat{e}_1, \hat{x}_1\right)} - \frac{1-\beta+\frac{\beta-\alpha}{1+t\left(\hat{e}_1, \hat{x}_1\right)}}{(1+\nu)\frac{\hat{\lambda}_1}{\hat{\lambda}_1+\nu}} \\
&= \left(1-\beta+\frac{\beta-\alpha}{1+t\left(\hat{e}_1, \hat{x}_1\right)}\right)\left(1-\frac{1+\frac{\nu}{\hat{\lambda}_1}}{(1+\nu)}\right) < 0,
\end{aligned}
$$

where the last inequality is from $\hat{\lambda}_1 < 1$ and $t \doteq \frac{A_g}{A_\pi}\left(\frac{e}{x}\right)^{\beta-\alpha}$. Similarly as before,

$$
\begin{aligned}
SW_x\left(\hat{e}_1, \hat{x}_0, \hat{\epsilon}_1\right) &\propto 1-\beta+\frac{\beta-\alpha}{1+\frac{A_g \hat{e}_1^{\beta-\alpha}}{A_\pi x^{\beta-\alpha}}} - \frac{c_x \hat{x}_0}{1+\nu} \\
&< 1-\alpha-\frac{c_x \hat{x}_0}{1+\nu} \\
&= \frac{\hat{\lambda}_0 c_x \hat{x}_0}{\hat{\lambda}_0+\nu} - \frac{c_x \hat{x}_0}{1+\nu} < 0.
\end{aligned}
$$

Also, from the regularity assumption of concave social welfare function, $SW_{xx}\left(e, x, \epsilon\right) \leq 0$. Therefore, for any $x \in [\hat{x}_0, \hat{x}_1]$,

$$
SW_x\left(\hat{e}_1, x, \hat{\epsilon}_1\right) \leq \sup\left\{SW_x\left(\hat{e}_1, \hat{x}_0, \hat{\epsilon}_1\right), SW_x\left(\hat{e}_1, \hat{x}_1, \hat{\epsilon}_1\right)\right\} < 0 \tag{56}
$$

FOC of $\hat{x}_1$ is

$$
\begin{aligned}
c_x &= \frac{\hat{\lambda}_1}{\left(\hat{\lambda}_1+\nu\right)\hat{x}_1}\left(1-\beta+\frac{\beta-\alpha}{1+t\left(\hat{e}_1,\hat{x}_1\right)}\right) \\
\Rightarrow \hat{x}_1 &= \frac{\hat{\lambda}_1}{\left(\hat{\lambda}_1+\nu\right)c_x}\left(1-\beta+\frac{\beta-\alpha}{1+t\left(\hat{e}_1,\hat{x}_1\right)}\right) \\
&> \frac{\hat{\lambda}_1}{\left(\hat{\lambda}_1+\nu\right)c_x}(1-\alpha) \\
&> \frac{\hat{\lambda}_0}{\left(\hat{\lambda}_0+\nu\right)c_x}(1-\alpha) = \hat{x}_0,
\end{aligned}
$$

where $t \doteq \frac{A_g}{A_\pi}\left(\frac{e}{x}\right)^{\beta-\alpha}$ and the first inequality is from $t\left(\hat{e}_1,\hat{x}_1\right)>0$ and the second inequality is from $\hat{\lambda}_1<\hat{\lambda}_0$.

From $\hat{x}_1>\hat{x}_0$ and inequality (56),

$$
SW\left(\hat{e}_1,\hat{x}_1,\hat{\epsilon}_1\right) < SW\left(\hat{e}_1,\hat{x}_0,\hat{\epsilon}_1\right).
$$

Combine the inequality above with (54) and (55),

$$
SW\left(\hat{e}_1,\hat{x}_1,\hat{\epsilon}_1\right) < SW\left(\hat{e}_1,\hat{x}_0,\hat{\epsilon}_1\right) < SW\left(\hat{e}_0,\hat{x}_0,\hat{\epsilon}_1\right) < SW(\hat{e}_0,\hat{x}_0,\hat{\epsilon}_0),
$$

which is equivalent to $SW(\hat{e}_1,\hat{x}_1,\hat{w}_1) < SW(\hat{e}_0,\hat{x}_0,\hat{w}_0)$ where the social welfare levels are uniquely solved by the equilibrium allocations. □

**Proof of Corollary 3.**

From Proposition 2, if condition (18) is satisfied under $A_g$,

$$
\begin{aligned}
\frac{b_g(A_g^{'})}{\beta}-\frac{b_\pi}{\alpha} > \frac{b_g(A_g)}{\beta}-\frac{b_\pi}{\alpha} \\
> \ln\left(\frac{\beta}{\alpha}\right)+\frac{1-\beta}{\beta}-\frac{1-\alpha}{\alpha},
\end{aligned}
$$

where $b_\pi$ and other parameters stay the same and $b_g(A_g^{'}) \doteq \ln\left(A_g^{'}\left(\frac{\beta}{c_e}\right)^{\beta}\left(\frac{1-\beta}{c_x}\right)^{1-\beta}\right)$.

Hence, $\hat{e}_1<\hat{e}_0$ under $A_g^{'}>A_g$. □

**Proof of Proposition 3.**

Step 1 establishes the necessary and sufficient conditions for all cases of the optimal subsidy's value, including $\mu^*=0$ and $\mu^*\in(0,1)$. The first part is the building block of this proof, and the second part applies to Proposition 4.

**Step 1:** Sufficient and necessary condition

From the continuity of the objective function (20) and the compactness of the feasible set of $e$ defined by constraint (21) and (22), the function $e_\mu$ is continuous and the optimal subsidy includes three cases,

(a) $\mu^* = 0$ if and only if $\lim_{\mu\to 0} \frac{\partial e(\mu)}{\partial \mu} \le 0$ and $\hat{e}_0 \ge \hat{e}_1$;

(b) $\mu^* \in (0,1)$ if and only if $\lim_{\mu\to 0} \frac{\partial e(\mu)}{\partial \mu} > 0$ and $\lim_{\mu\to 1} \frac{\partial e(\mu)}{\partial \mu} \le 0$.

(c) $\mu^* = 1$ if and only if $\lim_{\mu\to 1} \frac{\partial e(\mu)}{\partial \mu} > 0$ and $\hat{e}_0 < \hat{e}_1$.

**Step 2:** sufficient condition for (57)

From Proposition 2, conditions (18) and (19) ensure $\hat{e}_0 > \hat{e}_1$. According to the previous step, what remains to be proven is that under (23) and (24),

$$\lim_{\mu\to 0} \frac{\partial e(\mu)}{\partial \mu} \le 0. \tag{57}$$

Below proves the claim above in two sub-steps. **(i)** To show condition (57) is equivalent to $\hat{r}_0 \ge \frac{2c_x^2\nu}{c_e\left(\frac{(1-\alpha)^2}{\alpha}+b_2\right)}$.

Let $\tilde{A}_g \doteq \mu A_g$. The efforts level solving constraint (21) is equivalent to

$$\max_e \left\{ \nu \ln\left( \frac{\nu}{\nu+\lambda}\left( A_\pi e^\alpha x^{(1-\alpha)} + \tilde{A}_g e^\beta x^{(1-\beta)} \right)\right) - c_e e \right\}, \tag{58}$$

$$c_x x = \ln\left( \frac{\lambda}{\nu+\lambda}\left( A_\pi e^\alpha x^{(1-\alpha)} + \tilde{A}_g e^\beta x^{(1-\beta)} \right)\right), \forall e. \tag{59}$$

FOC of (58) with regard to $e$ is

$$\frac{e(\mu)c_e}{\lambda(\mu)+\nu} = \alpha + (\beta-\alpha)\frac{\frac{\mu A_g}{A_\pi} r^{(\beta-\alpha)}}{1+\frac{\mu A_g}{A_\pi} r^{(\beta-\alpha)}} = \beta - \frac{(\beta-\alpha)}{1+\frac{\mu A_g}{A_\pi} r^{(\beta-\alpha)}}.$$

$$e(\mu) = \frac{R_e(\tilde{A}_g)\left(\lambda(\tilde{A}_g)+\nu\right)}{c_e}, \text{where } R_e(\tilde{A}_g) \doteq \beta - \frac{(\beta-\alpha)}{1+\frac{\tilde{A}_g}{A_\pi} r^{(\beta-\alpha)}}.$$

Let $r \doteq \frac{e}{x}$ and $R_x \doteq (1-\beta) + \frac{(\beta-\alpha)}{1+\mu k}$. The FOC of $x$ and the constraint (59) imply that

$$\frac{c_e r}{c_x \lambda} = \frac{R_e}{1-R_e} \Rightarrow \lambda = \frac{(1-R_e)\,c_e r}{R_e c_x}.$$

Combining the above equation and $e = \left(\lambda(\tilde{A}_g)+\nu\right) R_e(\tilde{A}_g)$,

$$\frac{\partial e(\mu)}{\partial \mu} = A_g \frac{\partial \left( \left( \lambda(\tilde{A}_g, r) + \nu \right) R_e(\tilde{A}_g, r) \right)}{\partial \tilde{A}_g} \propto \frac{\partial \left( \left( \lambda(\tilde{A}_g, r) + \nu \right) R_e(\tilde{A}_g) \right)}{\partial \tilde{A}_g},$$

$$= R_e(\tilde{A}_g) \frac{d \left( \lambda(\tilde{A}_g, r) + \nu \right)}{d\tilde{A}_g} + \left( \lambda(\tilde{A}_g) + \nu \right) \frac{dR_e(\tilde{A}_g, r)}{d\tilde{A}_g},$$

$$= \frac{dR_e(\tilde{A}_g, r)}{d\tilde{A}_g} \left( \lambda(\tilde{A}_g, r) + \nu \right) + \lambda_{\tilde{A}_g}(\tilde{A}_g, r) R_e(\tilde{A}_g) + \frac{\partial r}{\partial \tilde{A}_g} \left[ \frac{dR_e}{dr} \left( \lambda(\tilde{A}_g) + \nu \right) + \lambda'(r) R_e(\tilde{A}_g) \right]. \tag{60}$$

where $r \doteq \frac{e}{x}$. $R_e'(\tilde{A}_g),\ \lambda'(\tilde{A}_g),\ R_e'(r),\ \lambda'(r)$ are

$$\frac{dR_e}{dr} = \frac{(\beta - \alpha)^2 \frac{\tilde{A}_g}{A_\pi} r^{(\beta-\alpha)-1}}{\left( 1 + \frac{\tilde{A}_g}{A_\pi} r^{(\beta-\alpha)} \right)^2} \geq 0, \tag{61}$$

$$\frac{dR_e}{d\tilde{A}_g} = \frac{(\beta - \alpha) r^{(\beta-\alpha)}}{A_\pi \left( 1 + \frac{\tilde{A}_g}{A_\pi} r^{(\beta-\alpha)} \right)^2} > 0, \tag{62}$$

$$\lambda'(\tilde{A}_g) = \frac{d\lambda}{dR_e} \left( \frac{dR_e}{d\tilde{A}_g} \right) = -\frac{c_e r}{R_e^2(\tilde{A}_g) c_x} \frac{dR_e}{d\tilde{A}_g}, \tag{63}$$

$$\frac{\partial \lambda}{\partial r} = \frac{d\lambda}{dR_e} \left( \frac{dR_e}{dr} \right) + \frac{d\lambda}{dr} = -\frac{c_e r}{R_e^2 c_x} \left( \frac{dR_e}{d\tilde{A}_g} \frac{\tilde{A}_g}{r} \right) + \frac{(1 - R_e) c_e}{R_e c_x}, \tag{64}$$

where $\tilde{y} = A_\pi r^\alpha + \tilde{A}_g r^\beta$, and $R_x = 1 - R_e(\tilde{A}_g)$.

Plugging the above equation into (60) which is denoted as $f(\tilde{A}_g)$ below

$$\lim_{\tilde{A}_g \to 0} f(\tilde{A}_g) = \lim_{\tilde{A}_g \to 0} \left\{ \frac{dR_e(\tilde{A}_g)}{d\tilde{A}_g} \left( \hat{\lambda}_0 + \nu \right) - \frac{c_e r R_e}{R_e^2(\tilde{A}_g) c_x} \frac{dR_e}{d\tilde{A}_g} + \frac{\partial r}{\partial \tilde{A}_g} \left[ \frac{dR_e}{dr} \left( \hat{\lambda}_0 + \nu \right) + \lambda'(r) R_e(\tilde{A}_g) \right] \right\},$$

$$= \lim_{\tilde{A}_g \to 0} \left\{ \frac{dR_e(\tilde{A}_g)}{d\tilde{A}_g} \left( \hat{\lambda}_0 + \nu \right) - \frac{c_e r \alpha}{R_e^2 c_x} \frac{dR_e}{d\tilde{A}_g} + \frac{\partial r}{\partial \tilde{A}_g} \left[ \frac{dR_e}{d\tilde{A}_g} \frac{\tilde{A}_g \left( \hat{\lambda}_0 + \nu \right)}{r(\beta - \alpha)} + \frac{(1 - R_e) c_e}{c_x} \right] \right\},$$

$$= \lim_{\tilde{A}_g \to 0} \left\{ \frac{dR_e(\tilde{A}_g)}{d\tilde{A}_g} \left( \hat{\lambda}_0 + \nu \right) - \frac{c_e \hat{r}_0}{\alpha c_x} \frac{dR_e}{d\tilde{A}_g} + \frac{\partial r}{\partial \tilde{A}_g} \left[ \frac{(1 - \alpha) c_e}{c_x} \right] \right\},$$

$$= \left( \hat{\lambda}_0 + \nu - \frac{c_e \hat{r}_0}{\alpha c_x} \right) \lim_{\tilde{A}_g \to 0} \frac{dR_e(\tilde{A}_g)}{d\tilde{A}_g} + \frac{(1 - \alpha) c_e}{c_x} \lim_{\tilde{A}_g \to 0} \frac{\partial r}{\partial \tilde{A}_g},$$

$$= \left( \frac{(1 - \alpha) c_e \hat{r}_0}{\alpha c_x} + \nu - \frac{c_e \hat{r}_0}{\alpha c_x} \right) \lim_{\tilde{A}_g \to 0} \frac{dR_e(\tilde{A}_g)}{d\tilde{A}_g} + \frac{(1 - \alpha) c_e}{c_x} \lim_{\tilde{A}_g \to 0} \frac{\partial r}{\partial \tilde{A}_g},$$

$$= \left( \nu - \frac{c_e \hat{r}_0}{c_x} \right) \lim_{\tilde{A}_g \to 0} \frac{dR_e(\tilde{A}_g)}{d\tilde{A}_g} + \frac{(1 - \alpha) c_e}{c_x} \lim_{\tilde{A}_g \to 0} \frac{\partial r}{\partial \tilde{A}_g},$$

$$= \left( \nu - \frac{c_e \hat{r}_0}{c_x} \right) \frac{(\beta - \alpha) \hat{r}_0^{(\beta-\alpha)}}{A_\pi} + \frac{(1 - \alpha) c_e}{c_x} \lim_{\tilde{A}_g \to 0} \frac{\partial r}{\partial \tilde{A}_g}.$$

The equality is from $\frac{dR_e}{dr} = \frac{dR_e}{d\tilde{A}_g}\frac{\tilde{A}_g(\beta-\alpha)}{r}$ at the laissez-faire equilibrium level.

From FOC and the participation constraint, $r(\tilde{A}_g)$ solves the equation below

$$\ln\left(\frac{\left(1-R_e(\tilde{A}_g)\right)}{c_x}\left(A_\pi r^\alpha + \tilde{A}_g r^\beta\right)\right) - \left(1-R_e(\tilde{A}_g)\right) - \frac{\nu c_x}{c_e r} R_e(\tilde{A}_g) = 0. \tag{65}$$

Take derivative with regard to $\tilde{A}_g$ and according to the implicit function theorem,

$$\frac{\partial r}{\partial \tilde{A}_g} = -\frac{R_x'(\tilde{A}_g)\left(\frac{R_e(\tilde{A}_g)}{1-R_e(\tilde{A}_g)} + \frac{\nu\left(1-R_e(\tilde{A}_g)\right)}{c_e r}\right) + \frac{\tilde{y}'(\tilde{A}_g)}{\tilde{y}}}{R_x'(r)\left[\frac{R_e(\tilde{A}_g)}{1-R_e(\tilde{A}_g)} + \frac{\nu c_x}{r c_e}\right] + \frac{\tilde{y}'(r)}{\tilde{y}} + \frac{\nu c_x}{c_e r^2} R_e(\tilde{A}_g)}, \tag{66}$$

where $\tilde{y} \doteq A_\pi r^\alpha + \tilde{A}_g r^\beta$.

$$\begin{aligned}
\lim_{\tilde{A}_g\to 0} r'(\tilde{A}_g) &= -\lim_{\tilde{A}_g\to 0}\frac{R_x'(\tilde{A}_g)\left(\frac{\alpha}{1-\alpha} + \frac{\nu(1-\alpha)}{c_e\hat{r}_0}\right) + \frac{\tilde{y}'(\tilde{A}_g)}{\tilde{y}}}{\frac{\tilde{y}'(r)}{\tilde{y}} + \frac{\alpha\nu c_x}{c_e r^2}} \\
&= -\lim_{\tilde{A}_g\to 0}\frac{-\frac{(\beta-\alpha)\hat{r}_0^{r^{\beta-\alpha}}}{A_\pi}\left(\frac{\alpha}{1-\alpha} + \frac{\nu(1-\alpha)}{c_e\hat{r}_0}\right) + \frac{1}{A_\pi \hat{r}_0^{\alpha-\beta}}}{\frac{\alpha}{\hat{r}_0} + \frac{\alpha\nu c_x}{c_e\hat{r}_0^2}} \\
&= \frac{\frac{\hat{r}_0^{\beta-\alpha}}{A_\pi}\left[1-(\beta-\alpha)\left(\frac{\alpha}{1-\alpha} + \frac{\nu(1-\alpha)}{c_e\hat{r}_0}\right)\right]}{\frac{\alpha}{\hat{r}_0} + \frac{\alpha\nu c_x}{c_e\hat{r}_0^2}}.
\end{aligned}$$

From the above representation of $\lim_{\tilde{A}_g\to 0} f(\tilde{A}_g)$ and $\lim_{\tilde{A}_g\to 0} r'(\tilde{A}_g)$, condition (57), i.e., $\lim_{\tilde{A}_g\to 0} f(\tilde{A}_g) \le 0$ is equivalent to,

$$\begin{aligned}
&\lim_{\tilde{A}_g\to 0}\frac{\partial r}{\partial \tilde{A}_g} \le \left(\frac{c_e\hat{r}_0}{c_x} - \nu\right)\frac{(\beta-\alpha)\, c_x \hat{r}_0^{\beta-\alpha}}{(1-\alpha)\, c_e A_\pi} \\
\Leftrightarrow &\frac{\frac{\hat{r}_0^{\beta-\alpha}}{A_\pi}\left[1-(\beta-\alpha)\left(\frac{\alpha}{1-\alpha} + \frac{\nu(1-\alpha)}{c_e\hat{r}_0}\right)\right]}{\frac{\alpha}{\hat{r}_0} + \frac{\alpha\nu c_x}{c_e\hat{r}_0^2}} \le \left(\frac{c_e\hat{r}_0}{c_x} - \nu\right)\frac{(\beta-\alpha)\, c_x \hat{r}_0^{(\beta-\alpha)}}{(1-\alpha)\, c_e A_\pi} \\
\Leftrightarrow &\frac{\left[1-(\beta-\alpha)\left(\frac{\alpha}{1-\alpha} + \frac{\nu(1-\alpha)}{c_e\hat{r}_0}\right)\right]}{1+\frac{\nu c_x}{c_e\hat{r}_0}} \le \left(\frac{c_e}{c_x} - \frac{\nu}{\hat{r}_0}\right)\frac{\alpha(\beta-\alpha)\, c_x}{(1-\alpha)\, c_e} \\
\Leftrightarrow &1-(\beta-\alpha)\left(\frac{\alpha}{1-\alpha} + \frac{\nu(1-\alpha)}{c_e\hat{r}_0}\right) \le \left(1+\frac{\nu c_x}{c_e\hat{r}_0}\right)\left(\frac{c_e}{c_x} - \frac{\nu}{\hat{r}_0}\right)\frac{\alpha(\beta-\alpha)\, c_x}{(1-\alpha)\, c_e}.
\end{aligned}$$

Let $t \doteq \frac{\nu}{\hat{r}_0}$ and $b_1 \doteq \frac{c_x}{c_e}$. The above inequality is transformed into

$$\begin{aligned}
&\frac{1-\alpha}{\alpha(\beta-\alpha)} - \left(1 + \frac{\hat{t}_0 (1-\alpha)^2}{c_e \alpha}\right) \le \left(1 + \hat{t}_0 b_1\right) \left(\frac{1}{b_1} - \hat{t}_0\right) b_1 \\
\Leftrightarrow &\frac{1-\alpha}{\alpha(\beta-\alpha)} - \left(1 + \frac{\hat{t}_0 (1-\alpha)^2}{c_e \alpha}\right) \le 1 - b_1^2 \hat{t}_0^2 \\
\Leftrightarrow &b_1^2 \hat{t}_0^2 - \frac{(1-\alpha)^2}{c_e \alpha} \hat{t}_0 + \frac{1-\alpha}{\alpha(\beta-\alpha)} - 2 \le 0.
\end{aligned}$$

$$\hat{t}_0 \in \left[\frac{c_e}{2c_x^2}\left(\frac{(1-\alpha)^2}{\alpha} - b_2\right), \frac{c_e}{2c_x^2}\left(\frac{(1-\alpha)^2}{\alpha} + b_2\right)\right] \cap \Re^+$$
$$\text{where } b_2 \doteq \sqrt{\frac{1-\alpha}{\alpha}\left(\frac{(1-\alpha)^3}{\alpha} - \frac{4c_x^2}{\beta-\alpha}\right)}.$$

Under condition (24), $\frac{c_e}{2c_x^2}\left(\frac{(1-\alpha)^2}{\alpha} - b_2\right) \le 0$, thus, $\hat{t}_0 \in \left[0, \frac{c_e}{2c_x^2}\left(\frac{(1-\alpha)^2}{\alpha} + b_2\right)\right]$. The corresponding range of $\hat{r}_0$ is

$$\hat{r}_0 = \frac{\nu}{\hat{t}_0} \ge \frac{2c_x^2 \nu}{c_e\left(\frac{(1-\alpha)^2}{\alpha} + b_2\right)} \tag{67}$$

Hence, Condition (67) is the necessary and sufficient condition for (57).

**(ii)** Condition(23) and (24) sufficient for Condition (67).

Let $F(r)$ denote the left-hand side of equation (65) under $R_e(\tilde{A}_g) = \alpha$ and $\tilde{A}_g = 0$, i.e., $F(r) \doteq \ln\left(\frac{1-\alpha}{c_x} A_\pi r^\alpha\right) - (1-\alpha) - \frac{\nu \alpha c_x}{c_e r}$. First, $\hat{r}_0$ solves equation $F(r) = 0$. Second, $F(r)$ increases in $r$, because $\ln(A_\pi r^\alpha)$ increases in $r$ and $\frac{\nu c_x}{c_e r} R_e(\tilde{A}_g)$ decreases in $r$. Combing the above two facts, condition (67) is equivalent to

$$F\left(\frac{2c_x^2 \nu}{c_e\left(\frac{(1-\alpha)^2}{\alpha} + b_2\right)}\right) \le 0,$$

where $b_2 \doteq \sqrt{\frac{1-\alpha}{\alpha}\left(\frac{(1-\alpha)^3}{\alpha} - \frac{4c_x^2}{\beta-\alpha}\right)}$.

$$\frac{2c_x^2 \nu}{c_e\left(\frac{(1-\alpha)^2}{\alpha} + b_2\right)} \le \frac{2c_x^2 \nu}{c_e\left(\frac{(1-\alpha)^2}{\alpha} + \sqrt{\frac{1-\alpha}{\alpha}\left(\frac{(1-\alpha)^3}{\alpha}\right)}\right)} = \frac{\nu c_x^2 \alpha}{c_e (1-\alpha)^2}.$$

Thus, a sufficient condition for (67) is

$$F\left(\frac{\nu c_x^2\alpha}{c_e(1-\alpha)^2}\right)\le 0,$$
$$\Leftrightarrow \ln\left(\frac{A_\pi\nu^\alpha(1-\alpha)^{1-2\alpha}}{c_e^\alpha c_x^{1-2\alpha}}\right)\le(1-\alpha)+\frac{(1-\alpha)^2}{c_x},$$

where the first line is from $F\left(\frac{2c_x^2\nu}{c_e\left(\frac{(1-\alpha)^2}{\alpha}+b_2\right)}\right)\le F\left(\frac{\nu c_x^2\alpha}{c_e(1-\alpha)^2}\right)$. This proves the sufficiency of condition (23) and (24) for (57). Together with Step 1, the conditions are sufficient. □

**Proof of Proposition 4.**

According to Step 1 in Proposition 3, we only need to prove the given conditions is sufficient for

$$\lim_{\mu\to 0}\frac{\partial e(\mu)}{\partial\mu}\ge 0, \tag{68}$$
$$\lim_{\mu\to 1}\frac{\partial e(\mu)}{\partial\mu}\le 0. \tag{69}$$

**Step 1:** To show condition (27) sufficient for equation (68)

Let $\tilde{A}_g\doteq\mu A_g$. The constraint (21) is equivalent to solving the equilibrium efforts from the optimization problem below

$$\begin{aligned}\max_e&\left\{\nu\ln\left(\frac{\nu}{\nu+\lambda}\left(A_\pi e^\alpha x^{(1-\alpha)}+\tilde{A}_g e^\beta x^{(1-\beta)}\right)\right)-c_e e\right\},\\ c_x x=&\ln\left(\frac{\lambda}{\nu+\lambda}\left(A_\pi e^\alpha x^{(1-\alpha)}+\tilde{A}_g e^\beta x^{(1-\beta)}\right)\right),\forall e.\end{aligned} \tag{70}$$

FOC of (70) with regard to $e$ is

$$\frac{e(\mu)c_e}{\lambda(\mu)+\nu}=\alpha+(\beta-\alpha)\frac{\frac{\mu A_g}{A_\pi}r^{(\beta-\alpha)}}{1+\frac{\mu A_g}{A_\pi}r^{(\beta-\alpha)}}=\beta-\frac{(\beta-\alpha)}{1+\frac{\mu A_g}{A_\pi}r^{(\beta-\alpha)}}.$$

Thus, the equilibrium efforts satisfy

$$e(\mu)=\frac{R_e(\tilde{A}_g)\left(\lambda(\tilde{A}_g)+\nu\right)}{c_e},\text{ where } R_e(\tilde{A}_g)\doteq\beta-\frac{(\beta-\alpha)}{1+\frac{\tilde{A}_g}{A_\pi}r^{(\beta-\alpha)}}$$

Let $r\doteq\frac{e}{x}$ and $R_x\doteq(1-\beta)+\frac{(\beta-\alpha)}{1+\mu k}$. The FOC of $x$ and the constraint (59) imply that

$$\frac{c_e r}{c_x\lambda}=\frac{R_e}{1-R_e}\Rightarrow\lambda=\frac{(1-R_e)\,c_e r}{R_e c_x},$$

Combine the equation above and $e = \left(\lambda(\tilde{A}_g) + \nu\right) R_e(\tilde{A}_g)$,

$$\frac{\partial e(\mu)}{\partial \mu} \propto \frac{dR_e(\tilde{A}_g, r)}{d\tilde{A}_g} \left(\lambda(\tilde{A}_g, r) + \nu\right) + \lambda_{\tilde{A}_g}(\tilde{A}_g, r) R_e(\tilde{A}_g) + \frac{\partial r}{\partial \tilde{A}_g} \left[\frac{dR_e}{dr} \left(\lambda(\tilde{A}_g) + \nu\right) + \lambda'(r) R_e(\tilde{A}_g)\right],$$

where $r \doteq \frac{e}{x}$ and $R_e'(\tilde{A}_g),\ \lambda'(\tilde{A}_g),\ R_e'(r),\ \lambda'(r)$ are all partial derivatives given as before (61), (62), (63) and (64). where $\tilde{y} = A_\pi r^\alpha + \tilde{A}_g r^\beta$, and $R_x = 1 - R_e(\tilde{A}_g)$.

Let $f(\tilde{A}_g)$ denote the function (60) and $\hat{r}_0 \doteq \frac{\hat{e}_0}{\hat{x}_0}$. Pluggiing (61), (62), (63) and (64) in $f(\tilde{A}_g)$,

$$\lim_{\tilde{A}_g \to 0} f(\tilde{A}_g) = \left(\nu - \frac{c_e \hat{r}_0}{c_x}\right) \frac{(\beta - \alpha)\, \hat{r}_0^{(\beta-\alpha)}}{A_\pi} + \frac{(1-\alpha)\, c_e}{c_x} \lim_{\tilde{A}_g \to 0} \frac{\partial r}{\partial \tilde{A}_g},$$

where the equality is from $\frac{dR_e}{dr} = \frac{dR_e}{d\tilde{A}_g} \frac{\tilde{A}_g}{r(\beta-\alpha)}$ with variables at the laissez-faire equilibrium level.

From FOC and the participation constraint, $r(\tilde{A}_g)$ solves

$$\ln\left(\frac{\left(1 - R_e(\tilde{A}_g)\right)}{c_x} \left(A_\pi r^\alpha + \tilde{A}_g r^\beta\right)\right) = \left(1 - R_e(\tilde{A}_g)\right) - \frac{\nu c_x}{c_e r} R_e(\tilde{A}_g). \tag{71}$$

$$\lim_{\tilde{A}_g \to 0} r'(\tilde{A}_g) = \frac{\frac{\hat{r}_0^{\beta-\alpha}}{A_\pi} \left[1 - (\beta - \alpha) \left(\frac{\alpha}{1-\alpha} + \frac{\nu(1-\alpha)}{c_e \hat{r}_0}\right)\right]}{\frac{\alpha}{\hat{r}_0} + \frac{\alpha \nu c_x}{c_e \hat{r}_0^2}}.$$

Plug the expression of $\lim_{\tilde{A}_g \to 0} f(\tilde{A}_g)$ and $\lim_{\tilde{A}_g \to 0} r'(\tilde{A}_g)$ into condition (68), condition (68) is equivalent to

$$\begin{aligned}
&\lim_{\tilde{A}_g \to 0} \frac{\partial r}{\partial \tilde{A}_g} \geq \left(\frac{c_e \hat{r}_0}{c_x} - \nu\right) \frac{(\beta - \alpha)\, c_x \hat{r}_0^{(\beta-\alpha)}}{(1-\alpha)\, c_e A_\pi}, \\
\Leftrightarrow & 1 - (\beta - \alpha) \left(\frac{\alpha}{1-\alpha} + \frac{\nu\,(1-\alpha)}{c_e \hat{r}_0}\right) \geq \left(1 + \frac{\nu c_x}{c_e \hat{r}_0}\right) \left(\frac{c_e}{c_x} - \frac{\nu}{\hat{r}_0}\right) \frac{\alpha\,(\beta-\alpha)\, c_x}{(1-\alpha)\, c_e}.
\end{aligned}$$

Let $\hat{t}_0 \doteq \frac{\nu}{\hat{r}_0}$ and $b_1 \doteq \frac{c_x}{c_e}$. The above inequality is

$$\begin{aligned}
&\frac{1-\alpha}{\alpha\,(\beta-\alpha)} - \left(1 + \frac{\hat{t}_0\,(1-\alpha)^2}{c_e \alpha}\right) \geq \left(1 + \hat{t}_0 b_1\right) \left(\frac{1}{b_1} - \hat{t}_0\right) b_1 \\
\Leftrightarrow & b_1^2 \hat{t}_0^2 - \frac{(1-\alpha)^2}{c_e \alpha} \hat{t}_0 + \frac{1-\alpha}{\alpha\,(\beta-\alpha)} - 2 \geq 0.
\end{aligned}$$

Hence, $\hat{t}_0 \geq \frac{c_e}{2c_x^2}\left(\frac{(1-\alpha)^2}{\alpha}+b_2\right)$ is sufficient for the inequality above. The range of $\hat{r}_0$ is

$$\hat{r}_0 = \frac{\nu}{\hat{t}_0} \leq \frac{2c_x^2\nu}{c_e\left(\frac{(1-\alpha)^2}{\alpha}+b_2\right)}.$$

Let $F(r)$ denote the left-hand side of the equation (71) minus its right-hand side under $R_e(\tilde{A}_g) = \alpha$. As $F(r)$ increases in $r$ and $\hat{r}_0$ solves $F(r) = 0$, condition (67) is equivalent to

$$F\left(\frac{2c_x^2\nu}{c_e\left(\frac{(1-\alpha)^2}{\alpha}+b_2\right)}\right) \geq 0,$$

where $b_2 \doteq \sqrt{\frac{1-\alpha}{\alpha}\left(\frac{(1-\alpha)^3}{\alpha}-\frac{4c_x^2}{\beta-\alpha}\right)}$.

Plugging in the expression of $F(\cdot)$, the inequality above is equivalent to

$$\ln\left(\frac{A_\pi\left(2\nu\right)^\alpha\left(1-\alpha\right)}{c_e^\alpha c_x^{1-2\alpha}}\right) - \left(1-\alpha\right)\left(2-\alpha\right) \geq \alpha\ln\left(\frac{\left(1-\alpha\right)^2}{\alpha}+b_2\right) + \alpha b_2,$$

where $b_2 \doteq \sqrt{\frac{1-\alpha}{\alpha}\left(\frac{(1-\alpha)^3}{\alpha}-\frac{4c_x^2}{\beta-\alpha}\right)}$.

**Step 2:** Sufficient conditions for (69) For $\lim_{\mu\to 1} e'(\mu) \leq 0$,

$$\lim_{\tilde{A}_g\to A_g} f(\tilde{A}_g) = \lim_{\tilde{A}_g\to A_g}\left\{\frac{dR_e(\tilde{A}_g)}{d\tilde{A}_g}\left(\hat{\lambda}_1+\nu\right) - \frac{c_e r(\tilde{A}_g)R_e}{R_e^2(\tilde{A}_g)c_x}\frac{dR_e}{d\tilde{A}_g} + \frac{\partial r}{\partial \tilde{A}_g}\left[\frac{dR_e}{dr}\left(\hat{\lambda}_1+\nu\right)+\lambda'(r)R_e(\tilde{A}_g)\right]\right\},$$

(72)
$$= \frac{\left(\beta-\alpha\right)\hat{r}_1^{(\beta-\alpha)}}{A_\pi\left(1+\frac{A_g}{A_\pi}\hat{r}_1^{(\beta-\alpha)}\right)^2}\left[\left(\frac{\left(1-R_e\right)c_e\hat{r}_1}{R_e c_x}+\nu\right)-\frac{c_e\hat{r}_1}{R_e c_x}\right],$$

(73)
$$+ \lim_{\tilde{A}_g\to A_g}\frac{\partial r}{\partial \tilde{A}_g}\left\{\frac{\left(\beta-\alpha\right)\hat{r}_1^{(\beta-\alpha)}}{A_\pi\left(1+\frac{A_g}{A_\pi}\hat{r}_1^{(\beta-\alpha)}\right)^2}\frac{c_e\hat{r}_1}{R_e^2 c_x}\left(\frac{\left(1-R_e\right)c_e\hat{r}_1}{R_e c_x}+\nu-\frac{A_g R_e}{A_\pi\hat{r}_1}\right)+\frac{\left(1-R_e\right)c_e}{c_x}\right\},$$

where $\hat{r}_1 \doteq \frac{\hat{e_1}}{\hat{x}_1}$, $\hat{R}_{e,1} \doteq \frac{\alpha+\beta\hat{k}_1}{1+\hat{k}_1}$ and $\hat{k}_1 \doteq \frac{A_g}{A_\pi}\hat{r}_1^{\beta-\alpha}$.

Let $T_1(r)$ denote term (72) and $T_2(r)$ denote the big bracket term in (73). Then the following three steps prove condition (69) holds if $T_1(\hat{r}_1) \leq 0$, $\lim_{\tilde{A}_g\to A_g}\frac{\partial r}{\partial \tilde{A}_g} \leq 0$, and $T_2(\hat{r}_1) \geq 0$.

**(i)** To show $T_2(\hat{r}_1) \geq 0$ and condition $\frac{c_e\hat{r}_1}{c_x} \leq \nu$ ensures $T_1(r) \leq 0$.

We prove $T_2\left(\hat{r}_1\right) > 0$ by contradiction. Suppose there exists parameter value such that $T_2\left(\hat{r}_1\right) \geq$

0. From $T_2(\hat{r}_1) = \frac{A_g}{r} T_1(r) + \frac{(1-\hat{R}_{e,1})c_e}{c_x}$ with $T_1(r)$ and $T_2(r)$ defined in term (72) and (73).

$$
\begin{aligned}
& T_2(\hat{r}_1) \leq 0 \\
\Leftrightarrow\ & T_1(\hat{r}_1) \leq -\frac{c_e \hat{r}_1 \left(1-\hat{R}_{e,1}\right)}{c_x A_g} = -\frac{\hat{\lambda}_1 \hat{R}_{e,1}}{A_g}.
\end{aligned}
$$

If $\frac{c_e \hat{r}_1}{c_x} \leq \nu$, then

$$
\begin{aligned}
T_1(r) + \frac{\hat{R}_{e,1}\hat{\lambda}_1}{A_g} &= \frac{d\hat{R}_{e,1}}{dA_g}\left(\hat{\lambda}_1 + \nu\right) + \hat{R}_{e,1}\frac{d\lambda}{dA_g} + \frac{\hat{\lambda}_1 \hat{R}_{e,1}}{A_g} \\
&= \frac{d\hat{R}_{e,1}}{dA_g}\left(\hat{\lambda}_1 + \nu - \frac{c_e \hat{r}_1}{\hat{R}_{e,1} c_x}\right) + \frac{\hat{R}_{e,1}}{A_g}\hat{\lambda}_1 \\
&\propto \xi_e \left(\hat{\lambda}_1 + \nu - \frac{\hat{\lambda}_1}{1-\hat{R}_{e,1}}\right) + \hat{\lambda}_1 \\
&= \xi_e \left(\nu - \frac{\hat{R}_{e,1}\hat{\lambda}_1}{1-\hat{R}_{e,1}}\right) + \hat{\lambda}_1 \\
&= \xi_e \left(\nu - \frac{c_e \hat{r}_1}{c_x}\right) + \frac{c_e \hat{r}_1 \left(1-\hat{R}_{e,1}\right)}{c_x \hat{R}_{e,1}} \\
&= \xi_e \left(\nu - \frac{c_e \hat{r}_1}{c_x}\right) + \frac{c_e \hat{r}_1 \left(1-\hat{R}_{e,1}\right)}{c_x \hat{R}_{e,1}}.
\end{aligned}
$$

Thus, $T_2(\hat{r}_1) \leq 0$ is equivalent to

$$
\begin{aligned}
& \xi_e \left(\nu - \frac{c_e \hat{r}_1}{c_x}\right) + \frac{c_e \hat{r}_1 \left(1-\hat{R}_{e,1}\right)}{c_x \hat{R}_{e,1}} \leq 0, \\
\Leftrightarrow\ & \frac{\hat{R}_{e,1}\xi_e}{1-\hat{R}_{e,1}} \geq \frac{\frac{c_e \hat{r}_1}{c_x}}{\frac{c_e \hat{r}_1}{c_x} - \nu} = \frac{1}{1-\frac{\nu c_x}{c_e \hat{r}_1}}, \\
\Leftrightarrow\ & \frac{(\beta-\alpha)\left(1-\frac{\nu c_x}{c_e \hat{r}_1}\right)}{\left(1+\frac{1}{\hat{k}_1}\right)\left(1-\alpha+(1-\beta)\hat{k}_1\right)} \geq 1,
\end{aligned}
$$

(74) $$(\beta-\alpha)\left(1-\frac{\nu c_x}{c_e \hat{r}_1}\right) \geq \left(1+\frac{1}{\hat{k}_1}\right)\left(1-\alpha+(1-\beta)\hat{k}_1\right),$$

where the third line is from plugging the expression of $\xi_e$ and $\hat{R}_{e,1}$ in the second line.

Let $LH(r)$ and $RH(r)$ denote the left-hand side and the right-hand side of inequality (74):

$$\begin{aligned}
LH(r) &\doteq (\beta-\alpha)\left(1-\frac{\nu c_x}{c_e r}\right) \\
RH(r) &\doteq \left(1+\frac{1}{\hat{k}_1}\right)\left(1-\alpha+(1-\beta)\,\hat{k}_1\right) \\
&= 2-(\alpha+\beta)+(1-\beta)\,k+\frac{1-\alpha}{k} \\
&= 2-(\alpha+\beta)+\frac{(1-\beta)\,A_g r^{\beta-\alpha}}{A_\pi}+\frac{(1-\alpha)\,A_\pi r^{\alpha-\beta}}{A_g}.
\end{aligned}$$

Take the first-order derivatives,

$$\begin{aligned}
&LH'(r)>0,\ \forall r \\
&RH'(r)\le 0,\ \text{if } r\le\left(\sqrt{\frac{1-\alpha}{1-\beta}\frac{A_\pi}{A_g}}\right)^{\frac{1}{\beta-\alpha}} \\
&RH'(r)>0,\ \text{if } r>\left(\sqrt{\frac{1-\alpha}{1-\beta}\frac{A_\pi}{A_g}}\right)^{\frac{1}{\beta-\alpha}}.
\end{aligned}$$

Therefore, $\lim_{r\to 0} LH(r) < \lim_{r\to 0} RH(r)$ and $\lim_{r\to+\infty} LH(r) < \lim_{r\to+\infty} RH(r)$. It follows that

$$\begin{aligned}
LH\left(\left(\sqrt{\frac{1-\alpha}{1-\beta}\frac{A_\pi}{A_g}}\right)^{\frac{1}{\beta-\alpha}}\right) &< \beta-\alpha \\
&< 2-(\alpha+\beta)+2\sqrt{(1-\alpha)\,(1-\beta)} \\
&= RH\left(\left(\sqrt{\frac{1-\alpha}{1-\beta}\frac{A_\pi}{A_g}}\right)^{\frac{1}{\beta-\alpha}}\right),
\end{aligned}$$

where the second inequality is because $2-\alpha-\beta>\beta-\alpha$ directly from $\beta\in(0,1)$ and the first inequality is from $1-\frac{\nu c_x}{c_e r}\in(0,1)$. Because $LH(r)<RH(r), \forall r$ contradicts (74), $T_2(r)>0$.

For $T_1(r)$, if $\frac{c_e \hat{r}_1}{c_x} \leq \nu$,

$$\begin{aligned}
T_1(r) &= \frac{d\hat{R}_{e,1}}{dA_g}\left(\hat{\lambda}_1 + \nu\right) + \hat{R}_{e,1}\frac{d\lambda}{dA_g} \\
&= \frac{d\hat{R}_{e,1}}{dA_g}\left(\hat{\lambda}_1 + \nu - \frac{c_e \hat{r}_1}{\hat{R}_{e,1} c_x}\right) \\
&\propto \xi_e \left(\hat{\lambda}_1 + \nu - \frac{\hat{\lambda}_1}{1 - \hat{R}_{e,1}}\right) \\
&= \xi_e \left(\nu - \frac{\hat{R}_{e,1}\hat{\lambda}_1}{1 - \hat{R}_{e,1}}\right) \\
&= \xi_e \left(\nu - \frac{c_e \hat{r}_1}{c_x}\right) \leq 0,
\end{aligned}$$

where the inequality is from $\hat{r}_1 \geq \frac{\nu c_x}{c_e}$.

**(ii)** Sufficient conditions of $\hat{r}_1$ for $\lim_{\tilde{A}_g \to A_g} \frac{\partial r}{\partial \tilde{A}_g} \geq 0$

From the previous results, $\frac{\partial r}{\partial A_g}$ is

$$\frac{\partial r}{\partial A_g} = -\frac{R_x'(\tilde{A}_g)\left(\frac{R_e(\tilde{A}_g)}{1-R_e(\tilde{A}_g)} + \frac{\nu\left(1-R_e(\tilde{A}_g)\right)}{c_e r}\right) + \frac{\tilde{y}'(\tilde{A}_g)}{\tilde{y}}}{R_x'(r)\left[\frac{R_e(\tilde{A}_g)}{1-R_e(\tilde{A}_g)} + \frac{\nu c_x}{r c_e}\right] + \frac{\tilde{y}'(r)}{\tilde{y}} + \frac{\nu c_x}{c_e r^2} R_e(\tilde{A}_g)}.$$

Hence,

$$\frac{\partial r}{\partial A_g} \leq 0 \Leftrightarrow \frac{R_x'(\tilde{A}_g)\left(\frac{R_e(\tilde{A}_g)}{1-R_e(\tilde{A}_g)} + \frac{\nu\left(1-R_e(\tilde{A}_g)\right)}{c_e r}\right) + \frac{\tilde{y}'(\tilde{A}_g)}{\tilde{y}}}{R_x'(r)\left[\frac{R_e(\tilde{A}_g)}{1-R_e(\tilde{A}_g)} + \frac{\nu c_x}{r c_e}\right] + \frac{\tilde{y}'(r)}{\tilde{y}} + \frac{\nu c_x}{c_e r^2} R_e(\tilde{A}_g)} \geq 0. \tag{75}$$

Let $N(\hat{r}_1)$ and $D(\hat{r}_1)$ denote the limits of the numerator and denominator at $\tilde{A}_g \to A_g$, i.e., the left-hand side of (75)

$$\begin{aligned}
N(\hat{r}_1) &= \lim_{\tilde{A}_g \to A_g}\left\{R_x'(\tilde{A}_g)\left(\frac{R_e(\tilde{A}_g)}{1 - R_e(\tilde{A}_g)} + \frac{\nu\left(1 - R_e(\tilde{A}_g)\right)}{c_e r}\right) + \frac{\tilde{y}'(\tilde{A}_g)}{\tilde{y}}\right\}, \\
&= -\frac{d\hat{R}_{e,1}}{dA_g}\left(\frac{\hat{R}_{e,1}}{1 - \hat{R}_{e,1}} + \frac{\nu\left(1 - \hat{R}_{e,1}\right)}{c_e \hat{r}_1}\right) + \frac{\hat{r}_1^{\beta-\alpha}}{A_\pi + A_g \hat{r}_1^{\beta-\alpha}} \\
D(\hat{r}_1) &= \lim_{\tilde{A}_g \to A_g}\left\{R_x'(r)\left[\frac{R_e(\tilde{A}_g)}{1 - R_e(\tilde{A}_g)} + \frac{\nu c_x}{r c_e}\right] + \frac{\tilde{y}'(r)}{\tilde{y}} + \frac{\nu c_x}{c_e r^2} R_e(\tilde{A}_g)\right\}, \\
&= -\frac{d\hat{R}_{e,1}}{dr}\left[\frac{\hat{R}_{e,1}}{1 - \hat{R}_{e,1}} + \frac{\nu c_x}{c_e \hat{r}_1}\right] + \frac{\left(\alpha A_\pi + \beta A_g \hat{r}_1^{\beta-\alpha}\right)}{\hat{r}_1\left(A_\pi + A_g \hat{r}_1^{\beta-\alpha}\right)} + \frac{\nu c_x}{c_e \hat{r}_1}\frac{\hat{R}_{e,1}}{\hat{r}_1},
\end{aligned}$$

where $\hat{R}_{e,1} \doteq \frac{\alpha+\beta\hat{k}_1}{1+\hat{k}_1}$, $\hat{k}_1 \doteq \frac{A_g}{A_\pi}\hat{r}_1^{\beta-\alpha}$ and $\hat{r_1} \doteq \frac{\hat{e}_1}{\hat{x}_1}$.

$$
\begin{aligned}
D(\hat{r}_1) =& \frac{1}{\hat{r}_1}\left[-\frac{d\hat{R}_{e,1}}{dA_g}A_g\left[\frac{\hat{R}_{e,1}}{1-\hat{R}_{e,1}}+\frac{\nu c_x}{c_e\hat{r}_1}\right]+\frac{\alpha A_\pi+\beta A_g\hat{r}_1^{\beta-\alpha}}{\left(A_\pi+A_g\hat{r}_1^{\beta-\alpha}\right)}\left(1+\frac{\nu c_x}{c_e\hat{r}_1}\right)\right] \\
&\propto \left[-\frac{d\hat{R}_{e,1}}{dA_g}\frac{A_g}{\hat{R}_{e,1}}\left[\frac{\hat{R}_{e,1}}{1-\hat{R}_{e,1}}+\frac{\nu c_x}{c_e\hat{r}_1}\right]+\left(1+\frac{\nu c_x}{c_e\hat{r}_1}\right)\right] \\
&= -\xi_e\left[\frac{\hat{R}_{e,1}}{1-\hat{R}_{e,1}}+\frac{\nu c_x}{c_e\hat{r}_1}\right]+\left(1+\frac{\nu c_x}{c_e\hat{r}_1}\right),
\end{aligned}
$$

where $\xi_e \doteq \frac{\frac{d\hat{R}_{e,1}}{dA_g}}{\frac{\hat{R}_{e,1}}{A_g}}$. The first and the second line are from $\hat{R}_{e,1} = \frac{\alpha A_\pi+\beta A_g\hat{r}_1^{\beta-\alpha}}{A_\pi+A_g\hat{r}_1^{\beta-\alpha}}$, and the third line is by plugging $\xi_e$, the elasticity of $R_e$ with regard to $A_g$ in the second line.

$$
\begin{aligned}
D(\hat{r}_1) \geq 0 \Leftrightarrow & 1+\frac{\nu c_x}{c_e\hat{r}_1} \geq \xi_e\left[\frac{\hat{R}_{e,1}}{1-\hat{R}_{e,1}}+\frac{\nu c_x}{c_e\hat{r}_1}\right] \\
\Leftrightarrow & \xi_e \leq \frac{1+\frac{\nu c_x}{c_e\hat{r}_1}}{\frac{\hat{R}_{e,1}}{1-\hat{R}_{e,1}}+\frac{\nu c_x}{c_e\hat{r}_1}}.
\end{aligned}
$$

Next, we show that $\xi_e \leq \frac{1+\frac{\nu c_x}{c_e\hat{r}_1}}{\frac{\hat{R}_{e,1}}{1-\hat{R}_{e,1}}+\frac{\nu c_x}{c_e\hat{r}_1}}$ also ensures $N\left(\hat{r}_1\right) \geq 0$.

Plugging the expression of $\frac{d\hat{R}_{e,1}}{dA_g}$ in (62) into $N(\hat{r}_1)$ and let $\hat{R}_{e,1} \doteq \frac{\alpha+\beta\hat{k}_1}{1+\hat{k}_1}$ with $\hat{k}_1 \doteq \frac{A_g}{A_\pi}\hat{r}_1^{\beta-\alpha}$,

$$
\begin{aligned}
& N(\hat{r}_1) \geq 0, \\
\Leftrightarrow & \frac{d\hat{R}_{e,1}}{dA_g}\left(\frac{1+\hat{k}_1}{1-\alpha+(1-\beta)\hat{k}_1}+\frac{\left(1-\hat{R}_{e,1}\right)\nu}{c_x\hat{\lambda}_1\hat{R}_{e,1}}\right) \leq \frac{\hat{k}_1\left(1+\hat{k}_1\right)}{\left(\alpha+\beta\hat{k}_1\right)A_g}, \\
\Leftrightarrow & \xi_e\hat{R}_{e,1}\left[\frac{1}{1-\hat{R}_{e,1}}+\frac{\nu\left(1-\hat{R}_{e,1}\right)}{\hat{R}_{e,1}\frac{c_e\hat{r}_1\left(1-\hat{R}_{e,1}\right)}{\hat{R}_{e,1}}}\right] \leq \frac{\hat{k}_1}{\alpha+\beta\hat{k}_1}, \\
\Leftrightarrow & \xi_e \leq \frac{\frac{\hat{k}_1}{\alpha+\beta\hat{k}_1}}{\frac{\hat{R}_{e,1}}{1-\hat{R}_{e,1}}+\frac{\nu\hat{R}_{e,1}}{c_e\hat{r}_1}}, \qquad (76)
\end{aligned}
$$

where $\xi_e \doteq \frac{\frac{d\hat{R}_{e,1}}{dA_g}}{\frac{\hat{R}_{e,1}}{A_g}}$ denote the elasticity of $R_e$ with regard to $A_g$ and the third line is from $\frac{c_e\hat{r}_1\left(1-\hat{R}_{e,1}\right)}{\hat{R}_{e,1}} = c_x\hat{\lambda}_1$ and $\frac{d\hat{R}_{e,1}}{dA_g}A_g = \xi_e\hat{R}_{e,1}$.

Hence, $\frac{\partial r}{\partial A_g} \leq 0$ under condition (76).

Combine the previous results, (76) is sufficient for $N(\hat{r}_1) \geq 0$, and $\hat{r}_1 \geq \frac{\nu c_x}{c_e}$ sufficient for

$T_1(\hat{r}_1) \leq 0$.

Plugging expression of $\frac{d\hat{R}_{e,1}}{dA_g}$ and rearranging (76) into the following

$$\frac{\beta-\alpha}{\left(1+\frac{1}{\hat{k}_1}\right)\left(1+\hat{k}_1\right)}\left(\frac{1+\hat{k}_1}{1-\alpha+(1-\beta)\hat{k}_1}+\frac{\nu}{c_e\hat{r}_1}\right) \leq \frac{\hat{k}_1}{\alpha+\beta\hat{k}_1}$$

$$\Leftrightarrow \frac{(\beta-\alpha)\hat{k}_1}{\left(1+\hat{k}_1\right)}\left(\frac{1}{1-\alpha+(1-\beta)\hat{k}_1}+\frac{\nu}{c_e\hat{r}_1\left(1+\hat{k}_1\right)}\right) \leq \frac{\hat{k}_1}{\alpha+\beta\hat{k}_1}$$

$$\Leftrightarrow \frac{(\beta-\alpha)}{\left(1+\hat{k}_1\right)}\left(\frac{1}{1-\alpha+(1-\beta)\hat{k}_1}+\frac{\nu}{c_e\hat{r}_1\left(1+\hat{k}_1\right)}\right) \leq \frac{1}{\alpha+\beta\hat{k}_1}$$

$$\Leftrightarrow \frac{1}{1-\alpha+(1-\beta)\hat{k}_1}+\frac{\nu}{c_e\hat{r}_1\left(1+\hat{k}_1\right)} \leq \frac{1+\hat{k}_1}{(\beta-\alpha)\left(\alpha+\beta\hat{k}_1\right)}. \tag{77}$$

From $\hat{r}_1 \geq \frac{\nu c_x}{c_e}$,

$$\frac{\nu}{c_e\hat{r}_1\left(1+\hat{k}_1\right)} \leq \frac{1}{c_x\left(1+\hat{k}_1\right)}.$$

A sufficient condition for (c) is thus

$$\frac{1}{1-\alpha+(1-\beta)\hat{k}_1}+\frac{1}{c_x\left(1+\hat{k}_1\right)} \leq \frac{1+\hat{k}_1}{(\beta-\alpha)\left(\alpha+\beta\hat{k}_1\right)}$$

$$\Leftrightarrow c_x\left(1+\hat{k}_1\right)^2\left(1-\alpha+(1-\beta)\hat{k}_1\right) \geq (\beta-\alpha)\left(\alpha+\beta\hat{k}_1\right)\left(1+c_x-\alpha+(1+c_x-\beta)\hat{k}_1\right)$$

$$\Leftrightarrow c_x\left(1+\hat{k}_1\right) \geq (\beta-\alpha)\left(\frac{\alpha+\beta\hat{k}_1}{1+\hat{k}_1}\right)\left(\frac{1+c_x-\alpha+(1+c_x-\beta)\hat{k}_1}{\left(1-\alpha+(1-\beta)\hat{k}_1\right)}\right)$$

$$= (\beta-\alpha)\left(\frac{\alpha+\beta\hat{k}_1}{1+\hat{k}_1}\right)\left(1+\frac{c_x\left(1+\hat{k}_1\right)}{\left(1-\alpha+(1-\beta)\hat{k}_1\right)}\right)$$

$$= (\beta-\alpha)\hat{R}_{e,1}\left(1+\frac{c_x}{1-\hat{R}_{e,1}}\right),$$

where the last line is from $\hat{R}_{e,1} \doteq \frac{\alpha+\beta\hat{k}_1}{1+\hat{k}_1}$, $\hat{k}_1 \doteq \frac{A_g}{A_\pi}\hat{r}_1^{\beta-\alpha}$ and $\hat{r}_1 \doteq \frac{\hat{e}_1}{\hat{x}_1}$.

Let $LHS(\hat{k}_1) \doteq c_x\left(1+\hat{k}_1\right)$ and $RHS(\hat{k}_1) \doteq (\beta-\alpha)\hat{R}_{e,1}\left(1+\frac{c_x}{1-\hat{R}_{e,1}}\right)$.

$$\lim_{\hat{k}_1\to+\infty} RHS(\hat{k}_1) = (\beta-\alpha)\beta\left(1+\frac{c_x}{1-\beta}\right) < \lim_{\hat{k}_1\to+\infty} LHS(\hat{k}_1)$$

$$\lim_{\hat{k}_1\to 0} RHS(\hat{k}_1) = (\beta-\alpha)\alpha\left(1+\frac{c_x}{1-\alpha}\right) < c_x = \lim_{\hat{k}_1\to 0} LHS(\hat{k}_1),$$

where the last line is because $(\beta-\alpha)\alpha < c_x\left(1-\frac{(\beta-\alpha)\alpha}{1-\alpha}\right)$ from condition (25).

From the above two inequalities and $LHS(\hat{k}_1)-RHS(\hat{k}_1)$ increasing in $\hat{k}_1$, $LHS(\hat{k}_1)-RHS(\hat{k}_1) > 0,\ \forall \hat{k}_1 \in (0,+\infty)$ and hence equation (77) always holds given $\hat{r}_1 \geq \frac{\nu c_x}{c_e}$.

**Step 3:** Show condition (26) sufficient for $\hat{r}_1 \geq \frac{\nu c_x}{c_e}$.

Let $G(r)$ denote the function where the left-hand-side minus the right-hand side of (71) with $\tilde{A}_g = A_g$. Because the left-hand side of (71) increases in $r$ and its right-hand side decreases in $r$, $G(r)$ increases in $r$. Also,b ecause $\hat{r}_1$ solves $G(r)=0$, $\hat{r}_1 \geq \frac{\nu c_x}{c_e}$ is equivalent to $G\left(\frac{\nu c_x}{c_e}\right) < 0$, i.e.,

$$\ln\left(\frac{\left((1-\alpha)A_\pi + (1-\beta)A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta-\alpha}\right)\left(A_\pi\left(\frac{\nu c_x}{c_e}\right)^{\alpha} + A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta}\right)}{c_x\left(A_\pi + A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta-\alpha}\right)}\right),$$

$$\leq \frac{\left((1-\alpha)A_\pi + (1-\beta)A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta-\alpha}\right)}{\left(A_\pi + A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta-\alpha}\right)} + \frac{\alpha A_\pi + \beta A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta-\alpha}}{\left(A_\pi + A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta-\alpha}\right)}\frac{\nu c_x}{c_e\frac{\nu c_x}{c_e}}.$$

Rearranging the inequality above,

$$\ln\left(\frac{\left((1-\alpha)A_\pi\left(\frac{\nu c_x}{c_e}\right)^{\alpha} + (1-\beta)A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta}\right)}{c_x}\right) \leq \frac{\left(A_\pi + A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta-\alpha}\right)}{\left(A_\pi + A_g\left(\frac{\nu c_x}{c_e}\right)^{\beta-\alpha}\right)}$$

.

It holds under condition (26). □

### Proof of Proposition 5.

We prove it by contradiction. Suppose the first-best effort pair is restored by $\mu^*$. We first show an equivalent equation system of equation (20) to characterize $\mu^*$. Take the FOC of the principal's best response and plug in the first-best effort levels, we have equation (78) from (15) and (16), and equation (80) as the transformation of (22) under the first-best efforts.

$$\left(\lambda^f + \nu\right)\left(\beta - (\beta-\alpha)\frac{1}{1+\mu t^*}\right) = c_e e^* \tag{78}$$

$$\lambda^f\left((1-\beta) + (\beta-\alpha)\frac{1}{1+\mu t^*} - c_x x^*\right) = \nu\left((1-\beta) + (\beta-\alpha)\frac{1}{1+\mu t^*}\right) \tag{79}$$

$$\lambda^f\left(\pi(e^*, x^*) + \mu g(e^*, x^*)\right) - \exp(c_x x^*) = \nu \exp(c_x x^*), \tag{80}$$

where $(e^*, x^*)$ denote the first-best efforts and $\lambda^f$ is the Lagrangian multiplier of the problem (20).

The three-equation system above has two unknowns, $\mu^*$ and $\lambda^f$, and the three equations are linearly independent. Hence, $\mu^*$ and $\lambda^f$ are over-identified. In general, there does not exist $\left(\mu^*, \lambda^f\right)$ such that (78), (16) and (80) hold simultaneously. □

**Proof of Proposition 6.**

We first characterize the FOC of $w$ from (28). With FOCs, we simplify the optimization problem by showing the equivalence between subsidies and reducing the number of choice variables. The equivalence is proved in two steps. Lastly, we prove by contradiction that the first-best cannot be restored.

**Step 1:** FOC characterization. FOC of (28) with regard to $w$ is

$$\frac{\nu}{\tilde{Y}_s(e,x)+\tau_{e,\pi}\pi(e,x)+\tau_{e,g}g(e,x)-w}=\frac{\lambda}{w+\tau_{x,\pi}\pi(e,x)+\tau_{x,g}g(e,x)} \tag{81}$$
$$\Rightarrow w=\frac{\lambda\left[\tilde{Y}_s(e,x)+\tau_{e,\pi}\pi(e,x)+\tau_{e,g}g(e,x)\right]}{(\nu+\lambda)}-\frac{\nu}{(\nu+\lambda)}\left[\tau_{x,\pi}\pi(e,x)+\tau_{x,g}g(e,x)\right].$$

**Step 2:** Simplify the choice space of the social planner.

For any arbitrary $(e,x)$, suppose $(\mu,\tau_{e,\pi},\tau_{e,g},\tau_{x,\pi},\tau_{x,g})$ implements $(e,x)$, where $\tau_{e,\pi}>0$. Because the terms of $\tau_{e,\pi}$ and $\tau_{e,g}$ in the FOC of (28) with regard to $e$ are perfect substitutes, $\tilde{\tau}_{p,g}$ is a linear combination of $\tau_{e,\pi},\tau_{e,g}$. Hence, there must be another subsidy rate tuple $(\mu,\tilde{\tau}_{p,g},\tau_{x,\pi},\tau_{x,g})$, where all subsidy given to the principal is imposed on the knowledge spillovers value implements $(e,x)$.

Similarly, the equivalence of $\tau_{x,\pi},\tau_{x,g}$. Therefore the optimization problem is reduced to the following form,

$$\max_{(\mu,\tau_{e,g},\tau_{x,g})}\left\{\left[\nu\ln\left(\frac{\nu}{\nu+1}\tilde{Y}(\dot{e}x)\right)+\ln\left(\frac{\tilde{Y}(\dot{e}x)}{\nu+1}\right)\right]-c_e\dot{e}-c_x x\right\},$$
$$\text{s.t. }\dot{e}=\arg\max_{e,w}\left\{\nu\ln\left(\tilde{Y}_s+\tau_{e,g}g(e,x)-w\right)-c_e e\right\} \tag{82}$$
$$c_x x=\ln\left(w+\tau_{x,g}g(e,x)\right),\forall e,$$
$$\tilde{Y}_s\doteq A_\pi e^\alpha x^{(1-\alpha)}+\mu A_g e^\beta x^{(1-\beta)},$$

Correspondingly, the optimal transfer from the principal to the agent is

$$w=\frac{\lambda}{(\nu+\lambda)}\left[\tilde{Y}_s(e,x)+\tau_{e,g}g(e,x)\right]-\frac{\nu}{(\nu+\lambda)}\tau_{x,g}g(e,x).$$

From the expression above, the subsidy $\mu$ given to the startup is a perfect substitute to $\tau_{e,g}$ given to the principal. They are both defined as a portion of knowledge value. The income share of the startup subsidy going to the principal adds to the principal's individual subsidy to her total income, according to the principal's income and utility represented in (82).

For any arbitrary $(e, x)$, suppose $(\mu, \tau_{e,g}, \tau_{x,g})$ implements $(e, x)$, where $\mu > 0$, there must be another subsidy rate tuple $(\tilde{\tau}_{p,g}, \tilde{\tau}_{a,g})$, where all subsidy given to the principal is imposed on the knowledge spillovers value.

From the result above, the social planner's optimization is further simplified as follows

$$\max_{(\tau_{e,g},\tau_{x,g})} \left\{ \left[ \nu \ln \left( \frac{\nu}{\nu+1} \tilde{Y}(\dot{e}x) \right) + \ln \left( \frac{\tilde{Y}(\dot{e}x)}{\nu+1} \right) \right] - c_e \dot{e} - c_x x \right\} \tag{83}$$

$$\text{s.t. } \dot{e} = \arg\max_{e,w} \left\{ \nu \ln \left( \pi(e,x) + \tau_{e,g} g(e,x) - w \right) - c_e e \right\} \tag{84}$$

$$c_x x = \ln \left( w + \tau_{x,g} g(e,x) \right), \forall e,$$

**Step 3:** Prove that the first-best cannot be restored by contradiction.

Given the social planner's problem (83), simplify condition (81),

$$\begin{aligned} w =& \frac{\lambda}{\nu+\lambda} \left[ \pi(e^* x^*) + \tau_{e,g} g(e^* x^*) \right] - \frac{\nu}{\nu+\lambda} \tau_{x,g} g(e^* x^*) \\ =& \frac{\lambda \pi(e^* x^*)}{\nu+\lambda} + \frac{(\lambda \tau_{e,g} - \nu \tau_{x,g}) g(e^* x^*)}{\nu+\lambda}, \end{aligned}$$

where $\lambda$ is the Lagrangian multiplier.

The Lagrangian is

$$\begin{aligned} \mathcal{L} =& \nu \ln \left( \frac{\nu}{\nu+\lambda} \pi(e,x) + \frac{\nu}{\nu+\lambda} (\tau_{e,g} + \tau_{x,g}) g(e,x) \right) - c_e e \\ &+ \lambda \left[ \ln \left( \frac{\lambda \pi(e^* x^*)}{\nu+\lambda} + \frac{\lambda (\tau_{e,g} + \tau_{x,g}) g(e^* x^*)}{\nu+\lambda} \right) - c_x x \right]. \end{aligned}$$

Taking the derivative of the principal's best response and plugging in the first-best efforts into the constraints,

$$\begin{aligned} (\nu+\lambda) \left[ \beta - (\beta-\alpha) \sum_s \frac{q_s}{1 + (\tau_{e,g} + \tau_{x,g}) t^*} \right] &= c_e e^* \\ \lambda \left( (1-\beta) + (\beta-\alpha) \sum_s \frac{q_s}{1 + (\tau_{e,g} + \tau_{x,g}) t^*} - c_x x^* \right) &= \nu \left( (1-\beta) + (\beta-\alpha) \frac{1}{1+\mu t^*} \right) \\ \lambda \left( \left( \pi(e^* x^*) + (\tau_{e,g} + \tau_{x,g}) g(e^* x^*) \right) - \exp(c_x x^*) \right) =& \nu \exp(c_x x^*). \end{aligned}$$

The individual subsidy rate $\tau_{e,g}$ and $\tau_{x,g}$ are perfect substitutes, and hence their sum is a sufficient statistic denoted as $\tau \doteq (\tau_{e,g} + \tau_{x,g})$ solving the planner's problem. The equations are reduced into two unknowns, $\tau$ and $\lambda$, and represented as three linearly independent equations. Hence, there does not exist $\lambda > 0$ and $\tau > 0$ to solve the equations simultaneously. □

**Proof of Proposition 7-(a).**

We first prove parts (ii) and (iii), then move to part (i).

**Step 1:** Equilibrium characterization

Similar to Lemma 4, the Lagrangian is

$$\mathcal{L} = V\left(Y - w\right) - c_e e + \lambda\left[U\left(w\right) - c_x x - \underline{u}\right].$$

The equilibrium, $(\hat{w}_1(\underline{u}), \hat{e}_1(\underline{u}), \hat{x}_1(\underline{u}))$, is characterized by FOCs:

$$\frac{c_e \hat{e}_1(\underline{u})}{\hat{\lambda}_1(\underline{u}) + \nu} = \alpha + (\beta - \alpha)\frac{\hat{t}_1(\underline{u})}{1 + \hat{t}_1(\underline{u})} \tag{85}$$

$$\frac{\hat{\lambda}_1 c_x \hat{x}_1(\underline{u})}{\hat{\lambda}_1 + \nu} = (1 - \alpha) - (\beta - \alpha)\frac{\hat{t}_1(\underline{u})}{1 + \hat{t}_1(\underline{u})} \tag{86}$$

$$\hat{w}_1 = \frac{\hat{\lambda}_1(\underline{u})}{\nu + \hat{\lambda}_1(\underline{u})}\hat{Y}_1(\underline{u}) \tag{87}$$

$$\ln\left(\frac{\hat{\lambda}_1(\underline{u})}{\nu + \hat{\lambda}_1(\underline{u})}\hat{Y}_1(\underline{u})\right) = c_x \hat{x}_1(\underline{u}) + \underline{u}, \tag{88}$$

where $\hat{\lambda}_1(\underline{u})$ denote the Lagrangian multiplier at optimum, $\hat{t}_1(\underline{u}) \doteq \frac{A_g}{A_\pi}\left(\frac{\hat{e}_1(\underline{u})}{\hat{x}_1(\underline{u})}\right)^{(\beta-\alpha)}$, and $Y(e,x) = A_\pi e^\alpha x^{(1-\alpha)} + A_g e^\beta x^{(1-\beta)}$.

**Step 2:** Prove $\frac{\hat{\lambda}_1(\underline{u})}{\hat{\lambda}_1(\underline{u})+\nu} < \frac{1}{1+\nu}$ by contradiction

The first-best equity share is $\frac{1}{1+\nu}$ and the equilibrium equity share is $\frac{\hat{\lambda}_1(\underline{u})}{\hat{\lambda}_1(\underline{u})+\nu}$ from equation (88). It means that the equilibrium equity share is below the first best, if and only if $\hat{\lambda}_1(\underline{u}) < 1$. We prove $\hat{\lambda}_1(\underline{u}) < 1$ by contradiction.

From the uniqueness of optimum, $\hat{\lambda}_1(\underline{u}) \neq 1$.

Suppose $\hat{\lambda}_1(\underline{u}) > 1$. Let $F(e; x, \lambda) \doteq \frac{c_e e}{\lambda+\nu} - \left[\alpha + (\beta - \alpha)\frac{t}{1+t}\right]$, where $t \doteq \frac{A_g}{A_\pi}\left(\frac{x}{e}\right)^{(1-\beta)-(1-\alpha)}$. $F(e; x, \lambda)$ decreases in $e$ regardless of the value of $\lambda$.

Because the right-hand side of equation (78) is the same as that of equation (4) and $F(e; x, \lambda)$ decreases in $e$,

$$\hat{\lambda}_1(\underline{u}) > 1 \Rightarrow \hat{e}_1 > e^*$$

Similarly, comparing (86) and (5),

$$\hat{\lambda}_1(\underline{u}) > 1 \Rightarrow \hat{x}_1 < x^*$$

Construct a hypothetical equity share $\tilde{\epsilon}$, such that

$$U\left(\tilde{\epsilon}(\underline{u})Y^*\right) = c_x x^* + \underline{u} \text{ where } Y^* \doteq Y(e^*, x^*).$$

Let $\hat{\epsilon}_1(\underline{u}) \doteq \frac{\hat{\lambda}_1(\underline{u})}{\hat{\lambda}_1(\underline{u})+\nu}$. Now the allocation $(\tilde{\epsilon}(\underline{u})Y^*, e^*, x^*)$ is feasible for the principal's problem. From the optimality of the principal's problem at $(\hat{w}_1(\underline{u}), \hat{e}_1(\underline{u}), \hat{x}_1(\underline{u}))$, the principal obtains a larger net utility from allocation $(\hat{w}_1(\underline{u}), \hat{e}_1(\underline{u}), \hat{x}_1(\underline{u}))$ than $(\tilde{\epsilon}(\underline{u})Y^*, e^*, x^*)$, i.e.,

$$\begin{aligned} V\left((1-\tilde{\epsilon}(\underline{u}))\,Y^*)\right) - c_e e^* &< V\left((1-\hat{\epsilon}_1(\underline{u}))\,\hat{Y}_1(\underline{u})\right) - c_e \hat{e}_1(\underline{u}) \\ (1-\tilde{\epsilon}(\underline{u}))\,Y^* &< (1-\hat{\epsilon}_1(\underline{u}))\,\hat{Y}_1(\underline{u}) \\ \frac{(1-\tilde{\epsilon}(\underline{u}))}{(1-\hat{\epsilon}_1(\underline{u}))} &< \frac{\hat{Y}_1(\underline{u})}{Y^*}, \end{aligned}$$

where, the second inequality is because $\hat{e}_1(\underline{u}) > e^*(\underline{u})$.

Because $\hat{x}_1(\underline{u}) < x^*(\underline{u})$ and the agent's net utility is $(\underline{u})$ at both allocations,

$$\begin{aligned} U\left((\tilde{\epsilon}(\underline{u}))\,Y^*)\right) - c_x x^* &= U\left((\hat{\epsilon}_1(\underline{u}))\,\hat{Y}_1(\underline{u})\right) - c_x \hat{x}_1(\underline{u}) \\ U\left((\tilde{\epsilon}(\underline{u}))\,Y^*)\right) &> U\left((\hat{\epsilon}_1(\underline{u}))\,\hat{Y}_1(\underline{u})\right) \\ \tilde{\epsilon}(\underline{u})Y^* &> \hat{\epsilon}_1(\underline{u})\hat{Y}_1(\underline{u}) \\ \frac{\tilde{\epsilon}(\underline{u})}{\hat{\epsilon}_1(\underline{u})} &> \frac{\hat{Y}_1(\underline{u})}{Y^*}. \end{aligned}$$

Combining the two inequalities,

$$\frac{\tilde{\epsilon}(\underline{u})}{\hat{\epsilon}_1(\underline{u})} > \frac{1-\tilde{\epsilon}(\underline{u})}{1-\hat{\epsilon}_1(\underline{u})} \Rightarrow \hat{\epsilon}_1(\underline{u}) < \tilde{\epsilon}(\underline{u}). \tag{89}$$

From the construction of $\tilde{\epsilon}$,

$$\begin{aligned} U\left(\tilde{\epsilon}(\underline{u})Y^*\right) - c_x x^* &= 0, \\ U\left(\frac{1}{1+\nu}Y^*\right) - c_x x^* &> 0, \end{aligned}$$
$$\Rightarrow \tilde{\epsilon}(\underline{u}) < \frac{1}{1+\nu}. \tag{90}$$

Combining (90) and (89), we have $\hat{\epsilon}_1(\underline{u}) < \frac{1}{1+\nu}$, contradicting $\hat{\lambda}_1(\underline{u}) > 1$. Hence, $\frac{\hat{\lambda}_1(\underline{u})}{\hat{\lambda}_1(\underline{u})+\nu} < \frac{1}{1+\nu}$.

**Step 3:** Prove that $\hat{e}_1(\underline{u}) < e^*$ and $\hat{x}_1(\underline{u}) > x^*$.

Rearrange the FOC (85) and (86), and let $t \doteq \frac{A_g}{A_\pi} r^{(\beta-\alpha)}$ for the equilibrium input ratio $\hat{r}_1(\underline{u})$ and the first-best input ratio $r^*$. Then

$$\frac{c_e}{c_x \hat{\lambda}_1(\underline{u})} = \frac{\alpha + (\beta-\alpha)\frac{\hat{t}_1(\underline{u})}{1+\hat{t}_1(\underline{u})}}{\hat{r}_1(\underline{u})\left[(1-\alpha) - (\beta-\alpha)\frac{\hat{t}_1(\underline{u})}{1+\hat{t}_1(\underline{u})}\right]} \tag{91}$$

$$\frac{c_e}{c_x} = \frac{\alpha + (\beta-\alpha)\frac{r^*}{1+r^*}}{r^*\left[(1-\alpha) - (\beta-\alpha)\frac{r^*}{1+r^*}\right]}. \tag{92}$$

Therefore,

$$\begin{aligned}
&\hat{\lambda}_1(\underline{u}) < 1 \\
\Leftrightarrow &\frac{\alpha + (\beta-\alpha)\frac{\hat{t}_1(\underline{u})}{1+\hat{t}_1(\underline{u})}}{\hat{r}_1(\underline{u})\left[(1-\alpha)-(\beta-\alpha)\frac{\hat{t}_1(\underline{u})}{1+\hat{t}_1(\underline{u})}\right]} > \frac{\alpha + (\beta-\alpha)\frac{r^*}{1+r^*}}{r^*\left[(1-\alpha)-(\beta-\alpha)\frac{r^*}{1+r^*}\right]} \\
&\Leftrightarrow \hat{r}_1(\underline{u}) < r^*,
\end{aligned}$$

where the first inequality is from (91) and (92). The second inequality is because the ratio $\mathcal{R}(r) \doteq \frac{\alpha+(\beta-\alpha)\frac{r}{1+r}}{r\left[(1-\alpha)-(\beta-\alpha)\frac{r}{1+r}\right]}$ decreases in $r$, i.e.,$\mathcal{R}'(r) < 0$.

Compare the right-hand side of $e$'s FOC in equilibrium and under the first-best,

$$\begin{aligned}
\alpha + (\beta-\alpha)\frac{\hat{t}_1(\underline{u})}{1+\hat{t}_1(\underline{u})} &< \alpha + (\beta-\alpha)\frac{r^*}{1+r^*} \\
\Leftrightarrow \frac{c_e\hat{e}_1(\underline{u})}{\hat{\lambda}_1(\underline{u})+\nu} &< \frac{c_e e^*}{1+\nu} \\
\Rightarrow \hat{e}_1(\underline{u}) &< e^*,
\end{aligned}$$

where first inequality is from $\hat{r}_1(\underline{u}) < r^*$, the second is from FOCs (85) and (4), and the third inequality is from $\hat{\lambda}_1(\underline{u}) < 1$. Similarly,

$$\begin{aligned}
(1-\alpha)-(\beta-\alpha)\frac{\hat{t}_1(\underline{u})}{1+\hat{t}_1(\underline{u})} &> (1-\alpha)-(\beta-\alpha)\frac{r^*}{1+r^*} \\
\Leftrightarrow \frac{\hat{\lambda}_1(\underline{u})c_x}{\hat{\lambda}_1(\underline{u})+\nu}\hat{x}_1(\underline{u}) &< \frac{c_x x^*}{1+\nu} \\
\Rightarrow \hat{e}_1(\underline{u}) &> e^*
\end{aligned}$$

The first inequality is from $\hat{r}_1(\underline{u}) < r^*$, the second is from FOCs (86) and (5), and the third inequality from $\hat{\lambda}_1(\underline{u}) < 1$.

The analyses above complete the proof of part (b). To see part (a), that FOC characterization shown in the first step still holds under $\underline{u} = u_x^*$. Combine the condition $\underline{u} = u_x^*$ with the FOCs and the agent's utility determination (88), $\hat{\lambda}_1(u_x^*) = 1$.

Plugging $\hat{\lambda}_1(u_x^*) = 1$ into FOCs (87), (91), and (92), we have $\hat{e}_1(u_x^*) = e^*$, $\hat{x}_1(u_x^*) = x^*$, and $\hat{w}_1(u_x^*) = w^*$. □

## Proof of Proposition 7-(b).

Let $\hat{r}_1(\underline{u}) \doteq \frac{\hat{e}_1(\underline{u})}{\hat{x}_1(\underline{u})}$ denote the ratio of effort pair $(\hat{e}_1, \hat{x}_1)$ in the Pigouvian subsidy equilibrium where the agent's outside option value is $\underline{u}$. According to the proof of Proposition 7-(a), equation (85), (86), and (88) characterize equilibrium effort pair $(\hat{e}_1, \hat{x}_1)$. We transform these equations into the following equation with $\hat{r}_1(\underline{u})$ as the only variable, i.e.,

$$\frac{\underline{u}}{\alpha+(\beta-\alpha)\frac{\hat{t}_1(\hat{r}_1)}{1+\hat{t}_1(\hat{r}_1)}}=\frac{A_\pi\hat{r}_1(\underline{u})^\alpha+A_g\hat{r}_1(\underline{u})^\beta}{c_x}-1-\frac{\nu c_x}{\hat{r}_1(\underline{u})c_e}\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right), \tag{93}$$

where $\hat{t}_1(\hat{r}_1)\doteq\frac{A_g}{A_\pi}\left(\hat{r}_1(\underline{u})\right)^{\beta-\alpha}$.

We treat $\hat{r}_1(\underline{u})$ as the only unknown and $\underline{u}$ as its parameter for equation (93). We first show how $\hat{r}_1(\underline{u})$ changes with regard to $\underline{u}$. Second, we represent $\hat{e}_1(\underline{u})$ and $\hat{x}_1(\underline{u})$ from equation (85) and (86). Third step is the comparative statics of $\hat{e}_1(\underline{u})$ and $\hat{x}_1(\underline{u})$ with regard to $\underline{u}$. Note that the first-best effort level $e^*$ and $x^*$ can be represented as $e^*\doteq\hat{e}_1(u^*)$ and $x^*\doteq\hat{x}_1(u^*)$, where the equilibrium we analyze here is $\underline{u}<u^*$. Thus, comparing $\hat{e}_1^{'}(\underline{u})$ and $\hat{x}_1^{'}(\underline{u})$ to their counterpart with agent's utility at the first-best level yields the comparative statics of effort distortion with regard to $\underline{u}$.

Let $L(\hat{r}_1;\underline{u})\doteq\frac{\underline{u}}{\alpha+(\beta-\alpha)\frac{\hat{t}_1(\hat{r}_1)}{1+\hat{t}_1(\hat{r}_1)}}$ and $R(\hat{r}_1;\underline{u})\doteq\frac{A_\pi\hat{r}_1(\underline{u})^\alpha+A_g\hat{r}_1(\underline{u})^\beta}{c_x}-1-\frac{\nu c_x}{\hat{r}_1(\underline{u})c_e}\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right)$ denote the left-hand side and right-hand side of equation (93). To demonstrate their monotonicity, we take first-order derivative of $\ln L(\hat{r}_1;\underline{u})$ and $R(\hat{r}_1;\underline{u})$,

$$\begin{aligned}
\frac{d\left(\ln L(\hat{r}_1;\underline{u})\right)}{d\hat{r}_1}=&\frac{d\left(\alpha+(\beta-\alpha)\frac{\hat{t}_1(\hat{r}_1)}{1+\hat{t}_1(\hat{r}_1)}\right)}{d\hat{r}_1}\left(-\left(\alpha+(\beta-\alpha)\frac{\hat{t}_1(\hat{r}_1)}{1+\hat{t}_1(\hat{r}_1)}\right)^{-2}\right)\\
&\propto-\frac{d\left(\alpha+(\beta-\alpha)\frac{\hat{t}_1(\hat{r}_1)}{1+\hat{t}_1(\hat{r}_1)}\right)}{d\hat{r}_1}\\
&\propto-\frac{d\left(\frac{\hat{t}_1(\hat{r}_1)}{1+\hat{t}_1(\hat{r}_1)}\right)}{d\hat{r}_1}\\
&\propto\frac{d\left(\hat{t}_1(\hat{r}_1)\right)}{d\hat{r}_1}\\
&=\frac{A_g}{A_\pi}(\beta-\alpha)\left(\hat{r}_1(\underline{u})\right)^{\beta-\alpha-1}>0\\
\Rightarrow\ \frac{dL(\hat{r}_1;\underline{u})}{d\hat{r}_1}>0&
\end{aligned}$$

$$
\begin{aligned}
\frac{d\left(R(\hat{r}_1;\underline{u})\right)}{d\hat{r}_1} =& \frac{d\left(\frac{A_\pi \hat{r}_1(\underline{u})^\alpha + A_g \hat{r}_1(\underline{u})^\beta}{c_x}\right)}{d\hat{r}_1} + \frac{\nu c_x}{\hat{r}_1^2(\underline{u})c_e}\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right) \\
&- \frac{\nu c_x}{\hat{r}_1(\underline{u})c_e}\frac{d\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right)}{d\hat{r}_1} \\
>& \frac{\nu c_x}{\hat{r}_1^2(\underline{u})c_e}\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right) - \frac{\nu c_x}{\hat{r}_1(\underline{u})c_e}\frac{d\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right)}{d\hat{r}_1} \\
\propto& \frac{1}{\hat{r}_1(\underline{u})}\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right) - \frac{d\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right)}{d\hat{r}_1} \\
\propto& \frac{1}{\hat{r}_1(\underline{u})}\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right) - \frac{d\left(\frac{-\frac{\beta^2}{\alpha}+\alpha}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right)}{d\hat{r}_1} \\
=& \frac{1}{\hat{r}_1(\underline{u})}\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right) + \frac{d\left(\frac{\frac{\beta^2}{\alpha}-\alpha}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right)}{d\hat{r}_1} \\
=& \frac{1}{\hat{r}_1(\underline{u})}\left(\frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\right) - \frac{\beta^2-\alpha^2}{\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)^2}\frac{d\left(\hat{t}_1(\hat{r}_1)\right)}{d\hat{r}_1} \\
\propto& \frac{\alpha+\beta\hat{t}_1(\hat{r}_1)}{\hat{r}_1(\underline{u})} - \frac{\beta^2-\alpha^2}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\frac{d\left(\hat{t}_1(\hat{r}_1)\right)}{d\hat{r}_1} \\
\propto& \left(\alpha+\beta\hat{t}_1(\hat{r}_1)\right) - \frac{\beta^2-\alpha^2}{\beta+\alpha\hat{t}_1(\hat{r}_1)}\left(\frac{A_g\left(\beta-\alpha\right)\left(\hat{r}_1(\underline{u})\right)^{\beta-\alpha}}{A_\pi}\right) \\
\propto& \left(\alpha^2+\beta^2\right)\hat{t}_1(\hat{r}_1) + \alpha\beta\left(1+\hat{t}_1(\hat{r}_1)\right) - \left(\beta^2-\alpha^2\right)\hat{t}_1(\hat{r}_1) \\
=& 2\alpha^2\hat{t}_1(\hat{r}_1) + \alpha\beta\left(1+\hat{t}_1(\hat{r}_1)\right) > 0,
\end{aligned}
$$

where $\hat{t}_1(\hat{r}_1) \doteq \frac{A_g}{A_\pi}\left(\hat{r}_1(\underline{u})\right)^{\beta-\alpha}$.

Thus, both $L(\hat{r}_1;\underline{u})$ and $R(\hat{r}_1;\underline{u})$ monotonically increases in $\hat{r}_1 \in \Re^+$ and hence for any given $\underline{u}$, the solution $\hat{r}_1(\underleftarrow{u})$ is unique. Below shows that for any $r < \hat{r}_1$, $L(r) > R(r)$ and vice versa.

$$
\lim_{r\to 0} L(r) = \frac{u}{\beta} > \lim_{r\to 0} R(r) = -\infty
$$
$$
\lim_{r\to +\infty} L(r) = \frac{u}{\alpha} < \lim_{r\to\infty} R(r) = +\infty
$$

Therefore, $LHS(r) - RHS(r)$ monotonically decreases in $\hat{r}_1$, i.e., $\frac{d(LHS(r)-RHS(r))}{dr} < 0$. For

$\frac{d\hat{r}_1(\underline{u})}{d\underline{u}}$,

$$\frac{d\left(LHS(r)-RHS(r)\right)}{d\underline{u}} = \frac{1}{\alpha+(\beta-\alpha)\frac{\hat{t}_1(\hat{r}_1)}{1+\hat{t}_1(\hat{r}_1)}} > 0$$

$$\Rightarrow \frac{d\hat{r}_1(\underline{u})}{d\underline{u}} = -\frac{\frac{d(LHS(r)-RHS(r))}{d\underline{u}}}{\frac{d(LHS(r)-RHS(r))}{dr}} > 0.$$

From equation (85) and (86), the equilibrium efforts pairs, $(\hat{e}_1, \hat{x}_1)$, can be expressed as

$$\hat{x}_1 = \frac{1}{c_x}\left(\alpha+\frac{\beta-\alpha}{1+\hat{t}_1(\hat{r}_1)}\right)\left(1+\frac{\nu c_x\left(\alpha+\beta\hat{t}_1(\hat{r}_1)\right)}{c_e\hat{r}_1\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)}\right)$$

$$\hat{e}_1 = \frac{1}{c_e}\left(\beta+\frac{\alpha-\beta}{1+\hat{t}_1(\hat{r}_1)}\right)\left(\nu+\frac{c_e\hat{r}_1\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)}{c_x\left(\alpha+\beta\hat{t}_1(\hat{r}_1)\right)}\right),$$

where $\hat{t}_1(\hat{r}_1) \doteq \frac{A_g}{A_\pi}\left(\hat{r}_1(\underline{u})\right)^{\beta-\alpha}$.

Take the first-order derivative of $\hat{x}_1$ with regard to $\hat{r}_1$,

$$\begin{aligned}
\frac{\partial\hat{x}_1}{\partial\hat{r}_1} \propto & \hat{t}_1'(\hat{r}_1)\left[-\frac{(\beta-\alpha)\left(1+\frac{\nu c_x\left(\alpha+\beta\hat{t}_1(\hat{r}_1)\right)}{c_e\hat{r}_1\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)}\right)}{\left(1+\hat{t}_1(\hat{r}_1)\right)^2}+\left(\alpha+\frac{\beta-\alpha}{1+\hat{t}_1(\hat{r}_1)}\right)\frac{\nu c_x}{c_e\hat{r}_1}\frac{\frac{\beta^2}{\alpha}-\alpha}{\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)^2}\right] \\
& -\left(\alpha+\frac{\beta-\alpha}{1+\hat{t}_1(\hat{r}_1)}\right)\frac{\nu c_x\left(\alpha+\beta\hat{t}_1(\hat{r}_1)\right)}{c_e\hat{r}_1^2\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)} \\
\propto & \hat{t}_1'(\hat{r}_1)\left[\frac{-(\beta-\alpha)\left(\beta+\alpha\hat{t}_1\right)c_e\hat{r}_1+\nu c_x\left(\alpha+\beta\hat{t}_1(\hat{r}_1)\right)}{\left(1+\hat{t}_1(\hat{r}_1)\right)}+\frac{\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)\nu c_x\left(\frac{\beta^2}{\alpha}-\alpha\right)}{\left(\beta+\alpha\hat{t}_1\right)}\right] \\
& -\frac{\nu c_x\left(\alpha+\beta\hat{t}_1(\hat{r}_1)\right)\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)}{\hat{r}_1} \\
= & \hat{t}_1'(\hat{r}_1)\left[\frac{-(\beta-\alpha)\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)c_e\hat{r}_1+\nu c_x\left(\beta\hat{t}_1-\frac{\beta^2}{\alpha}+\alpha-\hat{t}_1\frac{\beta^2}{\alpha}\right)}{\left(1+\hat{t}_1(\hat{r}_1)\right)}\right]-\frac{\nu c_x\left(\alpha+\beta\hat{t}_1(\hat{r}_1)\right)\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)}{\hat{r}_1} \\
\propto & \hat{t}_1'(\hat{r}_1)\left[\frac{-(\beta-\alpha)\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)c_e\hat{r}_1+\frac{\nu c_x}{\alpha}(\alpha-\beta)\left(\beta\hat{t}_1+\beta+\alpha\right)}{\left(1+\hat{t}_1(\hat{r}_1)\right)}\right]-\frac{\nu c_x\left(\alpha+\beta\hat{t}_1(\hat{r}_1)\right)\left(\beta+\alpha\hat{t}_1(\hat{r}_1)\right)}{\hat{r}_1} \\
< & 0,
\end{aligned}$$

where $\hat{t}_1(\hat{r}_1) \doteq \frac{A_g}{A_\pi}\left(\hat{r}_1(\underline{u})\right)^{\beta-\alpha}$ and the last inequality is due to $\hat{t}_1'(\hat{r}_1) = \frac{A_g}{A_\pi}(\beta-\alpha)\left(\hat{r}_1(\underline{u})\right)^{\beta-\alpha-1} > 0$.

Combining $\frac{\partial\hat{x}_1}{\partial\hat{r}_1} < 0$ with $\frac{d\hat{r}_1(\underline{u})}{d\underline{u}} > 0$,

$$\frac{\partial\hat{x}_1(\hat{r}_1(\underline{u}))}{\partial\underline{u}} < 0 \tag{94}$$

Similarly, $\begin{cases} \frac{\partial \hat{e}_1(\hat{r}_1)}{\partial \hat{r}_1} > 0 \\ \frac{d\hat{r}_1(\underline{u})}{d\underline{u}} > 0 \end{cases}$ and hence

$$\frac{\partial \hat{e}_1(\hat{r}_1(\underline{u}))}{\partial \underline{u}} > 0. \tag{95}$$

From Proposition (7)-(a), $\hat{e}_1(\underline{u}) < e^*$ and $\hat{x}_1(\underline{u}) > x^*$.

Together from the regularity assumption $\underline{u} < u^*$, the distortion's dependence on $\underline{u}$ is

$$\begin{aligned} \frac{\partial (e^* - \hat{e}_1(\underline{u}))}{\partial \underline{u}} &= \frac{\partial (e^* - \hat{e}_1(\underline{u}))}{\partial \underline{u}} = -\frac{\partial \hat{e}_1(\hat{r}_1(\underline{u}))}{\partial \underline{u}} < 0 \\ \frac{\partial (\hat{x}_1(\underline{u}) - x^*)}{\partial \underline{u}} &= \frac{\partial (\hat{x}_1(\underline{u}) - x^*)}{\partial \underline{u}} = \frac{\partial \hat{x}_1(\hat{r}_1(\underline{u}))}{\partial \underline{u}} < 0, \end{aligned}$$

where the equality is from Proposition (7-(a)) and the inequalities are from (94) and (95). □

**Proof of Proposition 8-(a).**

We prove $\check{e}_1 > \check{e}_0$ by contradiction. Suppose $\check{e}_1 \leq \check{e}_0$. FOCs of laissez-faire equilibrium are

$$\begin{aligned} \frac{\check{\lambda}_0 c_e}{\check{\lambda}_0 + \nu} &= \frac{\alpha}{\check{e}_0} \\ \frac{c_x}{\check{\lambda}_0 + \nu} &= \frac{(1-\alpha)}{\check{x}_0}, \end{aligned} \tag{96}$$

Let $\epsilon \doteq \frac{\lambda}{\lambda+\nu}$. Rearrange FOC and combine with the participation constraint (29), the laissez-faire equilibrium is characterized by,

$$\frac{\check{\lambda}_0 c_e}{c_x} = \left(\frac{\alpha}{(1-\alpha)}\right) \frac{\check{x}_0}{\check{e}_0} \tag{97}$$

$$\frac{\alpha}{\check{\epsilon}_0} = \ln\left(\frac{A_\pi \alpha}{c_e}\right) + (1-\alpha)\ln\left(\frac{c_e(1-\alpha)}{c_x \alpha}\right). \tag{98}$$

Similarly, FOCs of the market equilibrium under the Pigouvian subsidy are,

$$\begin{aligned} \frac{\check{\lambda}_1 c_e}{\check{\lambda}_1 + \nu} &= \frac{1}{\check{e}_1}\left[\alpha + (\beta - \alpha)\frac{\frac{A_g}{A_\pi}\left(\frac{\hat{e}_{b,1}}{\hat{x}_{b,1}}\right)^{(\beta-\alpha)}}{1 + \frac{A_g}{A_\pi}\left(\frac{\hat{e}_{b,1}}{\hat{x}_{b,1}}\right)^{(\beta-\alpha)}}\right] \\ \frac{c_x}{\check{\lambda}_1 + \nu} &= \frac{1}{\check{x}_1}\left[(1-\alpha) + ((1-\beta) - (1-\alpha))\frac{\frac{A_g}{A_\pi}\left(\frac{\hat{e}_{b,1}}{\hat{x}_{b,1}}\right)^{(\beta-\alpha)}}{1 + \frac{A_g}{A_\pi}\left(\frac{\hat{e}_{b,1}}{\hat{x}_{b,1}}\right)^{(\beta-\alpha)}}\right], \end{aligned} \tag{99}$$

Let $\check{r}_1 \doteq \frac{\check{x}_1}{\check{e}_1}$ and $\epsilon \doteq \frac{\lambda}{\lambda+\nu}$. The equilibrium is characterized by,

$$\frac{1-f(\check{r}_1)}{\check{\epsilon}_1} = \ln\left(\frac{A_\pi(1-f(\check{r}_1))}{c_e}\right) + (1-\alpha)\ln(\check{r}_1) + \ln\left(1+\frac{A_g}{A_\pi}\check{r}_1^{(1-\beta)-(1-\alpha)}\right) \tag{100}$$

$$\check{r}_1 = \frac{\check{\lambda}_1 c_e}{c_x}\frac{f(\check{r}_1)}{1-f(\check{r}_1)}, \tag{101}$$

where $f(\check{r}_1) \doteq (1-\alpha) + ((1-\beta)-(1-\alpha))\frac{\frac{A_g}{A_\pi}\left(\frac{\hat{e}_{b,1}}{\hat{x}_{b,1}}\right)^{(\beta-\alpha)}}{1+\frac{A_g}{A_\pi}\left(\frac{\hat{e}_{b,1}}{\hat{x}_{b,1}}\right)^{(\beta-\alpha)}}$.

Because $(1-\beta) < (1-\alpha)$ from the definition of production functions, for any $r \doteq \frac{e_p}{e_a}$,

$$1-f(r) \doteq \alpha - ((1-\beta)-(1-\alpha))\frac{\frac{A_g}{A_\pi}r^{(1-\beta)-(1-\alpha)}}{1+\frac{A_g}{A_\pi}r^{(1-\beta)-(1-\alpha)}} > \alpha. \tag{102}$$

Let $\epsilon \doteq \frac{\lambda}{\lambda+\nu}$. If $\check{e}_1 \leq \check{e}_0$,

$$\begin{aligned} \frac{1-f(\check{r}_1)}{\check{\epsilon}_1} &= \hat{e}_1 < \hat{e}_0 = \frac{\alpha}{\check{\epsilon}_0}, \\ &\Rightarrow \check{\epsilon}_1 > \check{\epsilon}_0, \\ &\Rightarrow \check{\lambda}_1 > \check{\lambda}_0, \end{aligned} \tag{103}$$

where the first and the last equality arise from equation (99) and (96). The second inequality is from (102) and the last inequality is from $\epsilon \doteq \frac{\lambda}{\lambda+\nu}$.

From (98) and (100),

$$\begin{aligned} \ln\left(A_\pi\check{r}_0^{(1-\alpha)}\right) + \ln\left(\frac{\alpha}{c_e}\right) &= c_e\check{e}_0, \\ &> c_e\check{e}_1, \\ &= \ln\left(A_\pi\check{r}_1{}^{(1-\alpha)} + A_g\check{r}_1{}^{(1-\beta)}\right) + \ln\left(\frac{1-f(\check{r}_1)}{c_e}\right), \\ &\geq \ln\left(A_\pi\check{r}_1^{(1-\alpha)} + A_g\check{r}_1^{(1-\beta)}\right) + \ln\left(\frac{\alpha}{c_e}\right), \\ &> \ln\left(A_\pi\check{r}_1^{(1-\alpha)}\right) + \ln\left(\frac{\alpha}{c_e}\right), \\ \Rightarrow \check{r}_1 &< \check{r}_0, \end{aligned} \tag{104}$$

where the first inequality is from $\check{e}_1 \leq \check{e}_0$ and the second is from (102). The equations are from the rearrangement of participation constraints (98) and (100).

Compare equation (97) and (101), if $\check{e}_1 \leq \check{e}_0$,

$$\begin{aligned}\frac{1}{\check{r}_1} =& \frac{\check{\lambda}_1 c_e}{c_x}\frac{f(\check{r}_1)}{(1-f(\check{r}_1))},\\ &\geq \frac{f(\check{r}_1)}{1-f(\check{r}_1)}\frac{\check{\lambda}_0 c_e}{c_x},\\ &> \frac{(1-\alpha)}{\alpha}\frac{\check{\lambda}_0 c_e}{c_x},\\ &= \frac{1}{\check{r}_0}\\ \Rightarrow \check{r}_1 <& \check{r}_0,\end{aligned} \tag{105}$$

where the first inequality is from $\check{\lambda}_1 > \check{\lambda}_0$ in (103) and the second is from (102).

Because (104) contradicts (105), $\check{e}_1 > \check{e}_0$. □

**Proof of Proposition 8-(b).**

The proof consists of three steps. First, we construct a hypothetical equilibrium allocation. Let $\tilde{\epsilon}_b$ and $\check{\epsilon}_0$ denote the equilibrium equity share in the hypothetical equilibrium and in laissez-faire. We first show $\tilde{\epsilon}_b < \check{\epsilon}_0$. Second, let $\check{\epsilon}_1$ denote the agent's equilibrium equity share under the Pigouvian subsidy. We show $\check{\epsilon}_1 < \tilde{\epsilon}_b$ and hence $\check{\epsilon}_1 < \check{\epsilon}_0$. At last, we show $\check{x}_1 < \check{x}_0$.

**Step 1** Let $(\tilde{\epsilon}_b, \tilde{r}_b)$ solves the following equation system

$$\frac{c_x}{c_e} r_b = \frac{(1-\beta)\lambda}{\beta} \tag{106}$$

$$\epsilon \doteq \frac{\lambda}{\lambda+\nu},$$

$$\ln\left(\frac{\beta}{c_e}\left(A_\pi r_b^{(1-\alpha)} + A_g r_b^{(1-\beta)}\right)\right) = \frac{\beta}{\epsilon}. \tag{107}$$

For the laissez-faire equilibrium, define the agent's equity share and the effort ratio by $\check{\epsilon}_0 \doteq \frac{\check{\lambda}_0}{\check{\lambda}_0+\nu}$ and $\check{r}_0 \doteq \frac{\check{x}_0}{\check{e}_0}$. Let $r_b \doteq \frac{x}{e}$ for each scenario.

Below we prove that $\tilde{\epsilon}_b < \check{\epsilon}_0$,given $A_\pi \geq c_e$.

From FOC characterization, $(\check{\epsilon}_0, \hat{r}_b)$ as the solution of the following equation system.

$$\frac{c_x}{c_e} r_b = \frac{(1-\alpha)\lambda}{\alpha}, \tag{108}$$

$$\ln\left(\frac{\alpha}{c_e} A_\pi r_b^{(1-\alpha)}\right) = \frac{\alpha}{\epsilon}. \tag{109}$$

Then we define $F(r)$ and $\tilde{F}(r)$ from equation (106) and (108), and $G(r)$ and $\tilde{G}(r)$ from equation

(107) and (109), respectively.

$$
\begin{aligned}
F(r) &\doteq \frac{(1-\alpha)}{\alpha r_b}\frac{c_e}{c_x} + 1 \\
\tilde{F}(r) &\doteq \frac{(1-\beta)}{\beta r_b}\frac{c_e}{c_x} + 1, \\
G(r) &\doteq \frac{1}{\alpha}\ln\left(\frac{\alpha}{c_e}A_\pi r_b^{(1-\alpha)}\right), \\
\tilde{G}(r) &\doteq \frac{1}{\beta}\ln\left(\frac{\beta}{c_e}\left(A_\pi r_b^{(1-\alpha)} + A_g r_b^{(1-\beta)}\right)\right),
\end{aligned} \tag{110}
$$

Thus, $\hat{r}$ is the solution of $F(r) = G(r)$, and $\tilde{r}_b$ is the solution of $\tilde{F}(r) = \tilde{G}(r)$, i.e., $\frac{1}{\tilde{\epsilon}_b} = \tilde{F}(\tilde{r}_b) = \tilde{G}(\tilde{r}_b)$. Define $r_{b,g}$ and $r_{b,f}$ as the solution of $G(r_{b,g}) = \frac{1}{\tilde{\epsilon}_b}$ and $F(r_{b,f}) = \frac{1}{\tilde{\epsilon}_b}$.

**(i)** To show $r_{b,f} < r_{b,g} \Rightarrow \tilde{\epsilon}_b < \check{\epsilon}_0$.

Proof of the claim above. Take derivatives of $F(r)$, $\tilde{F}(r)$, $G(r)$ and $\tilde{G}(r)$,

$$
\begin{aligned}
F'(r) < 0,\ \tilde{F}'(r) < 0 \\
G'(r) > 0,\ \tilde{G}'(r) > 0 \\
\frac{d\left(F(r) - G(r)\right)}{dr} < 0.
\end{aligned}
$$

Because $r_{b,f} < r_{b,g}$, $G(r_{b,f}) < G(r_{b,g}) = \frac{1}{\tilde{\epsilon}_b}$ and $F(r_{b,g}) < F(r_{b,f}) = \frac{1}{\tilde{\epsilon}_b}$. Thus,

$$
\begin{aligned}
F(r_{b,g}) - G(r_{b,g}) < 0 \\
F(r_{b,f}) - G(r_{b,f}) > 0.
\end{aligned}
$$

From definition of $F(r)$ and $G(r)$, $F(\hat{r}) - G(\hat{r}) = 0$.

Hence,

$$
\begin{aligned}
F(r_{b,g}) - G(r_{b,g}) < F(\hat{r}) - G(\hat{r}) < F(r_{b,f}) - G(r_{b,f}) \\
\Rightarrow r_{b,f} < \check{r}_0 < r_{b,g}.
\end{aligned}
$$

Because $F(r_{b,f}) = \frac{1}{\tilde{\epsilon}_b}$, $F(\hat{r}) = \frac{1}{\check{\epsilon}_0}$ and $F'(r) < 0$,

$$
\frac{1}{\check{\epsilon}_0} < \frac{1}{\tilde{\epsilon}_b} \Rightarrow \check{\epsilon}_0 > \tilde{\epsilon}_b.
$$

**(ii)** To show that given $A_\pi \geq c_e$, $r_{b,f} < r_{b,g}$.

From definition $F(r_{b,f}) = \frac{1}{\tilde{\epsilon}_b}$,

$$
\tilde{r}_b = r_{b,f}\frac{(1-\alpha)}{\alpha}\frac{\beta}{(1-\beta)}. \tag{111}
$$

Plug $r_{b,f}$ in $\tilde{G}(r)$,

$$
\begin{aligned}
G(r_{b,f}) &= \frac{1}{\alpha}\ln\left(\frac{\alpha}{c_e}A_\pi\right) + \frac{(1-\alpha)}{\alpha}\ln r_{b,f} \\
&= \frac{1}{\alpha}\ln\left(\frac{\alpha}{c_e}A_\pi\right) + \frac{(1-\alpha)}{\alpha}\left(\ln\left(\frac{\alpha}{1-\alpha}\frac{1-\beta}{\beta}\right) + \ln\tilde{r}_b\right) \\
&< \frac{1}{\alpha}\ln\left(\frac{\alpha}{c_e}A_\pi\right) + \frac{(1-\alpha)}{\alpha}\ln\tilde{r}_b \\
&= \frac{1-\beta}{(1-\alpha)\,\beta}\ln\left(\frac{\alpha}{c_e}A_\pi\right) + \frac{(1-\beta)}{\beta}\ln\tilde{r}_b \\
&< \frac{1}{\beta}\ln\left(\frac{A_\pi}{c_e}\right) + \frac{1-\beta}{(1-\alpha)\beta}\ln\alpha + \frac{(1-\beta)}{\beta}\ln\tilde{r}_b \\
&< \frac{1}{\beta}\ln\left(\frac{A_\pi}{c_e}\right) + \frac{1}{\beta}\ln\alpha + \frac{(1-\beta)}{\beta}\ln\tilde{r}_b \\
&< \frac{1}{\beta}\ln\left(\frac{A_\pi}{c_e}\right) + \frac{1}{\beta}\ln\beta + \frac{(1-\beta)}{\beta}\ln\tilde{r}_b \\
&< \frac{1}{\beta}\ln\left(\frac{\beta}{c_e}A_\pi\right) + \frac{(1-\beta)}{\beta}\ln\tilde{r}_b + \frac{\ln\left(1+\frac{A_g}{A_\pi}\tilde{r}_b^{(\beta-\alpha)}\right)}{\beta} \\
&= \tilde{G}\left(\tilde{r}_b\right) = G(r_{b,g}).
\end{aligned}
$$

The first inequality is from $\alpha < \beta$ and its resulting $1-\beta > 1-\alpha$; the second equation is from multiplying the previous line with $\frac{\alpha(1-\beta)}{\beta(1-\alpha)}$; the second inequality stems from $A_\pi \geq c_e$ and $\frac{1-\beta}{1-\alpha} < 1$; the third inequality is from $\frac{1-\beta}{1-\alpha} < 1$; the fourth inequality is from $\ln\beta > \ln\alpha$. The last two equations are from the definition of $\tilde{r}_b$ and $r_{b,g}$.

Combine $G(r_{b,f}) < G(r_{b,g})$ and $G'(r) > 0$, $r_{b,f} < r_{b,g}$. Combine with the previous step, $\tilde{\epsilon}_b < \check{\epsilon}_0$.

**Step 2.** Prove that $\tilde{\epsilon}_b > \check{\epsilon}_1$, where $\tilde{\epsilon}_b$ denotes the solution of equation (107) and (106). Let $\check{\epsilon}_1 \doteq \frac{\check{\lambda}_1}{\check{\lambda}_1+\nu}$.

We prove $\tilde{\epsilon}_b > \check{\epsilon}_1$ by contradiction. Suppose $\tilde{\epsilon}_b \leq \check{\epsilon}_1$. From the construction of hypothetical equilibrium $(\tilde{\epsilon}_b, \tilde{r}_b, \tilde{e}_b)$ and FOC of $\check{x}_1$ in equation (99),

$$
\tilde{\epsilon}_b c_e \tilde{e}_b = \beta > \alpha + (\beta-\alpha)\frac{\check{t}_1}{1+\check{t}_1} = \check{\epsilon}_1 c_e \check{e}_1, \tag{112}
$$

where the inequality is from $\check{t}_1 \doteq \frac{A_g}{A_\pi}\left(\frac{\hat{e}_{b,1}}{\hat{x}_{b,1}}\right)^{(1-\beta)-(1-\alpha)} > 0$.

Combine (112) with $\tilde{\epsilon}_b \leq \check{\epsilon}_1$,

$$
\tilde{e}_b > \check{e}_1 \tag{113}
$$

From $\epsilon \doteq \frac{\lambda}{\lambda+\nu}$, $\tilde{\lambda}_b \leq \check{\lambda}_1$.

From equation (99) and equation (107),

$$
\begin{aligned}
\frac{c_x \check{r}_1}{c_e \check{\lambda}_1} =& \frac{(1-\alpha) - (\beta-\alpha)\frac{\check{t}_1}{1+\check{t}_1}}{\alpha + (\beta-\alpha)\frac{\check{t}_1}{1+\check{t}_1}} \\
>& \frac{(1-\beta)}{\beta} \\
=& \frac{c_x \tilde{r}_b}{c_e \tilde{\lambda}_b} \\
\Rightarrow \frac{\check{r}_1}{\tilde{r}_b} >& \frac{\check{\lambda}_1}{\tilde{\lambda}_b} \geq 1 \\
\Rightarrow \check{r}_1 >& \tilde{r}_b
\end{aligned}
$$

Combine $\tilde{\epsilon}_b \leq \check{\epsilon}_1$ and $\check{r}_1 > \tilde{r}_b$,

$$
\begin{aligned}
c_e \check{e}_1 - \ln \check{e}_1 =& \ln(\check{\epsilon}_1) + \ln\left(A_\pi (\check{r}_1)^{(1-\alpha)} + A_g (\check{r}_1)^{(1-\beta)}\right) \\
>& \ln(\tilde{\epsilon}_b) + \ln\left(A_\pi (\tilde{r}_b)^{(1-\alpha)} + A_g (\tilde{r}_b)^{(1-\beta)}\right) \\
=& c_e \tilde{e}_b - \ln \tilde{e}_b
\end{aligned}
$$

From regularity assumptions, in equilibrium $c_e e - \ln e$ increases in $e$ such that the effort provision increases in her wage as derived from the participation constraint. Therefore, $c_e \check{e}_1 - \ln \check{e}_1 > c_e \tilde{e}_b - \ln \tilde{e}_b$ implies $\check{e}_1 > \tilde{e}_b$. This contradicts equation (113).

These two steps imply $\check{\epsilon}_0 > \tilde{\epsilon}_b > \check{\epsilon}_1$. Then $\check{\epsilon}_0 > \check{\epsilon}_1 \Rightarrow \check{\lambda}_0 > \check{\lambda}_1$. From equations (96) and (99),

$$
\begin{aligned}
c_x \check{x}_0 =& (1-\alpha)\left(\check{\lambda}_0 + \nu\right) \\
<& \left((1-\alpha) - (\beta-\alpha)\frac{\check{t}_1}{1+\check{t}_1}\right)\left(\check{\lambda}_0 + \nu\right) \\
<& \left((1-\alpha) - (\beta-\alpha)\frac{\check{t}_1}{1+\check{t}_1}\right)\left(\check{\lambda}_1 + \nu\right) = c_x \check{x}_1
\end{aligned}
$$

where the first inequality arises from $(1-\alpha) > (1-\beta)$ and $\check{t}_1 \doteq \frac{A_g}{A_\pi}\left(\frac{\hat{e}_{b,1}}{\hat{x}_{b,1}}\right)^{(\beta-\alpha)} > 0$. Hence, $\check{x}_0 < \check{x}_1$. □

**Proof of Proposition 8-(c).**

Let $\check{r}_0 \doteq \frac{\check{x}_0}{\check{e}_0}$. From the characterization of laissez-faire equilibrium in (97) and (98),

$$
\begin{aligned}
\ln\left(\frac{\check{\lambda}_0 \check{e}_0}{\check{\lambda}_0 + \nu}\right) + \ln\left(A_\pi \check{r}_0^{1-\alpha}\right) = c_e \check{e}_0 \\
\frac{\check{\lambda}_0 \check{e}_0 c_e}{\check{\lambda}_0 + \nu} = \alpha
\end{aligned}
$$

Combine the two equations, equilibrium $\check{r}_0$ uniquely solves the following equation

$$F_0(\check{r}_0) \doteq \ln\left(\frac{\alpha A_\pi}{c_e}\right) + (1-\alpha)\ln\check{r}_0 - \alpha - \frac{\nu(1-\alpha)c_e}{c_x\check{r}_0} = 0 \tag{114}$$

From the characterization of equilibrium $(\check{e}_1, \check{x}_1)$ and let $\check{r}_1 \doteq \frac{\check{x}_1}{\check{e}_1}$,

$$\ln\left(\frac{\check{\lambda}_0\check{e}_0}{\check{\lambda}_0+\nu}\right) + \ln\left(A_\pi\check{r}_0^{1-\alpha} + A_g\check{r}_0^{1-\beta}\right) = c_e\check{e}_0,$$

$$\frac{\check{\lambda}_1\check{e}_1c_e}{\check{\lambda}_1+\nu} = R_e\left(\check{r}_1\right),$$

$$R_e\left(\check{r}_1\right) \doteq \frac{\alpha+\beta\frac{A_g}{A_{]\pi}}\check{r}_1^{\alpha-\beta}}{1+\frac{A_g}{A_{]\pi}}\check{r}_1^{\alpha-\beta}} \tag{115}$$

The equation uniquely characterizing $\check{r}_1$ is thus given by

$$F_1(\check{r}_1) \doteq \ln\left(\frac{R_e(\check{r}_1)A_\pi}{c_e}\right) + (1-R_e(\check{r}_1))\ln\check{r}_1 + \ln\left(1+\frac{A_g}{A_\pi}\check{r}_1^{\alpha-\beta}\right) - R_e(\check{r}_1) - \frac{\nu(1-R_e(\check{r}_1))c_e}{c_x\check{r}_0} = 0$$

where $R_e(\check{r}_1)$ is given in equation (115). Note that $F_1'(r) > 0$.

From FOC characterizations in laissez-faire equilibrium and equilibrium under the Pigouvian subsidy,

$$\frac{c_x\check{r}_0}{c_e} = \frac{1-\alpha}{\alpha}\check{\lambda}_0$$

$$\frac{c_x\check{r}_1}{c_e} = \frac{1-R_e\left(\check{r}_1\right)}{R_e\left(\check{r}_1\right)}\check{\lambda}_1$$

where $R_e(\check{r}_1)$ is given in equation (115). The above equations imply

$$\check{\lambda}_0 > \check{\lambda}_1 \Leftrightarrow \check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right) > \check{r}_1. \tag{116}$$

Because $F_1(z)$ increases in $z$ and $F_1(\check{r}_1) = 0$, inequality (116) is equivalent to

$$F_1\left(\check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)\right) > 0.$$

Let $\tilde{R}_e \doteq R_e\left(\check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)\right)$ and $\ddot{r} \doteq \check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)$. Because $\ln R_e - R_e$ increases in $R_e \in (0,1)$ and $\tilde{R}_e \in (\alpha, \beta)$,

$$
\begin{aligned}
F_1\left(\check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)\right) =& \ln\left(\frac{A_\pi \tilde{R}_e}{c_e}\right) - \tilde{R}_e - \frac{\nu\left(1-\tilde{R}_e\right)c_e}{c_x \check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)},\\
&+(1-\alpha)\ln\left(\check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)\right) + \ln\left(1+\frac{A_g}{A_\pi}\ddot{r}^{\alpha-\beta}\right),\\
>& \ln\left(\frac{\alpha A_\pi}{c_e}\right) - \alpha - \frac{\nu\left(1-\tilde{R}_e\right)c_e}{c_x \check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)},\\
&+(1-\alpha)\ln\left(\check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)\right) + \ln\left(1+\frac{A_g}{A_\pi}\ddot{r}^{\alpha-\beta}\right),\\
>& \ln\left(\frac{\alpha A_\pi}{c_e}\right) - \alpha - \frac{\nu(1-\alpha)c_e}{c_x \check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)},\\
&+(1-\alpha)\ln\left(\check{r}_0\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)\right) + \ln\left(1+\frac{A_g}{A_\pi}\ddot{r}^{\alpha-\beta}\right),\\
=& -(1-\alpha)\ln\check{r}_0 + \frac{\nu(1-\alpha)c_e}{c_x\check{r}_0}\left(1-\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right),\\
&+(1-\alpha)\ln(\check{r}_0) + (1-\alpha)\ln\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right) + \ln\left(1+\frac{A_g}{A_\pi}\ddot{r}^{\alpha-\beta}\right),\\
=& \frac{\nu(1-\alpha)c_e}{c_x\check{r}_0}\left(1-\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right) + (1-\alpha)\ln\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right) + \ln\left(1+\frac{A_g}{A_\pi}\ddot{r}^{\alpha-\beta}\right),\\
=& \frac{\nu(1-\alpha)c_e}{c_x\check{r}_0}\left(1-\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right) + \ln\left(\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1-\alpha} + \frac{A_g}{A_\pi}\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1-\beta}\check{r}_0^{\alpha-\beta}\right),\\
=& \left[\ln\left(\frac{\alpha A_\pi}{c_e}\right) + (1-\alpha)\ln\check{r}_0 - \alpha\right]\left(1-\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right),\\
&+\ln\left(\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1-\alpha} + \frac{A_g}{A_\pi}\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1-\beta}\check{r}_0^{\alpha-\beta}\right),\\
>& \left[\ln\left(\frac{\alpha A_\pi}{c_e}\right) + (1-\alpha)\ln\check{r}_0 - \alpha\right]\left(1-\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right) + \ln\left(\frac{A_g}{A_\pi}\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1-\beta}\check{r}_0^{\alpha-\beta}\right), \quad (117)\\
\geq& \left[\ln\left(\frac{\alpha A_\pi}{c_e}\right) + (1-\alpha)\ln\check{r}_0 - \alpha\right]\left(1-\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right),\\
\geq& 0,
\end{aligned}
$$

where the first inequality is from $\ln R_e - R_e$ increases in $R_e \in (0,1)$, the second inequality is from $1-\tilde{R}_e < 1-\alpha$, and the second equation is because $(1-\alpha)\ln\check{r}_0 = \alpha + \frac{\nu(1-\alpha)c_e}{c_x\check{r}_0} - \ln\left(\frac{\alpha A_\pi}{c_e}\right)$ transformed from (114). (117) is because $\frac{\alpha(1-\beta)}{\beta(1-\alpha)}^{1-\alpha} > 0$ and $\ln(\cdot)$ is increasing function. The second last inequality is because $\check{r}_0 \leq 1$ derived from inequality (118) and $\frac{A_g}{A_\pi}\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1-\beta} \geq 1$ from the second given condition in Proposition 8-c. The final inequality is because $\frac{\beta(1-\alpha)}{\alpha(1-\beta)} > 1$ and

$$\ln\left(\frac{\alpha A_\pi}{c_e}\right) + (1-\alpha)\ln \check{r}_0 - \alpha \le \ln\left(\frac{\alpha A_\pi}{c_e}\right) - \alpha \le 0$$

Plug $z = 1$ into function $F_0(z)$ defined in equation (114),

$$\begin{aligned} F_0(1) &= \ln\left(\frac{\alpha A_\pi}{c_e}\right) + (1-\alpha)\ln 1 - \alpha - \frac{\nu\,(1-\alpha)\,c_e}{c_x}, \\ &= \ln\left(\frac{\alpha A_\pi}{c_e}\right) - \alpha - \frac{\nu\,(1-\alpha)\,c_e}{c_x} < 0, \end{aligned} \tag{118}$$

Because $F_0(z)$ increases in $z$, inequality (118) leads to $\check{r}_0 < 1$ and inequality (118) is from the given condition $\ln\left(\frac{\alpha A_\pi}{c_e}\right) \le \alpha$. □

**Proof of Corollary 4.**

Given Proposition 8, $\check{\lambda}_1 < \check{\lambda}_0$ and $\check{e}_1 < \check{e}_0$ under the conditions that $\ln\left(\frac{\alpha A_\pi}{c_e}\right) \le \alpha$, $\frac{\beta(1-\alpha)}{\alpha(1-\beta)} \le \left(\frac{A_g}{A_\pi}\right)^{\frac{1}{1-\beta}}$, and $A_\pi \ge c_e$. Let $\hat{\epsilon}_1 \doteq \frac{\hat{w}_1}{\hat{Y}_1}$ and $\hat{\epsilon}_0 \doteq \frac{\hat{w}_0}{\hat{\pi}_0}$, then

$$\check{\epsilon}_1 = \frac{\check{\lambda}_1}{\check{\lambda}_1 + \nu} < \frac{\check{\lambda}_0}{\check{\lambda}_0 + \nu} = \check{\epsilon}_0.$$

The social welfare function is

$$SW\,(e, x, \epsilon) = (1+\nu)\ln Y + \nu \ln(1-\epsilon) + \ln \epsilon - e c_e - x c_x.$$

The first-order derivatives are

$$\begin{aligned} SW_\epsilon(\check{e}_0, \check{x}_0, \epsilon) &= \frac{\nu}{1-\epsilon} + \frac{1}{\epsilon} > 0,\ \forall \epsilon \in [\check{\epsilon}_1, \check{\epsilon}_0] \\ SW_{\epsilon\epsilon}(\check{e}_0, \check{x}_0, \epsilon) &\le 0,\ \forall \epsilon \in [\check{\epsilon}_1, \check{\epsilon}_0], \end{aligned}$$

where inequality is from $\check{\epsilon}_0 < \frac{1}{1+\nu}$. Because $\check{\epsilon}_1 < \check{\epsilon}_0$,

$$SW(\check{e}_0, \check{x}_0, \check{\epsilon}_1) < SW(\check{e}_0, \check{x}_0, \check{\epsilon}_0). \tag{119}$$

Similarly, let $t \doteq \frac{A_g}{A_\pi}\left(\frac{e}{x}\right)^{\beta-\alpha}$

$$\begin{aligned}
SW_e(\check{e}_0, \check{x}_0, \epsilon) &= \frac{(1+\nu)}{\check{e}_0}\left(\beta - \frac{\beta-\alpha}{1+t\left(\check{e}_0, \check{x}_0\right)}\right) - c_e \\
&\propto \beta - \frac{\beta-\alpha}{1+t\left(\check{e}_0, \check{x}_0\right)} - \frac{c_e \check{e}_0}{1+\nu} \\
&= \beta - \frac{\beta-\alpha}{1+t\left(\check{e}_0, \check{x}_0\right)} - \frac{\alpha\left(\nu+\check{\lambda}_0\right)}{1+\nu} \\
&> \beta - \frac{\beta-\alpha}{1+t\left(\check{e}_0, \check{x}_0\right)} - \alpha \\
&= (\beta-\alpha)\left(1 - \frac{1}{1+t\left(\check{e}_0, \check{x}_0\right)}\right) > 0,
\end{aligned}$$

where the inequality is because $\check{\lambda}_0 < 1$.

Then for $\forall e \in [\check{e}_0, \check{e}_1]$,

$$\begin{aligned}
SW_e\left(e, \check{x}_0, \epsilon\right) &\propto \beta - \frac{\beta-\alpha}{1+\frac{A_g e^{\beta-\alpha}}{A_\pi \check{x}_0^{\beta-\alpha}}} - \frac{c_e e}{1+\nu} \\
&\geq \beta - \frac{\beta-\alpha}{1+t\left(\check{e}_0, \check{x}_0\right)} - \frac{c_e \check{e}_0}{1+\nu} > 0 \\
SW_{ee}\left(e, \check{x}_0, \epsilon\right) &\leq 0,
\end{aligned}$$

where the last inequality of second-order derivative is from the regularity assumption that social welfare function is concave at each variable. Therefore, from $\check{e}_1 < \check{e}_0$,

(120) $$SW\left(\check{e}_1, \check{x}_0, \check{\epsilon}_1\right) < SW\left(\check{e}_0, \check{x}_0, \check{\epsilon}_1\right).$$

Take the first-order derivative of social welfare function with regard to $x$,

$$\begin{aligned}
SW_x\left(\check{e}_1, \check{x}_0, \check{\epsilon}_1\right) &= \frac{(1+\nu)}{\check{x}_1}\left(1-\beta + \frac{\beta-\alpha}{1+t\left(\check{e}_1, \check{x}_1\right)}\right) - c_x \\
&\propto 1-\beta + \frac{\beta-\alpha}{1+t\left(\check{e}_1, \check{x}_1\right)} - \frac{c_x \check{x}_1}{1+\nu} \\
&= 1-\beta + \frac{\beta-\alpha}{1+t\left(\check{e}_1, \check{x}_1\right)} - \frac{1-\beta+\frac{\beta-\alpha}{1+t(\check{e}_1, \check{x}_1)}}{(1+\nu)\frac{\check{\lambda}_1}{\check{\lambda}_1+\nu}} \\
&= \left(1-\beta + \frac{\beta-\alpha}{1+t\left(\check{e}_1, \check{x}_1\right)}\right)\left(1 - \frac{1+\frac{\nu}{\check{\lambda}_1}}{(1+\nu)}\right) < 0,
\end{aligned}$$

where the last inequality is from $\check{\lambda}_1 < 1$ and $t \doteq \frac{A_g}{A_\pi}\left(\frac{e}{x}\right)^{\beta-\alpha}$. Similarly,

$$
\begin{aligned}
SW_x\left(\check{e}_1, \check{x}_0, \check{\epsilon}_1\right) &\propto 1-\beta+\frac{\beta-\alpha}{1+\frac{A_g \check{e}_1^{\beta-\alpha}}{A_\pi \check{x}_0^{\beta-\alpha}}}-\frac{c_x \check{x}_0}{1+\nu} \\
&< 1-\alpha-\frac{c_x \check{x}_0}{1+\nu} \\
&= \frac{\check{x}_0 c_x \check{x}_0}{\check{\lambda}_0+\nu}-\frac{c_x \check{x}_0}{1+\nu} < 0.
\end{aligned}
$$

Also, from the regularity assumption of concave social welfare function, $\forall\left(e, x, \epsilon\right)$, $SW_{xx}\left(e, x, \epsilon\right) \leq 0$. Therefore, for any $x$ in the interval between $\check{x}_0$ and $\check{x}_1$,

$$
SW_x\left(\check{e}_1, x, \check{\epsilon}_1\right) \leq \sup\left\{SW_x\left(\check{e}_1, \check{x}_0, \check{\epsilon}_1\right), SW_x\left(\check{e}_1, \check{x}_1, \check{\epsilon}_1\right)\right\} < 0. \tag{121}
$$

FOC of $\check{x}_1$ is

$$
\begin{aligned}
c_x &= \frac{\check{\lambda}_1}{\left(\check{\lambda}_1+\nu\right) \hat{x}_1}\left(1-\beta+\frac{\beta-\alpha}{1+t\left(\check{e}_1, \check{x}_1\right)}\right) \\
\Rightarrow \hat{x}_1 &= \frac{\check{\lambda}_1}{\left(\check{\lambda}_1+\nu\right) c_x}\left(1-\beta+\frac{\beta-\alpha}{1+t\left(\check{e}_1, \check{x}_1\right)}\right) \\
&> \frac{\check{\lambda}_1}{\left(\check{\lambda}_1+\nu\right) c_x}(1-\alpha) \\
&> \frac{\check{\lambda}_0}{\left(\check{\lambda}_0+\nu\right) c_x}(1-\alpha) = \check{x}_1,
\end{aligned}
$$

where $t \doteq \frac{A_g}{A_\pi}\left(\frac{e}{x}\right)^{\beta-\alpha}$ and the first inequality is from $t\left(\check{e}_1, \check{x}_1\right) > 0$, and the second inequality is from $\check{\lambda}_1 < \check{\lambda}_0$.

From $\check{x}_1 > \check{x}_0$ and inequality (121), $SW\left(\check{e}_1, \check{x}_1, \check{\epsilon}_1\right) < SW\left(\check{e}_1, \check{x}_0, \check{\epsilon}_1\right)$. Combine the inequality above with (119) and (120),

$$
SW\left(\check{e}_1, \check{x}_1, \check{\epsilon}_1\right) < SW\left(\check{e}_1, \check{x}_0, \check{\epsilon}_1\right) < SW\left(\check{e}_0, \check{x}_0, \check{\epsilon}_1\right) < SW(\check{e}_0, \check{x}_0, \check{\epsilon}_0).
$$

□

**Proof of Proposition 9.**

Proposition 9 is a simplified version of Proposition 11. Below we first characterize the equilibrium of the model with information friction. Then 9 follows immediately after the proof of Proposition 11.

## Extension Model: Information Friction

### First-best allocation

Let $(w_s^*, w_f^*)$ denote the transfer to the agent given the success state and the failure. The first-best allocation maximizes the social welfare:

$$\begin{aligned} SW(e, x, w_s^*, w_f^*) \quad &\doteq \quad q(e,x)\left[V(Y_s - w_s^*) + U(w_s^*)\right] - \frac{c_e e^2}{2} \\ &\quad +(1-q(e,x))\left[V(Y_f - w_f^*) + U(w_f^*)\right] - c_x x. \end{aligned}$$

One can easily solve the first-best allocation, $(e^*, x^*, w_s^*, w_f^*)$, as follows:

$$\begin{aligned} w_s^* = \tfrac{1}{\nu+1} Y_s \quad &\text{and} \quad w_f^* = \tfrac{1}{\nu+1} Y_f, \\ e^* = \tfrac{\sigma}{c_e}\left(\tfrac{c_x}{1-\sigma}\right)^{1-\frac{1}{\sigma}} &\left(A_q\left(1+\nu\right)\ln\tfrac{Y_s}{Y_f}\right)^{\frac{1}{\sigma}}, \\ x^* = \tfrac{\sigma}{c_e}\left(\tfrac{c_x}{1-\sigma}\right)^{1-\frac{2}{\sigma}} &\left(A_q\left(1+\nu\right)\ln\tfrac{Y_s}{Y_f}\right)^{\frac{2}{\sigma}}. \end{aligned}$$

### Laissez-faire equilibrium

Below is the characterization of the laissez-faire equilibrium. The principal's problem is given in (30). Given the constraint (32) and the IC constraint 31, the ex ante participation constraint is not binding. Because the agent's ex post utility is non-negative even at failing state and the IC ensures the net expected utility is at least as high as the non-hidden-action case to induce efforts, the agent's expected utility ex ante is positive, i.e., ex ante participation constraint (33) is not binding. The IC (31) and (32) under the failing state are binding.

The Lagrangian is

$$\begin{aligned} \mathcal{L} =& \mathbb{E}\left[V\left(e, w_{s,0}, w_{f,0}\right)\right] - \frac{c_e e^2}{2} + \lambda_s \ln\left(w_{s,0} - \underline{u}\right) + \lambda_f \ln\left(w_{f,0} - \underline{u}\right) \\ &+ \lambda_c\left(p_x\left(e,x\right)\left(U\left(w_{s,0}\right) - U\left(w_{f,0}\right)\right) - c_x\right), \end{aligned}$$

where the IC constraint is transformed into its FOC with regard to $x$.[25] We can then derive the equilibrium:

[25] This follows the widely-used first-order approach in hidden action contracting literature (Holmström, 1979). In the Appendix, we verify the validity of the first-order approach in our model.

**Lemma 6.** *The laissez-faire equilibrium is characterized by*

$$\hat{w}_{s,0} = \frac{\hat{\lambda}_{c,0}(1-\rho)}{\nu\hat{x}_0 + \hat{\lambda}_{c,0}(1-\rho)}\pi_s, \quad \text{and} \quad \hat{w}_{f,0} = 1 \tag{122}$$

$$\frac{c_e\hat{e}_0^2}{\rho} = \frac{\hat{\lambda}_{c,0}c_x(\rho\nu + 1 - \rho)}{(1-\rho)}, \tag{123}$$

$$\frac{\rho\hat{\lambda}_{c,0}}{\hat{x}_0}\frac{c_x}{A_p(1-\rho)\hat{r}_0^\rho} = \ln\left(\frac{\pi_s}{\pi_f - 1}\right) + \ln\left(\frac{\nu\hat{x}_0}{\nu\hat{x}_0 + \hat{\lambda}_{c,0}(1-\rho)}\right), \tag{124}$$

$$\frac{\hat{\lambda}_{c,0}(1-\rho)}{\nu\hat{x}_0 + \hat{\lambda}_{c,0}(1-\rho)}\pi_s = \exp\left(\frac{c_x}{A_p(1-\rho)\hat{r}_0^\rho}\right), \tag{125}$$

$$\text{where } \hat{r}_0 \doteq \frac{\hat{e}_0}{\hat{x}_0},$$

*and $\lambda_s = 0$ and $\lambda_f > 0$, where $\lambda_s$ and $\lambda_f$ denote the multipliers of constraint (32) under the success state and failure state and $\hat{\lambda}_{c,0}$ denote the Lagrangian multiplier of constraint (31).*

The contract in laissez-faire has standard features in the model hazard literature: the IC constraint is binding, and the participation constraint is binding at the failure state but not at the success state.

**Pigouvian Subsidy**

The Lagrangian is

$$\begin{aligned}\mathcal{L} =& \mathbb{E}\left[V\left(e, w_{s,1}, w_{f,1}\right)\right] - \frac{c_e e^2}{2} + \lambda_s \ln\left(w_{s,1} - \underline{u}\right) + \lambda_f \ln\left(w_{f,1} - \underline{u}\right) \\ &+ \lambda_c\left(q_x(e,x)\left(U\left(w_{s,1}\right) - U\left(w_{f,1}\right)\right) - c_x\right),\end{aligned}$$

where the IC is transformed into its first-order condition with regard to $x$.

**Lemma 7.** *The equilibrium under the Pigouvian subsidy is characterized by*

$$\hat{w}_{s,1} = \frac{\hat{\lambda}_{c,1}(1-\sigma)}{\nu\hat{x}_1 + \hat{\lambda}_{c,1}(1-\sigma)}Y_s, \quad \text{and} \quad \hat{w}_{f,1} = 1 \tag{126}$$

$$\frac{c_e\hat{e}_1^2}{\sigma} = \frac{\hat{\lambda}_{c,1}c_x(\sigma\nu + 1 - \sigma)}{(1-\sigma)}, \tag{127}$$

$$\frac{\sigma\hat{\lambda}_{c,1}}{\hat{x}_1}\frac{c_x}{A_q(1-\sigma)\hat{r}_1^\sigma} = \ln\left(\frac{Y_s}{Y_f - 1}\right) + \ln\left(\frac{\nu\hat{x}_1}{\nu\hat{x}_1 + \hat{\lambda}_{c,1}(1-\sigma)}\right), \tag{128}$$

$$\frac{\hat{\lambda}_{c,1}(1-\sigma)}{\nu\hat{x}_1 + \hat{\lambda}_{c,1}(1-\sigma)}Y_s = \exp\left(\frac{c_x}{A_q(1-\sigma)\hat{\lambda}_{c,1}^\sigma}\right), \tag{129}$$

$$\text{where } \hat{r}_1 \doteq \frac{\hat{e}_1}{\hat{x}_1},$$

*and $\lambda_s = 0$ and $\lambda_f > 0$, where $\lambda_s$ and $\lambda_f$ denote the multipliers of constraint (35) under the success state and failure state and $\hat{\lambda}_{c,1}$ denote the Lagrangian multiplier of constraint (34).*

Similarly to the laissez-faire equilibrium, the ex ante participation constraint is not binding, and the constraint (35) is binding at the failure state but not at the success state, and the IC constraint is binding.

### Pigouvian subsidy backfire

We now compare the two outcomes above. Let $\hat{\zeta}_0 \doteq \frac{\hat{w}_0}{\pi_s}$ and $\zeta_1 \doteq \frac{\hat{w}_1}{Y_s}$ denote the equity share under the laissez-faire equilibrium and the Pigouvian subsidy, respectively. For simplicity, assume the failure state yields $Y_f = 2$ and $\pi_f = 2$ in the following analyses.

Assume that the knowledge spillovers is not very high, i.e., the difference between $Y_s$ and $\pi_s$ is not large, and the researcher's contribution to the successful commercialization is limited. Formally,

$$\frac{\pi_s (2-\sigma)}{1-\sigma} > \frac{Y_s (2-\rho)}{1-\rho} \tag{130}$$

$$\rho\nu < 1. \tag{131}$$

Given the conditions (130) and (131), we characterize the economic environment such that the Pigouvian subsidy reduces the agent's equity share, which requires $Y_s$ close to $\pi_s$. The formal statement is as follows.

**Proposition 10.** *Given the economic environment (130) and (131), the agent's equilibrium equity shares decrease by the Pigouvian subsidy, i.e., $\hat{\zeta}_1 < \hat{\zeta}_0$, if the following conditions hold,*

$$\left(\frac{\pi_s (1-\rho)}{2-\rho}\right)^{\rho} < \left(\frac{Y_s (1-\sigma)}{2-\sigma}\right)^{\sigma}, \tag{132}$$

$$\left(\frac{(1-\rho)\,\bar{d}}{\nu}\right)^{\frac{\rho\bar{d}}{1-\rho\bar{d}}} > \pi_s \left(1 + \frac{(1-\rho)\,\bar{d}}{\nu}\right), \tag{133}$$

$$\textit{where } \bar{d} \doteq \frac{\nu}{Y_s - \pi_s}\left(\frac{\pi_s}{1-\sigma} - \frac{Y_s}{1-\rho}\right),$$

The sufficient conditions for the Pigouvian subsidy to reduce the agent's equity share imply that the social payoff is not very larger than the startup's payoff obtained from product market success. Conditions 132 and 133 strengthen the economic environment conditions 130 and 131.

Below first analyzes the economic environment for the Pigouvian subsidy to reduce research efforts. Strikingly, a more considerable role of research efforts, i.e., a large $\sigma$, could make the Pigouvian subsidy more likely to backfire.

**Proposition 11.** *The Pigouvian subsidy reduces research efforts, i.e., $\hat{e}_1 < \hat{e}_0$, under (130), (131),* *(132), (133) and the condition below,*

$$1 < \frac{A_q}{A_p} < \left(\frac{A_q}{c_x}\right)^{1-\frac{\rho}{\sigma}}. \tag{134}$$

The proposition above implies that even though the productivity of technology knowledge is larger than that of the commercial product, i.e., $A_q > A_p$, the Pigouvian subsidy compensates knowledge spillovers could reduce research efforts if the marginal effort cost of entrepreneur's business expansion effort is low, i.e., $c_x$ is small. This is because the Pigouvian subsidy favors the principal, the researcher, who exploits the favorable decision right to substitute own efforts with the entrepreneur's efforts.

Additionally, if the researcher's effort contributes very small to the product market or contributes dramatically to the knowledge spillovers, i.e., a low-valued $\rho$ or a large $\beta$, condition (134) is likely to hold, and strikingly, the Pigouvian subsidy still reduces researcher efforts by giving researcher more favorable decision right which enables him to exploit the agent more. However, intuitively, when the research efforts are influential to the success in social value, i.e., a large $\beta$, the research efforts reduction is more detrimental to the social welfare. Worse still, the government is more likely to subsidize under such circumstances. It thus demonstrates the policy relevance and significance of our argument. The intuitive common wisdom received by many governments to subsidize technology startups could backfire and call for a comprehensive review of the internal organizational structure of startups.

By comparing the outcomes above, we first show that the Pigouvian subsidy reduces the agent's equilibrium equity share and then demonstrate that the R&D efforts reduce simultaneously. In other words, the Pigouvian subsidy reduces the research effort and worsens production efficiency and inequality across players.

## Proof of Lemma 6

The participation constraint binds at the failure state, i.e., $U\left(w_{f,0}\right) = \underline{u}$ and hence $w_{f,0} = 1$.

By FOCs of the Lagrangian, we characterize the equilibrium. The FOC of the IC constraint (31),

$$c_x = A_p\left(1-\alpha\right)\hat{r}_0^{\alpha}\ln w_s, \tag{135}$$

$$\text{where } \hat{r}_0 \doteq \frac{\hat{e}_0}{\hat{x}_0}.$$

The FOC of $w_s$ deriving the agent's equilibrium equity share,

$$\begin{aligned}\frac{\nu p(e,x)}{\pi_s - w_s} &= \frac{\mu p_x(e,x)}{w_s} \\ \Leftrightarrow \frac{\nu A_p r^{\alpha} x}{\pi_s - w_s} &= \frac{\mu A_p (1-\alpha)\hat{r}_0^{\alpha}}{w_s} \\ \Leftrightarrow w_s &= \frac{\mu(1-\alpha)\pi_s}{\nu x + \mu(1-\alpha)}\end{aligned}$$

The FOC of $x$ is

$$\begin{aligned}x \ln\left(\frac{\pi_s - w_s}{\pi_f - 1}\right) &= \mu\alpha \ln w_s \\ \Rightarrow \frac{\ln\left(\frac{\pi_s - w_s}{\pi_f - 1}\right)}{\ln w_s} &= \frac{\mu\alpha}{x}\end{aligned}$$

FOC with regard to $e$ is

$$\begin{aligned}A_p\alpha\frac{r^{\alpha}}{e}\left[x\nu \ln\left(\frac{\pi_s - w_s}{\pi_f - 1}\right) + \mu(1-\alpha)\ln w_s\right] &= c_e e \\ \Rightarrow \left(x\nu\frac{\mu\alpha}{x} + \mu(1-\alpha)\right)\ln w_s &= \frac{c_e e^2}{A_p\alpha r^{\alpha}} \\ \Rightarrow \frac{\mu(\alpha\nu + 1 - \alpha)c_x}{A_p(1-\alpha)r^{\alpha}} &= \frac{c_e e^2}{A_p\alpha r^{\alpha}} \\ \Rightarrow \frac{\mu(\alpha\nu + 1 - \alpha)c_x}{(1-\alpha)} &= \frac{c_e e^2}{\alpha},\end{aligned}$$

where the second equation is from plugging (124) and (122) in the first equation. □The proof of 7 is similar as above.

**Proof of Proposition 10.**

Let $\hat{k}_0 \doteq \frac{\hat{\lambda}_{c,0}}{\hat{x}_0}$ and $\hat{k}_1 \doteq \frac{\hat{\lambda}_{c,1}}{\hat{x}_1}$, where $\lambda_{c,0},\ \lambda_{c,1}$ denote the Langrangian multiplier of the incentive compatibility constraint in laissez-faire equilibrium and the equlbrium with the Pigouvian subsidy. The first step shows that $\hat{k}_0 > \hat{k}_1$ is sufficient for $\hat{\zeta}_0 > \hat{\zeta}_1$, where $\zeta$ denotes the agent's equilibrium equity share and $\mu$ denote the Lagrangian multiplier of incentive-compatibility constraint. Second, we transform the equilibrium characterizations into equations with $\hat{k}_0$ and $\hat{k}_1$. Third, we demonstrate the necessary and sufficient condition of $\hat{k}_0 > \hat{k}_1$ from which we prove the sufficiency in the propostion.

**Step 1:** Prove that , $\hat{\zeta}_0 > \hat{\zeta}_1$ is equivalent to $\hat{k}_0 > \hat{k}_1$

From the characterization lemmas,

$$\begin{aligned}\hat{\zeta}_0 \doteq \frac{\hat{w}_{\xi,0}}{\pi_\xi} &= \frac{\hat{k}_0\,(1-\rho)}{\nu+\hat{k}_0\,(1-\rho)}\\ \hat{\zeta}_1 \doteq \frac{Y_\xi}{\hat{w}_1} &= \frac{\hat{k}_1(1-\sigma)}{\nu+\hat{k}_1(1-\sigma)}.\end{aligned}$$

Then $\hat{k}_0 > \hat{k}_1$ leads to the following condition,

$$\begin{aligned}\hat{\zeta}_0 =& \frac{1}{\frac{\nu}{\hat{k}_0(1-\rho)}+1}\\ >& \frac{1}{\frac{\nu}{\hat{k}_0(1-\sigma)}+1}\\ >& \frac{1}{\frac{\nu}{\hat{k}_1(1-\sigma)}+1} = \hat{\zeta}_1,\end{aligned}$$

where $k \doteq \frac{\lambda_c}{x}$ and the first inequality is from $\rho < \sigma$.

**Step 2:** Equilibrium characterization with variables $\hat{k}_0$ and $\hat{k}_1$.

The laissez-faire equilibrium is characterized by the following equation system

$$\begin{cases}\hat{e_0}^2 &= \frac{\hat{\lambda}_{c,0}(\rho\nu+1-\rho)\rho c_x}{(1-\rho)c_e}\\ \frac{\rho\hat{\lambda}_{c,0}}{\hat{x}_0} &= \frac{\ln\left(\frac{\pi_\xi-\hat{w}_{\xi,0}}{\pi_f-1}\right)}{\ln\hat{w}_{\xi,0}}\\ \hat{w}_{\xi,0} &= \frac{\hat{\lambda}_{c,0}(1-\rho)}{\nu\hat{x}_0+\hat{\lambda}_{c,0}(1-\rho)}\pi_\xi = \exp\left(\frac{c_x}{A_p(1-\rho)\hat{r}_0^\rho}\right).\end{cases}$$

Plug $k \doteq \frac{\lambda_c}{x}$ into the second and third equations,

$$\begin{aligned}\frac{\rho\lambda_c}{x} &= \rho k\\ \ln\left(\frac{\pi_\xi-w_\xi}{\pi_f-1}\right) &= \ln\frac{\pi_\xi}{\pi_f-1}+\ln\frac{\nu\hat{x}_0}{\nu\hat{x}_0+\hat{\lambda}_{c,0}\,(1-\rho)}\\ &= \ln\frac{\pi_\xi}{\pi_f-1}+\ln\frac{\nu}{\nu+k\,(1-\rho)}\\ \ln w_\xi &= \ln\pi_\xi+\ln\frac{k\,(1-\rho)}{\nu+k\,(1-\rho)}\\ \Rightarrow \rho k\left(\ln\pi_\xi+\ln\frac{k\,(1-\rho)}{\nu+k\,(1-\rho)}\right) &= \ln\frac{\pi_\xi}{\pi_f-1}+\ln\frac{\nu}{\nu+k\,(1-\rho)}\\ \left(\pi_\xi\frac{k\,(1-\rho)}{\nu+k\,(1-\rho)}\right)^{\rho k} &= \frac{\pi_\xi}{\pi_f-1}\frac{\nu}{\nu+k\,(1-\rho)}\end{aligned}$$

Assume $\pi_f = 2 = Y_f$ for simplicity.

$$\nu \pi_\xi^{1-\rho k} = (\nu + k(1-\rho))^{1-\rho k} (k(1-\rho))^{\rho k},$$

$$\Leftrightarrow \nu \pi_\xi^{1-\rho k} = (\nu + k(1-\rho)) \left( \frac{1}{\frac{\nu}{k(1-\rho)} + 1} \right)^{\rho k}. \tag{136}$$

$\hat{k}_0$ is uniquely solved by equation (136) and similarly, $\hat{k}_1$ is uniquely solved by

$$\nu Y_\xi^{1-\sigma k} = (\nu + k(1-\sigma)) \left( \frac{1}{\frac{\nu}{k(1-\sigma)} + 1} \right)^{\sigma k}. \tag{137}$$

**Step 3:** Sufficient and necessary condition for $\hat{k}_0 > \hat{k}_1$

Similarly, let $LS(k;\sigma,Y_s)$ and $RS(k;\sigma,Y_s)$ denote the left-hand side and the right-hand side of equation (137). $\nu Y_\xi^{1-\sigma k}$ decreases in $k$ and the monotonicity of $RS(k;\sigma,Y_s)$ with regard to $k$ are shown as below.

$$\begin{aligned}
\frac{dRS(k;\sigma,Y_s)}{dk} &= \frac{1-\sigma}{\nu + (1-\sigma)k} + \frac{\sigma\nu}{\nu + (1-\sigma)k} - \sigma \ln\left( \frac{\nu}{(1-\sigma)k} + 1 \right) \\
&= \frac{1 + \sigma(\nu - 1)}{\nu + (1-\sigma)k} + \sigma \ln\left( \frac{(1-\sigma)k}{\nu + (1-\sigma)k} \right) \\
&\propto \frac{\frac{1}{\sigma} + (\nu - 1)}{\nu + (1-\sigma)k} + \ln\left( \frac{(1-\sigma)k}{\nu + (1-\sigma)k} \right) \\
&= \frac{\frac{1}{\sigma} + (\nu - 1)}{\nu + (1-\sigma)k} - \ln\left( 1 + \frac{\nu}{(1-\sigma)k} \right) \\
&> \lim_{k\to\infty} \left[ \frac{\frac{1}{\sigma} + (\nu - 1)}{\nu + (1-\sigma)k} - \ln\left( 1 + \frac{\nu}{(1-\sigma)k} \right) \right] = 0,
\end{aligned}$$

where the last inequality is because $\frac{\frac{1}{\sigma}+(\nu-1)}{\nu+(1-\sigma)k} - \ln\left(1 + \frac{\nu}{(1-\sigma)k}\right)$ decreases in $k$.

Therefore, $LS(k;\sigma,Y_s) - RS(k;\sigma,Y_s)$ decreases in $k$, and $\hat{k}_0 > \hat{k}_1$ is equivalent to

$$LS(\hat{k}_0;\sigma,Y_s) - RS(\hat{k}_0;\sigma,Y_s) < LS(\hat{k}_1;\sigma,Y_s) - RS(\hat{k}_1;\sigma,Y_s) = 0$$

Plug in $\hat{k}_0$ in to $LS(\hat{k}_0;\sigma,Y_s) - RS(\hat{k}_0;\sigma,Y_s) < 0$ and rearrange it.

$$
\begin{aligned}
LS(\hat{k}_0;\sigma,Y_s) <& RS(\hat{k}_0;\sigma,Y_s),\\
&\Leftrightarrow \frac{\nu Y_\xi^{1-\sigma\hat{k}_0}}{\nu\pi_s^{1-\rho\hat{k}_0}} < \frac{RS(\hat{k}_0;\sigma,Y_s)}{\left(\nu+\hat{k}_0\left(1-\rho\right)\right)\left(\frac{1}{\frac{\nu}{\hat{k}_0(1-\rho)}+1}\right)^{\rho\hat{k}_0}},\\
&\Leftrightarrow \frac{Y_s}{\pi_s}\cdot\left(\frac{Y_s^\sigma}{\pi_s^\rho}\right)^{-\hat{k}_0} < \frac{\nu+\hat{k}_0\left(1-\sigma\right)}{\nu+\hat{k}_0\left(1-\rho\right)}\left(\frac{\left(1+\frac{\nu}{(1-\sigma)\hat{k}_0}\right)^\sigma}{\left(1+\frac{\nu}{(1-\rho)\hat{k}_0}\right)^\rho}\right)^{-\hat{k}_0},
\end{aligned}
$$

$$
\Leftrightarrow \frac{\frac{Y_s}{\nu+\hat{k}_0(1-\sigma)}}{\frac{\pi_s}{\nu+\hat{k}_0(1-\rho)}}\cdot\left(\frac{\left(\frac{Y_s}{1+\frac{\nu}{(1-\sigma)\hat{k}_0}}\right)^\sigma}{\left(\frac{\pi_s}{1+\frac{\nu}{(1-\rho)\hat{k}_0}}\right)^\rho}\right)^{-\hat{k}_0} < 1. \tag{138}
$$

where the first transformation is to divide each side by the left-hand side and the right-hand side of equation (136), and it holds due to $\hat{k}_0$ as the solution of (136).

Last, we derive the sufficient condition for (138). First, equation (138) holds given that each ratio in (138) is no more than 1, i.e.,

$$
\begin{aligned}
\frac{Y_s}{\nu+\hat{k}_0\left(1-\sigma\right)} &\leq \frac{\pi_s}{\nu+\hat{k}_0\left(1-\rho\right)}\\
\left(\frac{Y_s}{1+\frac{\nu}{(1-\sigma)\hat{k}_0}}\right)^\sigma &\geq \left(\frac{\pi_s}{1+\frac{\nu}{(1-\rho)\hat{k}_0}}\right)^\rho.
\end{aligned}
$$

Rearrange each inequality above, and

$$
\hat{k}_0 \leq \frac{\nu}{Y_s-\pi_s}\left(\frac{\pi_s}{1-\sigma}-\frac{Y_s}{1-\rho}\right), \tag{139}
$$

$$
\left(\frac{\pi_s}{1+\frac{\nu}{(1-\rho)\hat{k}_0}}\right)^\rho \leq \left(\frac{Y_s}{1+\frac{\nu}{(1-\sigma)\hat{k}_0}}\right)^\sigma, \tag{140}
$$

where condition (130) ensures (139) is well-defined.

Then we show that, under (131), $\hat{k}_0 > \nu$ is sufficient for (140), i.e., Under (139),

$$
\hat{k}_0 > \nu \Rightarrow \left(\frac{\pi_s}{1+\frac{\nu}{(1-\rho)\hat{k}_0}}\right)^\rho \leq \left(\frac{Y_s}{1+\frac{\nu}{(1-\sigma)\hat{k}_0}}\right)^\sigma. \tag{141}
$$

Below is the proof of (141). Let $x \doteq \frac{\nu}{k}$, inequality (140) is represented by

$$
\rho\ln\left(\frac{\pi_s\left(1-\rho\right)}{1-\rho+\hat{x}_0}\right) \leq \sigma\ln\left(\frac{Y_s\left(1-\sigma\right)}{1-\sigma+\hat{x}_0}\right).
$$

Let $W(x) \doteq \rho \ln(\pi_s(1-\rho)) - \sigma \ln(Y_s(1-\sigma)) + \sigma \ln(1-\sigma+x) - \rho \ln(1-\rho+x)$. $W(x)$ increases in $x$ because

$$W'(x) = \frac{\sigma}{1-\sigma+x} - \frac{\rho}{1-\rho+x} > 0, \quad \forall x \geq 0.$$

Thus, given $\hat{k}_0 > \nu$, $\hat{x}_0 < 1$ and hence

$$\begin{aligned}
W(\hat{x}_0) &< \rho \ln(\pi_s(1-\rho)) - \sigma \ln(Y_s(1-\sigma)) + \sigma \ln(2-\sigma) - \rho \ln(2-\rho) \\
&< \ln\left(\frac{\pi_s(1-\rho)}{2-\rho}\right)^{\rho} - \ln\left(\frac{Y_s(1-\sigma)}{2-\sigma}\right)^{\sigma} < 0,
\end{aligned}$$

where the last inequality is From (139).

Combine with (139)

$$\nu < \hat{k}_0 < \frac{\nu}{Y_s - \pi_s}\left(\frac{\pi_s}{1-\sigma} - \frac{Y_s}{1-\rho}\right). \tag{142}$$

Then we show that conditions (132) and (133) are sufficient for (142).

Let $R(k)$ denote the right-hand side of equation (136), i.e., $R(k) \doteq \ln(\nu + (1-\rho)k) - \rho k \ln\left(\frac{\nu}{(1-\rho)k} + 1\right)$.

Take derivative of $R(k)$ as follows

$$\begin{aligned}
R'(k) &= \frac{1-\rho}{\nu+(1-\rho)k} + \frac{\rho\nu}{\nu+(1-\rho)k} - \rho \ln\left(\frac{\nu}{(1-\rho)k} + 1\right) \\
&= \frac{1-\rho+\rho\nu}{\nu+(1-\rho)k} - \rho \ln\left(\frac{\nu}{(1-\rho)k} + 1\right) \\
&= \frac{1-\rho+\rho\nu}{\nu+(1-\rho)k} - \rho \ln(\nu+(1-\rho)k) + \rho \ln((1-\rho)k) \\
&= \frac{1+\rho(\nu-1)}{\nu+(1-\rho)k} + \rho \ln\left(\frac{(1-\rho)k}{\nu+(1-\rho)k}\right) \\
&\propto \frac{\frac{1}{\rho}+(\nu-1)}{\nu+(1-\rho)k} + \ln\left(\frac{(1-\rho)k}{\nu+(1-\rho)k}\right) \\
&= \frac{\frac{1}{\rho}+(\nu-1)}{\nu+(1-\rho)k} - \ln\left(1 + \frac{\nu}{(1-\rho)k}\right) \\
&> \lim_{k\to\infty}\left[\frac{\frac{1}{\rho}+(\nu-1)}{\nu+(1-\rho)k} - \ln\left(1 + \frac{\nu}{(1-\rho)k}\right)\right] = 0,
\end{aligned}$$

where the last inequality is because $\frac{\frac{1}{\rho}+(\nu-1)}{\nu+(1-\rho)k} + \ln\left(\frac{(1-\rho)k}{\nu+(1-\rho)k}\right)$ decreases in $k$.

Therefore, $R'(k) > 0$. equation (136) characterizing $\hat{k}_0$ is represented as $\nu\pi_s^{1-\rho k} = R(k)$. Let $NS(k) \doteq \ln\nu + (1-\rho k)\ln\pi_s - R(k)$.

$NS(k)$ decreases in $k$ because $(1-\rho k)\ln\pi_s$ decreases in $k$ under $\pi_s > 2$ and $R(k)$ increases in

$k$. Thus, condition (142) is equivalent to

$$NS(\nu) > NS(\hat{k}_0) > NS\left(\frac{\nu}{Y_s - \pi_s}\left(\frac{\pi_s}{1-\sigma} - \frac{Y_s}{1-\rho}\right)\right).$$

The first inequality holds under condition (131), which is proved as follows,

$$\begin{aligned} NS(\nu) =& \ln \nu + (1-\rho\nu)\ln \pi_s - R(\nu) \\ & > \ln \nu + \left(1-\rho\hat{k}_0\right)\ln \pi_s - R\left(\hat{k}_0\right) = 0 \\ \Leftrightarrow \left(\frac{\pi_s}{2-\rho}\right)^{1-\rho\nu} & > (1-\rho)^{\rho\nu} \\ \Leftrightarrow \left(\frac{\pi_s}{2-\rho}\right)^{\frac{1-\rho\nu}{\rho\nu}} & > 1-\rho \end{aligned}$$

where the last line is from $\pi_s > 2$ and $\rho v \in (0,1)$.

$$NS\left(\frac{\nu}{Y_s - \pi_s}\left(\frac{\pi_s}{1-\sigma} - \frac{Y_s}{1-\rho}\right)\right) < 0 \Rightarrow \left(1 + \frac{(1-\rho)\,\bar{d}}{\nu}\right)\left(\frac{(1-\rho)\,\bar{d}}{\nu}\right)^{\frac{\rho\bar{d}}{1-\rho\bar{d}}} > \pi_s,$$

where $\bar{d} \doteq \frac{\nu}{Y_s-\pi_s}\left(\frac{\pi_s}{1-\sigma} - \frac{Y_s}{1-\rho}\right)$.

To sum up, under conditions (130) and (131), the following conditions are well-defined and sufficient for $\hat{k}_0 > \hat{k}_1$,

$$\begin{aligned} \left(\frac{\pi_s}{2-\rho}\right)^{1-\rho\nu} & > (1-\rho)^{\rho\nu} \\ \left(\frac{(1-\rho)\,\bar{d}}{\nu}\right)^{\frac{\rho\bar{d}}{1-\rho\bar{d}}} & > \pi_s\left(1 + \frac{(1-\rho)\,\bar{d}}{\nu}\right), \end{aligned}$$

where $\bar{d} \doteq \frac{\nu}{Y_s-\pi_s}\left(\frac{\pi_s}{1-\sigma} - \frac{Y_s}{1-\rho}\right)$. □

**Proof of Proposition 11.**

We take the ratio $\frac{\hat{e}_0}{\hat{e}_1}$ and decompose it into three terms. We analyze three terms in three steps.
**Step 1:** From previous results, we know that the equilibrium efforts are characterized by

$$\hat{e}_0 = A_p^{\frac{1}{\rho}}(1-\rho)^{\frac{1}{\rho}-1}(\rho\nu + 1 - \rho)\,\rho\frac{c_x^{1-\frac{1}{\rho}}}{c_e}\left[\ln\left(\frac{(1-\rho)\,\pi_\xi}{\frac{\nu}{\hat{k}_0} + (1-\rho)}\right)\right]^{\frac{1}{\rho}}\hat{k}_0, \tag{143}$$

$$\hat{e}_1 = A_q^{\frac{1}{\sigma}}(1-\sigma)^{\frac{1}{\sigma}-1}(\sigma\nu + 1 - \sigma)\,\sigma\frac{c_x^{1-\frac{1}{\sigma}}}{c_e}\left[\ln\left(\frac{(1-\sigma)\,Y_\xi}{\frac{\nu}{\hat{k}_1} + (1-\sigma)}\right)\right]^{\frac{1}{\sigma}}\hat{k}_1. \tag{144}$$

The ratio between $\hat{e}_0$ and $\hat{e}_1$ is

$$\frac{\hat{e}_0}{\hat{e}_1} = \frac{B_\pi}{B_g} \frac{\left[\ln\left(\frac{(1-\rho)\pi_\xi}{\frac{\nu}{\hat{k}_0}+(1-\rho)}\right)\right]^{\frac{1}{\rho}}}{\left[\ln\left(\frac{(1-\rho)\pi_\xi}{\frac{\nu}{\hat{k}_0}+(1-\rho)}\right)\right]^{\frac{1}{\sigma}}} \frac{\hat{k}_0}{\hat{k}_1}.$$

From Proposition (10), $\frac{\hat{k}_0}{\hat{k}_1} > 1$, where $\hat{k}_0$ and $\hat{k}_1$ are solutions of equation (136) and (137), respectively.

**Step 2:** show that $\left[\ln\left(\frac{(1-\rho)\pi_\xi}{\frac{\nu}{\hat{k}_0}+(1-\rho)}\right)\right]^{\frac{1}{\rho}} > \left[\ln\left(\frac{(1-\sigma)Y_\xi}{\frac{\nu}{\hat{k}_1}+(1-\sigma)}\right)\right]^{\frac{1}{\sigma}}$.

From previous results, (139) is equivalent to

$$\frac{\pi_s}{\nu + \hat{k}_0 (1-\rho)} \geq \frac{Y_s}{\nu + \hat{k}_0 (1-\sigma)}.$$

Because $\rho < \sigma$ and $\hat{k}_0 > \hat{k}_1$, the inequality above leads to

$$\begin{aligned} \frac{\pi_s}{\frac{\nu}{\hat{k}_0} + (1-\rho)} &\geq \frac{Y_s}{\frac{\nu}{\hat{k}_0} + (1-\sigma)} \\ \Rightarrow \frac{\pi_s (1-\rho)}{\frac{\nu}{\hat{k}_0} + (1-\rho)} &\geq \frac{Y_s (1-\sigma)}{\frac{\nu}{\hat{k}_0} + (1-\sigma)} \\ \Rightarrow \frac{\pi_s (1-\rho)}{\frac{\nu}{\hat{k}_0} + (1-\rho)} &\geq \frac{Y_s (1-\sigma)}{\frac{\nu}{\hat{k}_1} + (1-\sigma)} \\ \Rightarrow \ln\left(\frac{(1-\rho)\,\pi_\xi}{\frac{\nu}{\hat{k}_0} + (1-\rho)}\right) &\geq \ln\left(\frac{(1-\rho)\,\pi_\xi}{\frac{\nu}{\hat{k}_0} + (1-\rho)}\right) \\ \Rightarrow \left[\ln\left(\frac{(1-\rho)\,\pi_\xi}{\frac{\nu}{\hat{k}_0} + (1-\rho)}\right)\right]^{\frac{1}{\rho}} &> \left[\ln\left(\frac{(1-\rho)\,\pi_\xi}{\frac{\nu}{\hat{k}_0} + (1-\rho)}\right)\right]^{\frac{1}{\sigma}}, \end{aligned}$$

where the first line is from $1-\rho > 1-\sigma$,the second is from $\hat{k}_0 > \hat{k}_1$, the third are due to $\ln(\cdot)$ as increasing function and the last is from $\frac{1}{\rho} > \frac{1}{\sigma}$.

**Step 3:** Show that $\frac{B_\pi}{B_g} \geq 1$ under $\frac{\hat{k}_0}{\hat{k}_1} > 1$ and equation (134), where $B_\pi$and $B_g$ denote the term independent of $k$ in equation (143) and (144), respectively,

$$\begin{aligned} B_\pi &\doteq A_p^{\frac{1}{\rho}} (1-\rho)^{\frac{1}{\rho}-1} (\rho\nu + 1 - \rho)\, \rho \frac{c_x^{1-\frac{1}{\rho}}}{c_e} \\ B_g &\doteq A_q^{\frac{1}{\rho}} (1-\sigma)^{\frac{1}{\rho}-1} (\sigma\nu + 1 - \sigma)\, \sigma \frac{c_x^{1-\frac{1}{\rho}}}{c_e}. \end{aligned}$$

We take the ratio $\frac{B_\pi}{B_g}$ as given

$$\frac{B_\pi}{B_g} = c_x^{\frac{1}{\sigma}-\frac{1}{\rho}} \frac{A_p^{\frac{1}{\rho}}}{A_q^{\frac{1}{\sigma}}} \left[ \frac{\rho\left(\frac{\rho\nu}{1-\rho}+1\right)}{\sigma\left(\frac{\sigma\nu}{1-\sigma}+1\right)} \frac{(1-\rho)^{\frac{1}{\rho}}}{(1-\sigma)^{\frac{1}{\sigma}}} \right].$$

Below we show that $c_x^{\frac{1}{\sigma}-\frac{1}{\rho}} \frac{A_p^{\frac{1}{\rho}}}{A_q^{\frac{1}{\sigma}}} > 1$ under condition (134).

$$\begin{aligned} c_x^{\frac{1}{\sigma}-\frac{1}{\rho}} \frac{A_p^{\frac{1}{\rho}}}{A_q^{\frac{1}{\sigma}}} &= A_q^{\frac{1}{\rho}-\frac{1}{\sigma}} \left(\frac{A_p}{A_q}\right)^{\frac{1}{\rho}} c_x^{\frac{1}{\sigma}-\frac{1}{\rho}}, \\ &= \left(\frac{A_q}{c_x}\right)^{\frac{1}{\rho}-\frac{1}{\sigma}} \left(\frac{A_p}{A_q}\right)^{\frac{1}{\rho}}, \\ &\propto \left(\frac{A_q}{c_x}\right)^{1-\frac{\rho}{\sigma}} \frac{A_p}{A_q} > 1, \end{aligned} \tag{145}$$

where the last inequality is from $\frac{A_q}{A_p} < \left(\frac{A_q}{c_x}\right)^{1-\frac{\rho}{\sigma}}$

The inequality above combines with (145) leads to

$$\frac{B_\pi}{B_g} = c_x^{\frac{1}{\sigma}-\frac{1}{\rho}} \frac{A_p^{\frac{1}{\rho}}}{A_q^{\frac{1}{\sigma}}} \left[ \frac{\rho\left(\frac{\rho\nu}{1-\rho}+1\right)}{\sigma\left(\frac{\sigma\nu}{1-\sigma}+1\right)} \frac{(1-\rho)^{\frac{1}{\rho}}}{(1-\sigma)^{\frac{1}{\sigma}}} \right] > 1,$$

i.e., $B_\pi > B_g$. Combining the three steps, $\frac{\hat{e}_0}{\hat{e}_1} > 1$. □